PERIOD FUNCTIONS FOR MAASS WAVE FORMS AND COHOMOLOGY

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ABSTRACT. We construct explicit isomorphisms between spaces of Maass wave forms and cohomology groups for discrete cofinite groups $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$.

In the case that Γ is the modular group $\mathrm{PSL}_2(\mathbb{Z})$ this gives a cohomological framework for the results in *Period functions for Maass wave forms. I,* of J. LEWIS and D. ZAGIER in Ann. Math. **153** (2001), 191–258, where a bijection was given between cuspidal Maass forms and period functions.

We introduce the concepts of *mixed parabolic cohomology group* and *semi-analytic vectors* in principal series representation. This enables us to describe cohomology groups isomorphic to spaces of Maass cusp forms, spaces spanned by residues of Eisenstein series, and spaces of all Γ-invariant eigenfunctions of the Laplace operator.

For spaces of Maass cusp forms we also describe isomorphisms to parabolic cohomology groups with smooth coefficients and standard cohomology groups with distribution coefficients. We use the latter correspondence to relate the Petersson scalar product to the cup product in cohomology.

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Introduction

These notes proceed from the ideas and results of [19], in which Maass forms for the full modular group were studied, but now treating arbitrary Fuchsian groups and stressing the cohomological interpretation. They can be read independently of [19]. The principal goal is to construct explicit isomorphisms between on the one hand spaces of Maass wave forms on discrete cofinite groups $\Gamma \subset G := \mathrm{PSL}_2(\mathbb{R})$ and on the other appropriate cohomology groups of Γ.

Recall that a *Maass wave form* (or simply *Maass form*) on Γ is a Γ-invariant function on $\mathfrak{H}$ satisfying $\Delta u = \lambda u$ for some $\lambda \in \mathbb{C}$, with polynomial growth. Here $\mathfrak{H}$ is the complex upper half-plane with the usual action of G and Δ is the hyperbolic Laplace operator $\Delta = -y^2(\partial_x^2 + \partial_y^2)$. The Maass wave forms which are small at the cusps (this is relevant only for $\Gamma\backslash\mathfrak{H}$ non-compact) we call *Maass cusp forms*.[1] The eigenvalue λ is most naturally written as $s(1-s)$ for some $s \in \mathbb{C}$ (*spectral parameter*), and our cohomological description of Maass wave forms will depend on picking one of the two roots of this equation. We assume throughout that $0 < \mathrm{Re}\,(s) < 1$.

[1] In [19] the word “wave form“ was used for cusp forms only.

In [19] we studied the case of the full modular group $\Gamma_1 := \mathrm{PSL}_2(\mathbb{Z})$ in detail and showed that the Maass cusp forms with eigenvalue $s(1-s)$ are canonically in one-to-one correspondence with the real-analytic functions $\psi : (0,\infty) \to \mathbb{C}$ which satisfy

$$\psi(x) \;=\; \psi(x+1) \;+\; (x+1)^{-2s}\,\psi\Bigl(\frac{x}{x+1}\Bigr) \qquad (x>0) \tag{1}$$

and for which both $\psi(x)$ and $x^{2s}\psi(x)$ are bounded. It turns out that any such function can be written (non-uniquely) as

$$\psi(x) \;=\; h(x) \;-\; x^{-2s}\,h(-1/x) \qquad (x>0) \tag{2}$$

for some real-analytic function $h : \mathbb{R} \to \mathbb{C}$ and that, when we do this, the map

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;\mapsto\; 0\,, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \;\mapsto\; \bigl(x \mapsto h(x+1) - h(x)\bigr)\,, \tag{3}$$

extends to a cocycle on Γ_1 with values in the analytic vectors $\mathcal{V}_s^\omega$ of a model of the principal series representation $\mathcal{V}_s$. Changing the choice of h changes this cocycle by a coboundary, and we get an isomorphism between the space $\mathsf{Maass}_s^0(\Gamma_1)$ of Maass cusp forms on Γ_1 and a specific subspace of $H^1(\Gamma_1;\mathcal{V}_s^\omega)$. Our goal in these notes is to give an analogous result for all Γ.

To achieve this, we will use several descriptions of the principal series: The model indicated above consists of functions on the boundary $\mathbb{P}^1_{\mathbb{R}}$ of the upper half-plane. We shall also use models of the principal series in functions on $\mathfrak{H}$ itself. The relevant investigations led to the paper [4]. We shall recapitulate the results we need in Sections 2 and 3. Specifically, the well known *Poisson transformation* realizes the principal series representation with spectral parameter s as the space $\mathcal{E}_s$ of all solutions on $\mathfrak{H}$ of the differential equation $\Delta u = s(1-s)u$. For the construction of the map from cohomology to Maass forms, we introduce a *transverse Poisson transformation* $\mathsf{P}_s^\dagger$, which provides us with a model of the principal series in a space of solutions of $\Delta u = s(1-s)u$ near the boundary of $\mathfrak{H}$ in $\mathbb{P}^1_{\mathbb{C}}$. Both Poisson transforms are given by integration against the kernel function $R(\,\cdot\,;z)^{1-s}$, where $R(t;z) = \operatorname{Im}(z)/(t-z)(t-\bar z)$, the integration being over $\mathbb{P}^1_{\mathbb{R}}$ for the usual Poisson transformation, and from z to $\bar z$ for the transverse one. We also need the inverse of the Poisson transform. It can be given explicitly by integration of the differential form $[u, R(t;\,\cdot\,)^s]$, where $[u,v] = \frac{\partial u}{\partial z} v\,dz + u\frac{\partial v}{\partial \bar z}\,d\bar z$ is the *Green's form*, already used in [19], which is closed if u and v are eigenfunctions of Δ with the same eigenvalue.

These facts are reviewed in Chapter I. See §1.3 for the Green's form and Section 2 for the principal series. The Poisson transformation is recalled in §2.2, and the transverse Poisson transformation is defined in §3.2.

In Chapter II we suppose that the discrete subgroup $\Gamma \subset G$ is cocompact. Here, $\mathsf{Maass}_s^0(\Gamma)$ is just the space $\mathcal{E}_s^\Gamma$ of all Γ-invariant solutions of $\Delta u = s(1-s)u$. Our first main result relates it to cohomology groups with values in the spaces of analytic, infinitely-often, and finitely-often differentiable functions in $\mathcal{V}_s$:

Theorem A. *For cocompact $\Gamma \subset G$ and $s \in \mathbb{C}$, $0 < \operatorname{Re} s < 1$, the space $\mathcal{E}_s^\Gamma$ is canonically isomorphic to the cohomology groups $H^1(\Gamma;\mathcal{V}_s^\omega)$, $H^1(\Gamma;\mathcal{V}_s^\infty)$, and $H^1(\Gamma;\mathcal{V}_s^p)$ for $p \in \mathbb{N}$, $p \geq 2$.*

To describe this isomorphism we associate to a given Maass form u the analytic cocycle

$$r_\gamma(t) \;=\; \int_{\gamma^{-1}z_0}^{z_0} [u, R(t;\,\cdot\,)^s]\ , \tag{4}$$

depending on a base point $z_0 \in \mathfrak{H}$. *In the other direction, the value of the Maass form* $u(z)$ *associated to a given analytic cocycle* $\{\varphi_\gamma\}$ *is given in any compact subset of* $\mathfrak{H}$ *by an explicit finite sum of terms of the form* $\mathsf{P}_s^\dagger(\varphi_\gamma)\,|\,\gamma'$ *with* γ, $\gamma' \in \Gamma$.

Bunke and Olbrich, [6], [7], proved that $\mathcal{E}_s^\Gamma \cong H^1(\Gamma;\mathcal{V}_s^\omega) \cong H^1(\Gamma;\mathcal{V}_s^\infty)$ in a more general setting (for automorphic forms on rank 1 symmetric spaces and torsion-free discrete cocompact groups). Our approach is more concrete and gives the isomorphism $\mathcal{E}_s^\Gamma \cong H^1(\Gamma;\mathcal{V}_s^\omega)$ much more explicitly. The integral in (4) gives the map from Maass forms to cohomology. For the map from cohomology to Maass forms, the starting point is the model $\mathcal{W}_s^\omega$ of the principal series in the solutions of $\Delta u = s(1-s)u$ near the boundary. We use a space $\mathcal{G}_s^\omega$ of functions on the whole of $\mathfrak{H}$ such that $\mathcal{G}_s^\omega \to \mathcal{W}_s^\omega$ is surjective. The kernel $\mathcal{N}^\omega$ of this morphism consists of compactly supported functions. A 1-cocycle on Γ with values in $\mathcal{W}_s^\omega$ gives rise to a 2-cochain with values in $\mathcal{N}^\omega$. Evaluation of this 2-cochain on a 2-cycle that represents the fundamental class in $H_2(\Gamma;\mathbb{Z})$ provides us with an element $f_0 \in \mathcal{N}^\omega$, which is unique up to linear combinations of the form $f\,|\,(1-\gamma)$ with $f \in \mathcal{N}^\omega$ and $\gamma \in \Gamma$. The locally finite sum $u(z) = \sum_{\gamma\in\Gamma} f_0(\gamma z)$ is independent of all choices, and is the Maass form we looked for.

The construction of maps in both directions between $\mathcal{E}_s^\Gamma$ and $H^1(\Gamma;\mathcal{V}_s^\omega)$ is the main result of Chapter II.

Chapter III presents results concerning $H^1(\Delta;\mathcal{V}_s^\omega)$ and related cohomology groups for infinite cyclic subgroups $\Delta \subset \mathrm{PSL}_2(\mathbb{R})$. It turns out, for instance, that the restriction map from $H^1(\Gamma;\mathcal{V}_s^\omega)$ to $H^1(\Delta;\mathcal{V}_s^\omega)$ in the theorem above, where Δ is the subgroup generated by any element $\gamma_0 \in \Gamma$ of infinite order, is injective, so that a Maass wave form $u \in \mathcal{E}_s^\Gamma$ can be reconstructed from the single element $r_{\gamma_0} \in \mathcal{V}_s^\omega$, without knowing the rest of the cocycle. The results for the case that Δ is generated by a parabolic (rather than a hyperbolic) element of $\mathrm{PSL}_2(\mathbb{R})$ are used in the following chapter.

In Chapter IV we consider groups Γ with cusps. Here the spaces $\mathsf{Maass}_s^0(\Gamma) \subset \mathsf{Maass}_s(\Gamma) \subset \mathcal{E}_s^\Gamma$ of, respectively, Maass cusp forms, Maass forms (at most polynomial growth), and arbitrary invariant eigenfunctions, are all different. The dimension of $\mathcal{E}_s^\Gamma$ is infinite, while the other two are finite dimensional. The approach used for cocompact groups has to be modified in several ways, as follows.

We have to look at more general cohomology groups. For Γ-modules V, the *parabolic cohomology group* $H^1_{\mathrm{par}}(\Gamma;V) \subset H^1(\Gamma;V)$ is given by cocycles $\{\psi_\gamma\}$ that are of the form $\psi_\pi = a_\pi\,|\,\pi - a_\pi$ for all parabolic $\pi \in \Gamma$, with $a_\pi \in V$. For the *mixed parabolic cohomology group* $H^1_{\mathrm{par}}(\Gamma;V,W)$ the cocycle $\{\psi_\gamma\}$ has values in V and the a_π are in a Γ-module $W \supset V$.

The example of the period functions for $\Gamma_1 = \mathrm{PSL}_2(\mathbb{Z})$ leads us to the space $\mathcal{V}_s^{\omega^*,\infty}$ of "semi-analytic vectors" in the principal series. This is a Γ-module satisfying $\mathcal{V}_s^\omega \subset \mathcal{V}_s^{\omega^*,\infty} \subset \mathcal{V}_s^\infty$, consisting, in the standard model of the principal series representation (functions on $\mathbb{P}^1_\mathbb{R}$), of smooth (C^∞) functions on $\mathbb{P}^1_\mathbb{R}$ that are real-analytic except for finitely many points.

With these modifications one has the following analogue of Theorem A for cusp forms on non-cocompact groups:

Theorem B. *For cofinite discrete subgroups* $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ *and* $0 < \operatorname{Re} s < 1$*, the spaces* $\mathsf{Maass}_s^0(\Gamma)$*,* $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\infty})$*,* $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*,\infty})$*,* $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\infty)$*, and* $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^p)$ *with* $p \in \mathbb{N}$*,* $p \geq 3$*, are canonically isomorphic. The relation between Maass cusp forms and the associated analytic cocycle has the same structure as in Theorem A.*

• *Example.* In the case $\Gamma_1 = \mathrm{PSL}_2(\mathbb{Z})$, the cocycle determined by (3) represents a class in the mixed parabolic cohomology group $H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\infty})$: its values are in $\mathcal{V}_s^\omega$, and its value on the parabolic generator $T = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_1$ is of the form $h|T - h$, with $h \in \mathcal{V}_s^{\omega^*,\infty}$. The period function ψ, on the other hand, is related to a class in $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*,\infty})$. It determines a cocycle with values in $\mathcal{V}_s^{\omega^*,\infty}$, defined on the standard generators by

$$\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{cases} \psi(x) & \text{if } x > 0\,, \\ -|x|^{-2s}\psi(-1/x) & \text{if } x < 0\,; \end{cases} \qquad \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto 0\,. \tag{5}$$

This cocycle vanishes on the parabolic element $\pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Finally, the last isomorphism in Theorem B, applied to the modular group, implies that any C^∞-function on $(0, \infty)$ satisfying (1) and the growth conditions given there is in fact real-analytic, giving a strengthening of the main result of [19].

We prove most of the isomorphisms in Theorem B in Chapter IV. The isomorphism with $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\infty)$ and $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^p)$ is established in Chapter V.

If $s \neq \frac{1}{2}$, the correspondence between Maass forms and cohomology classes in the Theorems A and B can be extended to the whole of $\mathcal{E}_s^\Gamma$. To do this, we introduce two further spaces $\mathcal{V}_s^{\omega^*,\mathrm{exc}} \supset \mathcal{V}_s^\omega$ and $\mathcal{V}_s^{\omega^0,\mathrm{exc}} \supset \mathcal{V}_s^\omega$. The first consists of functions on $\mathbb{P}^1_{\mathbb{R}}$ which are real analytic except for finitely many points and have singularities of a special type (Definition 9.17) at these points, and the second is the same except that the finitely many singularities must all be at cusps. Then we have:

Theorem C. *For cofinite discrete subgroups* $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ *and* $0 < \operatorname{Re} s < 1$*,* $s \neq \frac{1}{2}$*, the spaces* $\mathcal{E}_s^\Gamma$*,* $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{exc}})$ *and* $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^0,\mathrm{exc}})$ *are canonically isomorphic.*

For the modular group we show in Proposition 14.1 how $H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{\omega^0,\mathrm{exc}})$ can be described as a quotient of the space of all holomorphic functions on $\mathbb{C} \smallsetminus (-\infty, 0]$ that satisfy the three term equation (1). In Proposition 14.3 we show that the mixed parabolic cohomology group $H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{exc}})$ is a genuine subspace of $H^1(\Gamma_1; \mathcal{V}_s^\omega)$. We discuss briefly a notion of "quantum Maass forms" which provides us with a space of objects with a modular flavor that corresponds bijectively with $H^1(\Gamma_1; \mathcal{V}_s^\omega)$.

If Γ has no cusps then parabolic cohomology is standard cohomology. Theorems B and C give no more information than Theorem A in the cocompact case.

Bunke and Olbrich have shown that the space $\mathsf{Maass}_s^0(\Gamma)$ (which is equal to $\mathcal{E}_s^\Gamma$ for cocompact Γ) is isomorphic to $H^1(\Gamma; \mathcal{V}_s^{-\infty})$, where $\mathcal{V}_s^{-\infty}$ denotes the space of distribution vectors in $\mathcal{V}_s$. In Chapter VI we give an explicit realization of the isomorphism on the cocycle level and use it to express the Petersson scalar product in cohomological terms (Theorem 19.1).

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• *Notations and conventions.* We work with the standing assumption that $s \in \mathbb{C}$ satisfies $0 < \operatorname{Re} s < 1$, and use $\lambda_s = s(1-s)$. By $\mathbb{N}$ we denote the set $\{n \in \mathbb{Z} : n \geq 0\}$.

We denote by G the Lie group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm\mathrm{Id}\}$, and denote by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the element $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of G. We shall use various right representations of G, and use $g : v \mapsto v\,|\,g$ as a general notation for the action of G in a right G-module.

Chapter I. Eigenfunctions of the hyperbolic Laplace operator

This chapter has a preliminary character. It discusses concepts and results needed in the next chapters. In Sections 1–3 we recall results concerning eigenfunctions of the Laplace operator and principal series representations that we treat in more detail in [4]. The averaging operators in Section 4 form another important tool used in these notes.

1. Eigenfunctions on the hyperbolic plane. Maass forms are functions on the hyperbolic plane that satisfy $\Delta u = \lambda_s u$ and are invariant under a group of transformations. We define in this subsection the space of all such eigenfunctions of the Laplace operator and introduce several related spaces. An important result is Theorem 1.1, which plays for eigenfunctions of Δ the role of Cauchy's theorem for holomorphic functions.

1.1. *The hyperbolic plane.* By $\mathbb{H}$ we denote the hyperbolic plane. We use two realizations as a subset of $\mathbb{P}^1_{\mathbb{C}}$. The first is the upper half-plane model $\mathfrak{H} = \{z = x+iy : y > 0\}$, the other the disk model $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$. In the upper half-plane model, geodesics are Euclidean vertical half-lines and Euclidean half-circles with their center on the real axis. In the disk model, geodesics are given by Euclidean circles intersecting the boundary $\partial\mathbb{D} = \mathbb{S}^1 = \{\xi \in \mathbb{C} : |\xi| = 1\}$ orthogonally and Euclidean lines through 0. The real projective line $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the boundary of the upper half-plane. See Table 1 for a further comparison between both models.

• *The space of eigenfunctions.* By $\mathcal{E}_s$ we denote the space of solutions of

$$\Delta u = \lambda_s\, u \text{ in } \mathbb{H}, \qquad \lambda_s = s(1-s)\,. \tag{1.1}$$

The Laplace operator $\Delta = -y^2\partial_x^2 - y^2\partial_y^2$ is an elliptic differential operator with real-analytic coefficients. Hence all elements of $\mathcal{E}_s$ are real-analytic functions. This operator commutes with the action of the group G (on the right) given by

$$(u\,|\,g)(z) = u(gz)\,.$$

(We will use z to denote the coordinate in both $\mathfrak{H}$ and $\mathbb{D}$ when we make statements applying to both models of $\mathbb{H}$.) Obviously, $\mathcal{E}_s = \mathcal{E}_{1-s}$. If U is an open subset of $\mathbb{H}$, we denote by $\mathcal{E}_s(U)$ the space of solutions of $\Delta u = \lambda_s u$ on U, thus defining $\mathcal{E}_s$ as a sheaf on $\mathbb{H}$. We will refer to elements of $\mathcal{E}_s = \mathcal{E}_s(\mathbb{H})$ and of $\mathcal{E}_s(U)$ as *λ_s-eigenfunctions of* Δ.

So $\mathcal{E}_s$ denotes a sheaf as well as the space of global sections of that sheaf. For other sheaves we will allow ourselves a similar ambiguity.

model of $\mathbb{H}$	$\mathfrak{H}$	$\mathbb{D}$
coordinate	$z = x + iy = i\frac{1+w}{1-w}$	$w = \frac{z-i}{z+i}$
Laplace operator Δ	$-y^2(\partial_x^2 + \partial_y^2)$ $= (z-\bar{z})^2\partial_z\partial_{\bar{z}}$	$-(1-\|w\|^2)^2\partial_w\partial_{\bar{w}}$
infinitesimal distance	$y^{-1}\sqrt{(dx)^2 + (dy)^2}$	$\frac{2\sqrt{(d\,\mathrm{Re}\,w)^2+(d\,\mathrm{Im}\,w)^2}}{1-\|w\|^2}$
hyperbolic distanced $\rho(\,\cdot\,,\,\cdot\,) = \cosh(\mathrm{d}(\,\cdot\,,\,\cdot\,))$	$1 + \frac{\|z-z'\|^2}{2yy'}$	$1 + \frac{2\|w-w'\|^2}{(1-\|w\|^2)(1-\|w'\|^2)}$
volume element $d\mu$	$\frac{dx\,dy}{y^2}$	$\frac{4\,d\,\mathrm{Re}\,w\;d\,\mathrm{Im}\,w}{(1-\|w\|^2)^2}$
orientation preserving isometry group $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & -i \\ 1 & i\end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & -i \\ 1 & i\end{smallmatrix}\right]^{-1}$	$G = \mathrm{PSL}_2(\mathbb{R})$ $\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] : z \mapsto \frac{az+b}{cz+d}$	$G \cong \mathrm{PSU}(1,1)$ $\left[\begin{smallmatrix} A & B \\ \bar{B} & \bar{A}\end{smallmatrix}\right] : w \mapsto \frac{Aw+B}{\bar{B}w+\bar{A}}$
maximal compact subgroup K point fixed by K	PSO(2) i	PSU(1) 0
boundary $\partial\mathbb{H}$	$\partial\mathfrak{H} = \mathbb{P}^1_{\mathbb{R}}$	$\partial\mathbb{D} = \mathbb{S}^1$

TABLE 1. Upper half-plane model and disk model of the hyperbolic plane $\mathbb{H}$.

1.2. *Examples.* The functions $i_{s,0}(z) = y^s$ on $\mathfrak{H}$ is an element of $\mathcal{E}_s$ that occurs in the constant term of Fourier expansions of Maass forms. That term is a linear combination of $i_{s,0}$ and $i_{1-s,0}$, or of $i_{1/2,0}$ and $\ell_{1/2,0}(z) = y^{1/2}\ln y$ if $s = \frac{1}{2}$. This function $\ell_{1/2,0}$ is the value at $s = \frac{1}{2}$ of the family $\ell_{s,0} = \frac{1}{2s-1}\,(i_{s,0} - i_{1-s,0})$ of N-invariant elements of $\mathcal{E}_s$. The other terms of those Fourier expansions may contain the following elements of $\mathcal{E}_s$:

$$\begin{aligned} k_{s,\alpha}(z) &= \sqrt{y}\,K_{s-1/2}(|\alpha|y)\,e^{i\alpha x}\,, \\ i_{s,\alpha}(z) &= \frac{\Gamma(s+\frac{1}{2})}{|\alpha/2|^{s-1/2}}\,\sqrt{y}\,I_{s-1/2}(|\alpha|y)\,e^{i\alpha x}\,, \end{aligned} \tag{1.2}$$

for $\alpha \in \mathbb{R} \smallsetminus \{0\}$, with the modified Bessel functions $I_\mu(\,\cdot\,)$ and $K_\mu(\,\cdot\,)$. These functions on $\mathfrak{H}$ transform according to the character $\left[\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right] \mapsto e^{i\alpha x}$ of $N = \{\left[\begin{smallmatrix} 1 & * \\ 0 & 1\end{smallmatrix}\right]\} \subset G$.

The group $K = \mathrm{PSO}(2) \subset G$ has characters $\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{smallmatrix}\right] \mapsto e^{2in\theta}$ with $n \in \mathbb{Z}$. Functions transforming according to such a character are easiest described in the disk model, with Legendre functions or with hypergeometric functions:

$$\begin{aligned} P_{s,n}(re^{i\theta}) &= P^n_{s-1}\Big(\frac{1+r^2}{1-r^2}\Big)\,e^{in\theta} \\ &= \frac{\Gamma(s+n)}{|n|!\;\Gamma(s-|n|)}\,r^{|n|}\,{}_2F_1\Big(1-s, s; 1+|n|; \frac{r^2}{r^2-1}\Big)\,, \end{aligned} \tag{1.3a}$$

$$\begin{aligned} Q_{s,n}(re^{i\theta}) &= Q^n_{s-1}\Bigl(\frac{1+r^2}{1-r^2}\Bigr)e^{in\theta} \\ &= \frac{(-1)^n}{2}\,\frac{\Gamma(s)\,\Gamma(s+n)}{\Gamma(2s)}\,r^{-n}\,(1-r^2)^s\,{}_2F_1(s-n,s;2s;1-r^2)\,. \end{aligned} \tag{1.3b}$$

(Note the shift in the spectral parameter in P^n_{s-1} and $P_{s,n}$.) We have $P_{s,n}=P_{1-s,n}\in\mathcal{E}_s$ and $Q_{s,n}\in\mathcal{E}_s(\mathbb{D}\smallsetminus\{0\})$.

Section A.1.3 in the appendix of [4] gives also formulas for elements of $\mathcal{E}_s$ that transform according to a character of the group $A=\{\bigl[\begin{smallmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{smallmatrix}\bigr] \;:\; y>0\}\subset G$. Of these functions we will use $f^R_{s,\alpha}$ and $f^L_{s,\alpha}$. See (4.2).

Starting from the λ_s-eigenfunctions of Δ considered up till now, we can produce other ones by translating them. If $g\cdot 0=w'$ for $g\in G$ and $w'\in\mathbb{D}$, then $r=|g^{-1}w|$ satisfies $\frac{1+r^2}{1-r^2}=\rho(w,w')$, with ρ as in Table 1. The functions

$$\begin{aligned} p_s(w,w') &= P_{s,0}(g^{-1}w) = P^0_{s-1}(\rho(w,w'))\,, \\ q_s(w,w') &= Q_{s,0}(g^{-1}w) = Q^0_{s-1}(\rho(w,w'))\,, \end{aligned} \tag{1.4}$$

are *point-pair invariants*, i.e., they depend only on the hyperbolic distance between w and w'. Hence they are symmetric in w and w', and satisfy for all $g\in G$:

$$p_s(gz,gz')=p_s(z,z')\,,\qquad q_s(gz,gz')=q_s(z,z')\,. \tag{1.5}$$

They are λ_s-eigenfunctions of Δ in both variables. One calls q_s the *free-space resolvent kernel*. We have $p_s(\,\cdot\,,w')\in\mathcal{E}_s$, and $q_s(\,\cdot\,,w')\in\mathcal{E}_s(\mathbb{D}\smallsetminus\{w'\})$ for fixed $w'\in\mathbb{D}$.

Shifting $i_{s,0}:z\mapsto y^s$ by $\bigl[\begin{smallmatrix}0&1\\-1&t\end{smallmatrix}\bigr]$, with $t\in\mathbb{R}$, gives

$$R(t;z)^s = \Bigl(\mathrm{Im}\,\frac{1}{t-z}\Bigr)^s = \frac{y^s}{|t-z|^{2s}}\,. \tag{1.6}$$

We have $R(t;\,\cdot\,)^s\in\mathcal{E}_s$. Moving t off the real line gives rise to a multivalued function

$$R(\zeta;z)^s = \frac{y^s}{(\zeta-z)^s(\zeta-\bar z)^s}\,. \tag{1.7}$$

For $\zeta\in\mathbb{C}$ it is an element of $\mathcal{E}_s(U)$ for simply connected $U\subset\mathfrak{H}$ not containing ζ or $\bar\zeta$. We choose the branch such that $\arg(\zeta-z)+\arg(\zeta-\bar z)=0$ for $\zeta\in\mathbb{R}$.

For $\alpha\in\mathbb{R}$ and $\mathrm{Re}\,s>\frac12$, we can integrate $e^{i\alpha t}R(t;z)^s$ over $\mathbb{R}$ to obtain

$$e^{i\alpha x}\int_{-\infty}^{\infty} e^{i\alpha t}\frac{y^s\,dt}{(y^2+t^2)^s} = \begin{cases} \sqrt{\pi}\,\frac{\Gamma(s-\frac12)}{\Gamma(s)}\,i_{1-s,0}(z) & (\alpha=0)\,,\\ \sqrt{\pi}\,2^{3/2-s}\,\frac{|\alpha|^{s-1/2}}{\Gamma(s)}\,k_{s,\alpha}(z) & (\alpha\neq 0)\,. \end{cases} \tag{1.8}$$

This continues meromorphically, holomorphically in s if $\alpha\neq 0$, and having in the region $\mathrm{Re}\,s>0$ only a first order singularity at $s=\frac12$ if $\alpha=0$.

1.3. *Green's form.* We recall the bracket operation from [19], already mentioned in the introduction. There are two versions, which differ by an exact form:

$$[u,v] = u_z\,v\,dz + u\,v_{\bar z}\,d\bar z\,,\qquad \{u,v\} = 2i\,[u,v]-i\,d(uv)\,. \tag{1.9}$$

These formulas make sense in both the upper half-plane and the disk model of $\mathbb{H}$, and have the properties

$$[u\circ g,v\circ g] = [u,v]\circ g \quad \text{for each } g\in G\,, \tag{1.10a}$$

$$[u,v]+[v,u] = d(uv)\,, \tag{1.10b}$$

$$d[u,v] \;=\; \frac{1}{2i}(u\,\Delta v - v\,\Delta u)\,d\mu\,, \tag{1.10c}$$

$$\{v,u\} \;=\; -\{u,v\}\,. \tag{1.10d}$$

So $[u,v]$ is a closed 1-form on U if $u,v\in\mathcal{E}_s(U)$ for an open $U\subset\mathbb{H}$.

The bracket with q_s gives for elements of $\mathcal{E}_s$ a substitute for Cauchy's theorem:

Theorem 1.1. *Let C be a piecewise smooth simple closed curve in $\mathbb{H}$ and u an element of $\mathcal{E}_s(U)$, where $U\subset\mathbb{H}$ is some open set containing C and its interior. Then for $w\in\mathbb{H}\smallsetminus C$ we have*

$$\int_C [u, q_s(\,\cdot\,,w)] \;=\; \begin{cases} \pi i\,u(w) & \textit{if } w \textit{ is inside } C,\\ 0 & \textit{if } w \textit{ is outside } C,\end{cases} \tag{1.11}$$

where the curve C is traversed in the positive direction.

See Theorem 2.1 in [4].

2. Principal series. All the coefficient modules used in the cohomology groups mentioned in the introduction are spaces of vectors in the principal series representation associated to the spectral parameter s. The standard realizations of the principal series representation use spaces of functions on the boundary $\partial\mathbb{H}$ of the hyperbolic plane. With the Poisson transform we can also use a realization in $\mathcal{E}_s$.

We write $\mathcal{V}_s$ to denote "the" principal series representation when we do not want to specify precisely the space under consideration. Spaces $\mathcal{V}_s^\infty$ and $\mathcal{V}_s^\omega$ of smooth and analytic vectors are identified with the appropriate superscript.

In [4] we treat the material in this section in more depth. In particular, we study the various models more systematically. Each of the models of $\mathcal{V}_s$ has its advantages and disadvantages.

2.1. *Models of the principal series on the boundary of the hyperbolic plane.* We list some standard models of the principal series.

- *Line model.* In the introduction we already mentioned the well known model of $\mathcal{V}_s$, consisting of functions on $\mathbb{R}$ with the transformation behavior

$$\varphi\big|_{2s}\begin{bmatrix}a&b\\c&d\end{bmatrix}(x) \;=\; |cx+d|^{-2s}\varphi\Big(\frac{ax+b}{cx+d}\Big) \tag{2.1}$$

under $\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\in G$. To get a sensible result at $x=-\frac{d}{c}$, we need to require that φ behaves well as $|x|\to\infty$. By $\mathcal{V}_s^\infty$, the space of *smooth vectors* in $\mathcal{V}_s$ we denote the space of $\varphi\in C^\infty(\mathbb{R})$ that have an expansion

$$\varphi(t) \;\sim\; |t|^{-2s}\sum_{n=0}^{\infty} c_n\,t^{-n} \tag{2.2}$$

as $|t|\to\infty$. Similarly, the space $\mathcal{V}_s^\omega$ of *analytic vectors* consists of the $\varphi\in C^\omega(\mathbb{R})$ (real-analytic functions on $\mathbb{R}$) for which the series appearing on the right-hand side of (2.2) converges to $\varphi(x)$ for $|x|\geq x_0$ for some x_0. Analogously, we define $\mathcal{V}_s^p$, $p\in\mathbb{N}$, as the space of $\varphi\in C^p(\mathbb{R})$ satisfying (2.2) with the asymptotic expansion replaced by a Taylor expansion of order p.

We call this the *line model* of $\mathcal{V}_s$. It is well known and has a simple transformation formula (2.1) that reminds us of the transformation behavior of holomorphic automorphic forms. It has the disadvantages that we need to specify the behavior as $|x|\to\infty$

separately, and that it requires some work to check that the spaces $\mathcal{V}_s^\omega$, $\mathcal{V}_s^\infty$, ... are preserved under the action of G.

Often we shall write $\varphi\,|\,g$ instead of $\varphi|_{2s}\,g$ if there is no danger of confusion.

- *Projective model.* The relation

$$\varphi^{\mathbb{P}}(t) \;=\; (1+t^2)^s\varphi(t) \tag{2.3}$$

gives a model for which $\mathcal{V}_s^\omega$, V_s^∞, and the $\mathcal{V}_s^p$ correspond to respectively $C^\omega(\mathbb{P}^1_{\mathbb{R}})$, $C^\infty(\mathbb{P}^1_{\mathbb{R}})$ and $C^p(\mathbb{P}^1_{\mathbb{R}})$ of respectively real analytic, smooth and p times continuously differentiable functions on $\mathbb{P}^1_{\mathbb{R}}$. The action of G is described by the more complicated formula

$$f\big|^{\mathbb{P}}_{2s}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(t) \;=\; \Bigl(\frac{t^2+1}{(at+b)^2+(ct+d)^2}\Bigr)^s\, f\Bigl(\frac{at+b}{ct+d}\Bigr). \tag{2.4}$$

The factor $\bigl(\frac{t^2+1}{(at+b)^2+(ct+d)^2}\bigr)^s$ is real-analytic on the whole of $\mathbb{P}^1_{\mathbb{R}}$. Thus it is immediately clear that the action of G preserves real-analyticity, smoothness and p times continuous differentiability. A drawback is that the point i, corresponding to the choice of K as maximal compact subgroup, plays a special role. In §1.1, [4], we mention the plane model of the principal series that does not have this drawback.

- *Circle model.* The circle model is directly related to the projective model by the inverse transformations $\xi = \frac{t-i}{t+i}$ and $t = i\frac{1+\xi}{1-\xi}$, in $\mathbb{P}^1_{\mathbb{C}}$, identifying the projective line $\mathbb{P}^1_{\mathbb{R}}$ to the unit circle $\mathbb{S}^1$ in $\mathbb{C}$. This leads to the circle model of $\mathcal{V}_s$, in which the action of $g = \left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$ is described by $\tilde g = \left[\begin{smallmatrix} 1 & -i\\ 1 & i\end{smallmatrix}\right] g \left[\begin{smallmatrix} 1 & -i\\ 1 & i\end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} A & B\\ \bar B & \bar A\end{smallmatrix}\right]$ in $\mathrm{PSU}(1,1) \subset \mathrm{PSL}_2(\mathbb{C})$, with $A = \frac12(a+ib-ic+d)$, $B = \frac12(a-ib-ic-d)$:

$$f\big|^{\mathbb{S}}_{2s}\, g\,(\xi) = \;|A\xi+B|^{-2s} f\Bigl(\frac{A\xi+B}{\bar B\xi+\bar A}\Bigr) \qquad (|\xi|=1)\,. \tag{2.5}$$

The factor $|A\xi+B|$ is non-zero on the unit circle, since $|A|^2-|B|^2 = 1$. The relation with the previous models is given by

$$\varphi^{\mathbb{S}}(e^{-2i\theta}) \;=\; \varphi^{\mathbb{P}}(\cot\theta) \;=\; |\sin\theta|^{-2s}\,\varphi(\cot\theta)\,. \tag{2.6}$$

- *Realization of $\mathcal{V}_s^\omega$ in holomorphic functions.* The restriction of a holomorphic function on a neighborhood of $\mathbb{S}^1$ in $\mathbb{C}$ to $\mathbb{S}^1$ is real-analytic, and since every real-analytic function on $\mathbb{S}^1$ is such a restriction, $C^\omega(\mathbb{S}^1)$ can be identified with the space $\varinjlim \mathcal{O}(U)$, where U in the inductive limit runs over all open neighborhoods of $\mathbb{S}^1$ and where $\mathcal{O}(U)$ denotes the space of holomorphic functions on U. One can rewrite the automorphy factor in (2.5) as $\bigl((\bar A+\bar B\xi)(A+B\xi^{-1})\bigr)^{-s}$, which is holomorphic near $\mathbb{S}^1$. It can be extended to a holomorphic and one-valued function on a neighborhood of $\mathbb{S}^1$ in $\mathbb{P}^1_{\mathbb{C}}$, in fact, outside a path from 0 to $-B/A$ and a path from ∞ to $-\bar A/\bar B$. In other words, in the description of $\mathcal{V}_s^\omega$ as $\varinjlim_U \mathcal{O}(U)$, the action of G becomes

$$\varphi|^{\mathbb{S}}_{2s}\, g\ (w) \;=\; \bigl[(\bar A+\bar B w)(A+B/w)\bigr]^{-s}\,\varphi(\tilde g w)\ . \tag{2.7}$$

In the projective model, we have similar descriptions. Now U runs through neighborhoods of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. The action (2.4) can be rewritten as

$$f|^{\mathbb{P}}_{2s}\begin{bmatrix} a & b\\ c & d\end{bmatrix}(z) \;=\; (a^2+c^2)^{-s}\Bigl(\frac{z-i}{z-g^{-1}(i)}\Bigr)^s\Bigl(\frac{z+i}{z-g^{-1}(-i)}\Bigr)^s f\Bigl(\frac{az+b}{cz+d}\Bigr). \tag{2.8}$$

This unwieldy formula shows that the automorphy factor is holomorphic on $\mathbb{P}^1_{\mathbb{C}}$ minus a path from i to $g^{-1}i$ and a path from $-i$ to $g^{-1}(-i)$.

• *Topology.* We have not yet discussed topologies on the spaces in $\mathcal{V}_s$. For the cohomology groups, we will use $\mathcal{V}_s^\omega$, $\mathcal{V}_s^\infty$, and other spaces in $\mathcal{V}_s$, algebraically.

The natural topology on $\mathcal{V}_s^p$ is given by the finitely many seminorms $\|\varphi\|_j$, $0 \le j \le p$, where

$$\|\varphi\|_j \;=\; \sup_{x\in\partial\mathbb{H}} \left|\varphi|_{2s}\,\mathbf{W}^j(x)\right|, \tag{2.9}$$

and where $\mathbf{W} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ in the Lie algebra of G. By $\varphi \mapsto \varphi|_{2s}\,\mathbf{W}$ we denote the corresponding action in $\mathcal{V}_s$. In the circle model, $\mathbf{W}$ corresponds to the differential operator $2i\,\xi\,\partial_\xi$, in the projective model to $(1+t^2)\,\partial_t$, and in the line model to $(1+x^2)\,\partial_x + 2s\,x$. The natural topology on the space $\mathcal{V}_s^\infty = \bigcap_{p\in\mathbb{N}} \mathcal{V}_s^p$ is given by the collection of all seminorms $\|\cdot\|_p$, $p \in \mathbb{N}$.

The topology on $\mathcal{V}_s^\omega \cong \varinjlim \mathcal{O}(U)$ can be defined as the inductive limit topology given by the supremum norms on the sets U. The inclusion $\mathcal{V}_s^\omega \to \mathcal{V}_s^\infty$ is continuous with dense image. With these topologies, $\mathcal{V}_s^\omega$ and $\mathcal{V}_s^\infty$ are irreducible continuous representations of G. Here the restriction $0 < \operatorname{Re} s < 1$ is essential. Irreducibility does not hold when $s \in \mathbb{Z}$.

• *Hyperfunctions.* We put

$$\mathbf{H}(\mathbb{S}^1) \;=\; \varinjlim_U \mathcal{O}(U \smallsetminus \mathbb{S}^1)\,, \qquad \mathbf{H}(\mathbb{P}^1_{\mathbb{R}}) \;=\; \varinjlim_V \mathcal{O}(V \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \tag{2.10}$$

where U runs over the neighborhoods of $\mathbb{S}^1$ in $\mathbb{C}$ and V over the neighborhoods of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. The spaces $C^{-\omega}(\mathbb{S}^1)$ and $C^{-\omega}(\mathbb{P}^1_{\mathbb{R}})$ of *hyperfunctions* on $\mathbb{S}^1$, respectively $\mathbb{P}^1_{\mathbb{R}}$, are the quotients in the exact sequences

$$\begin{aligned} 0 \longrightarrow C^\omega(\mathbb{S}^1) \longrightarrow \mathbf{H}(\mathbb{S}^1) \longrightarrow C^{-\omega}(\mathbb{S}^1) \longrightarrow 0\,,\\ 0 \longrightarrow C^\omega(\mathbb{P}^1_{\mathbb{R}}) \longrightarrow \mathbf{H}(\mathbb{P}^1_{\mathbb{R}}) \longrightarrow C^{-\omega}(\mathbb{P}^1_{\mathbb{R}}) \longrightarrow 0\,. \end{aligned} \tag{2.11}$$

See, e.g., §1.1 of [22]. Actually, the quotients $\mathcal{O}(U \smallsetminus \mathbb{S}^1)/\mathcal{O}(U)$ and $\mathcal{O}(V \smallsetminus \mathbb{P}^1_{\mathbb{R}})/\mathcal{O}(V)$ do not depend on the choice of U, respectively V, so they give models for $C^{-\omega}(\mathbb{S}^1)$ and $C^{-\omega}(\mathbb{P}^1_{\mathbb{R}})$ for any choice of U, respectively V. Intuitively, a hyperfunction is the jump across $\mathbb{S}^1$, respectively $\mathbb{P}^1_{\mathbb{R}}$.

The actions in (2.7) and (2.8) make sense on the spaces of holomorphic functions in deleted neighborhoods $\mathcal{O}(U \smallsetminus \mathbb{S}^1)$ and $\mathcal{O}(V \smallsetminus \mathbb{P}^1_{\mathbb{R}})$. This gives an action of G on $\mathbf{H}(\mathbb{S}^1)$ and $\mathbf{H}(\mathbb{P}^1_{\mathbb{R}})$, and hence on the corresponding spaces of hyperfunctions. We call $\mathbf{H}_s$ the space $\mathbf{H}$ with this action, in the realizations $\mathbf{H}(\mathbb{S}^1)$ and $\mathbf{H}(\mathbb{P}^1_{\mathbb{R}})$, and $\mathcal{V}_s^{-\omega}$ the resulting representation of G in the hyperfunctions. Thus we have an exact sequence of G-modules

$$0 \longrightarrow \mathcal{V}_s^\omega \longrightarrow \mathbf{H}_s \longrightarrow \mathcal{V}_s^{-\omega} \longrightarrow 0\,, \tag{2.12}$$

realized in the circle model and in the projective model. The line model is inconvenient for hyperfunctions.

We can embed the space $C^\omega(\mathbb{S}^1)$ of analytic functions on $\mathbb{S}^1$ in the following way:

$$\varphi \in \mathcal{O}(U) \quad \mapsto \quad w \mapsto \begin{cases} \varphi(w) & \text{if } w \in U,\ |w| < 1\,,\\ 0 & \text{if } w \in U,\ |w| > 1\,. \end{cases} \tag{2.13}$$

Let $[f] \in C^{-\omega}(\mathbb{S}^1)$ be the hyperfunction represented by $f \in O(U \setminus \mathbb{S}^1)$. Then $[f] \in C^{\omega}(\mathbb{S}^1)$ if and only if the restrictions of f to $U \cap \{|w| < 1\}$ and $U \cap \{|w| > 1\}$ both extend holomorphically across the circle. In the projective model we have a similar embedding.

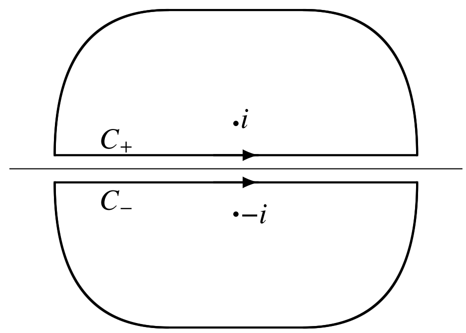
• *Duality.* Let $\varphi, \psi \in \mathbf{H}(\mathbb{S}^1)$ be represented by f, $h \in O(U \setminus \mathbb{S}^1)$ for some neighborhood U of $\mathbb{S}^1$. There is an annulus $e^{-a} \le |w| \le e^a$ contained in U. Let C_+ be a contour $|w| = c_+ \in [e^{-a}, 1)$ encircling 0 once in the positive direction, and let C_- be a similar contour $|w| = c_- \in (1, e^a]$. Then the integral

$$\langle \varphi, \psi \rangle \ = \ \frac{1}{2\pi i}\Big(\int_{C_+} - \int_{C_-}\Big) f(w)h(w)\,\frac{dw}{w} \tag{2.14}$$

is independent of the choice of the contours, as long as they are continuously deformed within $U \setminus \mathbb{S}^1$. So the actual neighborhood is not important. Moreover, if f and h are both in $O(U)$, then Cauchy's theorem gives $\langle \varphi, \psi \rangle = 0$. Thus, we get an induced pairing $C^{\omega}(\mathbb{S}^1) \times C^{-\omega}(\mathbb{S}^1) \to \mathbb{C}$, which we denote also by $\langle \cdot\,, \cdot \rangle$.

The description in the projective model is

$$\langle \varphi, \psi \rangle \ = \ \frac{1}{\pi}\Big(\int_{C_+} - \int_{C_-}\Big) \cdot \varphi(z)\, g(z)\, \frac{dz}{1+z^2}\,, \tag{2.15}$$



where $\varphi \in O(U)$ for some neighborhood U of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$, and $g \in O(U \setminus \mathbb{P}^1_{\mathbb{R}})$ represents $\psi \in C^{-\omega}(\mathbb{P}^1_{\mathbb{R}})$. The contours $C_+ \subset \mathfrak{H} \cap U$ and $C_- \subset \mathfrak{H}^- \cap U$ are homotopic with $\mathbb{P}^1_{\mathbb{R}}$. The orientation in $\mathbb{C}$ of C_+ is positive and the orientation of C_- negative. It turns out that for all $g \in G$

$$\langle \varphi|_{2-2s}\, g,\ \psi|_{2s}\, g \rangle \ = \ \langle \varphi, \psi \rangle\,. \tag{2.16}$$

Thus, we have a bilinear invariant pairing $\mathcal{V}^{\omega}_{1-s} \times \mathcal{V}^{-\omega}_s \to \mathbb{C}$.

From (2.14) we see that for fixed $h \in O(U \setminus \mathbb{S}^1)$ the map $f \mapsto \langle f, h \rangle$ is continuous with respect to the supremum norm of f on U. Thus $\varphi \mapsto \langle \varphi, \psi \rangle$ is a continuous linear form on $\mathcal{V}^{\omega}_{1-s}$ for fixed $\psi \in \mathcal{V}^{-\omega}_s$. It turns out that this gives an identification of $\mathcal{V}^{-\omega}_s$ with the continuous dual of $\mathcal{V}^{\omega}_{1-s}$.

It may happen that $\varphi \mapsto \langle \varphi, \psi \rangle$ can be continuously extended to $\mathcal{V}^{\infty}_s$ for the topology on $\mathcal{V}^{\infty}_s$. Such linear forms are distribution vectors in $\mathcal{V}_s$. The space $\mathcal{V}^{-\infty}_s$ consists of the distribution subspace of $\mathcal{V}^{-\omega}_s$. It can be identified with the continuous dual of $\mathcal{V}^{\infty}_{1-s}$.

• *Basis.* There are elements $\mathbf{e}_{s,n} \in \mathcal{V}^{\omega}_s$, $n \in \mathbb{Z}$, such that $\langle \mathbf{e}_{1-s,n}, \mathbf{e}_{s,m} \rangle = \delta_{n,-m}$:

$$\mathbf{e}_{s,n}(t) \ = \ (t^2+1)^{-s}\Big(\frac{t-i}{t+i}\Big)^n\,, \tag{2.17a}$$

$$\mathbf{e}^{\mathbb{P}}_{s,n}(t) \ = \ \Big(\frac{t-i}{t+i}\Big)^n\,, \tag{2.17b}$$

$$\mathbf{e}^{\mathbb{S}}_{s,n}(\xi) \ = \ \xi^n\,. \tag{2.17c}$$

Fourier theory gives an expansion $\varphi = \sum_{n=-\infty}^{\infty} c_n \mathbf{e}_{s,n}$ of each element $\varphi \in \mathcal{V}_s^{-\omega}$. We have

$$\begin{aligned} \mathcal{V}_s^{\omega} &= \{\sum c_n \mathbf{e}_{s,n} \;:\; c_n = \mathrm{O}(e^{-a|n|}) \text{ for some } a > 0\}, \\ \mathcal{V}_s^{\infty} &= \{\sum c_n \mathbf{e}_{s,n} \;:\; c_n = \mathrm{O}((1+|n|)^{-A}) \text{ for all } A > 0\}, \\ \mathcal{V}_s^{-\infty} &= \{\sum c_n \mathbf{e}_{s,n} \;:\; c_n = \mathrm{O}((1+|n|)^{a}) \text{ for some } a > 0\}, \\ \mathcal{V}_s^{-\omega} &= \{\sum c_n \mathbf{e}_{s,n} \;:\; c_n = \mathrm{O}(e^{A|n|}) \text{ for all } A > 0\}. \end{aligned} \tag{2.18}$$

• *Isomorphism.* For $0 < \mathrm{Re}\, s < 1$, the G-modules $\mathcal{V}_s^{-\omega}$ and $\mathcal{V}_{1-s}^{-\omega}$ are isomorphic. The intertwining operator $I_s : \mathcal{V}_s^{-\omega} \to \mathcal{V}_{1-s}^{-\omega}$ can be given on the basis vectors in (2.17):

$$I_s \mathbf{e}_{s,n} = \frac{\Gamma(s)\,\Gamma(1-s+n)}{\Gamma(1-s)\,\Gamma(s+n)}\, \mathbf{e}_{1-s,n}\,. \tag{2.19}$$

• *Sheaves.* The definitions of $\mathcal{V}_s^{\omega}$, $\mathcal{V}_s^{\infty}$ and the $\mathcal{V}_s^{p}$, are local. We can form the corresponding sheaves. We shall use this often for $\mathcal{V}_s^{\omega}$. We formulate this for the projective model and leave the analogous definitions in the other models to the reader.

For each open set $I \subset \mathbb{P}^1_{\mathbb{R}}$, we define

$$\mathcal{V}_s^{\omega}(I) = \varinjlim_{U} \mathcal{O}(U)\,, \tag{2.20}$$

where U runs through the neighborhoods of I in $\mathbb{P}^1_{\mathbb{C}}$. Note that we allow ourselves to write $\mathcal{V}_s^{\omega}$ instead of $\mathcal{V}_s^{\omega}(\mathbb{P}^1_{\mathbb{R}})$.

We also use the notation

$$\mathcal{V}_s^{\omega}[F] = \mathcal{V}_s^{\omega}\left(\mathbb{P}^1_{\mathbb{R}} \smallsetminus F\right) \tag{2.21}$$

where F is a finite subset of $\mathbb{P}^1_{\mathbb{R}}$. We will simply write $\mathcal{V}_s^{\omega}[\xi_1, \ldots, \xi_n]$ instead of $\mathcal{V}_s^{\omega}[\{\xi_1, \ldots, \xi_n\}]$. If we impose a condition on the sections of $\mathcal{V}_s^{\omega}$ at the points ξ_j we write $\mathcal{V}_s^{\omega,\mathrm{cond}}[\xi_1, \ldots, \xi_n]$. For instance, $\varphi \in \mathcal{V}_s^{\omega,\infty}[\xi_1, \ldots, \xi_n]$ is an element of $\mathcal{V}_s^{\infty}$ with analytic restriction to $\mathbb{P}^1_{\mathbb{R}} \setminus \{\xi_1, \ldots, \xi_n\}$.

The G-module $\mathcal{V}_s^{\omega}$ is naturally included in the G-module

$$\mathcal{V}_s^{\omega^*} = \varinjlim_{F} \mathcal{V}_s^{\omega}\left(\mathbb{P}^1_{\mathbb{R}} \smallsetminus F\right), \tag{2.22}$$

where F runs through the finite subsets of $\mathbb{P}^1_{\mathbb{R}}$. So $\mathcal{V}_s^{\omega^*}$ can be viewed as the union of all $\mathcal{V}_s^{\omega}[F]$, with identification of $f \in \mathcal{V}_s^{\omega}[F_1]$ with its image in $\mathcal{V}_s^{\omega}[F]$ if $F_1 \subset F$. The space $\mathcal{V}_s^{\omega^*}$ is not a subspace of $\mathcal{V}_s^{-\omega}$. We call $\mathcal{V}_s^{\omega^*}$ the space of *semi-analytic vectors* in the principal series representation. With an additional condition we write

$$\mathcal{V}_s^{\omega^*,\mathrm{cond}} = \varinjlim_{F} \mathcal{V}_s^{\omega,\mathrm{cond}}(\mathbb{P}^1_{\mathbb{R}} \smallsetminus F)\,. \tag{2.23}$$

By $\mathsf{BdSing}\,(f)$ for $f \in \mathcal{V}_s^{\omega^*}$ we denote the minimal finite set $\{\xi_1, \ldots, \xi_n\} \subset \mathbb{P}^1_{\mathbb{R}}$ such that $f \in \mathcal{V}_s^{\omega}[\xi_1, \ldots, \xi_n]$. We call these ξ_j the *singularities* of f.

• *Terminology.* Usually one denotes by $\mathcal{V}_s^{p}$ the space of p times differentiable vectors in a Hilbert space $\mathcal{V}_s^{L^2}$ in $\mathcal{V}_s$, for which the $\mathbf{e}_{s,n}$ form a complete orthogonal system. We use $\mathcal{V}_s^{p}$ to denote *functions* that are p times continuously differentiable in the projective of circle model. This space is smaller than the space of p times differentiable vectors in $\mathcal{V}_s^{L^2}$.

Our spaces $\mathcal{V}_s^\omega$, $\mathcal{V}_s^\infty$ coincide with the spaces of analytic and smooth vectors in $\mathcal{V}_s^{L^2}$, and similarly for $\mathcal{V}_s^{-\infty}$ and $\mathcal{V}_s^{-\omega}$. It seems hard and hardly interesting to characterize spaces like $\mathcal{V}_s^{\omega^*}$ in terms of the Hilbert space $\mathcal{V}_s^{L^2}$. To summarize: our upper indices in $\mathcal{V}_s^*$ refer to the behavior of *functions* in the circle and projective model, not to the behavior of *vectors* in a representation.

2.2. *Poisson transform.* The Poisson transform in this section provides us with $\mathcal{E}_s$ as a realization of $\mathcal{V}_s^{-\omega}$. It and its inverse can be described with the function $R(t;z)^{1-s}$ in (1.6) as the kernel function. (For more details see §2.3 of [4].)

On $\mathcal{V}_s^0$ the Poisson transformation is the linear G-equivariant map given in the line model by the simple formula

$$\mathsf{P}_s\alpha(z) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\Bigl(\Bigl(\frac{t-x}{y}\Bigr)^2+1\Bigr)^{s-1} y^{-1+s}\alpha(t)\,dt \;=\; \int_{-\infty}^{\infty} R(t;z)^{1-s}\alpha(t)\,\frac{dt}{\pi}\,. \tag{2.24}$$

The image is in $\mathcal{E}_s$, since $R(t;\,\cdot\,)^{1-s}$ is in $\mathcal{E}_s$ for all $t\in\mathbb{R}$. Since $R(\,\cdot\,;z)^{1-s}$ is an element of $\mathcal{V}_s^\omega$ (line model), the G-equivariance follows from

$$R(\,\cdot\,;gz)|_{2s}\,g \;=\; R(\,\cdot\,;z)^s \qquad \text{for all } g\in\Gamma\,. \tag{2.25}$$

A comparison of this invariance property with (1.5) shows that $R(\,\cdot\,;\,\cdot\,)^s$ is similar to p_s and q_s.

We can write the Poisson transform as

$$\mathsf{P}_s\alpha(z) = \langle R(\,\cdot\,;z)^{1-s},\alpha\rangle\,. \tag{2.26}$$

This can be used to define the Poisson transformation as a linear map $\mathsf{P}_s : \mathcal{V}_s^{-\omega}\to\mathcal{E}_s$, satisfying $\mathsf{P}_s(\alpha\,|\,g) = (\mathsf{P}_s\alpha)\,|\,g$ for all $g\in G$. The following diagram, involving the isomorphism I_s in (2.19), commutes:

$$\begin{tikzcd} \mathcal{V}_s^{-\omega} \arrow[d, "I_s"] \arrow[dr, "\mathsf{P}_s"] & \\ \mathcal{V}_{1-s}^{-\omega} \arrow[r, "\mathsf{P}_{1-s}"'] & \mathcal{E}_s = \mathcal{E}_{1-s} \end{tikzcd} \tag{2.27}$$

We also have

$$(\mathsf{P}_s R(\,\cdot\,;z')^s)(z) \;=\; p_s(z,z')\,; \tag{2.28}$$

$$I_s\,R(\,\cdot\,;z')^s \;=\; R(\,\cdot\,;z')^{1-s}\,. \tag{2.29}$$

(See (2.25) and (2.32) in [4].) In the other models, the Poisson kernel has the form:

$$R^{\mathbb{P}}(\zeta;z)^{1-s} \;=\; y^{s-1}\Bigl(\frac{\zeta-i}{\zeta-z}\Bigr)^{1-s}\Bigl(\frac{\zeta+i}{\zeta-\bar z}\Bigr)^{1-s} \;=\; \Bigl(\frac{R(\zeta;z)}{R(\zeta;i)}\Bigr)^{1-s}, \tag{2.30a}$$

$$R^{\mathbb{S}}(\xi;w)^{1-s} \;=\; \Bigl(\frac{1-|w|^2}{(1-w/\xi)(1-\bar w\xi)}\Bigr)^{1-s} \tag{2.30b}$$

• *Bijectivity.* Crucial for these notes is that $\mathsf{P}_s : \mathcal{V}_s^{-\omega}\to\mathcal{E}_s$ is an isomorphism of G-modules. This follows from the next result of Helgason (Theorem 4.3 in [12]) and the G-equivariance of P_s.

Theorem 2.1. *The Poisson transformation* $\mathsf{P}_s : \mathcal{V}_s \to \mathcal{E}_s$ *is an isomorphism of G-modules for all* $s \in \mathbb{C}$ *with* $0 < \operatorname{Re} s < 1$.

Thus, $\mathcal{E}_s$ is a model of the principal series representation. This model has several advantages: the action of G involves no automorphy factor at all, the model does not give a preferential treatment to any point, and all vectors correspond to actual functions, with no need to work with distributions or hyperfunctions.

Theorem 3.2 in [4] and the discussion preceding it give an explicit way to describe the inverse of the Poisson transformation:

Theorem 2.2. *Let* $u \in \mathcal{E}_s$, *and* $z_0 \in \mathfrak{H}$. *Then the hyperfunction* α *on* $\partial\mathfrak{H} = \mathbb{P}^1_{\mathbb{R}}$ *represented by the following function* g *on* $U \smallsetminus \mathbb{P}^1_{\mathbb{R}}$ *for a neighborhood* U *of* $\mathbb{P}^1_{\mathbb{R}}$ *in* $\mathbb{P}^1_{\mathbb{C}}$

$$g(\zeta) \;=\; \begin{cases} \int_{z_0}^{\zeta} [u, R^{\mathbb{P}}(\zeta;\,\cdot\,)^s] \;+\; u(z_0)\, R^{\mathbb{P}}(\zeta; z_0)^s & \text{if } \zeta \in \mathfrak{H}, \\ \int_{\bar\zeta}^{z_0} [R^{\mathbb{P}}(\zeta;\,\cdot\,)^s,\, u] & \text{if } \zeta \in \mathfrak{H}^-. \end{cases}$$

is independent of the choice of the base point z_0, *and* $u = \mathsf{P}_s\alpha$.

• *Polynomial growth.* We define $\mathcal{E}_s^{-\infty}$, $\mathcal{E}_s^{\infty}$ and $\mathcal{E}_s^{\omega}$ as the images under P_s of $\mathcal{V}_s^{-\infty}$, $\mathcal{V}_s^{\infty}$ and $\mathcal{V}_s^{\omega}$, respectively. For $\mathcal{E}_s^{-\infty}$ we can indicate here an independent characterization: We say that a function f on $\mathbb{D}$ has *polynomial growth* if $\left(1-|w|^2\right)^a f(w) = \mathrm{O}(1)$ as $|w| \uparrow 1$ for some $a \in \mathbb{R}$. For functions on $\mathfrak{H}$, this corresponds to $z \mapsto \left(\frac{y}{|z+i|^2}\right)^a f(z)$ being bounded for some a.

Theorem 2.3. (Lewis; Theorem 4.1 and Theorem 5.3 in [16]) *Let* $0 < \operatorname{Re} s < 1$. *The space* $\mathcal{E}_s^{-\infty} = \mathsf{P}_s(\mathcal{V}_s^{\infty})$ *consists of the functions in* $\mathcal{E}_s$ *having at most polynomial growth.*

3. Boundary germs and transverse Poisson transform. In a comparison of the eigenfunctions $P_{s,n}$ and $Q_{s,n}$ introduced in (1.3), a nice property of $P_{s,n}$ is that it is defined on the whole of $\mathbb{D}$, whereas $Q_{s,n}$ has a singularity at 0. On the other hand, near the boundary $\partial\mathbb{D}$ the expression in (1.3b) of $Q_{s,n}$ in terms of a hypergeometric function implies a simple asymptotic relation $Q_{s,n}(re^{i\theta}) \sim c\,(1-r^2)^s\, e^{in\theta}$ as $r \uparrow 1$, whereas $P_{s,n}(re^{i\theta})$ has a more complicated behavior at the boundary. We can observe a similar distinction between the eigenfunctions $i_{s,\alpha}$ and $k_{s,\alpha}$, with $\alpha \neq 0$, in (1.2). The asymptotic behavior of the modified Bessel functions implies that $k_{s,\alpha}(z)$ is quickly decreasing as $y \to \infty$, whereas $i_{s,\alpha}(z)$ has exponential growth. Near $\mathbb{R} \subset \partial\mathfrak{H}$ however we have $i_{s,\alpha}(z) \sim e^{i\alpha x}\, y^s$ as $y \downarrow 0$, whereas $k_{s,\alpha}$ has a more complicated behavior.

We capture the special boundary behavior of $i_{s,\alpha}$ and $Q_{s,n}$ by defining in Subsection 3.1 a space of eigenfunctions on $\Omega \cap \mathfrak{H}$ for a neighborhood Ω of $\partial\mathbb{H}$ in $\mathbb{P}^1_{\mathbb{C}}$ with a special behavior near the boundary. Actually, we use germs of such eigenfunctions by taking an inductive limit over all such neighborhoods Ω. In this way we define a space of boundary germs $\mathcal{W}_s^{\omega}$ isomorphic to $\mathcal{V}_s^{\omega}$. The isomorphism $\mathcal{V}_s^{\omega} \to \mathcal{W}_s^{\omega}$ is described explicitly in Subsection 3.2 by an operator that we call the "transverse Poisson transformation". In our study of cohomology groups the space $\mathcal{W}_s^{\omega}$ will turn out to be an excellent model of $\mathcal{V}_s^{\omega}$.

For cohomology with coefficients in $\mathcal{V}_s^{\infty}$ we shall also need an isomorphic space $\mathcal{W}_s^{\infty}$. This cannot be a space of germs of eigenfunctions. In Subsection 3.3 we will define it as a space of expansions.

This whole section is a brief presentation of results discussed in much more detail in §4 and §6 of [4].

3.1. *Boundary germs.* We turn to λ_s-eigenfunctions only defined near the boundary $\partial\mathbb{H}$ of the hyperbolic plane. Our aim is to use such functions to define a space $\mathcal{W}_s^\omega$ isomorphic to $\mathcal{V}_s^\omega$.

- *The space of all boundary germs.* Put

$$\mathcal{F}_s = \varinjlim_{\Omega} \mathcal{E}_s(\Omega \cap \mathbb{H}) , \tag{3.1}$$

where Ω runs over the neighborhoods of $\partial\mathbb{H}$ in $\mathbb{P}^1_{\mathbb{C}}$. This is a large space. The action of the group G is induced by $f\,|\,g(z) = f \circ g(z) = f(gz)$ on representatives f. We identify $\mathcal{E}_s$ with its image in $\mathcal{F}_s$.

Functions representing an element of $\mathcal{F}_s$ may grow fast near the boundary. We define a subspace $\mathcal{W}_s^\omega$ by prescribing the boundary behavior:

Definition 3.1. The space $\mathcal{W}_s^\omega$ is the subspace of $\mathcal{F}_s$ represented by functions $f \in \mathcal{E}_s(\Omega \cap \mathbb{D})$ for some neighborhood Ω of $\mathbb{S}^1$ in $\mathbb{P}^1_{\mathbb{C}}$ such that $f(w) = 2^{-2s}(1-|w|^2)^s f^{\mathbb{S}}(w)$, where $f^{\mathbb{S}} \in C^\omega(\Omega)$.

In the projective model there is a similar definition, with $f(z) = \left(\frac{y}{|z+i|^2}\right)^s f^{\mathbb{P}}(z)$ where $f^{\mathbb{P}}$ is real analytic on a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. The space $\mathcal{W}_s^\omega$ is invariant under the action of G in $\mathcal{F}_s$.

The definition can be localized to define $\mathcal{W}_s^\omega(I)$ for open sets $I \subset \partial\mathbb{H}$. Then $f^{\mathbb{S}}$ or $f^{\mathbb{P}}$ is real analytic on a neighborhood Ω of I in $\mathbb{P}^1_{\mathbb{C}}$. For $I \subset \mathbb{R}$, the line model is most convenient. Then each element of $\mathcal{W}_s^\omega(I)$ is represented by $f \in \mathcal{E}_s(\Omega \cap \mathfrak{H})$ for a neighborhood Ω of I in $\mathbb{C}$, and $f(z) = y^s \tilde{f}(z)$ on $\Omega \cap \mathfrak{H}$ for some real analytic function $\tilde{f}$ on Ω.

We use the notation $\mathcal{W}_s^\omega[F] = \mathcal{W}_s^\omega(\mathbb{H} \smallsetminus F)$ for finite subsets $F \subset \partial\mathbb{H}$, and define

$$\mathcal{W}_s^{\omega^*} = \varinjlim_{F} \mathcal{W}_s^\omega(\partial\mathbb{H} \smallsetminus F) , \tag{3.2}$$

like in (2.21) and (2.22). For a given $f \in \mathcal{W}_s^{\omega^*}$, the set of singularities $\mathsf{BdSing}\, f$ is the minimal finite set $F \subset \partial\mathbb{H}$ such that $f \in \mathcal{W}_s^\omega[F]$. The Bd in this notation reminds us that we consider singularities on $\partial\mathbb{H}$, and not at points of $\mathbb{H}$ near $\partial\mathbb{H}$ where the functions $f^{\mathbb{S}}$ or $f^{\mathbb{P}}$ may have a singularity.

- *Decomposition.* Suppose $f \in \mathcal{E}_s(U)$, with $U = \{w \in \mathbb{D} \;:\; 1-\delta < |w| < 1\}$, represents a germ in $\mathcal{F}_s$. Taking a closed curve C in the annulus U going round once in the positive direction, with the Green's form in (1.9) we form the integral

$$\int_C [f, q_s(\,\cdot\,, w)] .$$

This integral represents functions $u \in \mathcal{E}_s(I)$ and $v \in \mathcal{E}_s(E)$, where I is the region inside the curve C, and E the annulus outside C. In Proposition 4.2, [4], we show that v represents an element of $\mathcal{W}_s^\omega$, which vanishes if $f \in \mathcal{E}_s$. Moving the curve C closer and closer to $\mathbb{S}^1 = \partial\mathbb{D}$, we see that u extends to $\mathbb{D}$ and is an element $u \in \mathcal{E}_s$. Theorem 1.1 shows that $\frac{1}{\pi i} f = u - v$. Thus we have obtained $\mathcal{F}_s = \mathcal{E}_s + \mathcal{W}_s^\omega$. If the original function f is in $\mathcal{E}_s$, then $v = 0$ and $u = \frac{1}{\pi i} f$, also by Theorem 1.1. So $\mathcal{E}_s \cap \mathcal{W}_s^\omega = \{0\}$, and

$$\mathcal{F}_s \;=\; \mathcal{E}_s \oplus \mathcal{W}_s^\omega\,. \tag{3.3}$$

• *Restriction to the boundary.* If w represents an element of $\mathcal{W}_s^\omega$, then $f(w) = (1-|w|^2)^s f^{\mathbb{S}}(w)$ near the boundary, with $f^{\mathbb{S}}$ extending analytically across the boundary. Thus $\rho_s f(\xi) = f^{\mathbb{S}}(\xi)$ is a well defined analytic function on $\mathbb{S}^1$, which is an element of the circle model of $\mathcal{V}_s^\omega$. This restriction map ρ_s intertwines the actions of G in $\mathcal{W}_s^\omega$ and $\mathcal{V}_s^\omega$. We should note that $f^{\mathbb{S}}$ is real analytic on a neighborhood Ω of $\mathbb{S}^1$ in $\mathbb{P}^1_{\mathbb{C}}$, and that $\varphi = \rho_s f$ is a real analytic function on $\mathbb{S}^1$ extending as a holomorphic function on some neighborhood Ω_1 of $\mathbb{S}^1$ in $\mathbb{P}^1_{\mathbb{C}}$. These functions φ and $\tilde f$ coincide on $\mathbb{S}^1$, not on the whole intersection $\Omega_1 \cap \Omega$.

In the upper half plane model of $\mathbb{H}$, we obtain $\rho_s f = f^{\mathbb{P}}$ in the projective model of $\mathcal{V}_s^\omega$ on $\mathbb{P}^1_{\mathbb{R}}$. The restriction ρ_s also gives linear maps $\rho_s : \mathcal{W}_s^\omega(I) \to \mathcal{V}_s^\omega(I)$ for open $I \subset \partial\mathbb{H}$. In particular, for $I \subset \mathbb{R}$, we obtain $\rho_s f = \tilde f$ in the line model.

3.2. *Transverse Poisson map.* The restriction map $\rho_s : \mathcal{W}_s^\omega \to \mathcal{V}_s^\omega$ is bijective. In §4.2 of [4] we explicitly describe the inverse, in two different ways.

One way is by an integral transform, with the following expression in the three models:

$$(\mathsf{P}_s^\dagger \varphi)(z) \;=\; \frac{1}{i\,b(s)} \int_{\bar z}^{z} R(\zeta; z)^{1-s}\, \varphi(\zeta)\, d\zeta\,, \tag{3.4a}$$

$$(\mathsf{P}_s^\dagger \varphi^{\mathbb{P}})(z) \;=\; \frac{1}{i\,b(s)} \int_{\bar z}^{z} R^{\mathbb{P}}(\zeta; z)^{1-s} \varphi^{\mathbb{P}}(\zeta)\, \frac{d\zeta}{1+\zeta^2}\,, \tag{3.4b}$$

$$(\mathsf{P}_s^\dagger \varphi^{\mathbb{S}})(w) \;=\; \frac{1}{2\,b(s)} \int_{w}^{1/\bar w} R^{\mathbb{S}}(\eta; w)^{1-s} \varphi^{\mathbb{S}}(\eta)\, \frac{d\eta}{\eta}\,, \tag{3.4c}$$

$$\text{where} \qquad b(s) \;=\; \mathsf{B}\Big(s, \frac{1}{2}\Big) \;=\; \frac{\sqrt{\pi}\,\Gamma(s)}{\Gamma(s+\frac{1}{2})}\,. \tag{3.4d}$$

An element $\varphi \in \mathcal{V}_s^\omega(I)$ for some open $I \subset \partial\mathbb{H}$ extends holomorphically to some neighborhood Ω of I in $\mathbb{P}^1_{\mathbb{C}}$. The integrals in (3.4) define $\mathsf{P}_s^\dagger \varphi$ on $\Omega \cap \bar\Omega \cap \mathbb{H}$, representing a germ that can be shown to be an element of $\mathcal{W}_s^\omega(I)$. (By $\bar\Omega$ we denote the image under complex conjugation.) On the other hand, Theorem 4.7 in [4] gives also an integral representation of φ in terms of $u = \mathsf{P}_s^\dagger \varphi$, showing that if (in the line model) $u = y^s A$ with A real analytic on a simply connected open set $\Omega = \bar\Omega$ intersecting $\mathbb{R}$ then φ is holomorphic on Ω.

The integral transformation in (3.4) has the same kernel function as that in the Poisson transformation. The path of integration is different. We call $\mathsf{P}_s^\dagger$ the *transverse Poisson transformation*.

Theorem 3.2. *The restriction map $\rho_s : \mathcal{W}_s^\omega(I) \to \mathcal{V}_s^\omega(I)$ is an isomorphism for each open set $I \subset \partial\mathbb{H}$. Its inverse is given by $\mathsf{P}_s^\dagger$.*

The other way to describe the transverse Poisson transformation works locally with the line model. The action of G allows restriction to an interval $I \subset \mathbb{R}$. Any $f \in \mathcal{W}_s^\omega(I)$ is of the form $f(z) = y^s A(z)$, with A real-analytic on a neighborhood Ω of I in $\mathbb{R}$. Let

$\varphi = \rho_s f$. Then it turns out that the fact that $f \in \mathcal{E}_s(\Omega \cap \mathfrak{H})$ implies that A has the expansion

$$A(z) \;=\; \sum_{k=0}^{\infty} \frac{(-1/4)^k\,\Gamma(s+\frac{1}{2})}{k!\,\Gamma(s+\frac{1}{2}+k)}\,\varphi^{(2k)}(x)\,y^{2k}\,. \tag{3.5}$$

(See Theorem 4.6 in [4].) If $\Omega_x \subset \Omega$ is an open neighborhood of $x \in I$ on which the power series of φ at x converges, then (3.5) converges on Ω_x as well.

This relation between the expansions of A and φ illustrates that being a λ_s-eigenfunction of Δ is a very strong property. Note that the description in (3.4) shows that if φ is holomorphic on Ω, then A is real-analytic on Ω, but that conversely if A is real-analytic on Ω_1, we know only that φ is holomorphic on some neighborhood Ω of I that may be much smaller than Ω_1.

- *Examples.* For the functions in (1.2), (1.3a), (1.6) and (1.4) we have:

$$\mathsf{P}_s^{\dagger} e^{i\alpha x} \;=\; i_{s,\alpha} \qquad \text{on } \mathbb{R} \text{ in the line model}\,, \tag{3.6a}$$

$$\mathsf{P}_s^{\dagger} \mathbf{e}_{s,n} \;=\; \frac{(-1)^n\,\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(s+n)}\,Q_{s,n} \qquad \text{on } \partial\mathbb{H}\,, \tag{3.6b}$$

$$\mathsf{P}_s^{\dagger} R(\,\cdot\,;z)^s\,(z') \;=\; b(s)^{-1}\,q_s(z,z') \qquad \text{on } \mathbb{R} \text{ in the line model}\,. \tag{3.6c}$$

The first two examples are easily checked by computing the restriction ρ_s of the left hand side. The third example is equation (4.19) in [4]. See §A.3 in the appendix of [4] for more examples.

- *Splitting of $\mathcal{E}_s^{\omega}$.* For $s \neq \frac{1}{2}$, Proposition 6.3 in [4] gives the following description of $\mathcal{E}_s^{\omega} = \mathsf{P}_s \mathcal{V}_s^{\omega}$:

$$\begin{aligned} \mathsf{P}_s\varphi \;&=\; c(s)\,\mathsf{P}_s^{\dagger}\varphi + c(1-s)\,\mathsf{P}_{1-s}^{\dagger} I_s \varphi\,,\\ c(s) \;&=\; \frac{\tan \pi s}{\pi}\,b(s)\,, \end{aligned} \tag{3.7}$$

with the intertwining operator I_s in (2.19), and $b(s)$ as in (3.4d). This implies that each of the isomorphic G-spaces $\mathcal{E}_s^{\omega}$, $\mathcal{W}_s^{\omega}$ and $\mathcal{W}_{1-s}^{\omega}$ is contained in the sum of the other two, and that each two of these spaces have intersection $\{0\}$.

- *Duality.* The G-invariant duality of $\mathcal{V}_s^{\omega} \times \mathcal{V}_{1-s}^{-\omega} \to \mathbb{C}$ in (2.14) can be transported to a G-invariant duality $\mathcal{W}_s^{\omega} \times \mathcal{E}_{1-s} \to \mathbb{C}$ by

$$\langle \mathsf{P}_s^{\dagger}\varphi, \mathsf{P}_{1-s}\alpha\rangle \;=\; \langle \varphi, \alpha\rangle \qquad (\alpha \in \mathcal{V}_{1-s}^{-\omega}\,, \varphi \in \mathcal{V}_s^{\omega})\,. \tag{3.8}$$

In Proposition 4.8 in [4] it is shown that for $u \in \mathcal{E}_{1-s}$ and $f \in \mathcal{W}_s^{\omega}$ we can describe this duality with the Green's form of §1.3:

$$\langle f, u\rangle \;=\; \frac{b(s)}{\pi i}\int_C [f,u] \;=\; -\frac{b(s)}{\pi i}\int_C [u,f]\,, \tag{3.9}$$

where C is homotopic to $\partial\mathbb{H}$ in the domain of a representative of f, going around once in the positive direction. Actually, $\int_C [f,u]$ is well defined for $f \in \mathcal{F}_s$, $u \in \mathcal{E}_{1-s}$, being zero for f in the component $\mathcal{E}_s = \mathcal{E}_{1-s}$ of the splitting $\mathcal{F}_{1-s} = \mathcal{E}_{1-s} \oplus \mathcal{W}_{1-s}^{\omega}$ in (3.3). The second equality in (3.9) follows from (1.10b) and the fact that C is a closed curve.

3.3. *Boundary jets.* In Chapter V we study cohomology with differentiable coefficients. We need a substitute $\mathcal{W}_s^p$ for $p = 2, 3, \ldots, \infty$ for the space $\mathcal{W}_s^\omega$ of boundary germs. In §4.4 of [4] we have constructed these spaces as a quotient. We recall the definitions and main results. For $p = 2, \ldots, \infty, \omega$ we consider the space of functions $f \in C^2(\mathbb{D})$ for which $\tilde{f}(w) = (1 - |w|^2)^{-s} f(w)$ extends to a C^p function on some neighborhood of $\mathbb{S}^1$ in $\mathbb{C}$. By $\tilde{\Delta}_s$ we denote the differential operator on $\tilde{f}$ corresponding to $\Delta - \lambda_s$ on f. We define $\mathcal{G}_s^p$ and $\mathcal{N}_s^p$ as the subspaces of functions $f = (1 - |w|^2)^s \tilde{f}$ that satisfy the conditions

$$\tilde{\Delta}_s \tilde{f}(w) = \begin{cases} \mathrm{o}((1 - |w|^2)^p) & \text{if } p \in \mathbb{N}, \\ \mathrm{o}((1 - |w|^2)^q) & \text{for all } q \in \mathbb{N} \text{ if } p = \infty, \\ 0 & \text{if } p = \omega, \end{cases} \tag{3.10a}$$

respectively

$$\tilde{f}(w) = \begin{cases} \mathrm{o}((1 - |w|^2)^p) & \text{if } p \in \mathbb{N}, \\ \mathrm{o}((1 - |w|^2)^q) & \text{for all } q \in \mathbb{N} \text{ if } p = \infty, \\ 0 & \text{if } p = \omega, \end{cases} \tag{3.10b}$$

as $|w| \to 1$ in some annulus $1 - \varepsilon < |w| < 1$. In [4], Lemma 4.10 it is shown that $\mathcal{N}_s^p \subset \mathcal{G}_s^p$. We define the space of *boundary jets* $\mathcal{W}_s^p$ as the quotient $\mathcal{G}_s^p / \mathcal{N}_s^p$ for $p = 2, 3, \ldots, \infty$, so that the following sequence is exact by definition:

$$0 \longrightarrow \mathcal{N}_s^p \longrightarrow \mathcal{G}_s^p \longrightarrow \mathcal{W}_s^p \longrightarrow 0\,. \tag{3.11}$$

For $p = \omega$ this agrees with our previous definition of $\mathcal{W}_s^\omega$ because a function $f \in \mathcal{G}_s^\omega$ is in $\mathcal{N}^\omega = C_c^2(\mathbb{D})$ if and only if it represents the zero element of $\mathcal{W}_s^\omega$. The group G acts on $\mathcal{G}_s^p$ and $\mathcal{N}_s^p$ by $(f\,|\,g)(w) = f(gw)$. This induces an action in $\mathcal{W}_s^p$.

These definitions also work locally. For $I \subset \mathbb{S}^1$ open we define $\mathcal{G}_s^p(I)$, $\mathcal{N}_s^p(I)$ as above (with f still defined on all of $\mathbb{D}$) but with the extendability across $\mathbb{S}^1$ and the growth conditions (3.10) near $\mathbb{S}^1$ required only near I. Thus, $\mathcal{G}_s^p$ and $\mathcal{N}_s^p$ are sheaves on $\mathbb{S}^1$. We define $\mathcal{W}_s^p$ as the quotient sheaf. One can show that $\mathcal{W}_s^p(I) = \mathcal{G}_s^p(I)/\mathcal{N}_s^p(I)$ for all I. In the upper half plane model we have corresponding definitions with the factor $(1 - |w|^2)^s$ replaced by $(\frac{y}{|z+i|^2})^s$ (or simply by y^s on $I \subseteq \mathbb{P}^1_\mathbb{R}$ with $\infty \notin I$).

We define G-equivariant sheaf morphisms $\rho_s : \mathcal{G}_s^p \to \mathcal{V}_s^p$ by sending $f \in \mathcal{G}_s^p(I)$ to the restriction to I of a C^p extension of $2^{2s}(1 - |w|^2)^{-s} f(w)$ (resp. of $(y/|z+i|^2)^{-s} f(z)$). In Theorem 4.11 of [4] we prove:

Theorem 3.3. *The restriction ρ_s induces a sheaf isomorphism $\rho_s : \mathcal{W}_s^p \to \mathcal{V}_s^p$ for $p = 2, \ldots, \infty, \omega$.*

Notice that we have global representatives in $\mathcal{G}_s^p(I) \subset C^2(\mathbb{H})$ of elements of $\mathcal{W}_s^p(I)$, even if $I \subset \partial\mathbb{H}$ is a tiny interval. We impose twice differentiability in all of $\mathbb{H}$ in order to be able to apply Δ freely. Even in the analytic case $p = \omega$ the representatives $f \in \mathcal{G}_s^\omega(I)$ need not be analytic on all of $\mathbb{H}$, and satisfy $\Delta f = \lambda_s f$ only near the boundary.

Definition 3.4. For any $f : \mathbb{H} \to \mathbb{C}$ the set of *singularities* $\mathsf{Sing}\, f$ of f is the complement of the maximal open set $U \subset \mathbb{H}$ such that $f \in \mathcal{E}_s(U)$.

This is a rather broad notion of singularity. It depends on the spectral parameter s, and even an analytic function may have singularities in our sense. For $f \in \mathcal{G}_s^\omega$ the set

$\mathsf{Sing}\, f$ is a compact subset of $\mathbb{H}$. This set may be empty. The function $i_{s,0}(z) = y^s$ is an element of $\mathcal{E}_s$ and of $\mathcal{G}_s^\omega(\mathbb{R})$, with $\mathsf{Sing}\, i_{s,0} = \emptyset$. Note that $i_{s,0}$ represents an element of $\mathcal{W}_s^\omega(\mathbb{R})$ which can be considered as an element of $\mathcal{W}_s^{\omega^*}$, as defined in (3.2). As such $\mathsf{BdSing}\, i_{s,0} = \{\infty\}$.

4. Averages. For v in any G-module V all finite sums $v \,|\, \sum_i g_i = \sum_i v \,|\, g_i$ converge. Some infinite sums converge as well, for certain modules. In this section we discuss infinite sums that will be used in the next chapters. It has turned out that these operators form a useful tool when dealing with transfer operators. (See [5], [18].)

The infinite sums that we discuss in this section are

$$\mathrm{Av}_g^+ = \sum_{n=0}^{\infty} g^n , \quad \mathrm{Av}_g^- = -\sum_{n=-\infty}^{-1} g^n , \quad \mathrm{Av}_g = \sum_{n=-\infty}^{\infty} g^n = \mathrm{Av}_g^+ - \mathrm{Av}_g^- , \tag{4.1}$$

for $g \in G$ with infinite order. If we can make sense of the convergence of such sums, the *one-sided averages* Av_g^+ and Av_g^- provide us with a substitute for $(1-g)^{-1}$. The *average* Av_g produces a g-invariant vector.

The elements of G of infinite order are either hyperbolic or parabolic. We treat these two cases separately, and consider the one-sided averages and the spaces of invariants for spaces in $\mathcal{V}_s$, in particular for $\mathcal{V}_s^\omega$.

• *Notation.* We shall use both $f \,|\, \mathrm{Av}_g$ and $\mathrm{Av}_g(f)$ to denote the average of f over the powers of g. The latter notation emphasizes the average as an operator, whereas the former stresses that Av_g is an element of the completion of the group ring of Γ.

4.1. *Invariants and averages for hyperbolic elements.* We start with the easiest case, where g is hyperbolic.

Any hyperbolic $\eta \in G$ leaves fixed two points of $\mathbb{P}^1_{\mathbb{C}}$, which are situated on $\mathbb{P}^1_{\mathbb{R}}$: The *repelling fixed point* $\alpha(\eta)$ and the *attracting fixed point* $\omega(\eta)$. The latter is characterized by $\lim_{n\to\infty} \eta^n x = \omega(\eta)$ for all $x \in \mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\alpha(\eta)\}$. By conjugation in G, we can arrange $\eta = \left[\begin{smallmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{smallmatrix}\right]$ with $t > 1$. Then $\alpha(\eta) = 0$, $\omega(\eta) = \infty$.

By V^g we denote the elements of V invariant under $g \in G$ for any G-module V.

Proposition 4.1. *The spaces $(\mathcal{V}_s^p)^\eta$ are $\{0\}$ for $p = 2, \ldots, \infty, \omega$, and also the spaces $(\mathcal{V}_s^\omega[0])^\eta$ and $(\mathcal{V}_s^\omega[\infty])^\eta$ are zero.*

See (2.21) for the definition of $\mathcal{V}_s^\omega[\xi]$.

Proof. It suffices to consider $\mathcal{V}_s^0(\mathbb{R}) = \mathcal{V}_s^0[\infty]$ and $\mathcal{V}_s^0(\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{0\}) = \mathcal{V}_s^0[0]$. Let $f \in \mathcal{V}_s^0(\mathbb{R})$ be given in the line model. Then $t^s f(tx) = f(x)$ for all $x \in \mathbb{R}$. There is a periodic function p on $\mathbb{R}$ with period $\log t$ such that $f(x) = x^{-s} p(\log x)$ for $x \in (0,\infty)$. This implies that $\lim_{u\to-\infty} p(u) = \lim_{u\to-\infty} e^{su} f(e^u) = 0$. Hence $f = 0$ on $(0,\infty)$, and analogously on $(-\infty, 0)$. Conjugate with $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ to obtain the statement for $\mathcal{V}_s^0(\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{0\})$. ■

With the isomorphism $\mathsf{P}_s^\dagger$, we obtain also that the corresponding spaces $(\mathcal{W}_s^p)^\eta$ all vanish.

For larger spaces in the principal series, the spaces of η-invariants are large. They contain the spaces of functions transforming according to the character $\left[\begin{smallmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{smallmatrix}\right] \mapsto$

$y^{i\alpha}$ for all $\alpha \in \frac{2\pi}{\log t}\mathbb{Z}$. For each $\alpha \in \mathbb{R}$ we give in (A.20) of [4] functions $f^R_{s,\alpha}$ and $f^L_{s,\alpha}$ in $\mathcal{E}_s$ that form a basis for the invariant functions for the character specified by α. We have for $z = \rho e^{i\phi} \in \mathfrak{H}$, $\rho > 0$, $0 < \phi < 2\pi$:

$$\begin{aligned} f^R_{s,\alpha}(\rho e^{i\phi}) &= \frac{\sqrt{\pi}\,\Gamma(s+\frac12)}{\Gamma(\frac{s+i\alpha+1}{2})\,\Gamma(\frac{s-i\alpha+1}{2})}\,\rho^{i\alpha}\,(\sin\phi)^s\,{}_2F_1\Bigl(\frac{s+i\alpha}{2},\frac{s-i\alpha}{2};\frac12;(\cos\phi)^2\Bigr) \\ &\quad - \frac{2\sqrt{\pi}\,\Gamma(s+\frac12)}{\Gamma(\frac{s+i\alpha}{2})\,\Gamma(\frac{s-i\alpha}{2})}\,\rho^{i\alpha}\,\cos\phi\,(\sin\phi)^s \\ &\qquad \cdot {}_2F_1\Bigl(\frac{s+i\alpha+1}{2},\frac{s-i\alpha+1}{2};\frac32;(\cos\phi)^2\Bigr), \end{aligned} \tag{4.2}$$

and we obtain $f^L_{s,\alpha}$ by taking the sum of the two terms instead of the difference. We have chosen the basis such that $f^R_{s,\alpha}$ represents an element of $\mathcal{W}^\omega_s(0,\infty)$ and $f^L_{s,\alpha}$ an element of $\mathcal{W}^\omega_s(-\infty,0)$. We have $\mathsf{BdSing}\, f^R_{s,\alpha} = \{\infty\}\cup(-\infty,0]$, and $\mathsf{BdSing}\, f^L_{s,\alpha} = [0,\infty)\cup\{\infty\}$. Note that the first term in (4.2) is invariant under $z \mapsto -\bar z$, and the second term anti-invariant.

Let $\varphi \in \mathcal{V}^\omega_s$. For large $|x|$ it is of the form $\varphi(x) = |x|^{-2s}\varphi_\infty(1/x)$ with φ_∞ the real-analytic function on a neighborhood of 0 given in (2.2). For $x \neq 0$ we have

$$\varphi|_{2s}\,\eta^n(x) = t^{ns}\varphi(t^n x) = t^{-ns}|x|^{-2s}\varphi_\infty(t^{-n}x^{-1})$$

as $n \to \infty$. Since $\operatorname{Re} s > 0$ and $t > 1$, the series $\sum_{n=0}^\infty t^{ns}\varphi(t^n x)$ defining $\mathrm{Av}^+_\eta(\varphi)(x)$ converges with exponential rapidity for $x \neq 0$, so that this function is defined and real-analytic on $\mathbb{P}^1_\mathbb{R} \smallsetminus \{0\}$. It may have a singularity at 0, so in general it will belong to the larger space $\mathcal{V}^\omega_s[0]$. In fact, we may allow φ itself to belong to this larger space, since then the convergence goes through. For $\mathrm{Av}^-_\eta(\varphi)$ we proceed similarly. Now the point ∞ may be a singularity. We have obtained the following left inverses of $1-\eta$:

$$\mathrm{Av}^+_\eta : \mathcal{V}^\omega_s[0] \longrightarrow \mathcal{V}^\omega_s[0] \quad\text{and}\quad \mathrm{Av}^-_\eta : \mathcal{V}^\omega_s[\infty] \longrightarrow \mathcal{V}^\omega_s[\infty]\,. \tag{4.3}$$

If $\varphi \in \mathcal{V}^\omega_s$, then the total average $\mathrm{Av}_\eta(\varphi) = \mathrm{Av}^+_\eta(\varphi) - \mathrm{Av}^-_\eta(\varphi)$ is defined and belongs to $\mathcal{V}^\omega_s[0,\infty]^\eta$.

Proposition 4.2. *The following three statements are equivalent for $\varphi \in \mathcal{V}^\omega_s$:*

$$1: \ \mathrm{Av}^+_\eta(\varphi) \in \mathcal{V}^\omega_s\,, \qquad 2: \ \mathrm{Av}^-_\eta(\varphi) \in \mathcal{V}^\omega_s\,, \qquad 3: \ \mathrm{Av}_\eta(\varphi) = 0\,. \tag{4.4}$$

Proof. (1) $\Rightarrow$ (3): If $\mathrm{Av}^+_\eta(\varphi) \in \mathcal{V}^\omega_s$, then $\mathrm{Av}_\eta(\varphi) = \mathrm{Av}^+_\eta(\varphi) - \mathrm{Av}^-_\eta(\varphi) \in \mathcal{V}^\omega_s[\infty]^\eta = \{0\}$, by Proposition 4.1. (2) $\Rightarrow$ (3) goes similarly. (3) $\Rightarrow$ (1)&(2): If $\mathrm{Av}_\eta(\varphi) = 0$, then $\mathrm{Av}^+_\eta(\varphi) = \mathrm{Av}^-_\eta(\varphi) \in \mathcal{V}^\omega_s[0] \cap \mathcal{V}^\omega_s[\infty] = \mathcal{V}^\omega_s$. ■

Corollary 4.3. *The kernel of* $\mathrm{Av}_\eta : \mathcal{V}^\omega_s \to \mathcal{V}^\omega_s[0,\infty]^\eta$ *is equal to* $\mathcal{V}^\omega_s\,|\,(1-\eta)$.

Proof. Clearly, $\mathcal{V}^\omega_s\,|\,(1-\eta) \subset \operatorname{Ker}\mathrm{Av}_\eta$. If $\mathrm{Av}_\eta(\varphi) = 0$, then $\varphi = \varphi\,|\,\mathrm{Av}^+_\eta|(1-\eta) \in \mathcal{V}^\omega_s\,|\,(1-\eta)$. ■

For other hyperbolic elements we have

$$\mathrm{Av}^+_{g\eta g^{-1}} : f \mapsto f\,|\,g\,\mathrm{Av}^+_\eta\,g\,, \qquad \mathrm{Av}^+_{\eta^{-1}} = -\eta\,\mathrm{Av}^-_\eta = -\mathrm{Av}^-_\eta\,\eta\,. \tag{4.5}$$

• *Averages in* $\mathcal{G}_s^p$. With the transverse Poisson transformation these statements can be transformed into analogous statements for $\mathcal{G}_s^p$.

For $f \in \mathcal{G}_s^p$, with $p = 2, \ldots, \infty, \omega$, we use that, in the line model, $f(z) = y^s |z|^{-2s} f_\infty(1/z)$ on $\Omega \cap \mathfrak{H}$ where $f_\infty \in C^p(\Omega)$ for some neighborhood Ω of 0 in $\mathbb{C}$. This implies that $\operatorname{Av}_\eta^+ f(z) = y^s |z|^{-2s} \sum_{n=0}^{\infty} t^{-ns} f_\infty(1/t^n z)$ converges absolutely, uniformly on compact sets in $\mathfrak{H}$. We see that for each neighborhood Ω_1 of ∞ in $\mathbb{P}^1_{\mathbb{C}}$ not containing 0 there is $A \in \mathbb{N}$ such that for $z \in C$ the function

$$z \mapsto t^{-ns} f_\infty(1/t^n z)$$

is in $C^p(\Omega_1)$. This shows that $\operatorname{Av}_\eta^+ f \in \mathcal{G}_s^p(\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{0\})$. Similarly we get $\operatorname{Av}_\eta^- f \in \mathcal{G}_s^p(\mathbb{R})$. Observe that $\operatorname{Av}_\eta^+ f$ is in general not an element of $\mathcal{G}_s^p = \mathcal{G}_s^p(\mathbb{P}^1_{\mathbb{R}})$.

The following lemma will be needed for the proof of Theorem A (see Lemma 7.7).

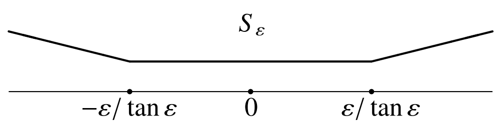
Lemma 4.4. *Suppose that for* $f \in \mathcal{G}_s^\omega$ *the set* $\mathsf{Sing}\operatorname{Av}_\eta^+ f$ *is compact in* $\mathfrak{H}$. *Then* $\operatorname{Av}_\eta^+ f \in \mathcal{G}_s^\omega$.

Proof. The singularities of $\operatorname{Av}_\eta^- f$ (see Definition 3.4) are contained in

$$\bigcup_{n \le -1} \eta^{-n} \mathsf{Sing}\, f\,.$$

Since $\mathsf{Sing}\, f$ is compact there exists $\varepsilon > 0$ such that

$$\mathsf{Sing}\operatorname{Av}_\eta^- f \quad \subset \quad S_\varepsilon := \{z \in \mathfrak{H} \;:\; y > \varepsilon,\; \varepsilon < \arg z < \pi - \varepsilon\}\,.$$



Decreasing $\varepsilon > 0$ if necessary, we arrange that the compact set $\mathsf{Sing}\operatorname{Av}_\eta^+ f$ is also contained in S_ε. So the η-invariant function $\operatorname{Av}_\eta f = \operatorname{Av}_\eta^+ f - \operatorname{Av}_\eta^- f$ determines an element $h \in \mathcal{E}_s(\mathfrak{H} \smallsetminus S_\varepsilon)$, which satisfies $h(tz) = h(z)$ whenever $z, tz \notin S_\varepsilon$. Hence h extends as an element of $\mathcal{E}_s(\mathfrak{H})^\eta$. Thus, h has a Fourier expansion with α running through $\frac{2\pi}{\log t}\mathbb{Z}$:

$$h(z) \;=\; \sum_\alpha h_\alpha(z)\,, \qquad h_\alpha(z) \;=\; \frac{1}{\log t}\int_0^{\log t} e^{-i\alpha u} h(e^u z)\, du\,. \tag{4.6}$$

For each such α, the function h_α is a linear combination of the functions $f^R_{s,\alpha}$ and $f^L_{s,\alpha}$ in (4.2).

The function h represents an element of $\mathcal{W}_s^\omega[0,\infty]$. Hence near $\mathbb{R}$ it has the form $h(z) = y^s \tilde{h}(z)$, with $\tilde{h}$ real analytic on a neighborhood of $\mathbb{R} \smallsetminus \{0\}$ in $\mathbb{C}$. For $x > 0$ and small values of $\theta > 0$:

$$\begin{aligned} h_\alpha(xe^{i\theta}) \;&=\; (x\sin\theta)^s \frac{1}{\log t}\int_0^{\log t} e^{-i\alpha u}\tilde{h}(e^u x e^{i\theta})\, du \\ &=\; (x\sin\theta)^s \cdot (\text{real-analytic on a neighborhood of } (0,\infty))\,. \end{aligned}$$

So h_α represents an element of $\mathcal{W}_s^\omega(0,\infty)$, and the coefficient of $f^L_{s,\alpha}$ is zero. Proceeding similarly near $(-\infty, 0)$ we obtain that the coefficient of $f^R_{s,\alpha}$ vanishes as well. This works for all $\alpha \in \frac{2\pi}{\log t}\mathbb{Z}$, and hence $h = 0$. This means that $\mathrm{Av}_\eta f \in \mathcal{G}_s^\omega(\mathbb{R})$. Since $\mathrm{Av}_\eta^- f \in \mathcal{G}^\omega(\mathbb{R})$, we have also $\mathrm{Av}_\eta^+ \in \mathcal{G}_s^\omega(\mathbb{R})$. For all $f \in \mathcal{G}_s^\omega$, we have $\mathrm{Av}_\eta^+ f \in \mathcal{G}_s^\omega(\mathbb{P}^1_\mathbb{R} \smallsetminus \{0\})$. Hence $\mathrm{Av}_\eta^+ f \in \mathcal{G}_s^\omega(\mathbb{P}^1_\mathbb{R}) = \mathcal{G}_s^\omega$. ■

4.2. *Invariants and averages for parabolic elements.* For parabolic elements of G the invariants in subspaces of $\mathcal{V}_s$ with high regularity vanish, like for hyperbolic elements. The convergence of the averages is more delicate than in the hyperbolic case.

Each parabolic element of G is conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ or to T^{-1}. Parabolic elements have only one fixed point, situated on $\mathbb{P}^1_\mathbb{R}$. The element T fixes $\infty \in \mathbb{P}^1_\mathbb{R}$.

Proposition 4.5. *The space $(\mathcal{V}_s^p)^T$ is zero for $p = 2, \ldots, \infty, \omega$.*

Proof. Each $\varphi \in \mathcal{V}_s^p$ (line model) satisfies $\varphi(x) = x^{-2s}\varphi(1/x) = \mathrm{o}(1)$ as $x \to \infty$. If φ is also periodic, then it must vanish identically. ■

For $\varphi \in \mathcal{V}_s^p$ (line model with $p = 2, \ldots, \infty, \omega$), the averages

$$\mathrm{Av}_T^+(\varphi)(x) \;=\; \sum_{n=0}^{\infty} \varphi(x+n)\,, \qquad \mathrm{Av}_T^-(\varphi)(x) \;=\; -\sum_{n=-\infty}^{-1} \varphi(x+n)\,, \tag{4.7}$$

converge if $\mathrm{Re}\, s > \frac{1}{2}$ or if $\mathrm{Re}\, s > 0$ and the number $C = \varphi^{\mathbb{P}}(\infty)$ vanishes. In general we have $\varphi(x) = C|x|^{-2s} + \mathrm{O}(|x|^{-2s-1})$ as $|x| \to \infty$ and we define (for $\mathrm{Re}\, s > 0$, $s \neq \frac{1}{2}$)

$$\begin{aligned} \mathrm{Av}_T^+(\varphi)(x) \;&=\; \sum_{n=0}^{\infty}\Bigl(\varphi(x+n) - \frac{C}{(n+1)^{2s}}\Bigr) \;+\; C\,\zeta(2s)\,, \\ \mathrm{Av}_T^-(\varphi)(x) \;&=\; -\sum_{n=1}^{\infty}\Bigl(\varphi(x-n) - \frac{C}{n^{2s}}\Bigr) \;-\; C\,\zeta(2s)\,. \end{aligned} \tag{4.8}$$

Since differentiation only improves the convergence, we see that if $\varphi \in \mathcal{V}_s^p$, then $\mathrm{Av}_T^\pm \varphi$ is in $C^p(\mathbb{R})$. So we have $\mathrm{Av}_T^\pm : \mathcal{V}_s^p \to \mathcal{V}_s^p[\infty] = \mathcal{V}_s^p(\mathbb{R})$, and more generally

$$\mathrm{Av}_T^+ \;:\; \mathcal{V}_s^p((b,a)_c) \;\to\; \mathcal{V}_s^p(b,\infty), \quad \mathrm{Av}_T^- \;:\; \mathcal{V}_s^p((b,a)_c) \;\to\; \mathcal{V}_s^p(-\infty, a+1) \tag{4.9}$$

for a, $b \in \mathbb{R}$ with $a < b$, where we use the convenient notation $(b,a)_c$ for the "cyclic interval" $(b,\infty) \cup \{\infty\} \cup (-\infty, a) \subset \mathbb{P}^1_\mathbb{R}$. It is clear that these one-sided averages satisfy

$$\varphi\,|\,(1-T)\,|\,\mathrm{Av}_T^\pm \;=\; \varphi\,|\,\mathrm{Av}_T^\pm\,|\,(1-T) \;=\; \varphi\,. \tag{4.10}$$

Furthermore, if we denote by C_m $(0 \leq m \leq p)$ the coefficient of x^m in the Taylor expansion of $\varphi|_{2s}\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right](x)$, then using the Euler-Maclaurin summation formula or arguing as in [19], Chap. III, §3, we find that the functions $\mathrm{Av}_T^+(\varphi)$ and $\mathrm{Av}_T^-(\varphi)$ have the one-sided asymptotic behavior

$$\mathrm{Av}_T^\pm(\varphi)(x) \;=\; |x|^{-2s} \sum_{m=-1}^{p-1} C_m^*\, x^{-m} \;+\; \mathrm{O}(|x|^{-2s-p}) \qquad \text{as } \pm x \to \infty \tag{4.11}$$

(in the line model) with the coefficients C_m^* in both cases given explicitly by

$$C_m^* \;=\; \frac{(-1)^{m+1}}{m+2s} \sum_{k=0}^{m+1} B_k\, C_{m+1-k} \binom{m+2s}{k}, \tag{4.12}$$

where B_k is the kth Bernoulli number. If $p = \infty$ or $p = \omega$, then (4.11) must be interpreted as an infinite asymptotic expansion (not necessarily convergent for any x, even in the analytic case).

For other parabolic $\pi \in G$, we define $\mathrm{Av}^{\pm}_{\pi}$ by conjugation and the relations $\mathrm{Av}^{+}_{\pi^{-1}} = -\pi\mathrm{Av}^{-}_{\pi}$ and $\mathrm{Av}^{-}_{\pi^{-1}} = -\pi^{-1}\mathrm{Av}^{+}_{\pi} = -\mathrm{Av}^{+}_{\pi}\pi$.

Using the transverse Poisson transformation $\mathsf{P}^{\dagger}_s$ in §3.2 we can transport the one-sided averages to $\mathcal{W}^p_s = \mathcal{G}^p_s/\mathcal{N}^p_s$ for $p = 2, \dots, \omega$. But we can also define the maps $\mathrm{Av}^{\pm}_T$ directly at the level of $\mathcal{G}^p_s$ in the obvious way (for instance, $\mathrm{Av}^{+}_T f(z)$ is defined as $\sum_{n=0}^{\infty} f(n+z)$ if $C = f^{\mathbb{P}}(\infty)$ vanishes and otherwise as $\sum_{n=0}^{\infty}\big(f(n+z) - Cy^s/(n+1)^{2s}\big) + C\zeta(2s)y^s$). We thus obtain maps as in (4.9) with $\mathcal{V}$ replaced by $\mathcal{W}$ or $\mathcal{G}$, still satisfying the relations (4.10). The new aspect is that, as partners of the asymptotic relations (4.11) on $\mathbb{R}$, we get new asymptotic relations for $\mathrm{Av}^{\pm}_T f(x+iy)$ as $y \to \infty$.

Lemma 4.6. *Let $f \in \mathcal{G}^p_s(I)$, with $p = 2, \dots, \infty, \omega$, for some interval $I \subset \mathbb{P}^1_{\mathbb{R}}$ containing ∞. For $s \neq \frac{1}{2}$, we have*

$$\mathrm{Av}^{\pm}_T f(z) \;=\; \pm\,\frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{2\,\Gamma(s)}\, f^{\mathbb{P}}(\infty)\, y^{1-s} \;+\; \mathrm{O}\big(f^{\mathbb{P}}(\infty)\, x\, y^{-s}\big) \;+\; \mathrm{O}(y^{-s})\,.$$

as $y \to \infty$, uniformly in x.

Proof. If $f^{\mathbb{P}}(\infty) = 0$, then $f(z) = \mathrm{O}(y^s|z|^{-2s-1})$ for $|z|$ large, so

$$\mathrm{Av}^{+}_T f(z) \;=\; \sum_{n=0}^{\infty} \mathrm{O}\Big(\frac{y^s}{|n+z|^{2s+1}}\Big) \;=\; \mathrm{O}\Big(y^s \int_{-\infty}^{\infty} \frac{dt}{(t^2+y^2)^{s+1/2}}\Big) \;=\; \mathrm{O}(y^{-s})\,.$$

To treat the general case, it suffices to consider one function f with $f^{\mathbb{P}}(\infty) \neq 0$. We choose $f = F_s$, where $F_s(z) = y^s|z|^{-2s}$. If $\mathrm{Re}\, s > \frac{1}{2}$ we have

$$\begin{aligned}\mathrm{Av}^{+}_T F_s(z) \;=\; y^s \sum_{n=0}^{\infty}\Big(&\frac{1}{((x+n)^2+y^2)^s} - \int_{x+n}^{x+n+1} \frac{dt}{(t^2+y^2)^s}\Big)\\ &+\; y^s \int_x^0 \frac{dt}{(t^2+y^2)^s} \;+\; c(s)\, y^{1-s}\,.\end{aligned}$$

where $c(s) = \int_0^{\infty}(t^2+1)^{-s}dt = \sqrt{\pi}\,\Gamma(s-\frac{1}{2})/2\,\Gamma(s)$. The sum on the right converges for $\mathrm{Re}\, s > 0$ and the formula remains true in this domain (for $s \neq \frac{1}{2}$) by uniqueness of meromorphic continuation. We then have the uniform estimate

$$\begin{aligned}\mathrm{Av}^{+}_T F_s(z) \,-\, c(s)\, y^{1-s} \;&\ll y^s \sum_{n\in\mathbb{Z}} \frac{|x+n| \,+\, 1}{((x+n)^2+y^2)^{s+1}} \;+\; y^s \int_x^0 y^{-2s}\, dt\\ &=\; \mathrm{O}(y^{-s}) \,+\, \mathrm{O}(xy^{-s})\,.\end{aligned}$$

This completes the proof for Av^{+}_T. The estimate for Av^{-}_T is exactly similar. ∎

Chapter II. Maass forms and analytic cohomology: cocompact groups

In this chapter, we define a map from the space $\mathcal{E}_s^\Gamma$ of Γ-invariant λ_s-eigenfunctions of the Laplace operator to the cohomology group $H^1(\Gamma; \mathcal{V}_s^\omega)$. In Section 5 this is carried out for any discrete $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$. If Γ has elements of infinite order, this map is injective. In Section 7 we prove that the map is a bijection if Γ is cocompact, thus proving part of Theorem A. As mentioned in the introduction, it is known that $\mathcal{E}_s^\Gamma$ and $H^1(\Gamma; \mathcal{V}_s^\omega)$ are isomorphic for cocompact Γ. Here we construct a map from cohomology to Maass forms explicitly. This also forms a preparation for Chapter IV, where groups with cusps are considered.

The constructions in Section 7 require a description of the cohomology using a complex based on the geometry of the action of Γ on $\mathbb{H}$. We discuss this description in Section 6.

5. From Maass forms to analytic cohomology. This section starts with a review of the standard definitions of group cohomology. In §5.2 we construct a map $\mathcal{E}_s^\Gamma \to H^1(\Gamma; \mathcal{V}_s^\omega)$ for any discrete $\Gamma \subset G = \mathrm{PSL}_2(\mathbb{R})$.

5.1. *Group cohomology.* See, e.g., [1], Chap. I and Chap. III, §1, for a general reference.

• *Resolutions.* For the moment let Γ denote an arbitrary group. We recall that the homology and cohomology groups of Γ with coefficients in a (right) $\mathbb{Q}[\Gamma]$-module V are defined with help of a projective resolution

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0$$

of the trivial $\mathbb{Q}[\Gamma]$-module $\mathbb{Q}$ as the (co)homology of the induced complexes

$$\cdots \xrightarrow{\partial_2} F_1 \otimes_{\mathbb{Q}[\Gamma]} V \xrightarrow{\partial_1} F_0 \otimes_{\mathbb{Q}[\Gamma]} V \longrightarrow 0\,,$$
$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Q}[\Gamma]}(F_0, V) \xrightarrow{d^0} \mathrm{Hom}_{\mathbb{Q}[\Gamma]}(F_1, V) \xrightarrow{d^1} \cdots\,,$$

namely

$$H_i(\Gamma; V) = \mathrm{Ker}\,(\partial_i)/\mathrm{Im}\,(\partial_{i+1})\,, \qquad H^i(\Gamma; V) = \mathrm{Ker}\,(d^i)/\mathrm{Im}\,(d^{i-1})\,. \tag{5.1}$$

We work with coefficients in $\mathbb{Q}$ because this gives us more freedom in the construction of projective resolutions (c.f. §6.1). These cohomology groups do not depend on the choice of the projective resolution. If $F_.$ and $\tilde{F}_.$ are two projective resolutions of the trivial $\mathbb{Q}[\Gamma]$-module $\mathbb{Q}$, the identity map $\mathbb{Q} \to \mathbb{Q}$ can always be lifted to an augmentation preserving chain map between the resolutions. This lift is unique up to homotopy, and induces isomorphisms of the homology and cohomology groups determined by the resolutions. The construction of such a chain map $F_. \to \tilde{F}_.$ may depend on many choices, so it may take work to describe the corresponding isomorphism of the (co)homology groups explicitly. For this reason it is important for explicit cohomological constructions to choose a specific resolution with good properties.

• *Standard resolution.* The standard model of group cohomology is obtained from the *standard resolution* $F_.^{\mathrm{gr}}$, where F_i^{gr} is the free $\mathbb{Q}[\Gamma]$-module $\mathbb{Q}[\Gamma^{i+1}]$. The boundary maps ∂_i, the augmentation ε, and the Γ-action are induced by

$$\begin{aligned}\partial_i(\gamma_0,\cdots,\gamma_i) &= \sum_{j=0}^{i}(-1)^j(\gamma_0,\cdots,\widehat{\gamma_j},\cdots,\gamma_i)\,,\\ \varepsilon(\Gamma_0) &= 1\,,\qquad (\gamma_0,\cdots,\gamma_i)\,|\,\gamma \;=\; (\gamma_0\gamma,\cdots,\gamma_i\gamma)\,.\end{aligned}\tag{5.2}$$

In this model, an i-cochain is represented by a Γ-equivariant map $c:\Gamma^{i+1}\to V$, which is then extended by linearity to $c: F_i^{\mathrm{gr}}\to V$. The equivariance implies that c is completely determined by its restriction ψ to $\Gamma^i\times\{1\}\subset\Gamma^{i+1}$, and one often uses this inhomogeneous version. (The last variable is then omitted from the notation and the definition of the coboundary map is modified in the obvious way.)

Each F_i^{gr} is a free $\mathbb{Q}[\Gamma]$-module. In dimension $i>0$ the rank is infinite if $|\Gamma|=\infty$. In dimension $i=0$, the cocycles satisfy $c(\gamma)=c(1)v\,|\,\gamma$, and hence are determined by $c(1)\in V^\Gamma$, and since there are no cochains in dimension -1 we have

$$H^0(\Gamma;V) \;=\; V^\Gamma \;=\; \{v\in V \;:\; v\,|\,\gamma = v \text{ for all } \gamma\in\Gamma\}\,. \tag{5.3}$$

For homology we find that $H_0(\Gamma;V)=V_\Gamma$, where V_Γ is the submodule of *coinvariants* $V/\langle v\,|\,(1-\gamma)\;:\; v\in V,\,\gamma\in\Gamma\rangle$.

In dimension $i=1$ the standard model gives homogeneous cocycles $(\gamma_0,\gamma_1)\mapsto c(\gamma_0,\gamma_1)\in V$ satisfying for all $\gamma_j\in\Gamma$:

$$c(\gamma_0,\gamma_1)\,|\,\gamma_2 \;=\; c(\gamma_0\gamma_2,\gamma_1\gamma_2) \quad\text{ and }\quad c(\gamma_0,\gamma_1)+c(\gamma_1,\gamma_2)=c(\gamma_0,\gamma_2)\,.$$

Such a 1-cocycle is a coboundary if $c(\gamma_0,\gamma_1)=f(\gamma_0)-f(\gamma_1)$ for some $f:\Gamma\to V$ satisfying $f(\gamma_1\gamma_2)=f(\gamma_1)\,|\,\gamma_2$. Going over to inhomogeneous cocycles $\gamma\mapsto\psi_\gamma=c(\gamma,1)$, we get the following well-known description of the first cohomology group:

$$\begin{aligned}H^1(\Gamma;V) &= Z^1(\Gamma;V)/B^1(\Gamma;V)\,,\\ Z^1(\Gamma;V) &= \left\{\psi:\Gamma\longrightarrow V \;:\; \psi_{\gamma\delta}=\psi_\gamma\,|\,\delta+\psi_\delta \text{ for all }\gamma,\delta\in\Gamma\right\}\,,\\ B^1(\Gamma;V) &= \{\gamma\mapsto v\,|\,(\gamma-1) \;:\; v\in V\}\,.\end{aligned}\tag{5.4}$$

5.2. *From invariant eigenfunctions to cohomology.* Let $\Gamma\subset G$ be an arbitrary subgroup, provided with the discrete topology. We now shall explicitly define a linear map $\mathbf{r}:\mathcal{E}_s^\Gamma\to H^1(\Gamma;\mathcal{V}_s^\omega)$, and show that it is injective if Γ has elements of infinite order.

• *Definition of analytic cocycles associated to invariant eigenfunctions.* Let $u\in\mathcal{E}_s^\Gamma$. We associate to it three inhomogeneous cocycles $(r_\gamma)\in Z^1(\Gamma;\mathcal{V}_s^\omega)$, $(p_\gamma)\in Z^1(\Gamma;\mathcal{E}_s^\omega)$ and $(q_\gamma)\in Z^1(\Gamma;\mathcal{W}_s^\omega)$ which correspond to one another under the isomorphisms $\mathcal{V}_s^\omega\cong\mathcal{E}_s^\omega\cong\mathcal{W}_s^\omega$ given by the Poisson transformation and the transverse Poisson transformation. These cocycles are obtained by integrating the Γ-invariant closed 1-forms

$$[u,R(\zeta;\,\cdot\,)^s]\,,\qquad [u,p_s(\,\cdot\,,z)]\,,\qquad [u,q_s(\,\cdot\,,z)]\,.$$

The Γ-invariance follows from (1.10a), (2.25) and (1.5). We choose a base point $z_0\in\mathbb{H}$, and integrate over a path from $\gamma^{-1}z_0$ to z_0:

$$r_\gamma(\zeta) \;=\; \int_{\gamma^{-1}z_0}^{z_0}[u,R(\zeta;\,\cdot\,)^s]\,, \tag{5.5a}$$

$$p_\gamma(z) \;=\; \int_{\gamma^{-1}z_0}^{z_0} [u, p_s(\,\cdot\,, z)]\,, \tag{5.5b}$$

$$q_\gamma(z) \;=\; \int_{\gamma^{-1}z_0}^{z_0} [u, q_s(\,\cdot\,, z)]\,. \tag{5.5c}$$

So $r_\gamma \in \mathcal{V}_s^\omega$, $p_\gamma \in \mathcal{E}_s$. We identify q_γ with the element of $\mathcal{W}_s^\omega$ represented by it. Changing the choice of the base point changes the cocycles by a coboundary. Thus, we find respectively $\mathbf{r}u \in H^1(\Gamma; \mathcal{V}_s^\omega)$, $\mathbf{p}u \in H^1(\Gamma; \mathcal{E}_s^\omega)$ and $\mathbf{q}u \in H^1(\Gamma; \mathcal{W}_s^\omega)$, which depend linearly on u. Relations (2.28) and (3.6c) imply that

$$\mathbf{p} \;=\; \mathsf{P}_s \mathbf{r}\,, \qquad \mathbf{q} = b(s) \mathsf{P}_s^\dagger \mathbf{r}\,, \tag{5.6}$$

with $b(s)$ as in (3.4d).

The Poisson kernel R in (5.5a) is considered to be general. Strictly speaking, (5.5a) defines the cocycle $\gamma \mapsto r_\gamma$ in the line model of $\mathcal{V}_s^\omega$. In the projective model, work with $R^{\mathbb{P}}$, and in the circle model with $R^{\mathbb{S}}$ and a base point in $\mathbb{D}$. See (2.30).

• *Symmetry $s \leftrightarrow 1-s$.* Since $\mathcal{E}_s^\omega = \mathcal{E}_{1-s}^\omega$, we can carry out the construction with s replaced by $1-s$. Denote the corresponding cocycles by $\hat{r}_\gamma$, $\hat{p}_\gamma$ and $\hat{q}_\gamma$. For $s \neq \frac{1}{2}$ we have $\hat{r}_\gamma = I_s r_\gamma$, $\hat{p}_\gamma = p_\gamma$, by (2.28) and (2.29), and $p_\gamma = \frac{\tan \pi s}{\pi}(q_\gamma - \tilde{q}_\gamma)$, by (3.6c) and (3.7).

$$\begin{tikzcd}
 & H^1\left(\Gamma; \mathcal{V}_{1-s}^\omega\right) \arrow[r, "b(1-s)\,\mathsf{P}_{1-s}^\dagger"] \arrow[dl, "\mathsf{P}_{1-s}"'] & H^1\left(\Gamma; \mathcal{W}_{1-s}^\omega\right) \\
H^1\left(\Gamma; \mathcal{E}_s^\omega\right) & \mathcal{E}_s^\Gamma \arrow[l, "\mathbf{p}"'] \arrow[u, "\hat{\mathbf{r}}"] \arrow[ur, "\hat{\mathbf{q}}"] \arrow[d, "\mathbf{r}"] \arrow[dr, "\mathbf{q}"] & \\
 & H^1\left(\Gamma; \mathcal{V}_s^\omega\right) \arrow[r, "b(s)\,\mathsf{P}_s^\dagger"] \arrow[ul, "\mathsf{P}_s"] \arrow[uu, bend right, "I_s"'] & H^1\left(\Gamma; \mathcal{W}_s^\omega\right)
\end{tikzcd} \tag{5.7}$$

• *Formulation with hyperfunctions.* Another point of view uses the isomorphism $\mathcal{E}_s^\Gamma \cong (\mathcal{V}_s^{-\omega})^\Gamma$ induced by the Poisson transformation. The short exact sequence (2.12) induces a long exact sequence of cohomology groups, which gives a connecting homomorphism $(\mathcal{V}_s^{-\omega})^\Gamma = H^0(\Gamma; \mathcal{V}_s^{-\omega}) \to H^1(\Gamma; \mathcal{V}_s^\omega)$. To describe this map explicitly, we choose for a given $\alpha \in (\mathcal{V}_s^{-\omega})^\Gamma$ a representative $g \in \mathbf{H}_s$. This gives the inhomogeneous cocycle $\gamma \mapsto \psi_\gamma = g\,|\,(\gamma - 1)$ with values in $\mathcal{V}_s^\omega$.

For a given $u \in \mathcal{E}_s^\Gamma$ let $\alpha \in (\mathcal{V}_s^{-\omega})^\Gamma$ be chosen such that $\mathsf{P}_s \alpha = u$. Theorem 2.2 gives an explicit choice (depending on u and a base point $z_0 \in \mathfrak{H}$) for a representative g of α, in the projective model of the principal series. Using the same base point in (5.5a), we find for $\zeta \in \mathfrak{H}$ near $\mathbb{P}^1_{\mathbb{R}}$ and $\gamma \in \Gamma$:

$$\begin{aligned}
\psi_\gamma^{\mathbb{P}}(\zeta) \;&=\; u(z_0)\, R^{\mathbb{P}}(\,\cdot\,; z_0)^s|^{\mathbb{P}}_{2s}\,\gamma\,(\zeta) + \int_{z_0}^{\gamma\zeta} \left[u(z'), R^{\mathbb{P}}(\,\cdot\,; z')^s|^{\mathbb{P}}_{2s}\,\gamma\,(\zeta)\right]_{z'} \\
&\qquad - u(z_0)\, R^{\mathbb{P}}(\zeta; z_0)^s - \int_{z_0}^{\zeta} [u, R^{\mathbb{P}}(\zeta; \,\cdot\,)^s] \\
&=\; u(z_0)\, R^{\mathbb{P}}(\,\cdot\,; z_0)^s|^{\mathbb{P}}_{2s}\,(\gamma - 1)\,(\zeta) + \int_{\gamma^{-1}z_0}^{\zeta} [u, R^{\mathbb{P}}(\zeta; \,\cdot\,)^s] - \int_{z_0}^{\zeta} [u, R^{\mathbb{P}}(\zeta; \,\cdot\,)^s]
\end{aligned}$$

$$= \; u(z_0)\, R^{\mathbb{P}}(\,\cdot\,; z_0)^s|^{\mathbb{P}}_{2s}\,(\gamma-1)\,(\zeta) + r^{\mathbb{P}}_\gamma(\zeta)\,,$$

where we have used (1.10a), (2.25) and the Γ-invariance of u. For ζ in the lower half-plane, near $\mathbb{P}^1_{\mathbb{R}}$, we also use (1.9) and (1.10d) to find the following equality.

$$\begin{aligned}
\psi^{\mathbb{P}}_\gamma(\zeta) \;&=\; \int_{\overline{\gamma\zeta}}^{z_0} [R^{\mathbb{P}}(\,\cdot\,; z')^s|^{\mathbb{P}}_{2s}\,\gamma\,(\zeta), u(z')] - \int_{\bar\zeta}^{z_0} [R^{\mathbb{P}}(\zeta;\,\cdot\,)^s, u] \\
&=\; \Big(\int_{\bar\zeta}^{\gamma^{-1}z_0} - \int_{\bar\zeta}^{z_0} \Big)\Big(-[u, R^{\mathbb{P}}(\zeta;\,\cdot\,)^s] + d\big(uR^{\mathbb{P}}(\zeta;\,\cdot\,)^s\big)\Big) \\
&=\; r^{\mathbb{P}}_\gamma(\zeta) + u(\gamma^{-1}z_0)\, R^{\mathbb{P}}(\zeta;\gamma^{-1}z_0)^s - u(z_0)\, R^{\mathbb{P}}(\zeta; z_0)^s \\
&=\; r^{\mathbb{P}}_\gamma(\zeta) + u(z_0)\; R^{\mathbb{P}}(\,\cdot\,; z_0)^s|^{\mathbb{P}}_{2s}\,(\gamma-1)\,(\zeta)\,,
\end{aligned}$$

which is the same expression as we obtained for $\zeta \in \mathfrak{H}$. By holomorphic continuation this description also holds for $\zeta \in \mathbb{P}^1_{\mathbb{R}}$. Thus, the cocycle $\gamma \mapsto \psi_\gamma$ represents the same cohomology class $\mathbf{r}u$ as $\gamma \mapsto r_\gamma$.

Proposition 5.1. *If the discrete subgroup* $\Gamma \subset G$ *is infinite, then* $\mathbf{r}$, $\mathbf{p}$ *and* $\mathbf{q}$ *are injective.*

Proof. For the injectivity it suffices to consider only $\mathbf{r}$, since $\mathbf{p}$ and $\mathbf{q}$ are isomorphic transforms of $\mathbf{r}$. The formulation with hyperfunctions shows that $\mathbf{r}$ corresponds to the connecting homomorphism δ in the part

$$\longrightarrow \mathbf{H}^\Gamma_s \longrightarrow (\mathcal{V}^{-\omega}_s)^\Gamma \xrightarrow{\;\delta\;} H^1(\Gamma; \mathcal{V}^\omega_s) \longrightarrow$$

of the long exact sequence associated to (2.12). Hence it suffices to show that $\mathbf{H}^\Gamma_s = \{0\}$.

We use the circle model. Let $g = g^{\mathbb{S}} \in \mathbf{H}^\Gamma_s \smallsetminus \{0\}$. Interchanging if necessary the roles of the interior and exterior of $\mathbb{S}^1$, we can assume that g is holomorphic and non-zero on the annulus $R = \{c \le |w| < 1\}$ for some $c < 1$. Then the 1-form $\omega = d\log g = \frac{g'(w)}{g(w)}\,dw$ is meromorphic on R, with integral residues. Since $\mathbb{D} \smallsetminus R$ is compact and Γ is infinite and discrete, we can choose $\gamma = \left[\begin{smallmatrix} A & B \\ \bar B & \bar A \end{smallmatrix}\right] \in \Gamma$ such that $R \cup \gamma R = \mathbb{D}$. The Γ-invariance implies $g(\gamma w) = g(w)\big((\bar A + \bar B w)(A + B w^{-1})\big)^s$, and hence

$$\omega\circ\gamma - \omega \;=\; s\Big(\frac{1}{w + \bar A/\bar B} \;-\; \frac{1}{w} \;+\; \frac{1}{w + B/A} \Big)$$

near $\partial\mathbb{D}$. It follows that ω has a meromorphic continuation to all of $\mathbb{D}$ and has integral residues everywhere except at $w = 0$, where its residue equals s. (Note that the point $-B/A = \gamma^{-1}0$ lies in R.) This is a contradiction since $\frac{1}{2\pi i}\int_{|w|=c} \omega = \frac{1}{2\pi i}\int_{|w|=c} d\log g \in \mathbb{Z}$ and $s \notin \mathbb{Z}$. ∎

6. Cohomology for cocompact groups. The description of group cohomology in §5.1 with the standard resolution $F^{\mathrm{gr}}_{\cdot}$ does not use the fact that Γ acts on the hyperbolic plane. We will mention in §6.1 and §6.2 several resolutions of geometrical nature, and describe group cohomology with these resolutions in §6.3. There we also formulate the linear maps $\mathbf{r}$, $\mathbf{p}$ and $\mathbf{q}$ in terms of these geometrical resolutions.

6.1. *Projective resolutions of geometric nature.* Group cohomology can be computed based on a free action of the group on a contractible set. Here we consider cocompact discrete $\Gamma \subset G$, i.e., discrete subgroups for which the quotient $\Gamma\backslash\mathbb{H}$ is compact. The space $\mathbb{H}$ is contractible. However the action of Γ on $\mathbb{H}$ is not free if Γ has elliptic elements, which have finite order. We circumvent this problem by working only with Γ-modules that are vector spaces over $\mathbb{Q}$.

First we discuss a resolution that is similar to $F^{\mathrm{gr}}_{\cdot}$ in §5.1, but rather large. More practical are smaller resolutions, based on a Γ-tesselation of $\mathbb{H}$.

• *Chain complex on* $\mathbb{H}$. The action of Γ on the contractible space $\mathbb{H}$ is taken into account in the complex $F^{\mathrm{hyp}}_{\cdot}$ defined by $F^{\mathrm{hyp}}_i = \mathbb{Q}[\mathbb{H}^{i+1}]$, with boundary maps ∂_i, augmentation ε, and group action determined by

$$\begin{aligned} \partial_i(P_0, \cdots, P_i) &= \sum_{j=0}^{i} (-1)^j (P_0, \cdots, \hat{P}_j, \cdots, P_i)\,, \\ \varepsilon(P_0) &= 1\,, \\ (P_0, \cdots, P_i)\,|\,\gamma &= (\gamma^{-1}P_0, \cdots, \gamma^{-1}P_i)\,. \end{aligned} \tag{6.1}$$

In low dimensions, we can think of the generators of $\mathbb{Q}[\mathbb{H}^{i+1}]$ as geometric objects: (P) corresponds to the point $P \in \mathbb{H}$, and (P, Q) corresponds to the geodesic segment oriented from P to Q (degenerate if $P = Q$). The generator (P, Q, R) corresponds to a (possibly degenerate) triangle with a numbering of it vertices.

The $\mathbb{Q}[\Gamma]$-modules $\mathbb{Q}[\mathbb{H}^{i+1}]$ need not be free if Γ has elliptic elements, which fix points in $\mathbb{H}$. To see that $\mathbb{Q}[\mathbb{H}^{i+1}]$ is a projective $\mathbb{Q}[\Gamma]$-module, we have to show that there is a $\mathbb{Q}[\Gamma]$-linear lift $s : \mathbb{Q}[\mathbb{H}^{i+1}] \to B$ for each given $t : \mathbb{Q}[\mathbb{H}^{i+1}] \to C$ in each exact sequence of $\mathbb{Q}[\Gamma]$-modules:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\ & & \mathbb{Q}[\mathbb{H}^{i+1}] \arrow[u, "s"] \arrow[ur, "t"'] & & \end{tikzcd} \tag{6.2}$$

This is done by taking lifts $b_x \in B$ of $t(x) \in C$ for a set of $x \in \mathbb{H}^{i+1}$ generating the $\mathbb{Q}[\Gamma]$-module $\mathbb{Q}[\mathbb{H}^{i+1}]$, and determining s by $s(x) = \frac{1}{|\Gamma_x|} \sum_{\gamma\in\Gamma_x} b_x\,|\,\gamma$, where Γ_x is the stabilizer of x in Γ. Then s can be extended $\mathbb{Q}[\Gamma]$-linearly. See [1], Chap. I, §8, for a further discussion of projective modules.

There are augmentation preserving chain maps $F^{\mathrm{gr}}_{\cdot} \to F^{\mathrm{hyp}}_{\cdot}$ and $F^{\mathrm{hyp}}_{\cdot} \to F^{\mathrm{gr}}_{\cdot}$ that induce isomorphisms in the cohomology groups. The latter requires uncountably many choices to be made. The former can be obtained with only one choice: Take a base point $P_0 \in \mathbb{H}$. This leads to the explicit chain map induced by $(\gamma_0, \cdots, \gamma_i) \mapsto (\gamma_0^{-1}P_0, \cdots, \gamma_i^{-1}P_0)$.

This description of (co)homology, with the chain complex on $\mathbb{H}$ as the projective resolution, can be used to describe group cohomology for any discrete subgroup $\Gamma \subset G$ for which the isotropy groups of elements of $\mathbb{H}$ are finite.

The symmetric group $\mathfrak{S}_{i+1}$ acts on F^{hyp}_i by $\pi(P_0, \ldots, P_i) = (P_{\pi 0}, \ldots, P_{\pi i})$. Let $F^{\mathrm{hyp}-}_i$ be the subspace on which $\mathfrak{S}_{i+1}$ acts by the sign character. These spaces form a subcomplex $F^{\mathrm{hyp}-}_{\cdot} \subset F^{\mathrm{hyp}}_{\cdot}$, which is also a projective resolution of $\mathbb{Q}$. A chain map

$F_\cdot^{\mathrm{hyp}} \to F_\cdot^{\mathrm{hyp}-}$ is the *antisymmetrization* $A_\cdot$ determined by

$$A_i : (P_0, \dots, P_i) \mapsto \frac{1}{(i+1)!} \sum_{\pi \in \mathfrak{S}_{i+1}} \mathrm{sign}(\pi)\, (P_{\pi 0}, \dots, P_{\pi i})\,. \tag{6.3}$$

This variant $F_\cdot^{\mathrm{hyp}-}$ (in which, for example, (P, Q, R) now corresponds to a triangle which is still oriented, but no longer has a numbering of its vertices) is often more convenient than the resolution $F_\cdot^{\mathrm{hyp}}$ itself. (One could avoid introducing denominators by defining $F_\cdot^{\mathrm{hyp}-}$ as a quotient complex rather than a subcomplex of $F_\cdot^{\mathrm{hyp}}$, but in any case they do not disturb us since we work over $\mathbb{Q}$.)

• *Resolutions based on a tesselation.* The models that we like best are geometrical, and finite in two ways: Each F_i is finitely generated as a $\mathbb{Q}[\Gamma]$-module, and F_i vanishes for $i > 2$. Here we use that $\Gamma \subset G$ is discrete and cocompact.

By a *tesselation* we mean a locally finite Γ-invariant covering $\mathcal{T}$ of $\mathbb{H}$ by compact polygons with geodesic boundary segments. The polygons overlap at most in their boundaries. Such a covering gives rise to the set $X_2 = X_2^{\mathcal{T}}$ of polygons of $\mathcal{T}$ (with the orientation inherited from that of $\mathbb{H}$), the set $X_1 = X_1^{\mathcal{T}}$ of oriented edges of $\mathcal{T}$ (with each element of X_1 arising as a boundary component of two neighboring elements of X_2), and the set $X_0 = X_0^{\mathcal{T}}$ of vertices. For vertices P we have $\varepsilon(P) = 1$. Each $e \in X_1$ is the oriented edge $e_{P,Q}$ (or $e(P, Q)$ when we want to avoid subscripts) joining some vertex P of $\mathcal{T}$ to a neighboring vertex Q. Thus $e_{Q,P} = -e$ in $\mathbb{Q}[X_1]$, so that we have chosen only one of the two possible orientations of the edge in defining X_1. We then define $\partial_1 : \mathbb{Q}[X_1] \to \mathbb{Q}[X_0]$ by $\partial_1 e = (Q) - (P)$. A polygon $V \in X_2$ has vertices $P_1, \cdots, P_l$, ordered corresponding to the orientation of $\mathbb{H}$. The boundary is $\partial_2 V = e_{P_1,P_2} + e_{P_2,P_3} + \cdots + e_{P_{l-1},P_l} + e_{P_l,P_1}$. The Γ-action is induced by $P \mapsto \gamma^{-1} P$ in $\mathbb{H}$. In this way, we have for each tesselation $\mathcal{T}$ a resolution $F_\cdot^{\mathcal{T}}$:

$$0 \longrightarrow \mathbb{Q}[X_2] \xrightarrow{\partial_2} \mathbb{Q}[X_1] \xrightarrow{\partial_1} \mathbb{Q}[X_0] \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0 \tag{6.4}$$

of the trivial $\mathbb{Q}[\Gamma]$-module $\mathbb{Q}$. So $F_i^{\mathcal{T}} = \mathbb{Q}[X_i]$ for $i = 0, 1, 2$, and $F_i^{\mathcal{T}} = \{0\}$ otherwise. As above, we can check that the $\mathbb{Q}[X_i]$ are projective.

If $f = \sum_l \alpha_l x_l \in F_i^{\mathcal{T}}$ with $x_l \subset B$ for all l for some subset $B \subset \mathbb{H}$, then we say that f is *supported* in B. The *support* $\mathrm{Supp}\, f$ of f is the intersection of all such B. It is a compact subset of $\mathbb{H}$, or empty if $f = 0$.

It is clear that $F_\cdot^{\mathcal{T}}$ is a complex. We check the exactness. We pick a base point $\tilde{P} \in X_0$. Suppose that $f = \sum_{P \in X_0} a_P\, (P)$ is in the kernel of ε. If $f \neq 0$, take $P \in \mathrm{Supp}\, f$ with maximal distance to $\tilde{P}$, where the distance is computed along edges in X_1. If there is a neighbor Q of P with smaller distance to $\tilde{P}$ subtract $\partial_1(a_P\, e_{Q,P})$ from f. This removes P from the support of f. Otherwise, choose an ordering of the points with maximal distance to $\tilde{P}$, and remove them successively. (Each of them has a neighbor that has at most the same distance from $\tilde{P}$.) This process stops when $f = 0$, or $\mathrm{Supp}\, f \subset \{\tilde{P}\}$, and then $\varepsilon f = 0$ also implies $f = 0$.

We call a 1-*chain* an element $C = \sum_{e \in X_1} a_e\, e \in \mathbb{Z}[X_1]$; so all $a_e \in \mathbb{Z}$. We may view C as a (possibly non-connected) path in $\mathbb{H}$ along edges of $\mathcal{T}$, with integral multiplicities. If $\partial_1 C = 0$, we call it a 1-*cycle*. In this case, the number of times that a point $P \in X_0^{\mathcal{T}}$ occurs as the terminal point of an edge in C (counted with multiplicities) is equal to the number of times that it occurs as the initial point of an edge in C. Thus, a 1-cycle corresponds to a combination of closed paths along edges in $X_1^{\mathcal{T}}$. To each $z \in \mathbb{H}$ not on the edges in C is associated a *winding number* $m_C(z) \in \mathbb{Z}$ of C around z. This function m_C has bounded support, and is constant on the interior of the polygons in $X_2^{\mathcal{T}}$. The 2-chain $D = \sum_{V\in X_1} m_C(V)\, V \in \mathbb{Z}[X_2^{\mathcal{T}}]$ satisfies $\partial_2 D = C$. In particular, the kernel of $\partial_1 : \mathbb{Z}[X_1] \to \mathbb{Z}[X_0]$ is equal to $\partial_2\mathbb{Z}[X_2]$. Since $\mathbb{Z}[X_i]$ has no torsion, tensoring with $\mathbb{Q}$ gives the exactness of $F_{\cdot}^{\mathcal{T}}$ at dimension 1.

If $f = \sum_{V\in X_2} a_V\, V \in \mathbb{Q}[X_2]$ is non-zero, consider an edge e in the boundary of $\operatorname{Supp} f$. So e is a boundary segment of only one $V \in X_2$, and $a_V \neq 0$. In $\partial_2 f$, this edge occurs with coefficient a_v or $-a_V$. This shows that ∂_2 is injective.

If the tesselation $\mathcal{S}$ is a refinement of the tesselation $\mathcal{T}$, there is an augmentation preserving chain map $F_{\cdot}^{\mathcal{T}} \to F_{\cdot}^{\mathcal{S}}$, where each $x \in X_i^{\mathcal{T}}$ is mapped to the sum of the $y \in X_i^{\mathcal{S}}$ into which it is subdivided. This induces an isomorphism in homology and cohomology. Since any two tesselations have a common refinement, this permits an explicit identification of the (co)homology groups constructed using the resolutions coming from distinct tesselations.

A *triangulation* is a tesselation for which all polygons $V \in X_2$ are triangles. Any tesselation can be refined to a triangulation. By $\Delta(P, Q, R)$ we denote the triangle with vertices P, Q and R ordered by the positive orientation of $\mathbb{H}$.

Let $\mathcal{T}$ be a Γ-invariant triangulation. An augmentation preserving chain map $a_{\cdot} : F_{\cdot}^{\mathcal{T}} \to F_{\cdot}^{\mathrm{hyp}-}$ can be defined by

$$\begin{aligned} a_0(P) &= (P) && (P \in X_0)\,,\\ a_1 e_{P,Q} &= \frac{1}{2}(P,Q) - \frac{1}{2}(Q,P) && (e_{P,Q} \in X_1)\,,\\ a_2\Delta(P,Q,R) &= \frac{1}{6}\sum_{\pi\in\mathfrak{S}_3} \operatorname{sign}(\pi)\,(\pi P, \pi Q, \pi R) && \Delta(P,Q,R) \in X_2)\,. \end{aligned} \tag{6.5}$$

Since every tesselation can be refined to a triangulation, this gives explicit chain maps between any $F_{\cdot}^{\mathcal{T}}$ and $F_{\cdot}^{\mathrm{hyp}}$.

Next we discuss an augmentation preserving chain map $f_{\cdot} : F_{\cdot}^{\mathrm{gr}} \to F_{\cdot}^{\mathcal{T}}$ for any tesselation $\mathcal{T}$. Choose a base point $P_0 \in X_0^{\mathcal{T}}$, and define $F_0^{\mathrm{gr}} \to F_0^{\mathcal{T}}$ by $f_0(\gamma) = \gamma^{-1}P_0$. In dimension 1, choose for each $\gamma \in \Gamma$ a path $p_\gamma \in \mathbb{Z}[X_1]$ from P_0 to $\gamma^{-1}P_0$ along edges in $X_1^{\mathcal{T}}$, and extend the definition $f_1(1,\gamma) = p_\gamma$ to F_1^{gr} in a $\mathbb{Q}[\Gamma]$-linear way. For $(1,\gamma,\delta) \in F_2^{\mathrm{gr}}$, the sum $C = f_1(1,\gamma) + f_1(\gamma,\delta) + f_1(\delta,1)$ is a 1-cycle in $\mathbb{Z}[X_1^{\mathcal{T}}]$. Take $D \in \mathbb{Z}[X_2^{\mathcal{T}}]$ such that $\partial_2 D = C$. The map f_2 is determined by $f_2(\varepsilon, \gamma\varepsilon, \delta\varepsilon) = \varepsilon^{-1}D$. For $i > 2$, put $f_i = 0$.

The resolutions coming from tesselations and the chain maps between them work with coefficients in $\mathbb{Z}$ instead of $\mathbb{Q}$. We need the order of elliptic elements as denominators for the projectivity, and we have used the denominators 2 and 3 in the construction of the chain map $a_{\cdot} : F_{\cdot}^{\mathcal{T}} \to F_{\cdot}^{\mathrm{hyp}}$.

6.2. *Choices of tesselations.* We shall use four special types of tesselations in particular, leading to four models of cohomology.

Tesselations of type Fd: Let $\mathfrak{F}$ be a connected closed *fundamental domain* of $\Gamma\backslash\mathbb{H}$ with finitely many geodesic sides. Use the tesselation $\mathcal{T}$ with the Γ-translates $\gamma\mathfrak{F}$ as the set of polygons $X_2 = X_2^{\mathcal{T}} = \{\gamma\mathfrak{F} \;:\; \gamma \in \Gamma\}$. If an elliptic fixed point occurs at the center of an edge e in $\partial\mathfrak{F}$, we add this point to X_0, and divide the edge e into two edges.

In the resulting resolution, $\mathbb{Q}[X_2]$ is a free $\mathbb{Q}[\Gamma]$-module of rank one, with basis $(\mathfrak{F})$. The fundamental domain has an even number $2n$ of of edges in its boundary. There exists a set E of n of these edges and a set of generators $\{\gamma_e \,:\, e \in E\}$ of Γ such that $\partial_2\mathfrak{F} = \sum_{e\in E}(e - \gamma_e^{-1}e)$. The module $\mathbb{Q}[X_1]$ is a free $\mathbb{Q}[\Gamma]$-module with the $e \in E$ as a basis. If Γ has elliptic elements, then $\mathbb{Q}[X_0]$ is not free.

Tesselations of type Dir: Fix a point $P_0 \in \mathbb{H}$ that is not an elliptic fixed point of Γ. Form the *Dirichlet fundamental domain* $\mathfrak{F}$ consisting of all points $P \in \mathbb{H}$ for which $\mathrm{d}(P, P_0) \leq \mathrm{d}(\gamma P, P_0)$ for all $\gamma \in \Gamma$. This fundamental domain $\mathfrak{F}$ shares sides with finitely many translates $\alpha^{-1}\mathfrak{F}$, $\alpha \in \Gamma$. These α's form a finite set $A = A^{-1}$ generating Γ. We take $X_0 = \Gamma P_0$, and $X_1 = \{\gamma^{-1} e_{P_0,\alpha^{-1}P_0} \,:\, \gamma \in \Gamma;\, \alpha \in A\}$. The elements of X_1 divide $\mathbb{H}$ into polygons indexed by the Γ-orbits of the vertices of $\mathfrak{F}$. This tesselation is dual to the tesselation of type **Fd** for the same Dirichlet fundamental domain. $\mathbb{Q}[X_0]$ is free on the generator (P_0). The fundamental domain $\mathfrak{F}$ is not necessarily the union of elements of X_2.

Tesselations of type Mix: Start with a tesselation $\mathcal{T}$ of type **Fd** for a Dirichlet fundamental domain $\mathfrak{F}$ with a base point P_0 in the interior of $\mathfrak{F}$. We add to $X_0^{\mathcal{T}}$ the Γ-translates of P_0, and to $X_1^{\mathcal{T}}$ the Γ-translates of the edges from P_0 to the vertices of $\mathfrak{F}$. We call the resulting refinement $\mathcal{S}$ of $\mathcal{T}$ a tesselation of type **Mix**. It is a triangulation.

Tesselations of type Mix′: This is a further refinement of a triangulation of type **Mix**. We add the Γ-translates of the intersection points of the geodesic segments from P_0 to $\alpha^{-1}P_0$, with $\alpha \in A$ as above, and the sides of $\mathfrak{F}$. We also add the Γ-translates of the resulting edges from P_0 to the new points. The resulting refinement is a triangulation.

• *Fundamental class.* Let $\mathcal{T}$ be a tesselation of type **Fd**. Since $V \otimes_{\mathbb{Q}} \mathbb{Q} = V_\Gamma$, for the trivial $\mathbb{Q}[\Gamma]$-module $\mathbb{Q}$, we have

$$H_2(\Gamma;\mathbb{Q}) \;\cong\; \mathrm{Ker}\left((F_2^{\mathcal{T}})_\Gamma \longrightarrow (F_1^{\mathcal{T}})_\Gamma\right) \;=\; (F_2^{\mathcal{T}})_\Gamma \;\cong\; \mathbb{Q}\,,$$

with the class of $(\mathfrak{F})$ in $(F_2^{\mathcal{T}})_\Gamma$ as its generator. This element of $H_2(\Gamma;\mathbb{Q})$ is the *fundamental class*. We denote it by $[\Gamma\backslash\mathbb{H}]$. If the fundamental domain $\mathfrak{F}$ on which $\mathcal{T}$ is based is a Dirichlet fundamental domain, we have the chain map constructed above to the resolution $F_\cdot^{\mathcal{S}}$ for the refinement $\mathcal{S}$ of $\mathcal{T}$ of type **Mix** or **Mix′**. This chain map induces an isomorphism in homology. The fundamental class is also represented by $(\mathfrak{F}) = \sum_{V\in X_2^{\mathcal{S}},\, V\subset\mathfrak{F}}(V)$ in the descriptions of these types. In the description of type **Dir**, the fundamental class is not represented by an element of $\mathbb{Z}[X_2]$.

6.3. *Cocycles.* In each of the models, the group cohomology of Γ with values in a right $\mathbb{Q}[\Gamma]$-module V is obtained up to isomorphism as the cohomology of the complex $C^\cdot(F_\cdot; V) = \mathrm{Hom}_{\mathbb{Q}[\Gamma]}(F_\cdot, V)$. If $F_i = \mathbb{Q}[X_i]$, this is equal to the complex $\mathrm{Map}(X_\cdot, V)^\Gamma$, where we define $\mathrm{Map}(X_i, V)$ as the $\mathbb{Q}$-linear space of all functions $X_i \to V$, with the action $f^\gamma(x) = f(\gamma x)\,|\,\gamma$. We denote by $Z^i(F_\cdot; V)$ the kernel of $d^i : C^i(F_\cdot; V) \to C^{i+1}(F_\cdot; V)$, and by $B^i(F_\cdot; V)$ the image $d^{i-1}\, C^{i-1}(F_\cdot; V)$.

• *Dimension* 0. In all models, it is easy to check that $H^0(\Gamma; V) = V^\Gamma$.

• *Dimension* 1. In the model based on a chain complex on $\mathbb{H}$, the 1-cochains are the maps $c : \mathbb{H}^2 \to V$ that satisfy

$$c(\gamma^{-1}P, \gamma^{-1}Q) = c(P, Q) \,|\, \gamma \qquad (\Gamma\text{-equivariance}). \tag{6.6}$$

For c to be a 1-cocycle, the additional condition is

$$c(P, Q) + c(Q, R) = c(P, R) \qquad (\text{cocycle relation}). \tag{6.7}$$

This implies that $c(P, P) = 0$ and that $c(Q, P) = -c(P, Q)$. The 1-cocycle c is a 1-coboundary if $c(P, Q) = f(P) - f(Q)$ for some Γ-equivariant map $f : \mathbb{H} \to V$.

In models built from a tesselation, the description is similar. A 1-cochain is determined by an equivariant map $c : X_1 \to V$. We can define $c(p)$ for any path p along edges in X_1 by linearity. If c is a cocycle, then $c(p)$ depends only on the end points of p, so we get a map $c : X_0^2 \to V$ satisfying (6.6) and (6.7). There is always a map $f : X_0 \to V$ such that $c(P, Q) = f(P) - f(Q)$. (Choose $f(P_0) \in V$ arbitrarily for some $P_0 \in X_0$ and define $f(P)$ as $f(P_0) + c(P, P_0)$.) The 1-cocycle c is a 1-coboundary if such an f can be found satisfying $f(\gamma^{-1}P) = f(P) \,|\, \gamma$ for all $\gamma \in \Gamma$.

For a 1-cocycle c in the model using a chain complex on $\mathbb{H}$, or in a model built on a tesselation, the choice of a base point P_0 gives a corresponding group cocycle $\psi_\gamma = c(\gamma^{-1}P_0, P_0)$ in the standard model in §5.1.

• *Dimension* 2. Here the most convenient choice is a projective resolution of type **Fd**, based on the tesselation $\mathcal{T}$ derived from a fundamental domain $\mathfrak{F}$, since $\mathbb{Q}[F_2]$ is free with basis $(\mathfrak{F})$. Any $v \in V$ determines a 2-cochain by $(\mathfrak{F}) \mapsto v$, which is automatically a 2-cocycle since $F_3 = \{0\}$. It is a 2-coboundary if there is $c \in \mathrm{Map}(X_1, V)^\Gamma$ such that

$$v = \sum_{e \in \partial\mathfrak{F}} c(e) \;=\; \sum_{e \in E} c(e) \,|\, (1 - \gamma_e).$$

Since the $c(e) \in V$ can be chosen arbitrarily and since the elements $\gamma \in E$ generate Γ, we have

$$H^2(\Gamma; V) \;\cong\; H^2(F_\cdot^{\mathcal{T}}; V) \;\cong\; V_\Gamma \;:=\; V \,/\, \textstyle\sum_{\gamma\in\Gamma} V|(1-\gamma). \tag{6.8}$$

The space V_Γ is called the space of coinvariants.

For general projective resolutions, the isomorphism (6.8) is obtained by evaluating a 2-cocycle b on a representative of the fundamental class. With the *cap product*

$$\langle \cdot\,, \cdot \rangle : H^2(\Gamma; V) \otimes H_2(\Gamma; \mathbb{Q}) \longrightarrow H_0(\Gamma; V) \;\cong\; V_\Gamma\,,$$

we can formulate this as $b \mapsto \langle [b], [\Gamma\backslash\mathbb{H}] \rangle$. (See, e.g., §3, Chap. V of [1].) This is a case of *Poincaré duality* for $\Gamma\backslash\mathbb{H}$, which holds since $\Gamma\backslash\mathbb{H}$ is a rational cohomology manifold.

In the model using the chain complex on $\mathbb{H}$, a 2-cocycle corresponds to a Γ-equivariant map $b : \mathbb{H}^3 \to V$ satisfying

$$b(P, Q, R) + b(P, R, S) \;=\; b(P, Q, S) + b(Q, R, S). \tag{6.9}$$

Such a cocycle is not necessarily an alternating function of its three arguments. If we antisymmetrize it by composition with A_2 in (6.3), then we get an alternating cocycle in the same cohomology class.

• *Cocycles associated to Maass forms.* In (5.5) we defined cocycles $r : \Gamma \to \mathcal{V}_s^\omega$, $p : \Gamma \to \mathcal{E}_s^\omega$ and $q : \Gamma \to \mathcal{W}_s^\omega$ associated to $u \in \mathcal{E}_s^\Gamma$ in the standard model of group cohomology. In the model built on the chain complex on $\mathbb{H}$, the corresponding cocycles are defined for $P, Q \in \mathbb{H}$ as follows:

$$r_{P,Q}(\zeta) = \int_P^Q [u, R(\zeta; \cdot)^s], \tag{6.10a}$$

$$p_{P,Q}(z) = \int_P^Q [u, p_s(\cdot, z)], \tag{6.10b}$$

$$q_{P,Q}(z) = \int_P^Q [u, q_s(\cdot, z)]. \tag{6.10c}$$

These cocycles describe the linear maps **r**, **p** and **q** of §5.2 in terms of the description of the cohomology groups with a tesselation. We identify $q_{P,Q}$ with the element of $\mathcal{W}_s^\omega$ represented by it. Note that the function $q_{P,Q}$ is not in $\mathcal{G}_s^\omega \subset C^2(\mathfrak{H})$ (it may jump across the path from P to Q). The Γ-equivariance follows from that of the bracket operator and that of the kernel functions. The cocycle relation (6.7) is ensured by the fact that $[u, v]$ is a closed form if u and v are λ_s-eigenfunctions of Δ. See (1.10c). The same formulas work for models based on a tesselation $\mathcal{T}$, provided $P, Q \in X_0^{\mathcal{T}}$.

6.4. *Algebraic description of cycles and chains.* This subsection gives some algebraic results expressing 1-chains and 1-cycles in terms of the group ring.

Let R denote the *group ring* $\mathbb{Z}[\Gamma]$ of an arbitrary group Γ and $R_0 \subset R$ the augmentation ideal, consisting of $\sum_j n_j \gamma_j \in R$ such that $\sum_j n_j = 0$. For a finite subset A of Γ we consider the map

$$\pi_A : R^A \to R_0, \qquad \xi \mapsto \sum_{\alpha \in A} (1-\alpha)\, \xi(\alpha).$$

Since R_0 is spanned by the elements $1-\gamma$ with $\gamma \in \Gamma$, the identity $1 - \gamma\alpha = (1-\alpha) + (1-\gamma)\alpha$ and an obvious induction show that the image of π_A is the kernel of the natural map from R to $\mathbb{Z}[\Gamma/\Delta]$, where Δ is the subgroup of Γ generated by A. In particular, π_A is surjective if (and only if) A generates Γ. As to its kernel, we have:

Lemma 6.1. *Suppose that A generates Γ and $\xi \in \operatorname{Ker} \pi_A$. Then*

$$\sum_{\alpha \in A} \psi_\alpha \,|\, \xi(\alpha) = 0 \tag{6.11}$$

holds for all $\mathbb{Q}[\Gamma]$-modules V and all cocycles $\psi \in Z^1(\Gamma; V)$.

Proof. For a coboundary $\psi = db$, we have $\sum_{\alpha\in A} \psi_\alpha \,|\, \xi(\alpha) = -b \,|\, \pi_A(\xi)$. So if $\pi_A(\xi) = 0$, then (6.11) holds for coboundaries. Any $\mathbb{Q}[\Gamma]$-module is a submodule of an injective $\mathbb{Z}[\Gamma]$-module I, for which $H^1(\Gamma; I) = \{0\}$. (See [10], §1.4, Théorème 1.2.2.) So any cocycle $\psi \in Z^1(\Gamma; V)$ is a coboundary in $B^1(\Gamma; I)$, and (6.11) holds for all cocycles. ∎

Now we again take Γ to be a discrete cocompact subgroup of G, and choose $A = A^{-1}$ to be the system of generators associated to a Dirichlet fundamental domain $\mathfrak{F}$ with base point P. Let $\mathcal{T}$ be the tesselation of type **Dir** associated to $\mathfrak{F}$. The edges in $X_1^{\mathcal{T}}$ starting from P are of the form $e_{P,\alpha^{-1}P}$ with α running through the set A. Since every oriented edge in $X_1^{\mathcal{T}}$ is the image under Γ of one of these, any 1-chain $C \in \mathbb{Z}[X_1^{\mathcal{T}}]$ can be written as $\sum_{\alpha\in A} e_{P,\alpha^{-1}P} \,|\, \xi(\alpha)$ for some $\xi \in R^A$. Then $\partial_1 C = \sum_{\alpha\in A} ([P] - [\alpha^{-1}P]) \,|\, \xi(\alpha) = [P] \,|\, \pi_A(\xi)$, so C is a 1-cycle if and only if $\xi \in \operatorname{Ker} \pi_A$.

7. From cohomology to Maass forms. In §5.2 we constructed an injective map from $\mathcal{E}_s^\Gamma$ to $H^1(\Gamma; \mathcal{V}_s^\omega)$ for any infinite discrete subgroup Γ of G. In this section we prove the bijectivity of this map when Γ is cocompact (Theorem A), and give explicit descriptions of the inverse map $H^1(\Gamma; \mathcal{V}_s^\omega) \to \mathcal{E}_s^\Gamma$.

The fact that $H^1(\Gamma; \mathcal{V}_s^\omega)$ and $\mathcal{E}_s^\Gamma$ are isomorphic for cocompact groups Γ follows from the work of Bunke and Olbrich. [2] Our approach is different and more elementary, and will also form the basis for the proofs in the non-cocompact case.

The map from Maass forms to cohomology was given in three versions **p**, **q** and **r** in §5.2 (defined by (5.5) for the standard model of cohomology and by (6.10) for the model based on a tesselation), depending whether we use the model $\mathcal{V}_s^\omega$, $\mathcal{W}_s^\omega$ or $\mathcal{E}_s^\omega$ for the analytic vectors in the principal series representation. For the inverse direction we will work with $\mathcal{W}_s^\omega$ and the map **q**. We construct an explicit one-sided inverse of **q** in §7.1, and give a second description of it in §7.2. The injectivity of this inverse map is proved in §7.3. Most of the proofs use the description of cohomology with resolutions based on a tesselation discussed in §6.1; in Theorem 7.2 we also give a formulation in terms of the standard model of group cohomology.

7.1. *Construction of a Maass form from a given cocycle.* We start with a cocycle $\psi \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{W}_s^\omega)$, given in a resolution based on a tesselation $\mathcal{T}$ as described in §6.1. This means that for each edge $e \in X_1^{\mathcal{T}}$ the boundary germ $\psi(e) \in \mathcal{W}_s^\omega = \mathcal{G}_s^\omega / \mathcal{N}^\omega$ is given. To make this concrete, we take representatives $\tilde\psi(e)$ in $\mathcal{G}_s^\omega$ of the $\psi(e)$. This can be done in a Γ-equivariant way: $\mathcal{F}_1^{\mathcal{T}}$ has a finite $\mathbb{Q}[\Gamma]$-basis $B \subset X_1^{\mathcal{T}}$, and we lift each $\psi(b) \in \mathcal{W}_s^\omega$ to $\tilde\psi(b) \in \mathcal{G}_s^\omega$ and then extend by Γ-equivariance to get a cochain in $C^1(F_\cdot^{\mathcal{T}}; \mathcal{G}_s^\omega) = \mathrm{Hom}_{\mathbb{Q}[\Gamma]}(X_1^{\mathcal{T}}; \mathcal{G}_s^\omega)$. This is in general not a cocycle, but the cocycle property $d\psi = 0$ implies that the values of $d\tilde\psi$ are in $\mathcal{N}^\omega = C_c^2(\mathbb{H})$. This defines an $\mathcal{N}^\omega$-valued 2-cocycle which we still denote by $d\tilde\psi$, although it is now no longer a coboundary.

We now construct a λ_s-eigenfunction u_ψ of Δ on $\mathbb{H}$. Recall that the set of singularities $\mathsf{Sing}\, f$ of an element $f \in \mathcal{G}_s^\omega$ is defined as the smallest subset of $\mathbb{H}$ outside of which f is a λ_s-eigenfunction of the Laplace operator, and is compact. For compact $x \subset \mathbb{H}$ we denote by $N_r(x)$ the r-neighborhood of x with respect to the hyperbolic distance. We choose R such that $\mathsf{Sing}\,\tilde\psi(b)$ is contained in $N_R(b)$ for each b in the finite set B. Then by Γ-equivariance it follows that $\mathsf{Sing}\,\tilde\psi(e)$ is contained in the R-neighborhood $N_R(e)$ of e for every edge $e \in X_1^{\mathcal{T}}$. For $z \in \mathbb{H}$ we define

$$u_{\tilde\psi}(z) \;=\; \frac{1}{\pi i}\,\tilde\psi(C)(z)\,, \tag{7.1}$$

[2] By Theorem 1.1 in [6] the hyperfunction cohomology group $H^1(\Gamma; \mathcal{V}_s^{-\omega})$ has finite dimension for torsion-free cocompact Γ. It is the dual of the analytic cohomology group $H^1(\Gamma; \mathcal{V}_{1-s}^{\omega})$; this is obtained in Proposition 5.2 with Poincaré duality. Proposition 7.2 in [7] gives the same duality for the distribution cohomology group $H^1(\Gamma; \mathcal{V}_s^{-\infty})$ and the smooth cohomology group $H^1(\Gamma; \mathcal{V}_{1-s}^{\infty})$. Corollary 7.3 states the equality $H^i(\Gamma; \mathcal{V}_s^{-\infty}) = H^i(\Gamma; \mathcal{V}_s^{-\omega})$ in all degrees i.

Without the assumption of cocompactness the isomorphism of $H^1(\Gamma; \mathcal{V}_s^{-\infty})$ with a space of Maass forms is derived in [7]. The vector bundle E in §2 is, in our situation, the constant vector bundle $\mathbb{C}$ over $G/K \cong \mathfrak{H}$. The operator B in §3 corresponds to $\Delta - \lambda_s$. The space $\mathcal{E}_Y$ in §6 corresponds to $C^\infty(\Gamma\backslash\mathfrak{H})$. The space $\mathcal{E}_Y(B)_{\mathrm{cusp}}$ at the bottom of p. 71 is the space $\mathcal{E}^\Gamma$ of invariant eigenfunctions if Γ is compact. If Γ has cusps it is the space $\mathsf{Maass}_s^0(\Gamma)$ of Maass cusp forms. Proposition 8.1 in [7] gives the isomorphism between $\mathcal{E}_Y(B)_{\mathrm{cusp}}$ and $H^1(\Gamma; \mathcal{V}_s^{-\infty})$.

where C is a cycle in $\mathbb{Z}[X_1^{\mathcal{T}}]$ going around z once in the positive direction at a distance greater than R. We claim that this is independent of the choice of C. Indeed, if C_1 is another 1-cycle with the same properties as C, we can deform C into C_1 in finitely many steps, where in each step we add to or subtract from C the boundary of a polygon $V \in X_2^{\mathcal{T}}$ whose distance to z is greater than R. The difference between $\tilde{\psi}(C)(z)$ and $\tilde{\psi}(C_1)(z)$ is the sum of contributions $\tilde{\psi}(\partial V)(z) = d\tilde{\psi}(V)(z)$. Each function $d\tilde{\psi}(V)$ is a λ_s-eigenfunction outside of $N_R(V)$ and is compactly supported, so vanishes identically outside of $N_R(V)$ (because $N_R(V)$ is simply connected). In particular, each $d\tilde{\psi}(V)$ vanishes near z, so (7.1) is the same for C and C_1. (An alternative argument would be to choose a larger R for which $d\tilde{\psi}(V)$ vanishes outside the R-neighborhood $N_R(V)$ for every $V \in X_2^{\mathcal{T}}$, which is possible by equivariance since $X_2^{\mathcal{T}}/\Gamma$ is finite. Then the vanishing of $d\tilde{\psi}(V)$ near z is immediate.)

The function $u_{\tilde{\psi}}$ satisfies $\Delta u_{\tilde{\psi}} = \lambda_s u_{\tilde{\psi}}$, since by the definition of R the point z lies outside the singularities of $\tilde{\psi}(e)$ for every e in C. It is also obviously Γ-invariant, since we can use the cycle γC in defining $u_{\tilde{\psi}}(\gamma z)$ and $\tilde{\psi}$ is equivariant. Moreover, $u_{\tilde{\psi}}$ is independent of the lifting $\tilde{\psi}$, and can hence be denoted simply u_ψ, because any two choices of $\tilde{\psi}$ differ by an equivariant $C_c^2(\mathbb{H})$-valued on $X_1^{\mathcal{T}}$, so that if we choose the cycle C far enough away from z the two values of $\tilde{\psi}(C)$ agree. Finally, u_ψ depends only on the cohomology class of ψ, because if we replace ψ by another cocycle $\psi_1 = \psi + dF$ in the same class, where F is an equivariant map from $X_0^{\mathcal{T}}$ to $\mathcal{W}_s^\omega$, then we can lift F equivariantly to a map $\tilde{F} : X_0^{\mathcal{T}} \to \mathcal{G}_s^\omega$ and hence, choosing C in (7.1) suitable for ψ and ψ_1, find $u_\psi(z) - u_{\psi_1}(z) = u_{\tilde{\psi}}(z) - u_{\tilde{\psi}+d\tilde{F}}(z) = \tilde{F}(\partial C) = 0$. This completes the construction of the map $H^1(\Gamma; \mathcal{W}_s^\omega) \to \mathcal{E}_s^\Gamma$.

We can also use the isomorphism between the various models of cohomology to write $u_{[\psi]}$ in terms of the standard model. We first observe that our construction is independent of the tesselation chosen, since any two tesselations are contained in a common one and a cycle C that works in (7.1) for a given tesselation also works for any finer one. If we use a tesselation of type **Dir**, with a set $A = A^{-1} \subset \Gamma$ of generators of Γ giving the transition from the Dirichlet fundamental domain $\mathfrak{F}$ to the adjacent fundamental domains $\alpha^{-1}\mathfrak{F}$, then every edge $e \in X_1^{\mathcal{T}}$ can be written uniquely as $\gamma^{-1} e_{\alpha^{-1}P_0, P_0}$ with $\gamma \in \Gamma$ and $\alpha \in A$, where $P_0 \in \mathring{\mathfrak{F}}$ is the base point of the tesselation, so we can associate to any group cocycle $c : \Gamma \to \mathcal{W}_s^\omega$ a cocycle ψ on $X_1^{\mathcal{T}}$ by setting $\psi(e) = c(\alpha)|\gamma$. It also follows that u_ψ has the property stated in Theorem A that (up to a constant factor depending on the normalization) it can be represented on any compact subset of $\mathbb{H}$ by a fixed finite $\mathbb{Z}$-linear combination of functions $\mathsf{P}_s^\dagger(\varphi_\gamma)|\gamma'$ with $\gamma, \gamma' \in \Gamma$, where $\varphi : \Gamma \to \mathcal{V}_s^\omega$ represents the cohomology class.

We now show that the map just constructed is a left inverse to $\mathbf{q}$. Start with $u \in \mathcal{E}_s^\Gamma$. The class $\mathbf{q}u$ is represented by the map $q : X_1^{\mathcal{T}} \to \mathcal{W}_s^\omega$ defined by

$$q(e)(z) = \int_e [u, q_s(\,\cdot\,, z)] \,.$$

Notice that the element $q(e)$ itself is not in $\mathcal{G}_s^\omega$, because it is singular on e. We choose a lifting of q to a map $\tilde{\psi} : X_1^{\mathcal{T}} \to \mathcal{G}_s^\omega$ by multiplying $q(b)$ for $b \in B$ by a smooth function that is 1 near $\partial\mathbb{H}$ and 0 near b and then extending equivariantly. Now we apply formula (7.1) with C chosen far enough from z that $\tilde{\psi}(C)$ and $q(C)$ agree near z, obtaining by Theorem 1.1 the identity

$$u_{\mathbf{q}u}(z) \;=\; u_{\tilde\psi}(z) \;=\; \frac{1}{\pi i}\,\tilde\psi(C)(z) \;=\; \frac{1}{\pi i}\int_C [u, q_s(\,\cdot\,, z)] \;=\; u(z)\,. \tag{7.2}$$

In summary, we have constructed an explicit map $\alpha_s^\omega : [\psi] \mapsto u_\psi$ from $H^1(\Gamma; \mathcal{W}_s^\omega)$ to $\mathcal{E}_s^\Gamma$ such that the restriction of the function u_ψ to any compact subset of $\mathbb{H}$ is a finite linear combination of translates $\psi_\gamma\,|\,\gamma'$ $(\gamma, \gamma' \in \Gamma)$, and such that $u_{\mathbf{q}u} = u$ for $u \in \mathcal{E}_s^\Gamma$.

7.2. *Construction of a Maass form from a cocycle as an average.* In the previous subsection, we constructed the Maass from u_ψ associated to a $\mathcal{W}_s^\omega$-valued cocycle ψ using the surjective map $\mathcal{G}_s^\omega \to \mathcal{W}_s^\omega$, where $\mathcal{G}_s^\omega$ is a space of functions defined on the whole of $\mathbb{H}$. (Recall that $\mathcal{W}_s^\omega$ is a space of boundary germs.) We now give an alternative description in which u_ψ is represented as the sum of the Γ-translates of a compactly supported function. This will be used in §7.3 for the proof of the injectivity of the map $[\psi] \mapsto u_\psi$.

We choose our tesselation $\mathcal{T}$ so that there is a fundamental domain $\mathfrak{F}$ for $\Gamma\backslash\mathbb{H}$ consisting of finitely many elements of $X_2^{\mathcal{T}}$. (This can be done by choosing $\mathcal{T}$ of type **Fd** or by refining any given tesselation appropriately.) By deforming the 1-cycle C used in the definition of u_ψ, we can assume that it bounds a region D consisting of finitely many Γ-translates of this fundamental domain. Then

$$\pi i\, u_\psi(z) \;=\; \tilde\psi(C)(z) \;=\; \tilde\psi(\partial D)(z) \;=\; d\tilde\psi(D)(z) \;=\; \sum_{\gamma\in\Gamma;\,\gamma^{-1}\mathfrak{F}\text{ inside }C} d\tilde\psi(\mathfrak{F})(\gamma z)\,.$$

But $d\tilde\psi(\gamma^{-1}\mathfrak{F})(z) = d\psi(\gamma^{-1}\mathfrak{F})(z) = 0$ for $\gamma^{-1}\mathfrak{F}$ outside C, because ψ is a cocycle. Hence

$$u_\psi(z) \;=\; \frac{1}{\pi i}\sum_{\gamma\in\Gamma} d\tilde\psi(\mathfrak{F})(\gamma z)\,. \tag{7.3}$$

Let us define the *averaging operator* on Γ for $f : \mathbb{H} \to \mathbb{C}$ as follows:

$$\mathrm{Av}_\Gamma(f) \;=\; f\,|\,\mathrm{Av}_\Gamma \;=\; \sum_{\gamma\in\Gamma} f\,|\,\gamma\,. \tag{7.4}$$

(This is not really an average since we do not divide by $|\Gamma|$, but the term is convenient.) If the sum converges absolutely on $\mathbb{H}$, the result is a Γ-invariant function. For compactly supported f the sum $\mathrm{Av}_\Gamma f$ is locally finite, and hence absolutely convergent. Note that the function $d\tilde\psi(\mathfrak{F})$ is compactly supported. We have obtained:

Proposition 7.1. *Suppose that the Γ-invariant tesselation $\mathcal{T}$ contains a fundamental domain $\mathfrak{F}$. Then for $\psi \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{W}_s^\omega)$ and all lifts $\tilde\psi \in C^1(F_\cdot^{\mathcal{T}}; \mathcal{G}_s^\omega)$ of ψ, we have*

$$u_\psi \;=\; \frac{1}{\pi i}\mathrm{Av}_\Gamma(d\tilde\psi(\mathfrak{F}))\,.$$

It is remarkable that $\mathrm{Av}_\Gamma(d\tilde\psi(\mathfrak{F}))$ is an analytic function on $\mathbb{H}$, whereas $d\tilde\psi(\mathfrak{F}) \in \mathcal{N}^\omega = C_c^2(\mathbb{H})$ is not (unless it is zero). We consider this a little more concretely.

Let f_b be representatives of $\psi(b)$ for $b \in B$ and choose a closed non-selfintersecting curve $\tilde C \subset \mathbb{H}$ (for instance, a circle near $\partial\mathbb{H}$) such that all the f_b are λ_s-eigenfunctions of Δ on and outside $\tilde C$. Now choose $\tilde\psi(b) \in \mathcal{G}_s^\omega$ which agrees with f_b outside $\tilde C$ and has singularities only in the ε-neighborhood $N_\varepsilon(\tilde C)$ of $\tilde C$ for some $\varepsilon \ll 1$. (For instance, we can multiply f_b by a smooth function which is 1 outside $\tilde C$ and vanishes on the bounded component of $\mathbb{H} \smallsetminus N_\varepsilon(\tilde{C})$.) We extend $\tilde{\psi}$ equivariantly as usual. Then all singularities of all $\tilde{\psi}(e)$ are contained in $\Gamma N_\varepsilon(\tilde{C})$, and the same holds for $d\tilde{\psi}(\mathfrak{F}) = \tilde{\psi}(\partial\mathfrak{F})$ and for $\mathrm{Av}_\Gamma(d\tilde{\psi}(\mathfrak{F}))$. Moving $\tilde{C}$ and changing the $\tilde{f}_b$ on $\tilde{C}_\varepsilon$ corresponds to adding to $\tilde{\psi}$ a cochain with values in $\mathcal{N}^\omega = C^2_c(\mathbb{H})$. This means for $d\tilde{\psi}$ that we add an element of $\sum_{\gamma\in\Gamma} \mathcal{N}^\omega \,|\, (1-\gamma)$, which is annihilated by Av_Γ, so $u_\psi = (\pi i)^{-1}\mathrm{Av}_\Gamma(d\tilde{\psi}(\mathfrak{F}))$ is unchanged. Since we can deform $\tilde{C}$ so that $N_\varepsilon(\tilde{C})$ avoids the Γ-orbit of any given point in $\mathbb{H}$, this makes it clear why u_ψ cannot be singular anywhere.

7.3. *Injectivity.* It remains to show that the map $\alpha_s^\omega : [\psi] \mapsto u_{[\psi]}$ of the previous section is injective. This map fits into the commutative diagram

$$\begin{array}{ccc} H^1(\Gamma;\mathcal{W}_s^\omega) & \xrightarrow{\ \delta\ } & H^2(\Gamma;\mathcal{N}^\omega) \\ \Big\downarrow{\scriptstyle \alpha_s^\omega} & & \Big\downarrow{\scriptstyle \cong\ \langle\,\cdot\,,(\mathfrak{F})\rangle} \\ & & (\mathcal{N}^\omega)_\Gamma = \mathcal{N}^\omega / \sum_\gamma \mathcal{N}^\omega|(1-\gamma) \\ & & \Big\downarrow{\scriptstyle \mathrm{Av}_\Gamma} \\ \mathcal{E}_s(\mathbb{H})^\Gamma & \overset{\subset}{\hookrightarrow} & C^2(\mathbb{H})^\Gamma \end{array} \tag{7.5}$$

in which δ is the connecting homomorphism in the long exact sequence

$$\cdots \longrightarrow H^1(\Gamma;\mathcal{G}_s^\omega) \longrightarrow H^1(\Gamma;\mathcal{W}_s^\omega) \xrightarrow{\ \delta\ } H^2(\Gamma;\mathcal{N}^\omega) \longrightarrow \cdots \tag{7.6}$$

corresponding to the exact sequence (3.11) with $p = \omega$ and the vertical isomorphism is the one given in (6.8). We will show the injectivity of Av_Γ in Proposition 7.3 and the vanishing of $H^1(\Gamma;\mathcal{G}_s^\omega)$ (and hence injectivity of δ) in Proposition 7.4. Together with what we have already done this gives a proof of the following theorem, which is a somewhat more detailed statement of Theorem A in the analytic case.

Theorem 7.2. *Let Γ be cocompact. There is an isomorphism $\alpha_s^\omega : H^1(\Gamma;\mathcal{V}_s^\omega) \to \mathcal{E}_s^\Gamma$, given by $\varphi \mapsto u_{\mathsf{P}_s\varphi}$ as defined in* (7.1)*, inverting* $\mathbf{q}$ *introduced in* §5.2*.*

In the description of cohomology based on a tesselation $\mathcal{T}$, the function $u_{\mathsf{P}_s^\dagger\varphi}$ associated to a given $\mathcal{V}_s^\omega$-valued cocycle φ can be given on each compact set in $\mathbb{H}$ by a finite linear combination of Γ-translates of $\mathsf{P}_s^\dagger\varphi(b)$ where b runs through a finite set of edges in $X_1^{\mathcal{T}}$. In the standard description of cohomology the function $u_{\mathsf{P}_s^\dagger\varphi}$ associated to a cocycle $\varphi : \Gamma \to \mathcal{V}_s^\omega$ can on any compact set in $\mathbb{H}$ be given as a multiple of a finite sum of translates $\mathsf{P}_s^\dagger\varphi_\gamma|\gamma'$ (γ, $\gamma' \in \Gamma$).

We prove the injectivity of Av_Γ in slightly more generality.

Proposition 7.3. *The map $\mathrm{Av}_\Gamma : C_c^p(\mathbb{H})_\Gamma \to C^p(\mathbb{H})^\Gamma$ is injective for $p = 0, 1, \ldots, \infty$.*

Proof. Let $f \in C_c^p(\mathbb{H})$ with $\mathrm{Av}_\Gamma(f) = 0$. We must show that $f \in \sum_{\gamma\in\Gamma} C_c^p(\mathbb{H}) \,|\, (1-\gamma)$. Choose $\chi \in C_c^\infty(\mathbb{H})$ such that $\sum_{\gamma\in\Gamma} \chi|\gamma = 1$ on $\mathbb{H}$. (For instance, choose $\Phi \in C_c^\infty(\mathbb{R})$ with $C \neq 0$, where C is the constant value of the integral $\int_{\mathbb{H}} \Phi(\rho(z,z'))\, d\mu(z')$, with $\rho(\,\cdot\,,\,\cdot\,) = \cosh \mathrm{d}(\,\cdot\,,\,\cdot\,)$ as in Table 1 in §1, and set $\chi(z) = \frac{1}{C} \int_{\mathfrak{F}} \Phi(\rho(z,z'))\, d\mu(z')$.) Then

$$f \;=\; f - \mathrm{Av}_\Gamma(f)\chi \;=\; \sum_{\gamma\in\Gamma} (\chi|\gamma^{-1} \cdot f \,-\, \chi \cdot f|\gamma) \;=\; \sum_{\gamma\in\Gamma} (\chi|\gamma^{-1} \cdot f) \,|\, (1-\gamma)\,.$$

In the last expression we can replace Γ by the set $\{\gamma \in \Gamma \,:\, \gamma^{-1}\mathrm{Supp}\,(\chi) \cap \mathrm{Supp}\,(f) \neq \emptyset\}$, which is finite because both χ and f have compact support. ■

Proposition 7.4. *If Γ is cocompact, then $H^1(\Gamma; \mathcal{G}_s^\omega) = \{0\}$.*

The proof of this proposition will occupy the rest of this subsection.

Let $\mathcal{T}$ be a Γ-invariant tesselation, as in §6. We put $X_0^H = X_0^{\mathcal{T}} \cup H$, where $H \subset \partial\mathbb{H}$ is the orbit of a fixed point of a hyperbolic element of Γ. A given element of $Z^1(F_{\cdot}^{\mathcal{T}}; \mathcal{G}_s^\omega)$ can be viewed as a Γ-equivariant function $c : X_0^{\mathcal{T}} \times X_0^{\mathcal{T}} \to \mathcal{G}_s^\omega$ satisfying the cocycle relation (6.7). We will extend c to a function c^H on $X_0^H \times X_0^H$ with the same properties, with values in a larger space.

For each $\xi \in H$ let η_ξ be the generator of Γ_ξ for which $\xi = \alpha(\eta_\xi)$, the repelling fixed point of η_ξ. We put

$$\mathcal{G}_s^{\omega^*,H} \;=\; \mathcal{G}_s^\omega + \sum_{\xi\in H} \mathrm{Av}_{\eta_\xi}^+ \mathcal{G}_s^\omega\,, \tag{7.7}$$

where $\mathrm{Av}_{\eta_\xi}^+$ is the one-sided average defined in §4.1. This average maps $\mathcal{G}_s^\omega$ into $\mathcal{G}_s^\omega(\partial\mathbb{H} \smallsetminus \{\xi\})$. Thus the elements of $\mathcal{G}_s^{\omega^*,H}$ represent germs in the space $\mathcal{W}_s^{\omega^*}$ defined in (3.2). Definition 3.4 gives $\mathsf{Sing}\,(f)$ for any $f : \mathbb{H} \to \mathbb{C}$ as the smallest set such that f is a λ_s-eigenfunction on $\mathbb{H} \smallsetminus \mathsf{Sing}\,(f)$. For $f \in \mathcal{G}_s^{\omega^*,H}$, the set $\mathsf{Sing}\,f$ is not necessarily compact in $\mathbb{H}$. Its closure in $\mathbb{P}^1_{\mathbb{C}}$ may contain points of H.

Lemma 7.5. *The map $c : X_0 \times X_0 \to \mathcal{G}_s^\omega$ corresponding to a cocycle $c \in \mathrm{Map}(X_1, \mathcal{G}_s^\omega)^\Gamma$ can be extended to a map $c^H : X_0^H \times X_0^H \to \mathcal{G}_s^{\omega^*,H}$ satisfying the conditions* (6.6) *and* (6.7) *for a Γ-cocycle.*

Proof. Let $\xi = \alpha(\eta) \in H$, with $\eta = \eta_\xi$ as above. For $P \in X_0$ we set

$$c^H(P,\xi) \;=\; -c^H(\xi,P) \;=\; \mathrm{Av}_\eta^+ c(P,\eta^{-1}P) \in \mathcal{G}_s^{\omega^*,H}\,. \tag{7.8}$$

Since the convergence of $\mathrm{Av}_\eta^+(c(P,\eta^{-1}P)(z)$ is absolute for each $z \in \mathbb{H}$, we have

$$\mathrm{Av}_\eta^+(c(P,\eta^{-1}P)(z) \;=\; \lim_{N\to\infty} \sum_{n=0}^N c(\eta^{-n}P, \eta^{-n-1}P)(z) \;=\; \lim_{N\to\infty} c(P,\eta^{-N-1}P)(z)\,,$$

where the second equality follows from the cocycle property. For $P, Q \in X_0$:

$$\begin{aligned} c^H(P,\xi) - c^H(Q,\xi) \;&=\; \lim_{N\to\infty} \sum_{n=0}^N \Big(c(\eta^{-n}P, \eta^{-n-1}P) - c(\eta^{-n}Q, \eta^{-n-1}Q) \Big) \\ &=\; \lim_{N\to\infty} \Big(c(Q,P)\,|\,\eta^{N+1} - c(Q,P) \Big) \;=\; -c(Q,P) \;=\; c(P,Q)\,. \end{aligned} \tag{7.9}$$

For ξ, $\xi_1 \in H$ we define

$$c^H(\xi,\xi_1) \;=\; c^H(P,\xi_1) - c^H(P,\xi)\,. \tag{7.10}$$

In (7.9) we see that this does not depend on the choice of $P \in X_0$. This defines c^H on $X_0^H \times X_0^H$ satisfying the cocycle relation.

Let $\xi_1 = \gamma^{-1}\xi$ with $\gamma \in \Gamma$. Then $\eta_{\xi_1} = \gamma^{-1}\eta_\xi\gamma$. The Γ-equivariance follows:

$$\begin{aligned} c^H(\gamma^{-1}P, \gamma^{-1}\xi) \;&=\; c(\gamma^{-1}P, \gamma^{-1}\eta^{-1}\gamma\gamma^{-1}P)\,|\,\mathrm{Av}_{\gamma^{-1}\eta\gamma}^+ \\ &=\; c(\gamma^{-1}P, \gamma^{-1}\eta^{-1}P)\,|\,\gamma^{-1}\,\mathrm{Av}_\eta^+\,\gamma \;=\; c(P,\eta^{-1}P)\,|\,\mathrm{Av}_\eta^+\,\gamma \;=\; c^H(P,\xi)\,|\,\gamma\,. \end{aligned}$$ ■

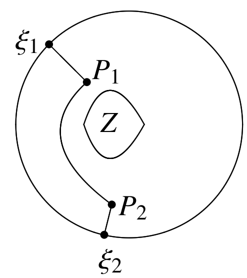
The construction of c^H shows that for $P \in X_0$ and $\xi, \xi_1 \in H$:

$$\overline{\mathsf{Sing}\, c(P,\xi)} \cap \partial\mathbb{H} \ \subset\ \{\xi\}\,, \qquad \overline{\mathsf{Sing}\, c(\xi,\xi_1)} \cap \partial\mathbb{H} \ \subset\ \{\xi,\xi_1\}\,. \tag{7.11}$$

We have defined $c^H(\,\cdot\,,\,\cdot\,)$ as a $\mathcal{G}_s^{\omega^*,H}$-valued function with the one-sided averaging operator. Of course, we think of $c^H(P,\xi)$ as c evaluated on an infinite path from P to ξ.

Lemma 7.6. *Let c^H be as in Lemma 7.5. Then $c^H(\xi_1,\xi_2) \in \mathcal{E}_s$ for all $\xi_1, \xi_2 \in H$.*

Proof. Write η_j for η_{ξ_j} ($j = 1, 2$). We look for a path p from ξ_1 to ξ_2 consisting of three pieces:
(1) The union $\bigcup_{n\ge 0} \eta_1^{-n} p_{\eta_1^{-1}P_1,P_1}$, with a chain $p_{\eta_1^{-1}P_1,P_1} \in \mathbb{Z}[X_1]$ from $\eta_1^{-1}P_1$ to P_1 for some $P_1 \in X_0$.
(2) a chain $p_{P_1,P_2} \in \mathbb{Z}[X_1]$ from P_1 to $P_2 \in X_0$.
(3) The union $\bigcup_{n\ge 0} \eta_2^{-n} p_{P_2,\eta_2^{-1}P_2}$ for a chain $p_{P_2,\eta_2^{-1}P_2}$ from P_2 to $\eta_2^{-1}P_2$.



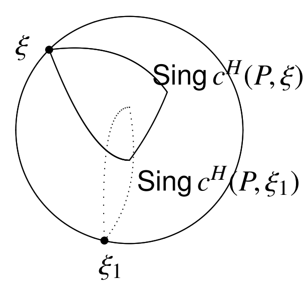
As in §7.1, there exists large $R > 0$ such that $\mathsf{Sing}\, e$ is contained in the R-neighborhood $N_R(e)$ for every $e \in X_1$. We can choose the path p such that $N_R(p) = \bigcup_{e\subset p} N_R(e)$ does not intersect any given compact set Z. Then the singularities of

$$c^H(p) \ = \ \mathrm{Av}^+_{\eta_1} c(\eta_1^{-1}P_1, P_1) + c(p_{P_1,P_2}) + \mathrm{Av}^+_{\eta_2} c(P_2, \eta_2^{-1}P_2)$$

do not meet Z. Since $c^H(p) = c^H(\xi_1,\xi_2)$ does not depend on the path p, but only on ξ_1 and ξ_2, there are no singularities at all. ■

Lemma 7.7. *Let c^H be as in Lemma 7.5. Then $c^H(P,\xi) \in \mathcal{G}_s^\omega$ for all $P \in X_0$, $\xi \in H$.*

Proof. We have $\overline{\mathsf{Sing}\,(c^H(P,\xi))} \cap \partial\mathbb{H} \subset \{\xi\}$. Let ξ_1 be another point of H. Then $c^H(P,\xi) = c^H(P,\xi_1) - c^H(\xi,\xi_1)$, so $\mathsf{Sing}\,\big(c^H(P,\xi)\big) = \mathsf{Sing}\,\big(c^H(P,\xi_1)\big)$ by Lemma 7.6. Hence $\overline{\mathsf{Sing}\,(c^H(P,\xi))} \cap \partial\mathbb{H} \subset \{\xi_1\} \cap \{\xi\} = \emptyset$. Thus $\mathsf{Sing}\,\big(c^H(P,\xi)\big)$ is a compact subset of $\mathbb{H}$. Now apply Lemma 4.4, using (7.8) and (4.3). ■



Lemma 7.8. *We have $c^H(\xi_1,\xi_2) = 0$ for all ξ_1, $\xi_2 \in H$.*

Proof. For $P \in X_0$ we have $c^H(\xi_1,\xi_2) = c^H(P,\xi_2) - c^H(P,\xi_1) \in \mathcal{E}_s \cap \mathcal{G}_s^\omega$ by the two preceding lemmas. But $\mathcal{E}_s \cap \mathcal{G}_s^\omega = \{0\}$ by virtue of the splitting (3.3). ■

Proof of Proposition 7.4. For a cocycle $c \in Z^1(F_{\cdot}^{\mathcal{T}}; \mathcal{G}_s^\omega)$, we have constructed the extension c^H to $X_0^H \times X_0^H$. For $P \in X_0^{\mathcal{T}}$ and $\xi \in H$ we have for all $\gamma \in \Gamma$

$$c(\gamma^{-1}P, P) \ = \ c^H(\gamma^{-1}P, \gamma^{-1}\xi) + c(\gamma^{-1}\xi, \xi) + c^H(\xi, P) \ = \ 0 + c^H(P,\xi)\,|\,(\gamma - 1)\,.$$

Lemma 7.7 shows that $c^H(P,\xi) \in \mathcal{G}_s^\omega$. Thus $\gamma \mapsto c(\gamma^{-1}P, P)$ is a coboundary, and the cohomology class $[c] \in H^1(\Gamma; \mathcal{G}_s^\omega)$ is trivial. ■

Chapter III. Cohomology of infinite cyclic subgroups of $\mathrm{PSL}_2(\mathbb{R})$

The general theme of these notes is the relation between Γ-invariant eigenfunctions and cohomology with values in the principal series, i.e., between the cohomology groups $H^0(\Gamma, \mathcal{E}_s)$ and subspaces of $H^1(\Gamma; \mathcal{E}_s)$, where Γ is a discrete and cofinite subgroup of $G = \mathrm{PSL}_2(\mathbb{R})$.

In this chapter we consider the corresponding question when Γ is replaced by an infinite cyclic subgroup $\Delta = \langle\gamma\rangle$ generated by a hyperbolic or parabolic element γ of G. This case is of course far easier, since the structure of Δ and the geometry of $\Delta\backslash\mathbb{H}$ are much simpler than those of Γ and $\Gamma\backslash\mathfrak{H}$, so that we can get very explicit descriptions of the corresponding cohomology groups. This will give information for the case of real interest, since the natural morphisms $H^i(\Gamma; \mathcal{E}_s) \to H^i(\Delta; \mathcal{E}_s)$ are injective for both $i = 0$ and $i = 1$ and we can therefore identify the Γ-invariant eigenfunctions and the cohomology groups of Γ with subspaces of explicit vector spaces. In particular, we show in §8.3 that a $\mathcal{V}_s^\omega$-valued 1-cocycle corresponding to an Γ-invariant eigenfunction in $\mathcal{E}_s^\Gamma$ can be reconstructed from its value on a single hyperbolic or parabolic element of Γ. And in the Propositions 9.11 and 9.15, and in Theorem 9.20 we give a cohomological characterization of various spaces of eigenfunctions invariant under a parabolic element of Γ. These results will be essential in Chapter IV, where Γ is a discrete subgroup with cusps. The results in the present chapter are the technical heart of these notes.

As in §4, we arrange by conjugation that in the parabolic case $\Delta = \langle T\rangle$, with $T = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ leaving fixed ∞, and in the hyperbolic case $\Delta = \langle\eta\rangle$, with $\eta = \left[\begin{smallmatrix}t&0\\0&1/t\end{smallmatrix}\right]$, $t > 1$, leaving fixed $\alpha(\eta) = 0$ and $\omega(\eta) = \infty$. By conjugation the results that we obtain are valid for general infinite cyclic $\Delta \subset G$. It is convenient to work in the upper half plane model of $\mathbb{H}$.

8. Invariants. The elements of $\mathcal{E}_s^\Delta$ have a periodicity under $z \mapsto z + 1$ (parabolic case) or $z \mapsto tz$ (hyperbolic case). In §8.1 we discuss the corresponding Fourier expansions, and in §8.2 we show how to associate Δ-invariant holomorphic functions to elements of $\mathcal{E}_s^\Delta$. In §8.3 we show how we can recover u from the value r_γ of a cocycle r representing $\mathbf{r}u$, where $\mathbf{r} : \mathcal{E}_s^\Delta \to H^1(\Delta; \mathcal{V}_s^\omega)$ is the injection given in §5.2.

8.1. *Fourier expansion.* Recall that Propositions 4.1 and 4.5 show that $(\mathcal{V}_s^0)^\Delta = \{0\}$. On the other hand, $(\mathcal{V}_s^{-\omega})^\Delta \cong \mathcal{E}_s^\Delta$ has infinite dimension. We now consider this space in more detail.

- *Parabolic case.* Each $u \in \mathcal{E}_s^T$ has an absolutely convergent Fourier expansion

$$u = A_0\, i_{1-s,0} + B_0\, i_{s,0} + \sum_{n\neq 0} \bigl(A_n\, k_{s,2\pi n} + B_n\, i_{s,2\pi n}\bigr). \tag{8.1}$$

with $k_{s,\nu}$, $i_{s,\nu}$ as in (1.2). For $s = \frac{1}{2}$ the Fourier term of order zero must be replaced by $a_0\, i_{s,0} + b_0\, \ell_{s,0}$ with $\ell_{s,0}$ defined as in §1.2. The terms with a factor B_n represent elements of $\mathcal{W}_s^\omega(\mathbb{R})$. We will sometimes write $A_n = A_n(u)$ and $B_n = B_n(u)$.

The modified Bessel functions have the asymptotic behavior $K_{s-1/2}(t) \sim e^{-t}\sqrt{\pi/2t}$ and $I_{s-1/2}(t) \sim e^t/\sqrt{2\pi t}$ as $t \to \infty$ ([23], §7.23). This implies the following necessary and sufficient conditions for the convergence of (8.1) in the upper half plane:

$$A_n = \mathrm{O}(e^{\varepsilon|n|}), \quad B_n = \mathrm{O}(e^{-|n|/\varepsilon}) \quad \text{for all } \varepsilon > 0. \tag{8.2}$$

For $s \neq \frac{1}{2}$, we write

$$\mathcal{E}_s^T = \mathcal{K}_s \oplus \mathcal{I}_s\,, \tag{8.3}$$

where elements of $\mathcal{I}_s$ have only terms with $i_{s,2\pi n}$ in their Fourier expansion, and elements of $\mathcal{K}_s$ have only $k_{s,2\pi n}$ and $i_{1-s,0}$. The space $\mathcal{K}_s^0$ inside $\mathcal{K}_s$ is characterized by the additional condition $B_0 = 0$. So $\mathcal{K}_s^0$ is the space of elements of $\mathcal{E}_s^T$ that have only $k_{s,2\pi n}$, $n \neq 0$, in their Fourier expansion. This characterization works also to define $\mathcal{K}_{1/2}^0$, whereas $\mathcal{K}_{1/2}$ is undefined.

The K-Bessel function and all its derivatives have exponential decay at ∞, as follows from §7.23 and §3.71 in [23]. This implies that if $u \in \mathcal{K}_s^0$ then $\partial_z^l \partial_{\bar z}^m u(z)$ has exponential decay for all choices of $l, m \in \mathbb{N}$.

- *Hyperbolic case.* Here we use the eigenfunctions $f_{s,\alpha}^R$ and $f_{s,\alpha}^L$ in (4.2). Each $u \in \mathcal{E}_s^\eta$ has an absolutely convergent Fourier expansion

$$u = \sum_\alpha \bigl(A_\alpha\, f_{s,\alpha}^R + B_\alpha\, f_{s,\alpha}^L\bigr)\,, \tag{8.4}$$

where α runs through $\frac{2\pi}{\log t}\mathbb{Z}$.

Lemma 8.1. *The coefficients in* (8.4) *satisfy*

$$A_\alpha\,,\ B_\alpha \ \ll\ e^{-\pi|\alpha|/2} \qquad (|\alpha| \to \infty)\,. \tag{8.5}$$

Proof. It is convenient not to use the basis $\{f_{s,\alpha}^R, f_{s,\alpha}^L\}$ of the space of functions transforming according to $\left[\begin{smallmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{smallmatrix}\right] \mapsto y^{i\alpha}$, but the basis $\{f_{s,\alpha}^+, f_{s,\alpha}^-\}$ in (A.17) of [4].

We write the Fourier terms as $p_\alpha f_{\alpha,s}^+ + q_\alpha f_{\alpha,s}^-$. For each $\varphi \in (0,\pi)$ the terms in the Fourier expansion of $u(\rho e^{i\varphi})$ are bounded, uniformly in $\alpha \in \frac{2\pi}{\log t}\mathbb{Z}$. The same holds for the derivatives with respect to φ. Then equation (A.18) in [4] implies that the p_α and q_α are bounded. We express A_α and B_α in p_α and q_α by inverting relation (A.20) in [4], and use of Stirling's formula to obtain (8.5). ■

8.2. *Holomorphic functions associated to periodic eigenfunctions.* The growth conditions (8.2) and (8.5) for the coefficients in the Fourier expansions give the possibility to encode elements of $\mathcal{E}_s^\Delta$ by a pair of holomorphic functions or by a holomorphic function and a hyperfunction.

- *Parabolic case.* Let $s \neq \frac{1}{2}$. For $u \in \mathcal{E}_s^T$, the formula

$$\beta(u)(\zeta) = \sum_{n\in\mathbb{Z}} B_n\, e^{2\pi i n\zeta} \tag{8.6}$$

defines $\beta(u) \in \mathcal{O}(\mathbb{C})^T$, and each element of $\mathcal{O}(\mathbb{C})^T$ occurs in this way. Alternatively, one may use $\sum_n B_n\, q^n \in \mathcal{O}(\mathbb{C}^*)$, with $q = e^{2\pi i\zeta}$. The coefficients A_n give rise to

$$\alpha(u) = \sum_{n\in\mathbb{Z}} A_n\, \mathbf{e}_{s,n}^{\mathbb{P}} \ \in\ C^{-\omega}(\mathbb{P}_{\mathbb{R}}^1)\,, \tag{8.7}$$

with $\mathbf{e}_{s,n}^{\mathbb{P}}$ as in (2.17b). Thus, the bijective correspondence $u \leftrightarrow (\alpha(u),\beta(u))$ codes elements of $\mathcal{E}_s^T$ as pairs consisting of a hyperfunction in $C^{-\omega}(\mathbb{P}_{\mathbb{R}}^1)$ and a holomorphic function in $\mathcal{O}(\mathbb{C})^T$. The following proposition shows that the function $\beta(u)$ can be related independently to the $\mathcal{V}_s^\omega$-valued cocycle r associated to u in (5.5a).

Proposition 8.2. *Let* $s \neq \frac{1}{2}$, $u \in \mathcal{E}_s^T$, $z_0 \in \mathfrak{H}$.

i) *With* $r_T(\zeta) = \int_{z_0-1}^{z_0} [u, R(\zeta;\,\cdot\,)^s]$, *and with the average* Av_T *in §4.2 and the gamma factor* $b(s)$ *in* (3.4d)*:*

$$\beta(u)(\zeta) \;=\; \frac{-b(s)}{\pi i}\mathrm{Av}_T(r_T)(\zeta) \;=\; \frac{-b(s)}{\pi i}\int_{iy_0-\infty}^{iy_0+\infty} [u, R(\zeta;\,\cdot\,)^s]\,. \tag{8.8}$$

ii) *Put* $R_s(t;\,\cdot\,) = \mathrm{Av}_T(R(t;\,\cdot\,)^s)$*. Then* $R_s(t,z) = R_s^1(z-t)$ *where* $R_s^1 \in \mathcal{E}_s^T$ *has the Fourier expansion*

$$R_s^1(z) \;=\; \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} i_{1-s,0}(z) + \frac{2\pi^s}{\Gamma(s)}\sum_{m\neq 0} |m|^{s-\frac{1}{2}} k_{s,2\pi m}(z)\,. \tag{8.9}$$

iii) *The functions* $\beta(u)$ *and* R_s *are related by*

$$\beta(u)(t) \;=\; \frac{-b(s)}{\pi i}\int_{z_0-1}^{z_0} [u, R_s(t;\,\cdot\,)]\,. \tag{8.10}$$

The integral in (8.8) converges absolutely if $\mathrm{Re}\, s > \frac{1}{2}$, and has to be understood in the regularized sense discussed in §4.2 if $0 < \mathrm{Re}\, s \le \frac{1}{2}$.

Before giving the proof of Proposition 8.2, we state the corresponding result for the hyperbolic case, which will be proved in §8.3.

- *Hyperbolic case.* Elements $u \in \mathcal{E}_s^\eta$ are coded by two holomorphic functions

$$\begin{aligned} \beta^R(u)(\zeta) \;&=\; \sum_\alpha A_\alpha\, \zeta^{i\alpha} &&\text{with } \mathrm{Re}\,\zeta > 0\,,\\ \beta^L(u)(\zeta) \;&=\; \sum_\alpha B_\alpha\, (-\zeta)^{i\alpha} &&\text{with } \mathrm{Re}\,\zeta < 0\,, \end{aligned} \tag{8.11}$$

where α runs through $\frac{2\pi}{\log t}\mathbb{Z}$. The Fourier coefficients A_α and B_α are those in (8.4). The holomorphic functions $\beta^R(u)$ and $\beta^L(u)$ are invariant under $\zeta \mapsto t\zeta$. We shall prove:

Proposition 8.3. *Let* $u \in \mathcal{E}_s^\eta$*, and* $z_0 = iy_0$*,* $y_0 > 0$*. With* $r_\eta(\zeta) = \int_{iy_0/t}^{iy_0} [u, R(\zeta;\,\cdot\,)^s]$*:*

$$\frac{-b(s)}{\pi i}\,\mathrm{Av}_\eta(r_\eta)(\zeta) \;=\; \begin{cases} \beta^R(u)(\zeta) & \textit{for } \mathrm{Re}\,\zeta > 0\,,\\ -\beta^L(u)(\zeta) & \textit{for } \mathrm{Re}\,\zeta < 0\,. \end{cases} \tag{8.12}$$

Proof of Proposition 8.2. We can assume that $\mathrm{Re}\, s > \frac{3}{2}$ since the Fourier expansion (8.1) with fixed coefficients A_n and B_n gives an analytic continuation of the function $u(z)$ (still belonging to $\mathcal{E}_s^T$), and hence also of $r_T(\zeta)$, to all complex values of s (with $\beta(u) \in \mathcal{O}(\mathbb{C})^T$ constant), and since all of the expressions in (8.8), (8.9) and (8.10) are meromorphic in s. For $\mathrm{Re}\, s > \frac{3}{2}$ the sum defining the average $\mathrm{Av}_T(r_T)$ converges absolutely and by (1.9) and partial integration we get for $|\mathrm{Im}\,\zeta| < y_0$

$$\begin{aligned} \mathrm{Av}_T(r_T)(\zeta) \;&=\; \int_{\mathrm{Im}\, z=y_0} [u, R(\zeta;\,\cdot\,)^s] \;=\; \int_{\mathrm{Im}\, z=y_0} \frac{-i}{2}\{u, R(\zeta;\,\cdot\,)^s\}\\ &=\; -\frac{i}{2}\int_{-\infty}^{\infty}\Big(R(\zeta; y_0+x)^s\,\frac{\partial u}{\partial y}(y_0+x) - u(y_0+x)\,\frac{\partial}{\partial y}R(\zeta; y_0+x)^s\Big)\,dx\,. \end{aligned}$$

The function u and its derivatives may be unbounded when z varies in $\mathfrak{H}$. On a horizontal line however, they stay bounded. We insert the Fourier expansion of u, and consider the term of order m, which has the form $f_m(y)\, e^{2\pi i m x}$.

$$\begin{aligned} -\frac{i}{2}\int_{-\infty}^{\infty} e^{2\pi i m x}\Big(f_m'(y_0)\, R(\zeta;z)^s - f_m(y_0)\,\frac{\partial}{\partial y} R(\zeta;z)^s\Big|_{y=y_0}\Big)\, dx \\ = -\frac{i}{2} e^{2\pi i m\zeta}\left(f_m'(y_0)\, L_m(y_0) - f_m(y_0)\, L_m'(y_0)\right), \end{aligned} \tag{8.13}$$

where

$$L_m(y) \;=\; \int_{-\infty}^{\infty} e^{2\pi i m x}\, R(0;z)^s\, dx \;=\; y^s \int_{-\infty}^{\infty} \frac{e^{2\pi i m x}}{(x^2+y^2)^s}\, dx\,.$$

The last expression is considered in (1.8). The contribution of the term of order m is given by a Wronskian, and can be computed with use of the definitions in (1.2) and (A.4) in [4]. We find:

$$\mathrm{Av}_T(r_T)(\zeta) \;=\; -\frac{\pi i\, \Gamma(2s)}{2^{2s-1}\,\Gamma(s)^2}\sum_m B_m e^{2\pi i m\zeta} \;=\; -\pi i\, b(s)^{-1}\, \beta(u)(\zeta)\,. \tag{8.14}$$

Now equation (8.9) is obtained with the summation formula of Poisson, and finally

$$\mathrm{Av}_T(r_T)(\zeta) \;=\; \sum_n \int_{z_0-n-1}^{z_0-n} [u, R(\zeta+n;\,\cdot\,)^s] \;=\; \int_{z_0-1}^{z_0} [u, R_s(\zeta,\,\cdot\,)]\,. \qquad \blacksquare$$

8.3. *Reconstruction.* Proposition 5.1 has shown that $u \mapsto r_\gamma$ is injective, both for $\gamma = T$ and $\gamma = \eta$. To reconstruct u explicitly from r_γ, we first pass to the image $q_\gamma = b(s)\mathsf{P}_s^\dagger r_\gamma$ under the transverse Poisson transform and then reconstruct u from q_γ.

With the base point $z_0 \in \mathfrak{H}$ as in the previous subsection, we have

$$\mathrm{Av}_T(q_T)(z) \;=\; \int_{\mathrm{Im}\, z'=z_0} [u, q_s(\,\cdot\,, z)]\,, \qquad \mathrm{Av}_\eta(q_\eta)(z) \;=\; \int_0^{i\infty} [u, q_s(\,\cdot\,, z)]\,.$$

The second integral converges absolutely for $\mathrm{Re}\, s > 0$, while the first has to be understood in the regularized sense, under the assumption $s \neq \frac{1}{2}$. Both integrals define λ_s-eigenfunctions of the Laplace operator outside the path of integration. Thus we have u_D, u_U, u_L, u_R in $\mathcal{E}_s(X)$, where X in each case is a component of $\mathfrak{H}$ minus the path of integration:

$$\int_{\mathrm{Im}\, z'=y_0} [u, q_s(\,\cdot\,, z)] \;=\; \begin{cases} u_D(z) & \text{for } z \text{ below the path}\,,\\ u_U(z) & \text{for } z \text{ above the path}\,,\end{cases} \tag{8.15a}$$

$$\int_0^{i\infty} [u, q_s(\,\cdot\,, z)] \;=\; \begin{cases} u_L(z) & \text{for } z \text{ on the left of the path}\,,\\ u_R(z) & \text{for } z \text{ on the right of the path}\,.\end{cases} \tag{8.15b}$$

Local deformation of the path of integration shows that these four functions extend to $\mathfrak{H}$, yielding four elements of $\mathcal{E}_s$. Theorem 1.1 implies that

$$u_U(z) - u_D(z) \;=\; u_L(z) - u_R(z) \;=\; \pi i\, u(z)\,. \tag{8.16}$$

In the *parabolic case*, u_D represents an element of $\mathcal{W}_s^\omega(\mathbb{R})$. We use that the restriction to the boundary ρ_s inverts the transverse Poisson transformation. With (5.6), §4.2 and Proposition 8.2, we have for $\xi \in \mathbb{R}$:

$$(\rho_s \mathrm{Av}_T q_T)(\xi) \;=\; b(s)\,(\rho_s \mathrm{Av}_T \mathsf{P}_s^\dagger r_T)(\xi) \;=\; b(s)\,(\mathrm{Av}_T r_T)(\xi) \;=\; -\pi i\, \beta(u)(\xi)\,.$$

Since $i_{s,2\pi n}(z) \sim y^s e^{2\pi i n x}$ as $y \downarrow 0$, this implies that

$$u_D(z) \;=\; -\pi i \sum_n B_n i_{s,2\pi n}(z)\,.$$

This Fourier expansion identifies $-u_D$ as the component of $\pi i\, u$ in $\mathcal{I}_s$ in the direct sum decomposition $\mathcal{E}_s^T = \mathcal{K}_s \oplus \mathcal{I}_s$ of (8.3). Then u_U is the component in $\mathcal{K}_s$. We have obtained:

Proposition 8.4. *Let $s \neq \frac{1}{2}$. Each function $u \in \mathcal{E}_s^T$ can be recovered as*

$$u \;=\; (\pi i)^{-1} u_U - (\pi i)^{-1} u_D \in \mathcal{K}_s \oplus \mathcal{I}_s \tag{8.17}$$

from the value $q_T \in \mathcal{W}_s^\omega$ of the cocycle q in (5.5c) *representing* $\mathbf{q}u$*, by the regularized integral in* (8.15a).

Remark. By (3.6c) the transverse Poisson transform of the function $R_s(\,\cdot\,, z')$ occurring in part ii) of Proposition 8.2 is the resolvent kernel function

$$Q_s(z,z') \;=\; b(s)^{-1}\left(\mathrm{Av}_T q_s(z,\,\cdot\,)\right)(z') \tag{8.18}$$

on $\{(z,z') \in \mathfrak{H}^2 \;:\; \operatorname{Im} z \neq \operatorname{Im} z'\}$. It satisfies $Q_s(z,z'+1) = Q_s(z,z') = Q_s(z',z)$. On $\operatorname{Im} z' < \operatorname{Im} z$ we use $\mathsf{P}_s e^{i\alpha x} = i_{s,\alpha}$ to obtain the expansion

$$\begin{aligned} Q_s(z,z') \;&=\; b(s)\left(\mathsf{P}_s R(z(\,\cdot\,,z)\right)(z') \\ &=\; \frac{1}{s-\frac{1}{2}}\, i_{1-s,0}(z)\, i_{s,0}(z') + \frac{2\pi^{s+\frac{1}{2}}}{\Gamma(s+\frac{1}{2})} \sum_{m\neq 0} |m|^{s-\frac{1}{2}}\, k_{2,s\pi m}(z)\, i_{s,-2\pi m}(z')\,. \end{aligned} \tag{8.19}$$

In the *hyperbolic case*, u_L represents an element of $\mathcal{W}_s^\omega(-\infty,0)$, and u_R an element of $\mathcal{W}_s^\omega(0,\infty)$. Thus $(\pi i)^{-1} u_L$ is given by the part of the Fourier expansion of u with $f_{s,\alpha}^L$, and $-(\pi i)^{-1} u_R$ by the part with $f_{s,\alpha}^R$:

$$u_L \;=\; \pi i \sum_\alpha B_\alpha f_{s,\alpha}^L\,, \qquad u_R \;=\; -\pi i \sum_\alpha A_\alpha f_{s,\alpha}^R\,. \tag{8.20}$$

Hence $\rho_s u_R(\xi) = -\pi i \beta^R(\xi)$ for $\xi > 0$, and $\rho_s u_L(\xi) = \pi i \beta^L(\xi)$ for $\xi < 0$. This proves Proposition 8.3. Furthermore, we have obtained:

Proposition 8.5. *Let $u \in \mathcal{E}_s^\eta$, and let $q \in Z^1(\langle\eta\rangle, \mathcal{W}_s^\omega)$ be a cocycle representing* $\mathbf{q}u$*. The integral in* (8.15b) *reconstructs u from q_η as $u = (\pi i)^{-1}\,(u_L - u_R)$, with the two terms corresponding as in* (8.20) *to the f^L and f^R terms in the Fourier expansion* (8.4).

9. Coinvariants. In this section the subject of study is $\mathcal{V}_s^\omega/(\mathcal{V}_s^\omega\,|\,(1-\gamma))$, for $\gamma = \eta$ and $\gamma = T$, as before. The parabolic case $\gamma = T$ is more complicated than the hyperbolic case $\gamma = \eta$. The parabolic case will lead us to consider several G-modules between $\mathcal{V}_s^\omega$ and $\mathcal{V}_s^{\omega^*}$, like $\mathcal{V}_s^{\omega^*,\infty}$ and $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ mentioned in the introduction.

The main theme in this section is the correspondence between various spaces of T-invariant eigenfunctions and cohomology groups. The main results are Proposition 9.11, Proposition 9.15 and Theorem 9.20. We will use these and other results from this section in Chapter IV, where we study the cohomological characterization of various spaces of Γ-invariant eigenfunctions for discrete subgroups $\Gamma \subset G$ with cusps.

9.1. *The first cohomology group and averaging operators.* Let $\gamma = T$ or η and $\Delta = \langle\gamma\rangle$ as in the previous section. We have $H^1(\Delta; \mathcal{V}_s^\omega) \cong (\mathcal{V}_s^\omega)_\Delta \cong \mathcal{V}_s^\omega/(\mathcal{V}_s^\omega|(1-\gamma))$, by associating to $v \in \mathcal{V}_s^\omega$ the cocycle ψ with $\psi_\gamma = v$. (This can be seen as a special case of Poincaré duality, since $V_\Delta = H_0(\Delta; V)$ and the classifying space of Δ is a circle.)

To apply the averaging operator Av_γ defined in §4, we assume in the parabolic case that $s \neq \frac{1}{2}$. Since $v|(1-\gamma)|\mathrm{Av}_\gamma^\pm = v$, the space $\mathcal{V}_s^\omega|(1-\gamma)$ is contained in the kernel of $\mathrm{Av}_\gamma = \mathrm{Av}_\gamma^+ - \mathrm{Av}_\gamma^- : \mathcal{V}_s^\omega \to (\mathcal{V}_s^{\omega^*})^\gamma$. So Av_γ induces a linear map $H^1(\Delta; \mathcal{V}_s^\omega) \to (\mathcal{V}_s^{\omega^*})^\Delta$.

The hyperbolic case is easy to treat. Here the image of Av_γ is contained $(\mathcal{V}_s^\omega[0,\infty])^\Delta \cong \mathcal{V}_s^\omega(-\infty,0)^\Delta \oplus \mathcal{V}_s^\omega(0,\infty)^\Delta$ (cf. (2.21)), and we have:

Proposition 9.1. *Let $\Delta = \langle\eta\rangle$ with η hyperbolic. Then the map $\mathrm{Av}_\eta : H^1(\Delta; \mathcal{V}_s^\omega) \to \mathcal{V}_s^\omega[0,\infty]^\Delta$ is injective, and the natural map $H^1(\Delta; \mathcal{V}_s^\omega) \to H^1(\Delta; \mathcal{V}_s^{\omega^*})$ is zero.*

Proof. The first statement is equivalent to Corollary 4.3. For $v \in \mathcal{V}_s^\omega$ we set $h = \mathrm{Av}_\eta(v) \in \mathcal{V}_s[0,\infty]^\Delta$ and define $f \in \mathcal{V}_s^{\omega^*}$ by $f(x) = -\frac{\log|x|}{\log t}\, h(x)$. Since $h|\eta = h$, the function f satisfies $f|(1-\eta) = h$. ∎

The parabolic case is more complicated. In that case the map $\mathrm{Av}_T : (\mathcal{V}_s^\omega)_\Delta \to (\mathcal{V}_s^\omega[\infty])^\Delta$ is no longer injective.

9.2. *Invariant eigenfunctions on subsets of the upper half-plane.* In the main theorems of these notes, we give isomorphisms between on the one hand spaces of Maass forms and larger spaces of invariant eigenfunctions, and on the other hand cohomology groups. In this section we study similar relations, but now between spaces of T-invariant eigenfunctions and cohomology groups for $\langle T\rangle$. This subsection serves to define such spaces of T-invariant eigenfunctions.

Definition 9.2. We put

$$\mathcal{E}_s^\uparrow = \varinjlim_Y \mathcal{E}_s(\{z \in \mathfrak{H} : \operatorname{Im} z > Y\}), \quad \mathcal{E}_s^\downarrow = \varinjlim_\varepsilon \mathcal{E}_s(\{z \in \mathfrak{H} : 0 < \operatorname{Im} z < \varepsilon\}). \tag{9.1}$$

Thus, elements of $\mathcal{E}_s^\uparrow$ may be viewed as eigenfunctions defined on some half-plane $\operatorname{Im} z > Y$, where Y may depend on the function. Similarly, elements of $\mathcal{E}_s^\downarrow$ are defined on some strip $0 < \operatorname{Im} z < \varepsilon$. Representatives of elements of $(\mathcal{E}_s^\uparrow)^T$ and $(\mathcal{E}_s^\downarrow)^T$ have Fourier expansions of the form indicated in (8.1) (modified as indicated there if $s = \frac{1}{2}$) converging on some half-plane or strip. The Fourier coefficients satisfy weaker growth conditions than those indicated in (8.2), namely

$$\begin{aligned} u \in (\mathcal{E}_s^\uparrow)^T &\Leftrightarrow A_n = \mathrm{O}(e^{\varepsilon|n|}) \text{ for some } \varepsilon > 0 \text{ and } B_n = \mathrm{O}(e^{-|n|/\varepsilon}) \text{ for all } \varepsilon > 0\,, \\ u \in (\mathcal{E}_s^\downarrow)^T &\Leftrightarrow A_n = \mathrm{O}(e^{\varepsilon|n|}) \text{ for all } \varepsilon > 0 \text{ and } B_n = \mathrm{O}(e^{-|n|/\varepsilon}) \text{ for some } \varepsilon > 0\,. \end{aligned} \tag{9.2}$$

Definition 9.3. For $s \neq \frac{1}{2}$, we define $\mathcal{K}_s^\uparrow = \mathcal{K}_s \cap \mathcal{E}_s^\uparrow$ as the subspace of elements of $(\mathcal{E}_s^\uparrow)^T$ represented by functions with only terms $k_{s,2\pi n}$, $n \neq 0$, and $i_{1-s,0}$ in their Fourier expansion. For all s we define $(K_s^0)^\uparrow$ as the subspace of $(\mathcal{E}_s^\uparrow)^T$ of elements represented by a Fourier expansion containing only terms $k_{s,2\pi n}$, $n \neq 0$, and $\mathcal{I}_s^\downarrow$ as the subspace of elements of $(\mathcal{E}_s^\downarrow)^T$ with representatives containing only terms $i_{s,2\pi n}$ in their Fourier expansion.

The cocycles r, p and q on $\langle T\rangle$ in (5.5) make sense for $u \in (\mathcal{E}_s^\uparrow)^T$, provided we take $\operatorname{Im} z_0$ sufficiently large. The proof of Proposition 8.2 can be extended to give:

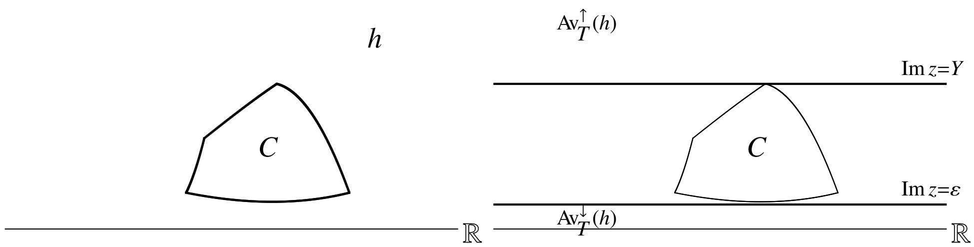
Lemma 9.4. *Let $s \neq \frac12$. Let $u \in (\mathcal{E}_s^\uparrow)^T$. The cohomology class* $\mathbf{r}u$ *is represented by the cocycle r determined by $r_T(\zeta) = \int_{z_0-1}^{z_0}[u, R(\zeta;\cdot)^s]$ for $\operatorname{Im} z_0$ sufficiently large. The average $\operatorname{Av}_T(r_T) \in \mathcal{V}_s^\omega[\infty]^T$ is represented by the holomorphic function $-\pi i\, b(s)^{-1}\beta(u)$, where $\beta(u) \in O(\mathbb{C})^T$ is defined by an expansion similar to* (8.6) *with B_n the coefficient of $i_{s,2\pi n}$ in the Fourier expansion of u.*

We note that although $u \in (\mathcal{E}_s^\uparrow)^T$ is represented by a function on some half-plane $\operatorname{Im} z > Y$, the corresponding series $\sum_n B_n i_{s,2\pi n}$ converges on all of $\mathfrak{H}$, and hence $\beta(u)$ is holomorphic on $\mathbb{C}$.

The average $\operatorname{Av}_T(r_T)$ in Lemma 9.4 is defined on a neighborhood of $\mathbb{R}$ in $\mathbb{C}$. The average $\operatorname{Av}_T(q_T)$ makes sense on two regions in $\mathfrak{H}$. We now consider the consequences of this fact.

Let h be a function on $\mathfrak{H}$ representing an element of $\mathcal{W}_s^\omega$. Such an element is in $\mathcal{E}_s(\mathfrak{H} \smallsetminus C)$ where $C \subset \mathfrak{H}$ is compact. The averaging operator defines two λ_s-eigenfunctions of the Laplace operator, on regions $\operatorname{Im} z > Y$ and $0 < \operatorname{Im} z < \varepsilon$, where $\varepsilon > 0$ and $Y > 0$ are such that the regions $\{z \in \mathfrak{H} \;:\; y \leq \varepsilon\}$ and $\{z \in \mathfrak{H} \;:\; y \geq Y\}$ are both contained in the domain of h. We obtain:

$$\begin{aligned} &\operatorname{Av}_T^\uparrow(h) \text{ represented by } \operatorname{Av}_T(h) \text{ on a region } \operatorname{Im} z \geq Y\,,\\ &\operatorname{Av}_T^\downarrow(h) \text{ represented by } \operatorname{Av}_T(h) \text{ on a region } 0 < \operatorname{Im} z < \varepsilon\,. \end{aligned} \tag{9.3}$$



Lemma 9.5. i) *If $s \neq \frac12$, then for all boundary forms $h \in \mathcal{W}_s^\omega$*

$$\operatorname{Av}_T^\downarrow(h) \in \mathcal{I}_s^\downarrow\,, \qquad \operatorname{Av}_T^\uparrow(h) \in \mathcal{K}_s^\uparrow\,.$$

ii) *If the boundary form $h \in \mathcal{W}_s^\omega$ satisfies $(\rho_s h)^{\mathbb{P}}(\infty) = 0$ then for all s with* $0 < \operatorname{Re} s < 1$

$$\operatorname{Av}_T^\downarrow(h) \in \mathcal{I}_s^\downarrow\,, \qquad \operatorname{Av}_T^\uparrow(h) \in (\mathcal{K}_s^0)^\uparrow\,.$$

Proof. Using the restriction $\rho_s : \mathcal{W}_s^\omega \to \mathcal{V}_s^\omega$, we put $\varphi = \rho_s h \in \mathcal{V}_s^\omega$. We follow the reasoning in the proof of Proposition 8.2 with u replaced by $\operatorname{Av}_T^\downarrow(h)$, keeping the integral on a line with $y_0 < \varepsilon$. Thus, we get a holomorphic 1-periodic function on $|\operatorname{Im}\zeta| < \varepsilon$. Applying $\mathsf{P}_s^\dagger$, we get a series expansion for $\operatorname{Av}_T^\downarrow(h)$ in the eigenfunctions $i_{s,2\pi n}$.

Next we turn to $\operatorname{Av}_T^\uparrow(h)$. There is no corresponding "$\operatorname{Av}_T^\uparrow(\varphi)$". Let first $s \neq \frac12$. Lemma 4.6 implies that $\operatorname{Av}_T(h)(z) = Cy^{1-s} + \mathrm{O}(y^{-s})$ as $y \to \infty$. This shows that the Fourier expansion of this periodic function consists of multiples of $i_{1-s,0}$ and $k_{s,2\pi n}$,

$n \neq 0$. If $(\rho_s h)^{\mathbb{P}}(\infty) = 0$, the same lemma gives an estimate $\mathrm{O}(y^{-s})$, which shows that we have only terms $k_{s,2\pi n}$, $n \neq 0$. ■

Proposition 9.6. *For $u \in (\mathcal{E}_s^{\uparrow})^T$ put $h(z) = \int_{z_0-1}^{z_0} [u, q_s(\,\cdot\,,z)]$. Then $\mathrm{Av}_T^{\downarrow}(h) \in \mathcal{I}_s$, and u is reconstructed from h by*

$$u(z) \;=\; (\pi i)^{-1}\mathrm{Av}_T^{\uparrow}(h)(z) - (\pi i)^{-1}\mathrm{Av}_T^{\downarrow}(h)(z) \qquad (\mathrm{Im}\, z \textit{ sufficiently large}), \tag{9.4}$$

giving the decomposition $(\mathcal{E}_s^{\uparrow})^T = \mathcal{K}_s^{\uparrow} \oplus \mathcal{I}_s$, which generalizes (8.17).

Proof. The definition of h generalizes the cocycle q in (5.5c). We have to take z_0 in the domain of a representative of u. The function h is in $\mathcal{W}_s^{\omega}$ by the properties of the kernel function q_s. Following the reasoning in §8.3 we get the decomposition (8.17), and obtain also that $\mathrm{Av}_T^{\downarrow}(h)(z)$ doe depend on z_0 with $\mathrm{Im}\, z_0 > \mathrm{Im}\, z$. This implies that $\mathrm{Av}_T^{\downarrow}(h) \in \mathcal{I}_s$. ■

9.3. *Smooth semi-analytic vectors.* It will turn out, in (12.6) and (14.8a), that the period functions ψ attached to modular Maass cusp forms in the introduction are elements of the space $\mathcal{V}_s^{\omega^*,\infty}$ defined in (2.23), which is equal to $\mathcal{V}_s^{\omega^*} \cap \mathcal{V}_s^{\infty}$. We call it the space of *smooth semi-analytic vectors* in $\mathcal{V}_s$.

For $\mathcal{W}$ we use the analogous notational convention $\mathcal{W}_s^{\omega^*,\mathrm{cond}}$ as for $\mathcal{V}$ in (2.23) and below (2.21), where "cond" denotes a condition imposed at the singularities.

The space $\mathcal{W}_s^{\omega^*,\infty}$ consists of the elements of $\mathcal{W}_s^{\omega^*}$ with representatives $f(z) = \left(\frac{y}{|z+i|^2}\right)^s a^{\mathbb{P}}(z)$ with $a^{\mathbb{P}}$ real analytic on an open set $\Omega \subset \mathbb{P}^1_{\mathbb{C}}$ with $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \Omega$ finite, such that $\rho_s : \xi \mapsto a^{\mathbb{P}}(\xi)$ on $\mathbb{P}^1_{\mathbb{R}} \cap \Omega$ extends as an element of $\mathcal{V}_s^{\infty}$.

The G-modules $\mathcal{V}_s^{\omega^*,\infty}$ and $\mathcal{W}_s^{\omega^*,\infty}$ are isomorphic with the inverse isomorphisms

$$\rho_s : \mathcal{W}_s^{\omega^*,\infty} \longrightarrow \mathcal{V}_s^{\omega^*,\infty}, \qquad \mathsf{P}_s^{\dagger} : \mathcal{V}_s^{\omega^*,\infty} \longrightarrow \mathcal{W}_s^{\omega^*,\infty}. \tag{9.5}$$

We can produce elements of $\mathcal{V}_s^{\omega^*,\infty}$ by the following integral, discussed in [19], Chapter 2, §2:

Proposition 9.7. *Let $u \in (\mathcal{K}_s^0)^{\uparrow}$. Then*

$$f_{z_0}(\zeta) \;=\; \int_{z_0}^{\infty} \left[u, R(\zeta;\,\cdot\,)^s\right], \tag{9.6}$$

with $\mathrm{Re}\, z_0$ *sufficiently large, defines $f_{z_0} \in \mathcal{V}_s^{\omega,\infty}[\infty]$, independent of the path of integration from z_0 to ∞, provided ∞ is approached along a vertical line.*

As z_0 tends to ∞ along a vertical half-line, then f_{z_0} tends to 0 *in the topology of $\mathcal{V}_s^{\infty}$, defined by the seminorms $\|\cdot\|_n$ in* (2.9) *for all $n \in \mathbb{N}$.*

The second part of this proposition is one of the few places in these notes where we mention the topology of principal series spaces. We will use this part in the proof of Theorem 19.1.

Proof. In part I, the first statement has been proved with the proposition in §2 of Chap. II and use of L-functions. Here we also consider the limit in the topology of $\mathcal{V}_s^{\infty}$.

It suffices to consider $z_0 = iy_0$ with $y_0 > 1$. We work in the projective model. From (2.30) and (1.9) we conclude for $\zeta \in \mathbb{R}$:

$$f_{iy_0}(\zeta) \;=\; \int_{y_0}^{\infty} y^s \left(\frac{\zeta^2+1}{\zeta^2+y^2}\right)^s \left(i\, u_z(iy) + \frac{s}{2y} u(iy) \frac{\zeta - iy}{\zeta + iy}\right) dy\,. \tag{9.7}$$

The exponential decay of u and u_z implies convergence. The coefficients in the expansion in $\zeta-\zeta_0$ of $\left(\frac{\zeta^2+1}{\zeta^2+y^2}\right)^s$ and of $y^{-1}\left(\frac{\zeta^2+1}{\zeta^2+y^2}\right)^s\frac{\zeta-iy}{\zeta+iy}$ at a given $\zeta_0\in\mathbb{R}$ are bounded functions of y. Hence after integration we get a converging power series representing $f_{iy_0}(\zeta)$ on a neighborhood of ζ_0. So $f_{iy_0}\in\mathcal{V}_s^\omega[\infty]$.

On a neighborhood of $\zeta=\infty$ we use the variable $\xi=\frac{1}{\zeta}$:

$$f_{iy_0}(1/\xi) \;=\; \int_{y_0}^{\infty} y^s\left(\frac{1+\xi^2}{1+y^2\xi^2}\right)^s\left(i\,u_z(iy)+\frac{s}{2y}\,u(iy)\,\frac{1-iy\xi}{1+iy\xi}\right)dy\,. \tag{9.8}$$

This converges as well. The coefficient of ξ^n in the expansion of the integrand at $\xi=0$ contains a term $y^{s+n}u(iy)$. So analyticity at ∞ seems out of the question. Differentiation with respect to ξ can be carried out inside the integral. This shows that $f_{iy_0}\in\mathcal{V}_s^\infty$.

Next we estimate the supremum norm $\|f_{iy_0}\,|\,\mathbf{W}^n\|_\infty$ on $\mathbb{R}$ for all $n\in\mathbb{N}$, where the differential operator $\mathbf{W}$ is given by $(1+\zeta^2)\partial_\zeta$ on $\mathbb{R}$, and by $-(1+\xi^2)\partial_\xi$ on a neighborhood of $\zeta=\infty$. From (9.7) we check that $((1+\zeta^2)\partial_\zeta)^n f_{iy_0}(\zeta)$ is a linear combination, with coefficients depending only on s and n, of integrals

$$\int_{y_0}^{\infty} f_j(y,\zeta)\,U_j(y)\,dy\,,$$

where

$$\begin{aligned}
f_j(y,\zeta) \;&=\; y^{s+a_j}\left(\frac{\zeta^2+1}{\zeta^2+y^2}\right)^s(\zeta-i)^{b_j}(\zeta+i)^{c_j}(\zeta-iy)^{-d_j}(\zeta+iy)^{-e_j}\,,\\
U_j(y) \;&=\; u_z(iy)\,,\ a_j \;=\; 0\,, \qquad \text{or} \quad U_j(y) \;=\; u(iy)\,,\ a_j \;=\; -1\,,\\
b_j,\,c_j \;&\in [0,n]\,,\quad -1\le d_j\le n\,,\quad 0\le e_j\le n+1\,,\\
b_j+c_j \;&=\; d_j+e_j+n\,.
\end{aligned}$$

For ζ in a bounded interval $[-A,A]$ and $y\ge y_0$ we have $f_j(y,\zeta)\ll_{s,n}A^ny^s$. With the exponential decay of u and its derivatives we obtain on $[-A,A]$:

$$((1+\zeta^2)\partial_\zeta)^n f_{iy_0}(\zeta)\;\ll_{s,n,\varepsilon} A^n y_0^s e^{-\varepsilon y_0}\,, \tag{9.9}$$

with $\varepsilon\in(0,2\pi)$.

For $|\xi|\le A^{-1}<1$, we have a similar linear combination of finitely many integrals $\int_{y_0}^\infty g_j(y,\xi)\,U_j(y)\,dy$ with U_j as before and

$$\begin{aligned}
g_j(y,\xi) \;&=\; y^{s+a_j}\left(\frac{1+\xi^2}{1+y^2\xi^2}\right)^s(1-i\xi)^{b_j}(1+i\xi)^{c_j}(1-iy\xi)^{-d_j}(1+iy\xi)^{-e_j}\,,\\
a_j,\,d_j &\ge -1\,,\quad b_j,\,c_j,\,e_j\in\mathbb{N}\,,\quad d_j+e_j\ge 0\,,\quad a_j\le n\,.
\end{aligned}$$

Now we have $g_j(y,\xi)\ll_{s,n}y^{s+a_j}\,(1+y^2\xi^2)^{-(d_j+e_j)/2}\ll y^{n+s}$. This leads to

$$(-(1+\xi^2)\partial_\xi)^n f_{iy_0}(1/\xi)\;\ll_{s,n}\; y_0^{n+s}e^{-\varepsilon y_0}\,. \tag{9.10}$$

This estimate and (9.9) show that $\lim_{y_0\to\infty}\left\|f_{iy_0}\,|\,\mathbf{W}^n\right\|_\infty=0$ for each $n\in\mathbb{N}$. ∎

Applying the transverse Poisson transformation $\mathsf{P}_s^\dagger$ to f_{z_0} in Proposition 9.7 and multiplying by the gamma factor $b(s)$, we obtain with use of (3.6c)

$$h_{z_0}(z)\;=\;\int_{z_0}^{\infty}[u,q_s(\,\cdot\,,z)]\,, \tag{9.11}$$

representing an element of $\mathcal{W}_s^{\omega,\infty}$, also called h_{z_0}.

Let q be a cocycle as in (5.5c), representing $\mathbf{q}u$, $u \in \mathcal{K}_s^0$. We would like to write $q_T = h_{z_0-1} - h_{z_0}$. To do that, we need the following extension of Theorem 1.1:

Proposition 9.8. *Suppose that C is a piecewise smooth positively oriented simple closed curve in $\mathfrak{H} \cup \{\infty\}$. Suppose that near ∞ the curve C consists of geodesic half-lines. For each $u \in \mathcal{K}_s^0$:*

$$\frac{1}{\pi i}\int_C [u, q_s(u_U, z)] \;=\; \begin{cases} u(z) & \text{if } z \text{ is inside } C\,, \\ 0 & \text{if } z \text{ is outside } C\,. \end{cases}$$

Proof. Approximate C by C_a, where near ∞ we have replaced the part of C in the region $y \geq a$ by a curve l_a along $\{x + ia \;:\; x \in \mathbb{R}\}$. Theorem 1.1 can be applied to C_a. The integral $\int_{l_a} [u, q_s(\,\cdot\,, z)]$ tends to zero as $a \to \infty$ for each $z \in \mathfrak{H}$, by the exponential decay of u and its derivatives. The same holds for the difference of the integrals over C and C_a. The results follows. ■

We apply Proposition 9.8 to the curve C that consists of geodesic paths from ∞ to $z_0 - 1$, to z_0, and then back to ∞. For z outside C:

$$\int_{z_0-1}^{z_0} [u, q_s(\,\cdot\,, z)] \;=\; h_{z_0-1}(z) - h_{z_0}(z)\,.$$

So $q_T = h_{z_0-1} - h_{z_0}$ in $\mathcal{W}_s^{\omega^*}$. Application of restriction map $\rho_s : \mathcal{W}_s^{\omega^*,\infty} \to \mathcal{V}_s^{\omega^*,\infty}$ gives for the cocycle r, with $f_{z_0} \in \mathcal{V}_s^{\omega^*,\infty}$ as in (9.6):

$$r_T \;=\; f_{z_0-1} - f_{z_0}\,. \tag{9.12}$$

Thus we have, with $\Delta = \langle T\rangle$:

$$\begin{aligned} \mathbf{r}\mathcal{K}_s^0 \;&\subset\; \operatorname{Ker}\big(H^1(\Delta;\mathcal{V}_s^{\omega}) \longrightarrow H^1(\Delta;\mathcal{V}_s^{\omega^*,\infty})\big)\,,\\ \mathbf{q}\mathcal{K}_s^0 \;&\subset\; \operatorname{Ker}\big(H^1(\Delta;\mathcal{W}_s^{\omega}) \longrightarrow H^1(\Delta;\mathcal{W}_s^{\omega^*,\infty})\big)\,. \end{aligned} \tag{9.13}$$

Notation. We will often deal with such kernels of natural maps between cohomology groups. For Δ-modules $V \subset W$ we put

$$H^1(\Delta; V, W) \;=\; \operatorname{Ker}\big(H^1(\Delta; V) \longrightarrow H^1(\Delta; W)\big)\,. \tag{9.14}$$

For $\Delta = \langle T\rangle$ one may view this as a mixed parabolic cohomology group, as we will discuss in Definition 10.1. We reformulate:

$$\mathbf{r}\mathcal{K}_s^0 \;\subset\; H^1(\Delta;\mathcal{V}_s^{\omega},\mathcal{V}_s^{\omega^*,\infty})\,, \qquad \mathbf{q}\mathcal{K}_s^0 \;\subset\; H^1(\Delta;\mathcal{W}_s^{\omega},\mathcal{W}_s^{\omega^*,\infty})\,. \tag{9.15}$$

Proposition 9.9. *The following statements are equivalent for $\varphi \in \mathcal{V}_s^{\omega}$:*

a) $\varphi^{\mathbb{P}}(\infty) = 0$ *and* $\mathrm{Av}_T^+(\varphi) = \mathrm{Av}_T^-(\varphi)$.

b) *There exists $h \in \mathcal{V}_s^{\omega^*,\infty}$ such that $h\,|\,(1-T) = \varphi$ in $\mathcal{V}_s^{\omega}$.*

If a) and b) hold, then there is only one $h \in \mathcal{V}_s^{\infty}$ as in b), namely $h = \mathrm{Av}_T^+(\varphi) = \mathrm{Av}_T^-(\varphi)$.

Proof. Suppose that a) is satisfied. Since $\varphi^{\mathbb{P}}(\infty) = 0$ the averaging operators Av_T^+ and Av_T^- converge absolutely on φ, without regularization and without the assumption $s \neq \frac{1}{2}$. In (4.12) the constant C_0 vanishes. Hence the expansions in (4.11) start at $m = 0$. From $\mathrm{Av}_T^+(\varphi) = \mathrm{Av}_T^-(\varphi)$ we conclude that the averages define an element $h \in \mathcal{V}_s^{\omega,\infty}[\infty]$, which satisfies b).

Suppose that b) is satisfied. This implies that $\varphi^{\mathbb{P}}(\infty) = 0$. Hence the averages converge without regularization. If $s \neq \frac{1}{2}$ we immediately obtain a) from Proposition 9.14 below. In a proof valid for all s with $0 < \operatorname{Re} s < 1$, we note that $\mathrm{Av}_T^+(\varphi)$ converges without regularization and satisfies $\mathrm{Av}_T^+(\varphi)(x) = \mathrm{O}(|x|^{-2s})$ as $x \to \infty$. Since h has the same behavior, and since $h - \mathrm{Av}_T^+(\varphi)$ is periodic we conclude that $h = \mathrm{Av}_T^+(\varphi)$. From the behavior as $x \downarrow -\infty$ we obtain $h = \mathrm{Av}_T^-(\varphi)$. ■

Corollary 9.10. *The kernel of* $\mathrm{Av}_T : \mathcal{V}_s^\omega \to \mathcal{V}_s^{\omega^*}$ *contains the space of* $\varphi \in \mathcal{V}_s^\omega$ *that satisfy the equivalent conditions a) and b) in Proposition 9.9, but is larger than this space.*

Proof. The first statement follows directly from Proposition 9.9. For the second statement, consider $u \in \mathcal{K}_s^0$, $u \neq 0$, and put $\varphi(\zeta) = \int_{z_0-1}^{z_0} [u, R(\zeta;\,\cdot\,)^s]$ for some $z_0 \in \mathfrak{H}$. Then $\varphi \in \mathcal{V}_s^\omega$, and it satisfies condition b) in Proposition 9.9 (by Proposition 9.7). Since $\varphi = r_T$ as defined in (5.5), the injectivity of $\mathbf{r}$ in Proposition 5.1 shows that the class $\mathbf{r}u$ in $H^1(\Gamma; \mathcal{V}_s^\omega)$ represented by r is non-zero. So $\varphi \notin \mathcal{V}_s^\omega \,|\, (1-T)$. ■

In §9.6 we shall prove:

Proposition 9.11. *The maps* $\mathbf{r}$ *and* $\mathbf{q}$ *give the following isomorphisms.*

$$(\mathcal{K}_s^0)^\uparrow \xrightarrow{\mathbf{r}} H^1(\langle T\rangle; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\infty}), \quad (\mathcal{K}_s^0)^\uparrow \xrightarrow{\mathbf{q}} H^1(\langle T\rangle; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\infty}).$$

9.4. *Semi-analytic vectors with simple poles.* We turn to a G-module between $\mathcal{V}_s^\omega$ and $\mathcal{V}_s^{\omega^*}$ that obtained by replacing the condition of smoothness at the singularities by the condition "simple", which allows simple singularities:

Definition 9.12. We define $\mathcal{V}_s^{\omega^*,\,\mathrm{simple}}$ as the space of $f \in \mathcal{V}_s^{\omega^*}$ such that for each $\xi \in \mathsf{BdSing}\,(f)$ the function $x \mapsto c_\xi(x) f^{\mathbb{P}}(x)$ is smooth at ξ, where c_ξ is a local coordinate on $\mathbb{P}^1_{\mathbb{R}}$ at ξ, e.g., $c_\infty(x) = \frac{1}{x}$, and $c_\xi(x) = x - \xi$ if $\xi \in \mathbb{R}$. We define $\mathcal{W}_s^{\omega^*,\,\mathrm{simple}}$ as the space of those elements of $\mathcal{W}_s^{\omega^*}$ for which $\rho_s f \in \mathcal{V}_s^{\omega^*,\,\mathrm{simple}}$.

The spaces $\mathcal{V}_s^{\omega^*,\,\mathrm{simple}}$ and $\mathcal{W}_s^{\omega^*,\,\mathrm{simple}}$ are isomorphic G-submodules of $\mathcal{V}_s^{\omega^*}$ respectively $\mathcal{W}_s^{\omega^*}$, by $\mathsf{P}_s^\dagger$ and ρ_s.

We have the following generalization of Proposition 4.5:

Proposition 9.13. *The spaces* $(\mathcal{V}_s^{\omega^*,\,\mathrm{simple}})^T$ *and* $(\mathcal{W}_s^{\omega^*,\,\mathrm{simple}})^T$ *are zero for* $s \neq \frac{1}{2}$.

Proof. Consider the expansion $\varphi(t) \sim |t|^{-2s} \sum_{n=-1}^\infty c_n t^{-n}$ at ∞ in the line model, and insert the T-invariance. If c_l is the first non-zero coefficient, it follows that $(l+2s)c_l = 0$. ■

Proposition 9.14. *Let* $s \neq \frac{1}{2}$. *For* $\varphi \in \mathcal{V}_s^\omega$ *the following statements are equivalent:*

a) $\mathrm{Av}_T(\varphi) = 0$.
b) $\mathrm{Av}_T^+(\varphi) = \mathrm{Av}_T^-(\varphi)$.
c) *There exists* $f \in \mathcal{V}_s^{\omega^*,\,\mathrm{simple}}$ *such that* $f \,|\, (1-T) = \varphi$.

If these statements holds, then f *in c) is unique, and is equal to* $\mathrm{Av}_T^\pm(\varphi)$.

Proof. The equivalence of a) and b) is clear. If a) and b) hold, then apply the asymptotic behavior in (4.11) to conclude that $f = \mathrm{Av}_T^+(\varphi) = \mathrm{Av}_T^-(\varphi) \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$. Conversely if $f \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$ satisfies $f\,|\,(1-T) = \varphi \in \mathcal{V}_s^{\omega}$, then $f - \mathrm{Av}_T^+(\varphi)$ has for $x \uparrow \infty$ an asymptotic behavior as indicated in (4.11). Since $f - \mathrm{Av}_T^+(\varphi) \in (\mathcal{V}_s^{\omega^*,\mathrm{simple}})^T = \{0\}$ (Proposition 9.13), this implies $f = \mathrm{Av}_T^+(\varphi)$. Proceed similarly for $\mathrm{Av}_T^-(\varphi)$. ∎

In §9.6 we shall prove:

Proposition 9.15. *If $s \neq \frac{1}{2}$, the maps* $\mathbf{r}$ *and* $\mathbf{q}$ *give isomorphisms*

$$\mathcal{K}_s^{\uparrow} \xrightarrow{\mathbf{r}} H^1(\langle T\rangle; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{simple}}), \quad \mathcal{K}_s^{\uparrow} \xrightarrow{\mathbf{q}} H^1(\langle T\rangle; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*,\mathrm{simple}}).$$

9.5. *Semi-analytic vectors with support condition on the singularities.* For a cohomological characterization analogous to Propositions 9.11 and 9.15 of the much larger spaces $\mathbf{r}(\mathcal{E}_s^{\uparrow})^T$ and $\mathbf{q}(\mathcal{E}_s^{\uparrow})^T$ we need to introduce yet another space of semi-analytic vectors.

To show the need for this larger space, we consider φ representing an element of $\mathcal{V}_s^{\omega}$, in the line model. The corresponding function $\varphi^{\mathbb{P}}$ in the projective model is holomorphic on a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. Hence φ is holomorphic at least on a strip $|\mathrm{Im}\,\zeta| < \varepsilon$ and on half-planes $\mathrm{Re}\,\zeta > \frac{1}{\varepsilon}$ and $\mathrm{Re}\,\zeta < -\frac{1}{\varepsilon}$ for some small positive ε. For $s \neq \frac{1}{2}$, the one-sided averages

$$\mathrm{Av}_T^+(\varphi)(\zeta) = \sum_{n=0}^{\infty} \varphi(\zeta+n), \qquad \mathrm{Av}_T^-(\varphi)(\zeta) = -\sum_{n=-\infty}^{-1} \varphi(\zeta+n)$$

are both holomorphic on the strip $|\mathrm{Im}\,\zeta| < \varepsilon$. Furthermore, $\mathrm{Av}_T^+(\varphi)$ is also holomorphic on $\mathrm{Re}\,\zeta > \frac{1}{\varepsilon}$, and $\mathrm{Av}_T^-(\varphi)$ on $\mathrm{Re}\,\zeta < -\frac{1}{\varepsilon}$, provided ε is sufficiently small. Suppose now that $\mathrm{Av}_T(\varphi)$ extends holomorphically as an element of $\mathcal{O}(\mathbb{C})^T$. (Proposition 8.2 shows that this is the case if $\varphi = r_T$ associated to $u \in \mathcal{E}_s^T$.) Then $\mathrm{Av}_T^+(\varphi) = \mathrm{Av}_T(\varphi) + \mathrm{Av}_T^-(\varphi)$ has a holomorphic extension to the half-plane $\mathrm{Re}\,\zeta < -\frac{1}{\varepsilon}$. Hence $\mathrm{Av}_T^+(\varphi) \in \mathcal{V}_s^{\omega^*}$, and also $\mathrm{Av}_T^-(\varphi) \in \mathcal{V}_s^{\omega^*}$ have representatives with large domains containing both a left and a right half-plane. They are elements of the space $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ that we now start to define.

Definition 9.16. Let $F \subset \mathbb{P}^1_{\mathbb{R}}$ be finite. We call a set $\Omega \subset \mathbb{P}^1_{\mathbb{C}}$ an *excised neighborhood* of $\mathbb{P}^1_{\mathbb{R}} \setminus F$ if it contains a set of the form

$$U \setminus \bigcup_{\xi \in F} W_\xi, \tag{9.16}$$

where U is a (usual) neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$ and where W_ξ is the set containing ξ and the sectors in $\mathfrak{H}$ and $\mathfrak{H}^-$ between two geodesic half-lines with final point ξ.

In the upper or lower half-plane, the sets W_∞ in this definition are the region between two vertical half-lines. For $\xi \in \mathbb{R}$ sets W_ξ are the regions between to half-circles through ξ with centers on $\mathbb{R}$ to the left and the right of ξ. See Figures 1 and 2 for sketches of excised neighborhoods.

Definition 9.17. Let $F \subset \mathbb{P}^1_{\mathbb{R}}$ be finite. We say that $\varphi \in \mathcal{V}_s^{\omega^*}[F]$ satisfies the condition "exc" if φ is represented by an element of $\mathcal{O}(\Omega)$ for an excised neighborhood of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus F$.

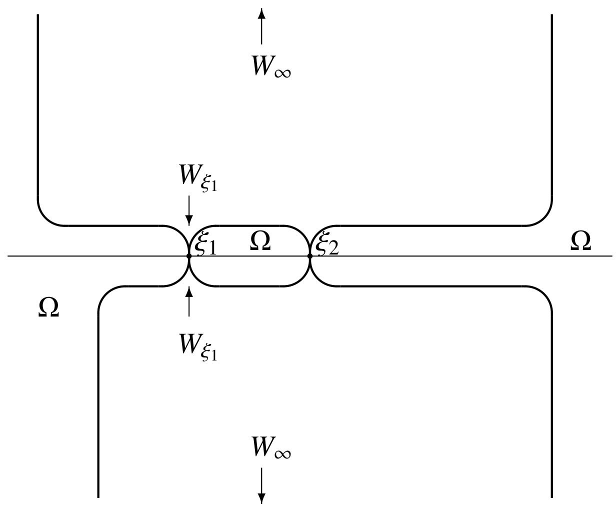


FIGURE 1. An excised neighborhood Ω of $\mathbb{P}^1_{\mathbb{R}} \setminus \{\xi_1, \xi_2, \infty\}$.

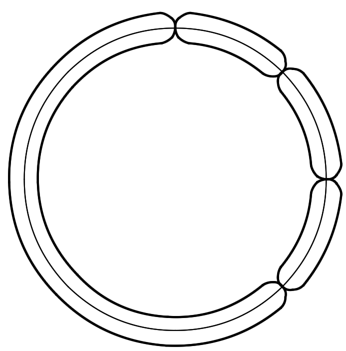


FIGURE 2. An excised neighborhood of $\mathbb{S}^1 \smallsetminus \{e^{-\pi i/4}, 1, e^{\pi n i/4}, i\}$ for the disk model.

This means that $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ is the direct limit $\varinjlim_{\Omega} \mathcal{O}(\Omega)$ where Ω runs over excised neighborhoods of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus F$ with F finite. Figure 3 depicts the relation of $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ with other principal series subspaces that we have defined up till now. The space $\mathcal{V}_s^{\omega^*,\mathrm{simple}}$ is defined only for $s \neq \frac{1}{2}$. The definition of $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ makes sense for $s = \frac{1}{2}$ as well.

The space $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ is a G-invariant subspace of $\mathcal{V}_s^{\omega^*}$. The elements $\mathrm{Av}_T^{\pm}(\varphi)$ discussed in the introduction of this subsection are in $\mathcal{V}_s^{\omega,\mathrm{exc}}[\infty]$. We have even more:

Proposition 9.18. *Let $s \neq \frac{1}{2}$. For $\varphi \in \mathcal{V}_s^{\omega}$ the following statements are equivalent:*

a) $\mathrm{Av}_T(\varphi) \in \mathcal{O}(\mathbb{C})^T$.
b) $\mathrm{Av}_T^{+}(\varphi)$ *and* $\mathrm{Av}_T^{-}(\varphi)$ *are elements of* $\mathcal{V}_s^{\omega,\mathrm{exc}}[\infty]$.

Proof. We have already discussed the implication a) $\Rightarrow$ b).

$$\begin{tikzcd}
\mathcal{V}_s^{\omega} \arrow[r, hook] \arrow[ddr, hook] & \mathcal{V}_s^{\omega^*,\infty} \arrow[r, hook] \arrow[dr, hook] & \mathcal{V}_s^{\infty} \arrow[r, hook] & \mathcal{V}_s^{-\infty} \arrow[r, hook] & \mathcal{V}_s^{-\omega} \\
 & & \mathcal{V}_s^{\omega^*,\,\mathrm{simple}} \arrow[dr, hook] & & \\
 & \mathcal{V}_s^{\omega^*,\,\mathrm{exc}} \arrow[rr, hook] & & \mathcal{V}_s^{\omega^*} &
\end{tikzcd}$$

FIGURE 3. Subspaces of the principal series space $\mathcal{V}_s$.

For a) $\Leftarrow$ b) we suppose that $\mathrm{Av}_T^+(\varphi)$ and $\mathrm{Av}_T^-(\varphi)$ are in $\mathcal{V}_s^{\omega,\mathrm{exc}}[\infty]$. By construction, they are in $\mathcal{V}_s^{\omega}[\infty]$. They are given by holomorphic functions on a region $\{\zeta\ :\ |\mathrm{Im}\,\zeta| < \varepsilon\} \cup \{\zeta\ :\ |\mathrm{Re}\,\zeta| > \varepsilon^{-1}\}$ for some $\varepsilon > 0$. On $|\mathrm{Im}\,\zeta| < \varepsilon$ we have $\mathrm{Av}_T(\varphi) = \mathrm{Av}_T^+(\varphi) - \mathrm{Av}_T^-(\varphi)$. So $\mathrm{Av}_T(\varphi)$ extends holomorphically to $\mathrm{Re}\,\zeta > \varepsilon^{-1}$, and the invariance under T stays valid by analytic continuation. The T-invariance on a right half-plane implies that $\mathrm{Av}_T(\varphi)$ extends holomorphically to $\mathbb{C}$. ■

We will now define the condition "exc" for the behavior of sections of $\mathcal{W}_\sigma^{\omega^*}$ at singularities:

Definition 9.19. An element of $\mathcal{W}_s^{\omega^*}$ satisfies the condition "exc" at the points of the finite set $F \subset \mathbb{P}^1_{\mathbb{R}}$ if it has a representative of the form $f(z) = \left(\frac{y}{|z+i|^2}\right)^s a^{\mathbb{P}}(z)$ on $\Omega \cap \mathfrak{H}$, where

a) Ω is an excised neighborhood of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus F$ for some finite set F.
b) $a^{\mathbb{P}}$ is real analytic on Ω.

The transverse Poisson transformation gives a G-equivariant isomorphism

$$\mathsf{P}_s^{\dagger} : \mathcal{V}_s^{\omega^*,\,\mathrm{exc}} \longrightarrow \mathcal{W}_s^{\omega^*,\,\mathrm{exc}}\,.$$

To see this we use the integral representations of $\mathsf{P}_s^{\dagger}$ and its inverse in Theorem 4.7 in [4]. To apply this we note that the intersection $\Omega \cap \bar{\Omega}$ of an excised neighborhood Ω of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus F$ is again an excised neighborhood of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus F$, for finite sets $F \subset \mathbb{P}^1_{\mathbb{R}}$.

Propositions 9.11 and 9.15 can be viewed partly as specializations of the following result:

Theorem 9.20. *If $s \neq \frac{1}{2}$, then the maps* $\mathbf{r}$ *and* $\mathbf{q}$ *give isomorphisms*

$$(\mathcal{E}_s^{\uparrow})^T \xrightarrow{\mathbf{r}} H^1(\langle T\rangle; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\,\mathrm{exc}})\,, \quad (\mathcal{E}_s^{\uparrow})^T \xrightarrow{\mathbf{q}} H^1(\langle T\rangle; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*,\,\mathrm{exc}})\,.$$

We shall give a proof in §9.6.

Definition 9.21. We define $\mathcal{G}_s^{\mathrm{exc}}$ as the space of $f \in C^2(\mathfrak{H})$ that are in $\mathcal{E}_s(\mathfrak{H} \cap \Omega)$ for an excised neighborhood Ω of $\mathbb{P}^1_{\mathbb{R}}$ minus a finite set, and we put

$$\mathcal{G}_s^{\omega^*,\,\mathrm{exc}} \;=\; \{f \in \mathcal{G}_s^{\mathrm{exc}}\ :\ f \text{ represents an element of } \mathcal{W}_s^{\omega^*}\}\,.$$

The minimal closed set $X \subset \mathfrak{H}$ such that $f \in \mathcal{E}_s(\mathfrak{H} \smallsetminus X)$ is the set of singularities $\mathsf{Sing}\,(f)$.

Examples: The function $i_{s,0}(z) = y^s$ is an element of $\mathcal{E}_s$ that represents an element of $\mathcal{W}_s^{\omega}(\mathbb{R})$, also called $i_{s,0}$, and $\mathsf{BdSing}\,(i_{s,0}) = \{\infty\}$. So $i_{s,0} \notin \mathcal{G}_s^{\omega}$. We have $i_{s,0}(z) =$

$\left(\frac{y}{|z+i|^2}\right)^s \left(x^2+(y+1)^2\right)^s$. We conclude that $a^{\mathbb{P}}(z) = \left(x^2+(y+1)^2\right)^s$ is real analytic on $\mathbb{C}$. Hence $i_{s,0} \in \mathcal{G}_s^{\omega^*,\mathrm{exc}}$, and $\mathsf{Sing}\,(i_{s,0}) = \emptyset$, $\mathsf{BdSing}\,(i_{s,0}) = \{\infty\}$. To get more examples of elements of $\mathcal{W}_s^{\omega^*,\mathrm{exc}}$, we consider $\varphi \in \mathcal{V}_s^\omega$ such that $\mathrm{Av}_T(\varphi) \in \mathcal{O}(\mathbb{C})^T$, as in Proposition 9.18. If $s \neq \frac{1}{2}$, then $\mathsf{P}_s^\dagger \mathrm{Av}_T^+(\varphi) = \mathrm{Av}_T^+(\mathsf{P}_s^\dagger\varphi)$ and $\mathsf{P}_s^\dagger \mathrm{Av}_T^-(\varphi) = \mathrm{Av}_T^-(\mathsf{P}_s^\dagger\varphi)$ are elements of $\mathcal{W}_s^{\omega^*,\mathrm{exc}}$.

For general $h \in \mathcal{W}_s^\omega$, the average $\mathrm{Av}_T h$ may very well have singularities in horizontal strips in the upper half-plane. Then the average is not in $\mathcal{G}_s^{\omega^*,\mathrm{exc}}$.

The example of $i_{s,0}$ shows that if $f \in \mathcal{G}_s^{\omega^*,\mathrm{exc}}$ represents $h \in \mathcal{W}_s^{\omega^*,\mathrm{exc}}$, then the set $\mathsf{BdSing}\,(h)$ can be larger than $\overline{\mathsf{Sing}\,(f)} \cap \partial\mathfrak{H}$. On the other hand, the zero element $n \in \mathcal{W}_s^\omega$ satisfies $\mathsf{BdSing}\,(n) = \emptyset$. It may be represented as an element of $\mathcal{W}_s^{\omega^*,\mathrm{exc}}$ by any $f \in C^2(\mathfrak{H})$ that is equal to 0 outside the region $|x| \leq 1$, $y \geq 1$, and equal to 1 on $|x| \leq \frac{1}{2}$, $y \geq 2$. In this example, $\overline{\mathsf{Sing}\,(f)} \cap \partial\mathfrak{H}$ is larger than $\mathsf{BdSing}\,(h)$. Definitions 9.19 and 9.21 imply that we can always represent $h \in \mathcal{W}_s^{\omega^*,\mathrm{exc}}$ by an element $f \in \mathcal{G}_s^{\omega^*,\mathrm{exc}}$ such that $\overline{\mathsf{Sing}\,(f)} \cap \partial\mathfrak{H} \subset \mathsf{BdSing}\,(h)$.

The exact sequence (3.11) extends to an exact diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{N}^\omega & \to & \mathcal{G}_s^\omega & \to & \mathcal{W}_s^\omega & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{N}^{\omega^*,\mathrm{exc}} & \to & \mathcal{G}_s^{\omega^*,\mathrm{exc}} & \to & \mathcal{W}_s^{\omega^*,\mathrm{exc}} & \to & 0 \end{array} \tag{9.17}$$

We recall that $\mathcal{N}^\omega = C_c^2(\mathfrak{H})$, and define $\mathcal{N}^{\omega^*,\mathrm{exc}}$ as the kernel in the lower row. The support of an element of $\mathcal{N}^{\omega^*,\mathrm{exc}}$ need not be compact; it may contain regions between geodesic half-lines to the same point of $\partial\mathbb{H}$. Siegel domains of $\Gamma_1 = \mathrm{PSL}_2(\mathbb{Z})$ are examples of such sets.

Lemma 9.22. *The spaces* $(\mathcal{W}_s^{\omega^*,\mathrm{exc}})^T \cong (\mathcal{G}_s^{\omega^*,\mathrm{exc}})^T$ *are equal to* $\mathcal{I}_s$*, and the space* $(\mathcal{N}^{\omega^*,\mathrm{exc}})^T$ *is zero.*

Proof. If $f \in (\mathcal{G}_s^{\omega^*,\mathrm{exc}})^T$, then the set $\mathsf{BdSing}\,(f)$ is a T-invariant finite subset of $\mathbb{P}^1_{\mathbb{R}}$, and hence is contained in $\{\infty\}$. The set $\mathsf{Sing}\,(f)$ is also T-invariant. It is contained in the union of a compact set and finitely many vertical regions. Hence $\mathsf{Sing}\,(f) = \emptyset$ and $f \in \mathcal{E}_s^T$. Since f represents an element of $\mathcal{W}_s^\omega[\infty]$, it is in $\mathcal{I}_s$. If $f \in (\mathcal{N}^{\omega^*,\mathrm{exc}})^T$, then $\mathsf{Sing}\,(f) = \emptyset$ implies $f = 0$.

We are left with the proof of $(\mathcal{W}_s^{\omega^*,\mathrm{exc}})^T \cong (\mathcal{G}_s^{\omega^*,\mathrm{exc}})^T$. Clearly, each element of $(\mathcal{G}_s^{\omega^*,\mathrm{exc}})^T = \mathcal{I}_s$ represents an element of $(\mathcal{W}_s^{\omega^*,\mathrm{exc}})^T$. Restriction of a given $h \in (\mathcal{W}_s^{\omega^*,\mathrm{exc}})^T$ gives $\rho_s h \in (\mathcal{V}_s^\omega[\infty])^T$. Hence h has a representative $f \in \mathcal{E}_s(\mathfrak{H} \smallsetminus W_\varepsilon)$, where

$$W_\varepsilon = \{z \in \mathfrak{H} : |\mathrm{Re}\, z| \leq \varepsilon^{-1},\ \mathrm{Im}\, z \geq \varepsilon\} \tag{9.18}$$

for some $\varepsilon > 0$. By extending f as a C^2-function on W_ε, we obtain a representative $f \in \mathcal{G}_s^{\omega^*,\mathrm{exc}}$. So $f\,|\,(1-T) \in \mathcal{N}^{\omega^*,\mathrm{exc}}$. After diminishing ε, we have $f(z) = f(z+1)$ on $\mathfrak{H} \smallsetminus W_\varepsilon$. Since f represents an element of $\mathcal{W}_s^\omega[\infty]$, it has a Fourier expansion with only $i_{s,2\pi n}$, and hence is in $\mathcal{I}_s^\downarrow$. This expansion converges on $\mathfrak{H}$ and defines another representative of h, which is in $\mathcal{I}_s$. ∎

Lemma 9.23. *Suppose that* $\hat q \in \mathcal{G}_s^{\omega^*,\mathrm{exc}}$ *satisfies:*

a) $\mathsf{Sing}\,(\hat q)$ *is a compact subset of* $\mathbb{H}$.
b) $\hat q\,|\,(1-T) \in \mathcal{G}_s^{\omega}$.

Then there exists $p \in \mathcal{I}_s$ *such that* $\hat q - p \in \mathcal{G}_s^{\omega}$.

Proof. From a) it follows that $\hat q$ represents an element q in the space $\mathcal{F}_s$ defined in (3.1). The direct sum decomposition (3.3) implies that there are unique $p \in \mathcal{E}_s$ and $f \in \mathcal{W}_s^{\omega}$ such that $q = p + f$. Condition b) implies that $q\,|\,(1-T) \in \mathcal{W}_s^{\omega}$. Hence $p\,|\,(1-T) = q\,|\,(1-T) - f\,|\,(1-T) \in \mathcal{W}_s^{\omega} \cap \mathcal{E}_s = \{0\}$. So $p \in \mathcal{E}_s^T$, and $p = q - f \in \mathcal{W}_s^{\omega^*,\mathrm{exc}}$. As in the proof of the previous lemma, this implies $p \in \mathcal{I}_s$. Now, $\hat q - p \in \mathcal{G}_s^{\omega^*,\mathrm{exc}}$ is a lift of $f \in \mathcal{W}_s^{\omega}$, for which $\mathsf{Sing}\,(\hat q - p) = \mathsf{Sing}\,(\hat q)$ is compact. Hence $\hat q - p \in \mathcal{G}_s^{\omega}$. ■

9.6. *T-invariant eigenfunctions and cohomology.* In this subsection we prove Theorem 9.20 and Propositions 9.11 and 9.15. We treat these three proofs in parallel, since the statements are closely related.

- *Images.* First we consider $u \in (\mathcal{E}_s^{\uparrow})^T$ and $s \neq \frac12$. We represent $\mathbf{r}u$ by the cocycle r determined by $r_T(\zeta) = \int_{z_0-1}^{z_0} [u, R(\zeta;\cdot)^s]$ with a suitable base point z_0. Relation (4.10), Lemma 9.4 and Proposition 9.18 show that $r_T = \mathrm{Av}_T^+(r_T)\,|\,(1-T)$ is in the space $\mathcal{V}_s^{\omega^*,\mathrm{exc}}\,|\,(1-T)$. So $\mathbf{r}$ maps $(\mathcal{E}_s^{\uparrow})^T$ to the subspace $H^1(\langle T\rangle; \mathcal{V}_s^{\omega}, V_s^{\omega^*,\mathrm{exc}})$. Applying the transverse Poisson transform, we see that $\mathbf{q}(\mathcal{E}_s^{\uparrow})^T$ is contained in the space $H^1(\langle T\rangle; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*,\mathrm{exc}})$.

If $u \in \mathcal{K}_s^{\uparrow}$, then Lemma 9.4 shows that $\mathrm{Av}_T(r_T) = 0$. Hence $\mathrm{Av}_T^+(r_T) = \mathrm{Av}_T^-(r_t)$, and then $\mathrm{Av}_T^+(r_T) \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$ by Proposition 9.14. This show that

$$\mathbf{r}\mathcal{K}_s^T \subset H^1(\langle T\rangle; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{exc}\,\&\,\mathrm{simple}}) \subset H^1(\langle T\rangle; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{exc}})\,.$$

Again we apply $\mathsf{P}_s^{\dagger}$ to get the analogous statement for $\mathbf{q}$. (The space $\mathcal{V}_s^{\omega^*,\mathrm{exc}\,\&\,\mathrm{simple}}$ is equal to $\mathcal{V}_s^{\omega^*,\mathrm{exc}} \cap \mathcal{V}_s^{\omega^*,\mathrm{simple}}$, since at the singularities both the conditions "exc" and "simple" are imposed.)

We have seen in (9.15) that $\mathbf{q}(\mathcal{K}_s^0)^T \subset H^1(\langle T\rangle; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*,\infty})$, even if $s = \frac12$. Since the restriction map ρ_s is an isomorphism on $\mathcal{W}_s^{\omega}$ and $\mathcal{W}_s^{\omega^*,\infty}$, the corresponding statement for $\mathbf{r}$ follows. The integrals in (9.6) and (9.11) show that $\mathbf{r}$ and $\mathbf{q}$ map $(\mathcal{K}_s^0)^{\uparrow}$ to $H^1(\langle T\rangle; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{exc}\,\&\,\infty})$, respectively $H^1(\langle T\rangle; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*,\mathrm{exc}\,\&\,\infty})$.

- *Comparison result.* For Propositions 9.11 and 9.15 it is important to have:

Lemma 9.24.

$$H^1(\langle T\rangle; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{exc}\,\&\,\infty}) \;=\; H^1(\langle T\rangle; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\infty})\,, \tag{9.19a}$$

$$H^1(\langle T\rangle; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*,\mathrm{exc}\,\&\,\infty}) \;=\; H^1(\langle T\rangle; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*,\infty})\,, \tag{9.19b}$$

and if $s \neq \frac12$

$$H^1(\langle T\rangle; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{exc}\,\&\,\mathrm{simple}}) \;=\; H^1(\langle T\rangle; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{simple}})\,, \tag{9.19c}$$

$$H^1(\langle T\rangle; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*,\mathrm{exc}\,\&\,\mathrm{simple}}) \;=\; H^1(\langle T\rangle; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*,\mathrm{simple}}) \tag{9.19d}$$

Proof. We have $H^1(\langle T\rangle; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\,\mathrm{simple}}) \supset H^1(\langle T\rangle; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\,\mathrm{exc\,\&\,simple}})$. Conversely, if $\psi_T \in \mathcal{V}_s^\omega$ is of the form $\psi_T = f\,|\,(1-T)$ with $f \in \mathcal{V}_s^{\omega^*,\,\mathrm{simple}}$, then $\mathrm{Av}_T(\psi_T) = 0$ (Proposition 9.14), and hence $\mathrm{Av}_T^+(\psi_T), \mathrm{Av}_T^-(\psi_T) \in \mathcal{V}_s^{\omega^*,\,\mathrm{exc}}$ (Proposition 9.18). This gives (9.19c). For (9.19a), we proceed similarly, with use of Proposition 9.9 to obtain $\mathrm{Av}_T(\psi_T) = 0$.

The transverse Poisson transformation provides us with an injection

$$\mathsf{P}_s^\dagger : \mathcal{V}_s^{\omega^*,\,\mathrm{exc\,\&\,simple}} \longrightarrow \mathcal{W}_s^{\omega^*,\,\mathrm{exc\,\&\,simple}},$$

and a bijection

$$\mathsf{P}_s^\dagger : \mathcal{V}_s^{\omega^*,\,\mathrm{simple}} \longrightarrow \mathcal{W}_s^{\omega^*,\,\mathrm{simple}}.$$

The resulting commuting diagram

$$\begin{tikzcd}
H^1(\langle T\rangle; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\,\mathrm{exc\,\&\,simple}}) \arrow[r, "= (9.19c)"] \arrow[d, "\mathsf{P}_s^\dagger"] & H^1(\langle T\rangle; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\,\mathrm{simple}}) \arrow[d, "\mathsf{P}_s^\dagger", "\cong"'] \\
H^1(\langle T\rangle; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\,\mathrm{exc\,\&\,simple}}) \arrow[r, hook] & H^1(\langle T\rangle; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\,\mathrm{simple}})
\end{tikzcd}$$

shows the equality in (9.19d). For (9.19b) we proceed similarly. ∎

- *Injectivity.* This is Proposition 5.1.
- *Surjectivity in Theorem 9.20.* Since $\mathcal{V}_s^{\omega^*,\,\mathrm{exc}}$ and $\mathcal{W}_s^{\omega^*,\,\mathrm{exc}}$ are isomorphic it suffices to prove that for $s \neq \frac{1}{2}$

$$\mathbf{q} : (\mathcal{E}_s^\uparrow)^T \longrightarrow H^1(\langle T\rangle; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\,\mathrm{exc}})$$

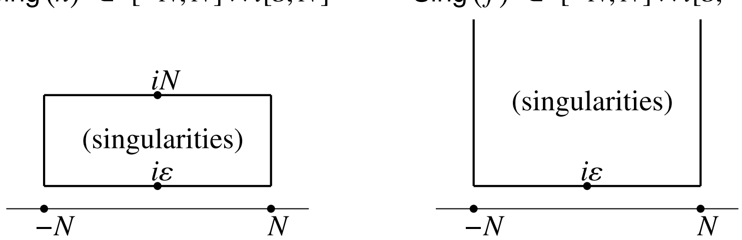
is surjective.

We recall that $\mathbf{q}$ associates to $u \in (\mathcal{E}_s^\uparrow)^T$ the cohomology class $q_T + \mathcal{W}_s^\omega\,|\,(1-T)$, where q_T can be represented by $\tilde{q}_T \in \mathcal{G}_s^\omega$ given by $\tilde{q}_T(z) = \int_{z_0-1}^{z_0} [u, q_s(\,\cdot\,, z)]$ outside a small neighborhood of the line segment from $z_0 - 1$ to z_0. The class does not depend on the choice of z_0 with $\mathrm{Im}\, z_0$ sufficiently large. In this proof we will mainly work with representatives in $\mathcal{G}_s^\omega$ and $\mathcal{N}^\omega$. See the diagram (9.17).

For the proof of the surjectivity we start with $f \in \S\omega^*$, exc$s$ such that $h := f\,|\,(1-T)$ belongs to $\mathcal{W}_s^\omega$. The aim is to construct $u \in (\mathcal{E}_s^\uparrow)^T$ whoe associated function q_T satisfies $h - \frac{1}{\pi i} q_T \in \mathcal{W}_s^\omega\,|\,(1-T)$.

Let $\hat{f} \in \mathcal{G}_s^{\omega^*,\,\mathrm{exc}}$ and $\hat{h} \in \mathcal{G}_s^\omega$ be representatives of f and h, respectively. Since the set $\mathsf{BdSing}\,(f)$ is finite and T-invariant, it is contained in $\{\infty\}$. We choose $N > 2$ sufficiently large and $\varepsilon \in (0,1)$ sufficiently small to achieve the following situation:

$\mathsf{Sing}\,(\hat{h}) \subset [-N, N] \times i[\varepsilon, N]$ $\qquad$ $\mathsf{Sing}\,(\hat{f}) \subset [-N, N] \times i[\varepsilon, \infty)$



(9.20)

The difference $k = \hat{h} - \hat{f}\,|\,(1-T)$ is an element of $\mathcal{N}^{\omega^*,\mathrm{exc}}$. It satisfies

$$\operatorname{Supp} k \subset [-N-1, N] \times i[\varepsilon, \infty)\,. \tag{9.21}$$

We set

$$u(z) \;=\; \mathrm{Av}_T(k)(z) \qquad \text{for } \operatorname{Im} z > N\,. \tag{9.22}$$

This will turn out to represent the element of $(\mathcal{E}_s^{\uparrow})^T$ corresponding to the cocycle on $\langle T \rangle$ given by h. We prove this in several steps.

Lemma 9.25. *The function u represents an element of $(\mathcal{E}_s^{\uparrow})^T$.*

Proof. The average $\mathrm{Av}_T k$ is given by a locally finite sum. For $z = x + iy \in [-1,1] \times i(0,\infty)$ we have

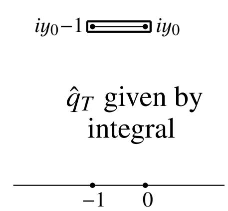
$$\mathrm{Av}_T(k)(z) \;=\; \sum_{n=-N-2}^{N+1} k(z+n) \;=\; \hat{f}(z+N+2) - \hat{f}(z-N-2) + \sum_{n=-N-2}^{N+1} \hat{h}(z+n)\,.$$

The terms $\hat{f}(z+N+2)$ and $\hat{f}(z-N-2)$ have no singularities in the region $\operatorname{Im} z = y > \varepsilon$, $|x| \le 1$, and $\hat{h}$ has no singularities in the region $y > N$. Hence $\mathrm{Av}_T(k)$ is a λ_s-eigenfunction of Δ on the region $y > N$, $|x| < 1$. Since it is 1-periodic, it is a λ_s-eigenfunction on $y > N$. ■

We take $y_0 > N + 2\varepsilon$, and define

$$\hat{q}_T(z) \;=\; \int_{iy_0-1}^{iy_0} [u, q_s(\,\cdot\,, z)] \tag{9.23}$$

outside the box $[-1-\varepsilon, \varepsilon] \times i[y_0-\varepsilon, y_0+\varepsilon]$, and extend $\hat{q}_T$ inside the box as a C^2-function. Thus we obtain $\hat{q}_T \in \mathcal{G}_s^{\omega}$ representing q_T. Our aim is to show that $\hat{h} - \frac{1}{\pi i}\hat{q}_T \in \mathcal{G}_s^{\omega}\,|\,(1-T) + \mathcal{N}^{\omega}$.



We apply Proposition 9.6, which writes

$$u \;=\; \frac{1}{\pi i}\,(u^{\uparrow} - u^{\downarrow}) \;=\; \frac{1}{\pi i}\Big(\mathrm{Av}_T^{\uparrow}(\hat{q}_T) - \mathrm{Av}_T^{\downarrow}(\hat{q}_T)\Big), \tag{9.24}$$

with $u^{\uparrow} := \mathrm{Av}_T^{\uparrow}(h) \in \mathcal{K}_s^{\uparrow}$ equal to $\mathrm{Av}_T(\hat{q}_T)$ on the region $\operatorname{Im} z > y_0 + \varepsilon$ and $u^{\downarrow} := \mathrm{Av}_T^{\downarrow}(\hat{q}_T) \in \mathcal{I}_s$ equal to $\mathrm{Av}_T(\hat{q}_T)$ on $\operatorname{Im} z < y_0 - \varepsilon$. Both functions do not depend on the choice of y_0.

The next step is to relate $\mathrm{Av}_T(\hat{q}_T)$ to $\mathrm{Av}_T(k)$, $\mathrm{Av}_T(\hat{h})$ and $\hat{f}$. To do this, we use the functions p_+ and p_- in the next lemma:

Lemma 9.26. *The following two functions $p_{\pm}$ belong to $\mathcal{I}_s$:*

$$p_+ \;=\; \mathrm{Av}_T^{+}(\hat{h}) - \mathrm{Av}_T^{+}(k) - \hat{f}\,, \qquad p_- \;=\; \mathrm{Av}_T^{-}(\hat{h}) - \mathrm{Av}_T^{-}(k) - \hat{f}\,.$$

Proof. $\mathrm{Av}_T^+(k)$ is given by a locally finite sum, and $\mathrm{Av}_T^+(\hat h)$ can be understood in regularized sense, since $\hat h \in \mathcal{G}_s^\omega$. Hence p_+ is defined on $\mathfrak{H}$, except for its singularities, which occur on a locally finite union of curves in the region $\{z \in \mathfrak{H} \,:\, x \le N\,,\ y \ge \varepsilon\}$. From $(\hat h - k)\,|\,\mathrm{Av}_T^+|(1-T) = \hat h - k = \hat f\,|\,(1-T)$ it follows that $p_+|(1-T) = 0$. Since p_+ has no singularities in the region $x > N$, the T-invariance implies that there are no singularities at all. Hence $p_+ \in \mathcal{E}_s^T$.

On $0 < y < \varepsilon$, we have $k = 0$. Moreover, $\mathrm{Av}_T^+(\hat h)$ represents an element of $\mathcal{W}_s^\omega[\infty]$, and $\hat f \in \mathcal{G}_s^{\omega,\mathrm{exc}}[\infty]$. So p_+ represents an element of $\mathcal{W}_s^\omega(\mathbb{R})$. Hence it is in $\mathcal{I}_s$.

The case of p_- goes similarly. ■

Lemma 9.27.

$$\frac{1}{\pi i}\mathrm{Av}_T(\hat q_T) \;=\; \begin{cases} \mathrm{Av}_T(\hat h) & \text{on} \quad \mathrm{Im}\, z > y_0 + \varepsilon\,, \\ p_+ - p_- & \text{on} \quad \mathrm{Im}\, z < y_0 - \varepsilon\,. \end{cases}$$

Proof. On the region $\mathrm{Im}\, z > N$ we have

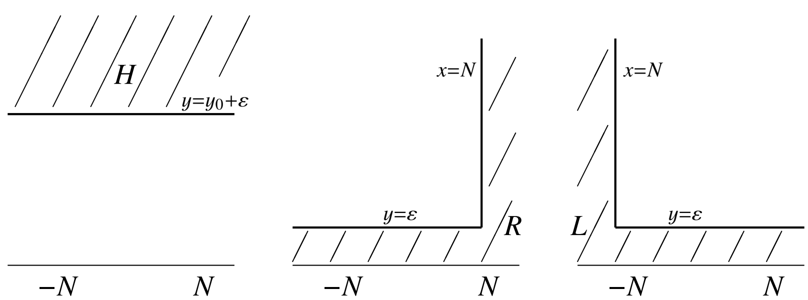
$$u \;=\; \mathrm{Av}_T \;=\; \mathrm{Av}_T(\hat h) + p_- - p_+\,.$$

Lemma 4.6 implies that $\mathrm{Av}_T(\hat h) = c_1 y^{1-s} + \mathrm{O}(y^{-s})$ as $y \to \infty$ for $|x| \le 1$. By T-invariance this estimates holds for all x, and shows that the restriction of $\mathrm{Av}_T(\hat h)$ to $y > N$ is in $\mathcal{K}_s^\uparrow$. Lemma 9.26 gives $p_+ - p_- \in \mathcal{I}_s$. Hence we have obtained the terms in the decomposition of u in $(\mathcal{E}_s^\uparrow)^T = \mathcal{K}s^\uparrow + \mathcal{I}_s$ given in Proposition 9.6:

$$u^+ \;=\; \mathrm{Av}_T^\uparrow(\hat q_T) \;=\; \pi i\,\mathrm{Av}_T(\hat h)\,, \quad u^- \;=\; \mathrm{Av}_T^\downarrow(\hat q_T) \;=\; \pi i\,(p_+ - p_-)\,.$$

The averages $\mathrm{Av}_T^\uparrow(\hat q_T)$ and $\mathrm{Av}_T^\downarrow(\hat q_T)$ are both given by $\mathrm{Av}_T(\hat q_T)$ on the regions $\mathrm{Im}\, z > y_0 + \varepsilon$ and $\mathrm{Im}\, z < y_0 - \varepsilon$, respectively. ■

We will choose a function $g \in C^2(\mathfrak{H})$ that will turn out to satisfy $g\,|\,(1-T) \in \hat h - \frac{1}{\pi i}\hat q_T + \mathcal{N}^\omega$. First we prescribe g on the union of the following three overlapping regions:



$H=\{z\in\mathfrak{H} : y>y_0+\varepsilon\}$ $R=\{z\in\mathfrak{H} : x>N \text{ or } y<\varepsilon\}$ $L=\{z\in\mathfrak{H} : x<-N \text{ or } y<\varepsilon\}$

$$g(\tau) \;=\; \begin{cases} \mathrm{Av}_T^+(\hat h) - \frac{1}{\pi i}\mathrm{Av}_T^+(\hat q_T) - p_+ & \text{on } H\,, \\ \hat f + \mathrm{Av}_T^+(k - \frac{1}{\pi i}\hat q_T) & \text{on } R\,, \\ \hat f + \mathrm{Av}_T^-(k - \frac{1}{\pi i}\hat q_T) + p_- - p_+ & \text{on } L\,. \end{cases} \tag{9.25}$$

To see that this is possible, we have to do some checking on the intersections. On $y < \varepsilon$, i.e., on $R \cap L$, we check:

$$\begin{aligned}
&\mathrm{Av}_T^+(k - \frac{1}{\pi i}\hat{q}_T) - (\mathrm{Av}_T^-(k - \frac{1}{\pi i}\hat{q}_T) + p_- - p_+) \\
&\qquad\qquad = 0 - \frac{1}{\pi i}\mathrm{Av}_T(\hat{q}_T) + p_+ - p_- \qquad \text{since } k = 0 \text{ on } y < \varepsilon \\
&\qquad\qquad = 0 - p_+ + p_- + p_- - p_+ = 0 \qquad \text{by Lemma 9.27}.
\end{aligned}$$

On $H \cap R$, i.e., for $y > y_0 + \varepsilon$ and $x > N$:

$$\hat{f} + \mathrm{Av}_T^+(k - \frac{1}{\pi i}\hat{q}_T) - (\mathrm{Av}_T^+(\hat{h} - \frac{1}{\pi i}\hat{q}_T) - p_+) = 0 \qquad \text{by Lemma 9.26}.$$

On $H \cap L$, i.e., for $y > y_0 + \varepsilon$ and $x < -N$:

$$\begin{aligned}
&\hat{f} + \mathrm{Av}_T^-(k - \frac{1}{\pi i}\hat{q}_T) + p_- - p_+ - \left(\mathrm{Av}_T^+(\hat{h} - \frac{1}{\pi i}\hat{q}_T) - p_+\right) \\
&= \mathrm{Av}_T^-(\hat{h}) - \frac{1}{\pi i}\mathrm{Av}_T^-(\hat{q}_T) - \mathrm{Av}_T^+(\hat{h}) + \frac{1}{\pi i}\mathrm{Av}_T^+(\hat{q}_T) \qquad \text{by Lemma 9.26} \\
&= -\mathrm{Av}_T(\hat{h}) + \frac{1}{\pi i}\mathrm{Av}_T(\hat{q}_T) = 0 \qquad \text{by Lemma 9.27}.
\end{aligned}$$

Thus, we have g on $H \cup R \cup L$, i.e, only on the exterior of the box $[-N, N] \times i[\varepsilon, y_0 + \varepsilon]$. We extend it by 0 on the box, and change it on an $(\varepsilon/2)$-neighborhood of the boundary to bring it into C^2.

Lemma 9.28. *We have $g \in \mathcal{G}_s^{\omega^*, \mathrm{exc}}$ and $g\,|\,(1 - T) \in \mathcal{G}_s^{\omega}$.*

Proof. The function $\hat{f} + \mathrm{Av}_T^+(k - \frac{1}{\pi i}\hat{q}_T) = \hat{f} - \frac{1}{\pi i}\mathrm{Av}_T^+(\hat{q}_T)$ on $R \cap L$ represents an element of $\mathcal{W}_s^{\omega}(\mathbb{R})$. The singularities of g are contained in the box $[-N, N] \times i[\varepsilon, y_0 + \varepsilon]$. Hence $g \in \mathcal{G}_s^{\omega^*, \mathrm{exc}}$.

On the region R:

$$g\,|\,(1 - T) = \hat{f}\,|\,(1 - T) + k - \frac{1}{\pi i}\hat{q}_T = \hat{h} - \frac{1}{\pi i}\hat{q}_T .$$

Since $\hat{h} - \frac{1}{\pi i}\hat{q}_T \in \mathcal{G}_s^{\omega}$, the equality $g\,|\,(1 - T) = \hat{h} - \frac{1}{\pi i}\hat{q}_T$ extends to the region in $\mathfrak{H}$ outside the box $[-N - 1, N] \times i[\varepsilon, y_0 + \varepsilon]$. So there is $g_1 \in \mathcal{N}^{\omega}$ such that

$$g\,|\,(1 - T) = \hat{h} - \frac{1}{\pi i}\hat{q}_T + g_1 \in \mathcal{G}_s^{\omega}. \quad \blacksquare \tag{9.26}$$

For the given $h \in \mathcal{W}_s^{\omega}$ and $f \in \mathcal{W}_s^{\omega^*, \mathrm{exc}}$ with $h = f\,|\,(1-T)$ we have given $u \in (\mathcal{E}_s^{\uparrow})^T$, and have given in (9.23) a representative $\hat{q}_T$ of the corresponding $q_T \in \mathcal{W}_s^{\omega}$ which determines the cocycle $\mathbf{q}u$. In Lemma 9.28 we see that $\hat{h} - \frac{1}{\pi i}\hat{q}_T \in \mathcal{N}^{\omega} + \mathcal{G}_s^{\omega^*, \mathrm{exc}}\,|\,(1-T)$. Hence $h - \frac{1}{\pi i}q_T \in \mathcal{W}_s^{\omega^*, \mathrm{exc}}\,|\,(1 - T)$. This completes the proof of the surjectivity in Theorem 9.20.

• *Surjectivity in Proposition 9.15.* We need only prove the surjectivity of $\mathbf{q}$. Applying Theorem 9.20 to a given $c \in H^1(\langle T \rangle; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*, \mathrm{exc}} \cap \mathcal{W}_s^{\omega^*, \mathrm{simple}})$, we obtain a unique $u \in (\mathcal{E}_s^{\uparrow})^T$ such that $\mathbf{q}u = c$. We have to check that $u \in \mathcal{K}_s^{\uparrow}$.

In the proof of the surjectivity that we have just given, we now have the additional information that $f \in \mathcal{W}_s^{\omega^*, \mathrm{exc}} \cap \mathcal{W}_s^{\omega^*, \mathrm{simple}}$. Hence $\mathrm{Av}_T(h) = 0$ (by Proposition 9.14), and $\mathrm{Av}_T(\hat{h})$ vanishes near $\mathbb{R}$. Since k also vanishes near $\mathbb{R}$, the difference

$p_+ - p_- \in \mathcal{I}_s$ in Lemma 9.26 vanishes near $\mathbb{R}$ as well, and hence $q_+ - p_- = 0$ everywhere. In Lemma 9.27 we see that $\mathrm{Av}_T(\hat{q}_T)^{\downarrow} = 0$. Hence (9.21) takes the form $u = \frac{1}{\pi i}\mathrm{Av}_T(\hat{q}_T)^{\uparrow} \in \mathcal{K}_s^{\uparrow}$.

- *Surjectivity in Proposition 9.11.* Now s may be equal to $\frac{1}{2}$. This forces a further review of the proof of the surjectivity of $\mathbf{q}$ for Theorem 9.20.

We start with $f \in \mathcal{W}_s^{\omega^*,\mathrm{exc}} \cap \mathcal{W}_s^{\omega^*,\infty}$ with $h = f|(1-T) \in \mathcal{W}_s^{\omega}$. This implies that $h(z) = \left(\frac{y}{|z+i|^2}\right)^s a^{\mathbb{P}}(z)$ with $a^{\mathbb{P}}(\infty) = 0$. So $\mathrm{Av}_T^{\pm}(h)$ and $\mathrm{Av}_T^{\pm}(\hat{h})$ converge without the need for regularization. The absolute convergence of $\mathrm{Av}_T^{\pm}(k)$ is clear anyhow. We have by Lemma 9.25 a function $u \in (\mathcal{E}_s^{\uparrow})^T$ given on $y > N$ by $\mathrm{Av}_T(k)$. We cannot yet use the one-sided averages of $\hat{q}_T$, but still have p_+ and p_- in $\mathcal{E}_s^T$ in Lemma 9.26. Proposition 9.9 and an application of the isomorphisms $\rho_s : \mathcal{W}_s^{\omega^*,\infty} \to \mathcal{V}_s^{\omega^*,\infty}$ and $\mathsf{P}_s^{\dagger} : \mathcal{V}_s^{\omega^*,\infty} \to \mathcal{W}_s^{\omega^*,\infty}$ imply that $\mathrm{Av}_T^+(\hat{h}) = \mathrm{Av}_T^-(\hat{h})$ near $\mathbb{R}$. Near $\mathbb{R}$, we also have $k = 0$. So $p_+ - p_- = \mathrm{Av}_T(\hat{h}) - \mathrm{Av}_T(k)$ vanishes first near $\mathbb{R}$, and then everywhere on $\mathfrak{H}$. On $y > N$ we have $\mathrm{Av}_T(\hat{h}) = \mathrm{Av}_T(k) + p_+ - p_- = \mathrm{Av}_T(k)$ as in the proof of Lemma 9.27. Hence $\mathrm{Av}_T(\hat{h}) = u$ on $y > N$. Lemma 4.6 gives $u(z) = \mathrm{Av}_T(\hat{h})(z) \ll y^{-s}$ as $y \to \infty$. So u can have only terms with $k_{s,2\pi n}$, $n \neq 0$, in its Fourier expansion. Thus, $u \in (\mathcal{K}_s^0)^{\uparrow}$.

With $\hat{q}_T$ as in (9.23), we use (9.11) and Proposition 9.8 to write $\hat{q}_T \in h_{iy_0}|(T-1) + \mathcal{N}^{\omega^*,\mathrm{exc}}$, with $h_{iy_0} \in \mathcal{G}_s^{\omega^*,\infty}$, $\mathsf{Sing}\,(h_{iy_0}) \subset i[y_0,\infty)$ and $\mathsf{BdSing}\,(h_{iy_0}) \subset \{\infty\}$. In particular, $(\rho_s\hat{q}_T)^{\mathbb{P}}(\infty) = 0$, and $\mathrm{Av}_T^+(\hat{q}_T)$ and $\mathrm{Av}_T^-(\hat{q}_T)$ converge absolutely. Now we can proceed as before.

Chapter IV. Maass forms and semi-analytic cohomology: groups with cusps

In this chapter we start the generalization of the results for the modular group $\mathrm{PSL}_2(\mathbb{Z})$ mentioned in the introduction to general cofinite discrete subgroups $\Gamma \subset G$ with cusps. We will prove those statements in Theorems B and C that concern cohomology groups with semi-analytic coefficients. The results concerning smooth (C^{∞}) and differentiable (C^p for some $p \in \mathbb{N}$) coefficients will be proved in Chapter V.

In Section 12 we consider the isomorphisms $\mathsf{Maass}_s^0(\Gamma) \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\infty})$ (in Theorem B), $\mathcal{E}_s^{\Gamma} \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{exc}})$ (in Theorem C), and a similar isomorphism using the space $\mathcal{V}_s^{\omega^*,\mathrm{simple}} \supset \mathcal{V}_s^{\omega^*,\infty}$ introduced in §9.4. The method of proof is the same as that followed for cocompact groups in Chapter II. The presence of cusps makes it necessary to look at geometrical models for cohomology again, especially in connection with parabolic cohomology. That is the subject of Section 11, where we also discuss an interpretation of our approach to parabolic cohomology in terms of sheaf cohomology.

In Section 13 we prove the isomorphisms $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\infty}) \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*,\infty})$ (in Theorem B) and $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{exc}}) \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^0,\mathrm{exc}})$ (in Theorem C), where $\mathcal{V}_s^{\omega^0,\mathrm{exc}}$ consists of the elements of $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ that have singularities in cusps only. This requires an analysis of the set of singularities of cocycles. In §13.2 we give a recapitulation of the proof of Theorem C.

This chapter generalizes results of [19], where Maass cusp forms on $\mathrm{SL}_2(\mathbb{Z})$ were related to "period functions". The link with the period function is discussed in Section 10, where we also give some general definitions, and in Section 14. In (8.6) we gave a holomorphic function associated to a λ_s-eigenfunctions invariant under the parabolic element $T = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ of G. Such linear maps to the space of 1-periodic holomorphic functions on $\mathbb{C}$ can also be defined for Γ-invariant eigenfunctions. That is the subject of Section 15.

10. Maass forms. Throughout this chapter the group $\Gamma \subset G = \mathrm{PSL}_2(\mathbb{R})$ is assumed to have cusps. For such groups we discuss several spaces of Maass forms and general invariant eigenfunctions, which coincide for cocompact groups discussed in Chapter II.

The image of the map from invariant eigenfunctions to cohomology with values in the space $\mathcal{V}_s^\omega$ of analytic vectors in the principal series is contained in a mixed parabolic subgroup, of which we will give a preliminary definition in this section.

Here the upper half-plane $\mathfrak{H}$ is the natural model of $\mathbb{H}$. A discrete subgroup $\Gamma \subset G$ is called *cofinite* if the quotient $\Gamma\backslash\mathfrak{H}$ has finite volume for the measure induced by the invariant measure $d\mu$ on $\mathfrak{H}$. The *cusps* of Γ are points $\kappa \in \mathbb{P}^1_{\mathbb{R}}$ for which the isotropy subgroup $\Gamma_\kappa = \{\gamma \in \Gamma \,:\, \gamma\kappa = \kappa\}$ is non-trivial, and hence infinite cyclic with a parabolic generator. We denote by C the set of cusps of Γ. This set depends on Γ. It is infinite, but consists of finitely many Γ-orbits.

For each $\kappa \in C$ we fix $g_\kappa \in G$ such that $g_\kappa\infty = \kappa$ and such that $\pi_\kappa = g_\kappa T g_\kappa^{-1}$ generates Γ_κ, with $T = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$. This leaves some freedom in the choice of the g_κ. We arrange the g_κ such that $g_{\gamma\kappa} \in \gamma g_\kappa T^{\mathbb{Z}}$ within each Γ-orbit of cusps.

The standard example is the modular group $\Gamma_1 = \mathrm{PSL}_2(\mathbb{Z})$, generated by T and $S = \left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$, with relations $S^2 = (TS)^3 = 1$. Its set of cusps $C = \mathbb{P}^1_{\mathbb{Q}}$ forms one Γ_1-orbit.

10.1. *Notations and terminology.* We call the elements of $\mathcal{E}_s^\Gamma$ *invariant eigenfunctions*, and reserve the notation $\mathsf{Maass}_s(\Gamma)$ for the finite dimensional space $(\mathcal{E}_s^{-\infty})^\Gamma$ of invariant eigenfunctions with polynomial growth, whose elements we call *Maass forms*. An invariant eigenfunction $u \in \mathcal{E}_s^\Gamma$ has *polynomial growth* if and only if

$$u\left(g_\kappa(x+iy)\right) \ll y^a \text{ as } y \to \infty \text{ for some } a \in \mathbb{R}, \text{ uniform in } x, \text{ for all } \kappa \in C. \tag{10.1a}$$

Inside $\mathsf{Maass}_s(\Gamma)$ the space $\mathsf{Maass}_s^0(\Gamma)$ of *(Maass) cusp forms* is determined by the stronger condition of *quick decay* at all cusps:

$$u\left(g_\kappa(x+iy)\right) \ll y^a \text{ as } y \to \infty \text{ for all } a \in \mathbb{R}, \text{ uniform in } x, \text{ for all } \kappa \in C. \tag{10.1b}$$

By the Γ-invariance, it suffices that these growth conditions hold for one representative κ of each Γ-orbit of cusps. In [19] we used Maass_s to denote the space which we now call $\mathsf{Maass}_s^0(\Gamma_1)$. For cocompact groups, the spaces $\mathsf{Maass}_s^0(\Gamma) \subset \mathsf{Maass}_s(\Gamma) \subset \mathcal{E}_s^\Gamma$ coincide.

Let $u \in \mathcal{E}_s^\Gamma$. For each $\kappa \in C$, the function $u\,|\,g_\kappa : z \mapsto u(g_\kappa z)$ is in $\mathcal{E}_s^T$, and has a Fourier expansion (8.1), with coefficients $A_n(u\,|\,g_\kappa)$ and $B_n(u\,|\,g_\kappa)$, and also $a_0(u\,|\,g_\kappa)$ and $b_0(u\,|\,g_\kappa)$. The space $\mathsf{Maass}_s(\Gamma)$ is characterized by $B_n(u\,|\,g_\kappa) = 0$ for $n \neq 0$ for all κ, and the space $\mathsf{Maass}_s^0(\Gamma)$ by the additional requirement $A_0(u\,|\,g_\kappa) = B_0(u\,|\,g_\kappa) = 0$ (for $s \neq \frac{1}{2}$), or $a_0(u\,|\,g_\kappa) = b_0(u\,|\,g_\kappa) = 0$ (for all s), for all κ. The form of the Fourier expansion implies that we can weaken (10.1b) by replacing "for all $a \in \mathbb{R}$" by "for some $a < \min(\mathrm{Re}\, s, 1 - \mathrm{Re}\, s)$".

For $s \neq \frac{1}{2}$ we define the space $\mathsf{Maass}^1_s(\Gamma) \subset \mathcal{E}^\Gamma_s$ by the condition that $B_n(u\,|\,g_\kappa) = 0$ for all $\kappa \in C$ and all $n \in \mathbb{Z}$. We have $\mathsf{Maass}^0_s(\Gamma) \subset \mathsf{Maass}^1_s(\Gamma) \subset \mathsf{Maass}_s(\Gamma)$. It may happen that $\mathsf{Maass}^1_{1-s}(\Gamma) \neq \mathsf{Maass}^1_s(\Gamma)$ (e.g., if $\Gamma = \Gamma_1$ and $2s$ is a zero of the Riemann zeta function), whereas $\mathsf{Maass}^0_s(\Gamma)$, $\mathsf{Maass}_s(\Gamma)$ and $\mathcal{E}^\Gamma_s$ are invariant under $s \mapsto 1 - s$.

10.2. *Invariant eigenfunctions and parabolic cohomology.* We start with an example. A 1-cocycle ψ on the modular group Γ_1 with values in a right $\mathbb{Q}[\Gamma_1]$-module V is, in the group model of cohomology, determined by ψ_T and ψ_S on the generators $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, subject to the relations $\psi_S\,|\,(1+S) = 0$ and $\psi_{TS}\,|\,(1 + TS + TSTS) = 0$ (and $\psi_{TS} = \psi_T\,|\,S + \psi_S$). There are various possibilities of normalization. We may for instance require that $\psi_S = 0$, which can be arranged by subtracting da from ψ, with $a \in V$ given by $\frac{1}{2}\,\psi_S$. Another normalization is by arranging $\psi_{TS} = 0$; then ψ is determined by $\psi_S = \psi_T$ satisfying $\psi_S\,|\,S = -\psi_S$. For the cocycle r in (5.5a) associated to an invariant eigenfunction, the former normalization is arranged by choosing the base point z_0 equal to i, and the latter by choosing $z_0 = \frac{1}{2}(1 + i\sqrt{3})$.

In general, it is impossible to choose ψ in its cohomology class such that $\psi_T = 0$. However, in the case of a cusp form $u \in \mathsf{Maass}^0_s(\Gamma_1)$, and r as in (5.5a), we know from (9.12) that r_T is of the form $f_{z_0}\,|\,(T-1)$, with $f_{z_0} \in \mathcal{V}^{\omega^*,\infty}_s$. Thus, subtracting df_{z_0} from r, we obtain a $\mathcal{V}^{\omega^*,\infty}_s$-valued cocycle satisfying $\psi_T = 0$. This cocycle is determined by its value ψ_S. Since $(\mathcal{V}^{\omega^*,\infty}_s)^T \subset (\mathcal{V}^{\infty}_s)^T = \{0\}$ (Proposition 4.5), this cocycle is unique. This motivates the following definition:

Definition 10.1. Let $V \subset W$ be $\mathbb{Q}[\Gamma]$-modules. We define the *mixed parabolic cohomology group* $H^1_{\mathrm{par}}(\Gamma; V, W)$ as $Z^1_{\mathrm{par}}(\Gamma; V, W)/B^1(\Gamma; V)$, where

$$Z^1_{\mathrm{par}}(\Gamma; V, W) \;=\; \{\psi \in Z^1\,(\Gamma; V) \;:\; \psi_\pi \in W\,|\,(\pi - 1) \text{ for all parabolic } \pi \in \Gamma\}$$

is the space of *mixed parabolic cocycles*.

We define the *parabolic cohomology group* $H^1_{\mathrm{par}}(\Gamma; V)$ as $H^1_{\mathrm{par}}(\Gamma; V, V)$, and call the elements of $Z^1_{\mathrm{par}}(\Gamma; V, V)$ *parabolic cocycles*.

It suffices to impose the parabolic condition only for $\pi = \pi_\kappa$ with κ running through a set of representatives of $\Gamma\backslash C$. The mixed parabolic cohomology group $H^1_{\mathrm{par}}(\Gamma; V, W)$ is the kernel of the natural map

$$H^1(\Gamma; V) \longrightarrow \bigoplus_{\kappa \in \Gamma\backslash C} H^1(\Gamma_\kappa; W)\,. \tag{10.2}$$

We may view the group $H^1(\langle T\rangle; V, W)$ in (9.14) as a mixed parabolic cohomology group.

Returning to the case $\Gamma = \Gamma_1$, we see that the $\mathcal{V}^{\omega^*,\infty}_s$-valued cocycle ψ with $\psi_T = 0$ associated above to a Maass cusp form u satisfies $\psi_S \in \mathcal{V}^{\omega^*,\omega^*,\mathrm{exc}\,\&\,\infty}_s$, since $f_{z_0} \in \mathcal{V}^{\omega^*,\omega^*,\mathrm{exc}\,\&\,\infty}_s$. (See Definition 9.17 for $\mathcal{V}^{\omega^*,\mathrm{exc}}_s$.) Actually, the singularities of $\psi_S = r_S - f_{z_0}\,|\,(S-1)$ can occur only in ∞ and $0 = S\infty$. So the cocycle ψ has values in $\mathcal{V}^{\omega^0,\mathrm{exc}\,\&\,\infty}_s$, where we use the following definition:

Definition 10.2. Let $\mathcal{V}^{\omega^0}_s$ be the Γ-submodule consisting of the $f \in \mathcal{V}^{\omega^*}_s$ such that $\mathsf{BdSing}\,(f) \subset C$. With a condition "cond" imposed on the singularities, we put $\mathcal{V}^{\omega^0,\mathrm{cond}}_s = \mathcal{V}^{\omega^0}_s \cap \mathcal{V}^{\omega^*,\mathrm{cond}}_s$.

For $\mathcal{W}^*_s$ we follow the same convention.

We recall that elements of $\mathcal{V}_s^{\omega^*}$ can have a finite number of arbitrary singularities on $\partial\mathfrak{H} = \mathbb{P}^1_{\mathbb{R}}$, those of $\mathcal{V}_s^{\omega^*,\text{simple}}$ (Definition 9.12) have a "simple pole" (i.e., $\tau \mapsto (\tau - \tau_0)(\text{smooth})$ at real points τ_0), and the singularities of elements of $\mathcal{V}_s^{\omega^*,\text{exc}}$ occur outside an excised neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ minus a finite set (Definition 9.16).

Proposition 10.3. *The injective maps* $\mathbf{r}$ *and* $\mathbf{q}$ *determined by* (5.5) *have images in the following equal cohomology groups.*

$$\mathbf{r}\,\mathsf{Maass}_s^0(\Gamma) \subset H^1_{\text{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^0,\infty}) = H^1_{\text{par}}(\Gamma, \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\infty}), \tag{10.3a}$$

$$\mathbf{q}\,\mathsf{Maass}_s^0(\Gamma) \subset H^1_{\text{par}}(\Gamma; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^0,\infty}) = H^1_{\text{par}}(\Gamma, \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\infty}). \tag{10.3b}$$

If $s \neq \frac{1}{2}$, *then we also have*

$$\mathbf{r}\,\mathsf{Maass}_s^1(\Gamma) \subset H^1_{\text{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^0,\text{simple}}) = H^1_{\text{par}}(\Gamma, \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\text{simple}}), \tag{10.3c}$$

$$\mathbf{q}\,\mathsf{Maass}_s^1(\Gamma) \subset H^1_{\text{par}}(\Gamma; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^0,\text{simple}}) = H^1_{\text{par}}(\Gamma, \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\text{simple}}), \tag{10.3d}$$

$$\mathbf{r}\,\mathcal{E}_s^\Gamma \subset H^1_{\text{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^0,\text{exc}}) = H^1_{\text{par}}(\Gamma, \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\text{exc}}), \tag{10.3e}$$

$$\mathbf{q}\,\mathcal{E}_s^\Gamma \subset H^1_{\text{par}}(\Gamma; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^0,\text{exc}}) = H^1_{\text{par}}(\Gamma, \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\text{exc}}). \tag{10.3f}$$

Proof. By definition, $H^1_{\text{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^0,\text{exc}}) \subset H^1_{\text{par}}(\Gamma\ \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\text{exc}})$. Consider a parabolic cocycle $c \in Z^1_{\text{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\text{exc}})$ and a parabolic element $\pi = \pi_\kappa$, $\kappa \in C$. Then $c_\pi \,|\, (1-\pi) \in \mathcal{V}_s^\omega$ implies that the π-invariant set $\mathsf{BdSing}\,(c_\pi)$ is contained in $\{\kappa\}$. So $[c] \in H^1_{\text{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^0,\text{exc}})$. This proves the equality of these cohomology groups. The same proof works for the other equalities of parabolic cohomology groups.

For the inclusion $\mathbf{r}\,\mathcal{E}_s^\Gamma$ in $H^1_{\text{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\text{exc}})$ we take $u \in \mathcal{E}_s^\Gamma$, and suppose that $s \neq \frac{1}{2}$. Let $z_0 \in \mathfrak{H}$ be the base point in the definition in (5.5a) of the cocycle $r : \Gamma \to \mathcal{V}_s^\omega$. Let $\kappa \in C$. The value r_{π_κ} on the generator $\pi_\kappa = g_\kappa T g_\kappa^{-1}$ of Γ_κ satisfies

$$r_{\pi_\kappa}(\zeta) = \int_{g_\kappa T^{-1} g_\kappa^{-1} z_0}^{z_0} [u, R(\zeta;\,\cdot\,)^s] = \int_{T^{-1} g_\kappa^{-1} z_0}^{g_\kappa^{-1} z_0} [u \,|\, g_\kappa, R(\,\cdot\,,\,\cdot\,)|_{2s}\, g_\kappa^{-1}\,(\zeta)],$$

where we have used the G-equivariance of the Green's form $[\,\cdot\,,\,\cdot\,]$ in (1.10a) and of $R(\,\cdot\,;\,\cdot\,)^s$ in (2.25). Hence

$$r_{\pi_\kappa} \,|\, g_\kappa\,(\zeta) = \int_{T^{-1} g_\kappa^{-1} z_0}^{g_\kappa^{-1} z_0} [u \,|\, g_\kappa, R(\zeta;\,\cdot\,)^s],$$

which shows that $r_{\pi_\kappa} \,|\, g_\kappa = r^\kappa_T$, where r^κ is a cocycle on $\langle T \rangle$ that represents $\mathbf{r}(u \,|\, g_\kappa) \in H^1(\langle T \rangle; \mathcal{V}_s^\omega)$, with the base point $g_\kappa^{-1} z_0$. Theorem 9.20 shows that

$$\mathbf{r}(u \,|\, g_\kappa) \in H^1(\langle T \rangle; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\text{exc}}),$$

as defined in (9.14). Hence there is $a_\kappa \in \mathcal{V}_s^{\omega^*,\text{exc}}$ such that $r^\kappa_T = a_\kappa \,|\, (T-1)$, and

$$r_{\pi_\kappa} = a_\kappa \,|\, (T-1) \,|\, g_\kappa^{-1} = (a_\kappa \,|\, g_\kappa^{-1}) \,|\, (\pi_\kappa - 1).$$

This works for all $\kappa \in C$, and hence $\mathbf{r}u \in H^1_{\text{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\text{exc}})$.

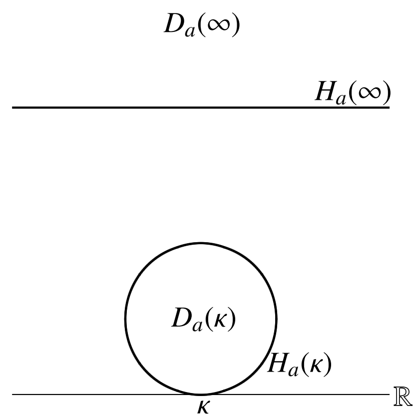
The other assertions go similarly, using also Propositions 9.11 and 9.15, and taking into account that $\mathsf{Maass}_s^1(\Gamma)$ is characterized in $\mathcal{E}_s^\Gamma$ by $u \,|\, g_\kappa \in \mathcal{K}_s$ for all cusps $\kappa \in C$, and $\mathsf{Maass}_s^0(\Gamma)$ by $u \,|\, g_\kappa \in \mathcal{K}_s^0$ for all $\kappa \in C$. ∎

We can use Proposition 8.4 to reconstruct $u \in \mathcal{E}_s^\Gamma$ from the value q_π of the cocycle q in (5.5c) on any parabolic $\pi \in \Gamma$.

11. Cohomology and parabolic cohomology for groups with cusps. We now turn to a more geometrical description of the cohomology of cofinite discrete groups with cusps, like we did in §6 for cocompact discrete subgroups of G. For the standard cohomology groups we do not work with resolutions built on $\mathfrak{H}$, but on a contractible subset $\mathfrak{H}_Y \subset \mathbb{H}$, where Y is a parameter. For the parabolic cohomology, we work on $\mathfrak{H}^* \supset \mathfrak{H}$, where all cusps of Γ are added to $\mathfrak{H}$. Using tesselations of $\mathfrak{H}^*$ we will define $H^i_{\mathrm{par}}(\Gamma; V, W)$ for all $i \geq 0$, extending Definition 10.1. In Proposition 11.8 we will relate these mixed parabolic cohomology groups $H^i_{\mathrm{par}}(\Gamma; V, W)$ to sheaf cohomology groups.

11.1. *Resolutions.* For $\kappa \in C$ and $a > 0$, we define the *horocycle* $H_a(\kappa)$ and the *open horocyclic disk* $D_a(\kappa)$ as follows:

$$\begin{aligned} H_a(\kappa) &= g_\kappa(ia + \mathbb{R}), \\ D_a(\kappa) &= \{g_\kappa z \ : \ \operatorname{Im} z > a\}. \end{aligned} \tag{11.1}$$



$D_a(\infty)$ is a euclidean half-plane, whereas $D_a(\kappa)$ is a euclidean disk touching $\mathbb{R}$ in κ if $\kappa \in \mathbb{R}$. We denote $\mathfrak{H}_a = \mathfrak{H} \smallsetminus \bigcup_{\kappa \in C} D_a(\kappa)$.

We provide the *extended upper half-plane* $\mathfrak{H}^* = \mathfrak{H} \cup C$ (depending on Γ via C) with its usual topology that induces the standard topology on $\mathfrak{H}$, and has the extended horocyclic disks $D_a(\kappa) \cup \{\kappa\}$, $a > 0$, as a basis of open neighborhoods of $\kappa \in C$. This topology is finer than that induced by the inclusion $\mathfrak{H}^* \subset \bar{\mathfrak{H}} \subset \mathbb{P}^1_{\mathbb{C}}$.

- *Fundamental domain.* We shall work with a fundamental domain $\mathfrak{F}$ for $\Gamma \backslash \mathfrak{H}$ that satisfies the following conditions: We require that $\mathfrak{F}$ is a Dirichlet fundamental domain, constructed from a base point P_0 that is not an elliptic fixed point. There is a finite set $A = A^{-1} \subset \Gamma$ such that $\alpha \mathfrak{F}$, $\alpha \in A$, runs through the Γ-translates of $\mathfrak{F}$ that have an edge in common with $\mathfrak{F}$. We add the cusps in the closure of $\mathfrak{F}$ in $\mathfrak{H}^*$ to $\mathfrak{F}$. We require that this extended fundamental domain meets each Γ-orbit of cusps exactly once. This is possible ([15], Chap. IV, §7G on p. 151). In particular, $\mathfrak{F}^{\mathrm{cu}} = \mathfrak{F} \cap C$ can and often will be used as a set of representatives for $\Gamma \backslash C$.

The standard fundamental domain $\mathfrak{F} = \{z \in \mathfrak{H} \ : \ |x| \leq \frac{1}{2},\ |z| \geq 1\}$ for the modular group satisfies these requirements, with $\mathfrak{F}^{\mathrm{cu}} = \{\infty\}$.

- *Tesselations.* For a fundamental domain $\mathfrak{F}$ as above, we choose $Y > 0$ large enough that all $D_Y(\kappa)$ are pairwise disjoint and that the following geodesic segments are contained in the interior of $\mathfrak{H}_Y$: the segments $e(P_0, \alpha^{-1}P_0)$ for $\alpha \in A$, all segments connecting P_0 to the vertices of $\mathfrak{F} \cap \mathfrak{H}_Y$, and all segments connecting those vertices.

In the sequel we will need tesselations of the following four types.

i) *Tesselation of type **Dir**.* With $X_0 = \Gamma P_0$, $X_1 = \Gamma \left\{ e(P_0, \alpha^{-1}P_0) \ : \ \alpha \in A \right\}$, and X_2 the collection of the closures of the connected regions enclosed by the edges in X_1, we get a tesselation of a contractible region contained in $\mathfrak{H}_Y$. This leads to a projective resolution $F_. = \mathbb{Q}[X_.]$, which computes the group homology and the group cohomology of Γ.

ii) *Tesselation of type **Fd**.* The fundamental domain $\mathfrak{F}$ gives a tesselation $\mathcal{T}$ of $\mathfrak{H}$. We add to the edges the Γ-translates of the intersections of $\mathfrak{F}$ with $\partial D_Y(\kappa)$ for $\kappa \in \mathfrak{F}^{\rm cu}$. These edges are not geodesic segments. In this way, $\mathfrak{F} = \mathfrak{F}_Y \cup \bigcup_{\kappa\in\mathfrak{F}^{\rm cu}} V_\kappa$, where $\mathfrak{F}_Y = \mathfrak{F} \cap \mathfrak{H}_Y$, and where $V_\kappa = \{g_\kappa z \;:\; \operatorname{Im} z \geq Y,\; x_\kappa - 1 \leq \operatorname{Re} z \leq x_\kappa\}$ for some $x_\kappa \in \mathbb{R}$, is a triangle with infinite height and finite hyperbolic area, with vertices κ, $P_\kappa = g_\kappa(x_\kappa + iY) \in \partial\mathfrak{H}_Y$ and $\pi_\kappa^{-1} P_\kappa = g_\kappa(x_\kappa - 1 + iY) \in \partial\mathfrak{H}_Y$.

We write $e_\kappa = e(P_\kappa, \kappa)$ and $f_\kappa = e(P_\kappa, \pi_\kappa^{-1} P_\kappa) \subset \partial H_Y(\kappa) \cap \mathfrak{F}$. So e_κ is a geodesic half-line and f_κ is a horocyclic segment. We have

$$\partial V_\kappa \;=\; e_\kappa - \pi_\kappa^{-1} e_\kappa - f_\kappa\,. \tag{11.2}$$

There is a finite set E of edges e of $\mathfrak{F}_Y$ and corresponding $\gamma_e \in \Gamma$ such that

$$\partial\mathfrak{F}_Y \;=\; \sum_{\kappa\in\mathfrak{F}^{\rm cu}} f_\kappa + \sum_{e\in E} e - \sum_{e\in E} \gamma_e^{-1} e\,. \tag{11.3}$$

We denote

$$B \;=\; E \sqcup \{e_\kappa,\, f_\kappa \;:\; \kappa \in \mathfrak{F}^{\rm cu}\}\,. \tag{11.4}$$

See Figure 4 for an illustration in the modular case.

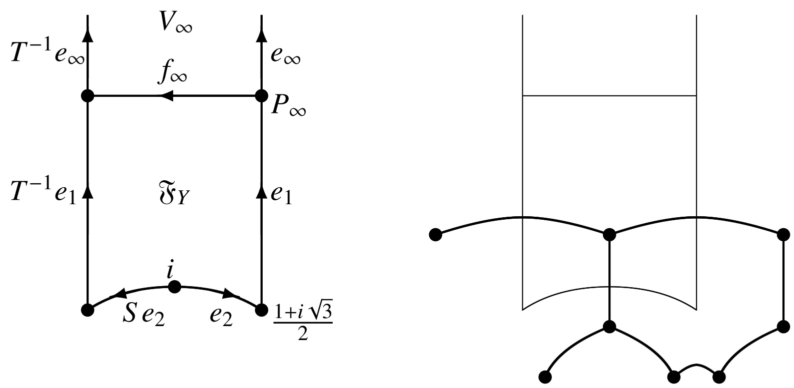


FIGURE 4. Modular group, parts of tesselations of type **Fd** (left) and **Dir** (right).

We put

$$\begin{aligned} X_0^Y &= \Gamma\{\text{vertices of } \mathfrak{F}_Y\}\,, & X_0 &= X_0^Y \sqcup C\,,\\ X_1^Y &= \Gamma\{f_\kappa \;:\; \kappa\in\mathfrak{F}^{\rm cu}\} \cup \Gamma E\,, & X_1 &= X_1^Y \sqcup \Gamma\{e_\kappa \;:\; \kappa\in\mathfrak{F}^{\rm cu}\}\,,\\ X_2^Y &= \Gamma(\mathfrak{F}_Y)\,, & X_2 &= X_2^Y \sqcup \Gamma\{V_\kappa \;:\; \kappa\in\mathfrak{F}^{\rm cu}\}\,. \end{aligned} \tag{11.5}$$

Here, and in the sequel, we consider the elements of the sets X_j as compact subsets of $\mathfrak{H}^*$. We have arranged that all elliptic fixed points are elements of X_0^Y.

The translates of $\mathfrak{F}_Y$ form a tesselation $\mathcal{T}$ of the contractible space $\mathfrak{H}_Y$, and $F_\cdot^{\mathcal{T},Y} = \mathbb{Q}[X_\cdot^Y]$ is a projective resolution of the $\mathbb{Q}[\Gamma]$-module $\mathbb{Q}$. It is contained in the chain complex $F_\cdot^{\mathcal{T}} = \mathbb{Q}[X_\cdot]$, which is not projective, due to the cusps in

$C \subset X_0$. The set B in (11.4) is a $\mathbb{Q}[\Gamma]$-basis of $F_1^{\mathcal{T}}$, and the following set is a $\mathbb{Q}[\Gamma]$-basis of $F_2^{\mathcal{T}}$:

$$\{\mathfrak{F}_Y\} \cup \{V_\kappa \ : \ \kappa \in \mathfrak{F}^{\mathrm{cu}}\} \,. \tag{11.6}$$

All other tesselations that we consider (apart from type **Dir**), are refinements of a tesselation of type **Fd**.

iii) *Tesselation of type **Mix***. Add to a tesselation $\mathcal{T}$ of type **Fd** based on a Dirichlet fundamental domain as above the interior base point P_0 and the edges from P_0 to the vertices of $\mathfrak{F}_Y$. Taking Γ-translates of the new points and edges we obtain a refinement of $\mathcal{T}$, which turns out to be a triangulation. See Figure 5.

iv) *Tesselation of type **Mix′***. Take the common refinement $\mathcal{T}$ of tesselations of type **Mix** and **Dir** built on the same Dirichlet fundamental domain. Add all Γ-translates of the geodesic half-lines from P_0 to the cusps in $\mathfrak{F}^{\mathrm{cu}}$ and the resulting additional vertices on the edges f_κ. We call the resulting triangulation a tesselation of type **Mix′**. See Figure 5.

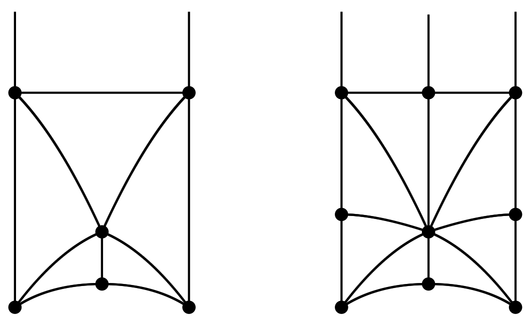


Figure 5. Modular group, part of tesselations of type **Mix** (left) and **Mix′** (right).

• *Chain complex on $\mathfrak{H}^*$*. The chain complex on $\mathfrak{H}^*$ gives the resolution $F_i^{\mathrm{hyp}} = \mathbb{Q}[(\mathfrak{H}^*)^{i+1}]$, with boundary maps, augmentation and group action as in (6.1). It has a subcomplex $F_i^{\mathrm{hyp},Y} = \mathbb{Q}[\mathfrak{H}_Y^{i+1}]$.

11.2. *Cohomology groups for* Γ. For all tesselations $\mathcal{T}$ that refine a tesselation of type **Fd**, the complex $F_\cdot^{\mathcal{T},Y}$ is a subcomplex of $F_\cdot^{\mathcal{T}}$ that gives a projective resolution of $\mathbb{Q}$ and can be used to compute the cohomology groups $H^\cdot(\Gamma; V)$. For this purpose, we can also use the complex corresponding to a tesselation of type **Dir**, and the complex $F_\cdot^{\mathrm{hyp},Y}$.

If there are cusps, then $H^2(\Gamma; V) = \{0\}$. In model **Fd**, $F_2^{\mathcal{T},Y}$ is generated by $\mathfrak{F}_Y$. For a cocycle c, the freedom in $c(\mathfrak{F}_Y)$ is determined by a coboundary $db(\mathfrak{F}_T) = \sum_{e\in E} b(e)\,|\,(1-\gamma_e) + \sum_{\kappa\in\mathfrak{F}^{\mathrm{cu}}} b(f_\kappa)$. The $b(f_\kappa) \in V$ can be freely chosen.

11.3. *Parabolic cohomology.* We will base the definition of parabolic cohomology groups on parabolic resolutions defined below. For the definition of mixed parabolic cohomology groups we use resolutions based on a tesselation of type **Fd** or a refinement of such a tesselation. To put these definitions in context, we shall prove (Proposition 11.8) that the resulting cohomology groups are isomorphic to certain cohomology groups in sheaf cohomology. Moreover, we will show that in dimension 1 these cohomology groups are isomorphic to those in Definition 10.1.

• *Parabolic resolutions.* For all resolutions obtained from a refinement of a tesselation of type **Fd** there is an exact sequence

$$\longrightarrow F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0$$

of $\mathbb{Q}[\Gamma]$-modules such that

a) F_0 has a set G_0 of generators over $\mathbb{Q}[\Gamma]$, such that for each $x \in G_0$ the subgroup $\Gamma_x \subset \Gamma$ fixing x is either finite, or equal to Γ_x with $x \in C$. (In the modular case, with a tesselation of type **Fd** based on the standard fundamental domain we may take G_0 consisting of i, $\frac{1+i\sqrt{3}}{2}$, P_∞ and ∞.)
b) For each $\kappa \in C$ the $\mathbb{Q}$-subspace $(F_0)^{\pi_\kappa}$ has dimension 1, and the augmentation ε is non-trivial on this subspace. (In the modular case, $(F_0)^T = \mathbb{Q}(\infty)$.)
c) The F_i, $i \geq 1$, are free $\mathbb{Q}[\Gamma]$-modules.

In resolutions coming from a tesselation of $\mathfrak{H}^*$, we have $F_i = 0$ for $i \geq 3$.

We call any resolution with the properties a)–c) a *parabolic resolution* of $\mathbb{Q}$. For the moment, we have only the example of resolutions based on a refinement of a tesselation of type **Fd**. In Chapter VI, we will use another example, obtained by taking tensor products.

Most of the properties of projective resolutions carry over:

Lemma 11.1. *If $f_\cdot : F_\cdot \to F_\cdot$ and $g_\cdot : F_\cdot \to F_\cdot$ are augmentation preserving chain maps of a parabolic resolution $F_\cdot$ of $\mathbb{Q}$, then they are homotopic.*

Proof. In dimension 0, we have for each $\kappa \in C$ a unique element $b_\kappa \in F_0$ such that $\pi_\kappa b_\kappa = b_\kappa$, and $\varepsilon b_\kappa = 1$. This forces $f_0 b_\kappa = g_0 b_\kappa$. From $\varepsilon \circ f_0 = \varepsilon \circ g_0$, we conclude that there is a Γ-equivariant map $h_0 : F_0 \to F_1$ such that $\partial_1 h_0 = f_0 - g_0$. It satisfies $h_0 b_\kappa = 0$ for all $\kappa \in C$. The further construction of a homotopy between $f_\cdot$ and $g_\cdot$ goes in the same way as for projective resolutions. See [1], Lemma 7.4 in Chap. I. ■

Lemma 11.2. *If $F_\cdot$ and $F'_\cdot$ are parabolic resolutions of $\mathbb{Q}$, then there exists an augmentation preserving chain map $f_\cdot : F_\cdot \to F'_\cdot$.*

Proof. For each $\kappa \in C$, we are forced to have $f_0 b_\kappa = b'_\kappa$. The further construction of the f_i on the generators can be carried out as for projective resolutions. See [1], §7 of Chap. I. ■

Definition 11.3. The *parabolic cohomology groups* $H^i_{\mathrm{par}}(\Gamma; V)$ of Γ with values in a $\mathbb{Q}[\Gamma]$-module V are the cohomology groups of the complex

$$\mathrm{Hom}_{\mathbb{Q}[\Gamma]}(F_\cdot, V) \cong \mathrm{Map}(X_\cdot, V)^\Gamma$$

for any parabolic resolution $F_\cdot$ of $\mathbb{Q}$.

The parabolic cohomology groups for different parabolic resolutions are canonically isomorphic.

In the case of the parabolic resolutions $F_{\cdot}^{\mathcal{T}}$ based on a refinement $\mathcal{T}$ of a tesselation of type **Fd**, there is a subcomplex $F_{\cdot}^{\mathcal{T},Y}$ which forms a projective resolution of $\mathbb{Q}$. We use such parabolic resolutions to define the mixed parabolic cohomology groups:

Definition 11.4. Let $V \subset W$ be an inclusion of $\mathbb{Q}[\Gamma]$-modules. We define $C^i(F_{\cdot}^{\mathcal{T}}; V, W)$ to be the space of Γ-equivariant maps $c : X_i^{\mathcal{T}} \to W$ such that $c(X_i^{\mathcal{T},Y}) \subset V$. We define coboundary maps $d^i : C^i(F_{\cdot}^{\mathcal{T}}; V, W) \to C^{i+1}(F_{\cdot}^{\mathcal{T}}; V, W)$ by $d^i c(x) = (-1)^{i+1} c(\partial_{i+1} x)$. We call the cohomology groups of the resulting complex

$$0 \longrightarrow C^0(F_{\cdot}^{\mathcal{T}}; V, W) \xrightarrow{d^0} C^1(F_{\cdot}^{\mathcal{T}}; V, W) \xrightarrow{d^1} C^2(F_{\cdot}^{\mathcal{T}}; V, W) \longrightarrow 0$$

the *mixed parabolic cohomology groups* $H^i_{\mathrm{par}}(\Gamma; V, W)$:

$$\begin{aligned} H^i_{\mathrm{par}}(\Gamma; V, W) &= Z^i(F_{\cdot}^{\mathcal{T}}; V, W)/B^i(F_{\cdot}^{\mathcal{T}}; V, W)\,, \\ Z^i(F_{\cdot}^{\mathcal{T}}; V, W) &= \mathrm{Ker}\left(d^i : C^i(F_{\cdot}^{\mathcal{T}}; V, W) \longrightarrow C^{i+1}(F_{\cdot}^{\mathcal{T}}; V, W)\right), \\ B^i(F_{\cdot}^{\mathcal{T}}; V, W) &= \begin{cases} d^{i-1}C^{i-1}(F_{\cdot}^{\mathcal{T}}; V, W) & \text{if } i > 0\,, \\ 0 & \text{if } i = 0\,. \end{cases} \end{aligned} \tag{11.7}$$

The definition is justified by Lemmas 11.1 and 11.2, which show that up to canonical isomorphisms the space $H^i_{\mathrm{par}}(\Gamma; V, W)$ is independent of the choices made. Note that $H^i_{\mathrm{par}}(\Gamma; V, W) = \{0\}$ for $i > 2$. In the case $V = W$ we denote $H^i_{\mathrm{par}}(\Gamma; V, V)$ by $H^i_{\mathrm{par}}(\Gamma; V)$. Since $F_{\cdot}^{\mathcal{T}}$ is a parabolic resolution, Definitions 11.3 and 11.4 lead to isomorphic parabolic cohomology groups. Finally, this definition is a redefinition in the case $i = 1$; in Proposition 11.5 we will see that Definitions 10.1 and 11.4 lead to isomorphic spaces $H^1_{\mathrm{par}}(\Gamma; V, W)$.

- *Dimension* 0. For all $V \subset W$, we have $H^0(\Gamma; V, W) = V^\Gamma$ (use that $X_0^{\mathcal{T},Y} \neq \emptyset$).
- *Dimension* 1. Consider the newly defined $H^1_{\mathrm{par}}(\Gamma; V, W)$ in a tesselation $\mathcal{T}$ of type **Mix**. If $c \in Z^1(F_1^{\mathcal{T}}; V, W)$, then $\psi_\gamma = c(\gamma^{-1}P_0, P_0)$ is a cocycle on Γ with values in V. For $\kappa \in C$:

$$\psi_{\pi_\kappa} = c(P_0, \kappa)\,|\,(\pi_\kappa - 1) \in W\,|\,(1 - \pi_\kappa)\,.$$

So ψ is a parabolic cocycle. If $c = db$ is a coboundary, then $\psi_\gamma = b(P_0)\,|\,(\gamma - 1)$, so $\psi \in B^1(\Gamma; V)$. This gives a map from $H^1_{\mathrm{par}}(\Gamma; V, W)$ defined here to the mixed parabolic cohomology group in Definition 10.1.

Conversely, since $F_{\cdot}^{\mathcal{T},Y}$ is a projective resolution, there is associated a cocycle $c \in Z^1(F_{\cdot}^{\mathcal{T},Y}; V)$ to each parabolic cocycle ψ on Γ. For each $P \in X_0^Y$, the cocycle $\gamma \mapsto c(\gamma^{-1}P, P)$ is in the same cohomology class as ψ, and hence is parabolic. For each cusp $\kappa \in \mathfrak{F}^{\mathrm{cu}}$, there is $w_\kappa \in W$ such that $c(f_\kappa) = c(\pi_\kappa^{-1}P_\kappa, P_\kappa) = w_\kappa\,|\,(\pi_\kappa - 1)$. Extend c in a $\mathbb{Q}[\Gamma]$-linear way to X_1 by defining $c(e_\kappa) = -w_\kappa$ for all $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. Then $c \in Z^1(F_{\cdot}^{\mathcal{T}}; V, W)$ corresponds to the parabolic cohomology class $[\psi]$. Thus, we have:

Proposition 11.5. *The mixed parabolic cohomology group $H^1(\Gamma; V, W)$ defined in Definition 11.4 is isomorphic to that defined in Definition 10.1.*

- *Dimension* 2. The second mixed parabolic cohomology groups do not necessarily vanish:

Proposition 11.6. *If* $|\Gamma\backslash C| = 1$*, then for each* $\kappa \in C$*:*

$$H^2_{\mathrm{par}}(\Gamma; V, W) \;\cong\; W\Big/\Big(W\,|\,(1-\pi_\kappa) + \sum_{\gamma\in\Gamma} V\,|\,(1-\gamma)\Big),$$

Proof. We work with a tesselation of model **Fd**. Let $\mathfrak{F}^{\mathrm{cu}} = \{\kappa\}$. Any cocycle $a \in Z^2(F^{\mathcal{T}}_{\cdot}; V, W) = C^2(F^{\mathcal{T}}_{\cdot}; V, W)$ is determined by its values $a(\mathfrak{F}_Y) \in V$ and $a(V_\kappa) \in W$. The freedom consists of adding dc with $c \in C^1(F^{\mathcal{T}}_{\cdot}; V, W)$. Choosing $c(e) \in V$ for $e \in E$ changes $a(\mathfrak{F}_Y)$ by $c(e)\,|\,(1-\gamma_e)$ and leaves $a(V_\kappa)$ unchanged. These elements generate $\sum_\gamma V\,|\,(1-\gamma)$. Choosing $c(e_\kappa) \in W$ changes $a(V_\kappa)$ by $c(e_\kappa)\,|\,(1-\pi_\kappa)$ and leaves $a(\mathfrak{F}_Y)$ unchanged. Finally, the choice of $c(f_\kappa) \in V$ changes $(a(\mathfrak{F}_T), a(V_\kappa))$ by $(c(f_\kappa), -c(f_\kappa))$. Thus, we can arrange $a(\mathfrak{F}_T) = 0$, and get $a(V_\kappa) \in W$ with freedom in $W\,|\,(1-\pi_\kappa) + \sum_{\gamma\in\Gamma}\,|\,V(1-\gamma)$. This completes the proof, and implies that the description is up to isomorphism independent of the choice of κ in its Γ-orbit C. To make this isomorphism explicit we consider $\delta \in \Gamma$, and note that $w \mapsto w|\delta$ leaves W and V invariant, and sends $W\,|\,(1-\pi_\kappa) + \sum_{\gamma\in\Gamma} V\,|\,(1-\gamma)$ to

$$W|\delta(1-\delta^{-1}\pi_\kappa\delta) + \sum_\gamma V|\delta(1-\delta^{-1}\gamma\delta) \;=\; W|(1-\pi_{\delta^{-1}\kappa}) + \sum_\gamma V|(1-\gamma)\,. \qquad \blacksquare$$

Along the same lines, we arrive at the following more complicated description for the general case:

Proposition 11.7. *For any cofinite group* Γ *with cusps:*

$$H^2_{\mathrm{par}}(\Gamma; V, W) \;\cong\; \Big(\bigoplus_{\kappa\in\mathfrak{F}^{\mathrm{cu}}} W/\Big(W\;\Big|\;(1-\pi_\kappa) + V\;\Big|\;\sum_{\gamma\in\Gamma}(1-\gamma)\Big)\Big)\Big/\mathrm{Ker}\,\sigma\,,$$

where $\sigma : \bigoplus_{\kappa\in\mathfrak{F}^{\mathrm{cu}}} V \to V_\Gamma : (v_\kappa) \mapsto \sum_\kappa v_\kappa + \sum_{\gamma\in\Gamma} V\,|\,(1-\gamma)$, $V_\Gamma = V/\sum_{\gamma\in\Gamma} V\,|\,(1-\gamma)$.

In the special case $V = 0$, we have $H^i_{\mathrm{par}}(\Gamma; 0, W) = \{0\}$ for $i = 0, 1$, and

$$H^2_{\mathrm{par}}(\Gamma; 0, W) \;=\; \bigoplus_{\kappa\in\mathfrak{F}^{\mathrm{cu}}} W\,/(W\,|\,(1-\pi_\kappa))\;\;. \tag{11.8}$$

If $V = W$, then

$$H^2_{\mathrm{par}}(\Gamma; V) \;\cong\; V_\Gamma\,. \tag{11.9}$$

This isomorphism is given by evaluation on the fundamental class represented by

$$(\mathfrak{F}) = (\mathfrak{F}_Y) + \sum_{\kappa\in\mathfrak{F}^{\mathrm{cu}}} (V_\kappa)\;\;. \tag{11.10}$$

Thus we have

$$\begin{array}{ccccccccc} H^0_{\mathrm{par}}(\Gamma; V, W) & = & H^0_{\mathrm{par}}(\Gamma; V) & = & H^0(\Gamma; V) & = & V^\Gamma\,, & & \\ H^1_{\mathrm{par}}(\Gamma; V) & \subset & H^1_{\mathrm{par}}(\Gamma; V, W) & \subset & H^1(\Gamma; V)\,, & & & & \\ H^2_{\mathrm{par}}(\Gamma; V, W) & \supset & H^2_{\mathrm{par}}(\Gamma; V) & = & V_\Gamma & \supset & H^2(\Gamma; V) & = & \{0\}\,. \end{array} \tag{11.11}$$

• *Sheaf cohomology.* We now show that the mixed parabolic cohomology groups can be identified with cohomology groups of certain sheaves on $\Gamma\backslash\mathfrak{H}^*$. This will then be used to give a long exact sequence for mixed parabolic cohomology groups (Proposition 11.9).

The topological space $\mathsf{X} := \Gamma\backslash\mathfrak{H}^*$ is compact. It contains the open subspace $\mathsf{Y} = \Gamma\backslash\mathfrak{H}$ and the closed subspace $\mathsf{Y}_Y := \Gamma\backslash\mathfrak{H}_Y$. We denote the projection map by $\pi : \mathfrak{H}^* \to \mathsf{X}$. Let $V \subset W$ be $\mathbb{Q}[\Gamma]$-modules. On $\mathfrak{H}^*$ we have the constant sheaf $W \times \mathfrak{H}^*$ with subsheaf $(V \times \mathfrak{H}_Y) \cup (W \times (\mathfrak{H}^* \smallsetminus \mathfrak{H}_Y))$. The group Γ acts by $(w, P) \,|\, \gamma = (w \,|\, \gamma, \gamma^{-1}P)$. The quotient

$$\mathcal{F}_{V,W} \;=\; \Big((V \times \mathfrak{H}_Y) \cup (W \times (\mathfrak{H}^* \smallsetminus \mathfrak{H}_Y))\Big)\Big/\Gamma \tag{11.12}$$

is a sheaf on X. For open $U \subset \mathsf{X}$ the space $\mathcal{F}_{V,W}(U)$ consists of all locally constant Γ-equivariant functions $\pi^{-1}(U) \to W$ that take values in V on the components of $\pi^{-1}(U)$ that intersect $\pi^{-1}(\mathsf{Y}_Y)$.

For a given $P \in \mathsf{X}$, choose $z \in \mathfrak{H}^*$ with $P = \pi z$. The stalk $(\mathcal{F}_{V,W})_P$ is isomorphic to V^{Γ_z} if $P \in \mathsf{Y}_Y$ and to W^{Γ_z} if $P \in \mathsf{X} \smallsetminus \mathsf{Y}_Y$. The isotropy group Γ_z is trivial for all but finitely many $P \in \mathsf{X}$.

Proposition 11.8. *Let $V \subset W$ be an inclusion of $\mathbb{Q}[\Gamma]$-modules. Then*

$$H^j_{\mathrm{par}}(\Gamma; V, W) \cong H^j(\mathsf{X}; \mathcal{F}_{V,W}) \qquad \textit{for all } j \geq 0\,.$$

Proof. For any refinement $\mathcal{T}$ of a tesselation of type **Fd**, we form for $z \in X_0$ the open set

$$\Omega_z \;=\; \{z\} \cup \{\mathring{e} \;:\; e \in X_1,\, z \in e\} \cup \Big\{\mathring{V} \;:\; V \in X_2,\, z \in V\Big\}\,.$$

in $\mathfrak{H}^*$. By $\mathring{e}$, we mean e minus its end points, not the (empty) interior as a subset of $\mathfrak{H}^*$. For $z \in X_0 \cap \mathfrak{H}$, the set Ω_z contains finitely many $\mathring{e}$ and $\mathring{V}$. If $\kappa \in C$, then Ω_κ is equal to $D_Y(\kappa) \cup \{\kappa\}$. If Γ_z is non-trivial, the set Ω_z necessarily contains Γ_z-equivalent points. We require that the tesselation $\mathcal{T}$ is such that all Ω_z contain no more Γ-equivalent points than necessary: If $\Gamma_z = \{1\}$, then Ω_z should not contain Γ-equivalent points; otherwise, if $z_1, \gamma^{-1}z_1 \in \Omega_z$ for $\gamma \in \Gamma$, then $\gamma \in \Gamma_z$. We also require that X_2 consists of triangles. A tesselation of type **Mix′** satisfies these conditions.

The set $\mathfrak{A} = \{\pi\Omega_z \;:\; z \in X_0\}$ is a finite open covering of X. The intersection of two different elements of $\mathfrak{A}$, if non-empty, contains the image $\pi\mathring{e}$ for exactly one Γ-orbit of $e \in X_1$. The non-empty intersection of three different elements of $\mathfrak{A}$ corresponds to the $\pi\mathring{V}$ for exactly one Γ-orbit of elements $V \in X_2$. In this way, we can check that the complex $C^{\cdot}(F_{\cdot}, V, W)$ is isomorphic to the complex $\big(C^i\,(\mathfrak{A}, \mathcal{F}_{V,W})\big)_i$ in Čech cohomology. This implies the following isomorphism:

$$H^i_{\mathrm{par}}(\Gamma; V, W) \;\cong\; \check{H}^i\,(\mathfrak{A}; \mathcal{F}_{V,W})\,.$$

See, e.g., [11], Chap. III, §4 for Čech cohomology.

Leray's theorem (Exercise 4.11, *loc. cit.*) states that $\check{H}^i\,(\mathfrak{A}; \mathcal{F}_{V,W}) \cong H^i\,(X; \mathcal{F}_{V,W})$ if $H^k\,(U; \mathcal{F}_{V,W}|_U) = \{0\}$ for all intersections U of elements of $\mathfrak{A}$ for all $k \geq 1$. To finish the proof, we have to check that this condition holds in the present situation.

We first consider a connected set U that does not contain the image of an elliptic or parabolic fixed point of Γ. It may happen that U is contained in Y_Y or in $\mathsf{X} \smallsetminus \mathsf{Y}_Y$. Then the restriction $\mathcal{G} = \mathcal{F}_{V,W}|_U$ is the constant sheaf $\mathbf{V}$ or $\mathbf{W}$, and $H^i\,(U; \mathcal{F}_{V,W}|_U) = 0$ for $i \geq 1$, since constant sheaves have trivial cohomology. The other possibility is that U is a neighborhood of $\pi\mathring{e}$ for some edge e contained in $\partial\mathfrak{H}_Y$. Then $\mathcal{G}(U_1) = W$ if $U_1 \subset \mathsf{X} \smallsetminus \mathsf{Y}_Y$ and $\mathcal{G}(U_1) = V$ otherwise. The set $U_0 = U \cap \mathsf{Y}_Y$ is closed in U. Let $k : U_0 \mapsto U$ denote the inclusion. We have an exact sequence of sheaves of $\mathbb{Q}$-vector spaces on U:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathbf{W} \longrightarrow k_*(\mathbf{W}/\mathbf{V}) \longrightarrow 0\,, \tag{11.13}$$

where $\mathbf{W}$ is the constant sheaf given by W on U, and $\mathbf{W}/\mathbf{V}$ the constant sheaf on U_0 given by W/V. For $i \geq 1$, we have $H^i(U;\mathbf{W}) = \{0\}$, and $H^i(U;k_*(\mathbf{W}/\mathbf{V})) \cong H^i(U_0;\mathbf{W}/\mathbf{V})$ by Lemma III.2.4, *loc. cit.*, and hence also $H^1(U;k_*(\mathbf{W}/\mathbf{V})) = \{0\}$. The long exact sequence corresponding to (11.13) starts with

$$\begin{array}{ccccccccccccc} 0 & \to & V & \to & W & \to & W/V & \to & H^1(U;\mathcal{G}) & \to & H^1(U;\mathbf{W}) & \to \\ &&&&&&&&&& \| & \\ &&&&&&&&&& 0 & \end{array}$$

Since $W \to W/V$ is surjective, this implies that $H^1(U;\mathcal{G}) = \{0\}$. The later parts

$$\begin{array}{ccccccccc} \to & H^{i-1}(U;k_*(\mathbf{W}/\mathbf{V}) & \to & H^i(U;\mathcal{G}) & \to & H^i(U;\mathbf{W}) & \to \\ & \| & & & & \| & \\ & 0 & & & & 0 & \end{array}$$

of the long exact sequence show that $H^i(U;\mathcal{G}) = 0$ for $i \geq 2$.

Suppose now that U contains the image of a parabolic or elliptic fixed point. Then $U = \pi\,\Omega_z$ where $z \in C$ or z is an elliptic fixed point. We treat the case $z = \kappa \in C$. The other case goes similarly.

Let $P = \pi\,\kappa$. The restriction of $\mathcal{G}$ to $U \smallsetminus \{P\}$ is the constant sheaf $\mathbf{W}$. With the injection $k : \{P\} \to U$, we have the following exact sequence of sheaves on U:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathbf{W} \longrightarrow k_*\,(\mathbf{W}/\mathbf{W}_\kappa) \longrightarrow 0\,,$$

where $\mathbf{W}_\kappa$ is the constant sheaf W/W^{Γ_κ} on $\{P\}$. We proceed as in the previous case. ■

This proposition shows that the concept of mixed parabolic cohomology in Definition 11.4 can be interpreted as sheaf cohomology. Since a sequence of sheaves is exact if all corresponding sequences of stalks are exact, we have:

Proposition 11.9. *Suppose that the rows in the following diagram of $\mathbb{Q}[\Gamma]$-modules are exact*

$$\begin{tikzcd} 0 \arrow[r] & V' \arrow[r] \arrow[d, hook] & V \arrow[r] \arrow[d, hook] & V'' \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & W' \arrow[r] & W \arrow[r] & W'' \arrow[r] & 0 \end{tikzcd} \tag{11.14}$$

and that for each $\kappa \in \mathfrak{F}^{\mathrm{cu}}$ the sequence

$$0 \longrightarrow (W')^{\Gamma_\kappa} \longrightarrow W^{\Gamma_\kappa} \longrightarrow (W'')^{\Gamma_\kappa} \longrightarrow 0 \tag{11.15}$$

is exact as well. Then there is a long exact sequence of mixed parabolic cohomology groups

$$\begin{aligned} \cdots \longrightarrow & H^{i-1}_{\mathrm{par}}(\Gamma; V'', W'') \longrightarrow H^i_{\mathrm{par}}(\Gamma; V', W') \longrightarrow H^i_{\mathrm{par}}(\Gamma; V, W) \\ \longrightarrow & H^i_{\mathrm{par}}(\Gamma; V'', W'') \longrightarrow H^{i+1}_{\mathrm{par}}(\Gamma; V, W) \longrightarrow \cdots \end{aligned}$$

Parabolic cohomology groups are sometimes defined as the kernel of the restriction to the boundary in the Borel-Serre compactification, e.g., in [8], §2. In dimension 1 this leads to parabolic cohomology groups isomorphic to the groups $H^1_{\mathrm{par}}(\Gamma; V)$ defined here.

The category with as its objects inclusions $V \subset W$ of $\mathbb{Q}[\Gamma]$-modules and the obvious morphisms is not abelian. To get a fully satisfactory cohomological treatment of mixed parabolic cohomology, one should extend Definition 11.4 to morphisms $V \to W$ of $\mathbb{Q}[\Gamma]$-modules. We refrain from carrying out this extension, and mention only one case, which will be used in Section 13. Let $V \hookrightarrow W$ be an inclusion of $\mathbb{Q}[\Gamma]$-modules. Define $\mathcal{Q}_{W/V}$ as the quotient in the exact sequence of sheaves on X

$$0 \longrightarrow \mathcal{F}_{V,W} \longrightarrow \mathcal{F}_{W,W} \longrightarrow \mathcal{Q}_{W/V} \longrightarrow 0\,. \tag{11.16}$$

So $\mathcal{Q}_{W/V}(U) = 0$ if the open set U is contained in $\mathsf{X} \smallsetminus \mathsf{Y}_Y$ and $\mathcal{Q}_{W/V}(U) = W/V$ otherwise. By generalizing the proof of Proposition 11.8 one sees that $H^i(X; \mathcal{Q}_{W/V}) \cong H^i(\Gamma; W/V)$ for all $i \geq 0$. As a consequence of the exactness of (11.16) we obtain a long exact sequence

$$\begin{aligned} \ldots \longrightarrow & H^{i-1}(\Gamma; W/V) \longrightarrow H^i_{\mathrm{par}}(\Gamma; V, W) \longrightarrow H^i_{\mathrm{par}}(\Gamma; W) \\ & \longrightarrow H^i(\Gamma; W/V) \longrightarrow H^{i+1}_{\mathrm{par}}(\Gamma; V, W) \longrightarrow \ldots \end{aligned} \tag{11.17}$$

12. Maass forms and cohomology. This section generalizes the results concerning the relation between Maass forms and analytic cohomology given in Chapter II for cocompact groups to groups with cusps. We follow the same method as for cocompact groups, with some modifications to handle the complications caused by the cusps. The presence of cusps brings also a simplification: The cusps are vertices of the tesselations situated on $\partial\mathbb{H}$. There is no need to extend cocycles to hyperbolic fixed points on $\partial\mathbb{H}$, like we needed to do in §7.3.

12.1. *From invariant eigenfunctions to parabolic cocycles.* The linear maps $\mathbf{r}$ and $\mathbf{q}$ from invariant eigenfunctions to cohomology classes have been described in (5.5) only in the group model of cohomology. In a model based on a tesselation $\mathcal{T}$ of type **Fd** or a refinement of it, cocycles r and q representing $\mathbf{r}u$, respectively $\mathbf{q}u$, for $u \in \mathcal{E}_s^\Gamma$, are determined by:

$$r(x)(\zeta) \;=\; \int_x [u, R(\zeta;\,\cdot\,)^s]\,, \qquad q(x)(z) \;=\; \int_x [u, q_s(\,\cdot\,, z)] \quad \text{for } x \in X_1^{\mathcal{T},Y}\,. \tag{12.1}$$

We know from Propositions 10.3 and 11.5 that $\mathbf{r}u$ is a parabolic cohomology class in $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{exc}})$, where $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ consists of elements with finitely many singularities of a special type, discussed in Definition 9.17. The following result gives explicit cocycles in the classes $\mathbf{r}u$ and $\mathbf{q}u$.

Proposition 12.1. *Let $\mathcal{T}$ be a tesselation refining a tesselation of type **Fd**.*

For a cusp form $u \in \mathsf{Maass}_s^0(\Gamma)$, the cocycles r and q in (12.1) *have unique extensions $r \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\infty})$ and $q \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\infty})$, given by the integrals in* (12.1) *for all $x \in X_1^{\mathcal{T}}$.*

If $u \in \mathsf{Maass}_s^1(\Gamma)$ and $s \neq \frac{1}{2}$, the cocycle r has a unique extension as an element of $Z^1(F_\cdot^{\mathcal{T}}; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{simple}})$ determined by

$$r(e_{P,\kappa}) \;=\; -\mathrm{Av}^+_{\pi_\kappa} r(e_{\pi_\kappa^{-1}P,P}) \;=\; -\mathrm{Av}^-_{\pi_\kappa} r(e_{\pi_\kappa^{-1}P,P})$$

for $\kappa \in C$ and $P \in X_1^{\mathcal{T}}$ on the horocycle $V_Y(\kappa)$. (See §4.2 for the one-sided averages.)

If $s \neq \frac{1}{2}$ and $u \in \mathcal{E}_s^\Gamma$ is a general invariant eigenfunction, then r can be extended, non-uniquely, as an element of $Z^1(F_\cdot^{\mathcal{T}}; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{exc}})$ *by defining*

$$r(e_{P,\kappa}) = -\mathrm{Av}^+_{\pi_\kappa} r(e_{\pi_\kappa^{-1}P,P})$$

for $\kappa \in C$ and for $P \in X_1^{\mathcal{T}}$ on the horocycle $V_Y(\kappa)$.

In all cases $q = b(s)\,\mathsf{P}_s^\dagger r$ gives an extension $q \in \mathrm{Z}^1(F_\cdot^{\mathcal{T}}; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\mathrm{exc}})$.

Proof. If u is a cusp form, then the integral in (12.1) converges also when one end point of x is a cusp. Use Proposition 9.7 and (9.12), and Proposition 4.5 for the uniqueness. In the general situation, we have $r(e_{\pi_\kappa^{-1}P,P}) \in \mathcal{V}_s^\omega$, and hence $\mathrm{Av}^+_{\pi_\kappa} r(e_{P,\pi_\kappa^{-1}P}) \in \mathcal{V}_s^\omega[\kappa]$. Lemma 9.4 and Proposition 9.18 imply, after conjugation, that $\mathrm{Av}^+_{\pi_\kappa} r(e_{\pi_\kappa^{-1}P,P}) \in \mathcal{V}_s^{\omega,\mathrm{exc}}[\kappa]$. The Γ-equivariance of r follows from $\pi_{\gamma^{-1}\kappa} = \gamma^{-1}\pi_\kappa\gamma$. For the cocycle property it suffices to consider dr on the triangles V_κ at the cusps $\kappa \in \mathfrak{F}^{\mathrm{cu}}$:

$$\begin{aligned} dr(V_\kappa) &= r(e_{P_\kappa,\kappa})\,|\,(1-\pi_\kappa) + r(e_{\pi_\kappa^{-1}P_\kappa,P_\kappa}) \\ &= \bigl(-\mathrm{Av}^+_{\pi_\kappa} r(e_{\pi_\kappa^{-1}P_\kappa,P_\kappa})\bigr)\,|\,(1-\pi_\kappa) + r(e_{\pi_\kappa^{-1}P_\kappa,P_\kappa}) = 0\,. \end{aligned}$$

If $u \in \mathsf{Maass}_s^1(\Gamma)$, then Proposition 9.15 implies that $r(e_{\pi_\kappa^{-1}P,P}) \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}\,|\,(1-\pi_\kappa)$, and then the choice $r(e_{P,\kappa}) = \mathrm{Av}^+_T r(e_{\pi_\kappa^{-1}P,P}) = \mathrm{Av}^-_T r(e_{\pi_\kappa^{-1}P,P})$ is unique. See Propositions 9.13 and 9.14. ∎

12.2. *From parabolic cocycles to invariant eigenfunctions.* The ideas in §7.1 and §7.2 can be applied, with some modifications. In this subsection, we construct an element of $\mathcal{E}_s^\Gamma$ starting from a parabolic $(\mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\mathrm{exc}})$-valued cocycle in two ways: a local representation as a sum of values of the cocycle, like in §7.1, and as an average over Γ, like in §7.2.

We work with a tesselation $\mathcal{T}$ of type **Fd** based on a Dirichlet fundamental domain $\mathfrak{F}$. A difference with the cocompact case is the presence of edges in $X_1^{\mathcal{T}}$ with a cusp as one of their end points. For the interior edges $e \subset \mathfrak{H}_Y$ we can work with R-neighborhoods $N_R(e)$, like in §7.1. The R-neighborhoods with respect to the hyperbolic distance of edges $e_{P,\kappa}$ with $\kappa \in C$ intersect infinitely many Γ-translates of $e_{P,\kappa}$. We define instead for $R > 0$

$$N_R(e_{P,\kappa}) = \{g_\kappa z \;:\; |\mathrm{Re}\,z| \leq R\,,\ \mathrm{Im}\,z \geq 1/R\}\,. \tag{12.2}$$

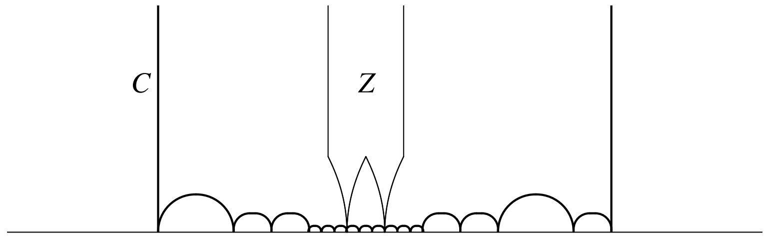
This is the set $g_\kappa W_{1/R}$ with $W_{1/R}$ as defined in (9.18). It is of the form $\mathfrak{H} \smallsetminus \Omega$ for an excised neighborhood Ω of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\kappa\}$. It contains $e_{P,\kappa}$ if $R \geq Y$.

For a given cocycle $\psi \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\mathrm{exc}})$ we choose a Γ-equivariant lift $\tilde\psi \in C^1(F_\cdot^{\mathcal{T}}; \mathcal{G}_s^\omega, \mathcal{G}_s^{\omega^*,\mathrm{exc}})$ by first choosing lifts of $\psi(b)$ for b in a $\mathbb{Q}[\Gamma]$-basis B of $F_1^{\mathcal{T}}$; e.g., $B = E \cup \{f_\kappa, e_\kappa \;:\; \kappa \in \mathfrak{F}^{\mathrm{cu}}\}$. For interior edges $e \in E$ or $e = f_\kappa$, we choose any lift $\tilde\psi(e) \in \mathcal{G}_s^\omega$ of $\psi(e) \in \mathcal{W}_s^\omega$. For the edge e_κ to the cusp $\kappa \in \mathfrak{F}^{\mathrm{cu}}$ it is sensible to take a lift of $\psi(e_\kappa) \in \mathcal{W}_s^{\omega^*,\mathrm{exc}}$ satisfying

$$\tilde\psi(e_\kappa) \in \mathcal{G}_s^{\omega^*,\mathrm{exc}}\,, \qquad \mathsf{Sing}\,\tilde\psi(e_\kappa) \subset N_R(e_\kappa) \quad \text{for some } R \geq Y\,. \tag{12.3}$$

To see that this is possible, we note that $\psi(e_\kappa)\,|\,(1-\pi_\kappa) = \psi(f_\kappa) \in \mathcal{W}_s^\omega$. Hence $\mathsf{BdSing}\,\psi(e_\kappa) \subset \{\kappa\}$.

Next we fix $R \geq Y$ large enough that $\mathsf{Sing}\,\tilde\psi(e) \subset N_R(e)$ for all $e \in X_1^{\mathcal{T}}$. Let Z be a finite union of Γ-translates of $\mathfrak{F}$. We can find cycles $C \in \mathbb{Z}[X_1^{\mathcal{T}}]$ with winding number 1 on Z such that $N_R(C) = \bigcup_{e\subset \mathrm{Supp}\, C} N_R(e)$ does not intersect Z.



The cycle C has to pass through the finitely many cusps in Z.

We define u_ψ on Z by

$$u_\psi(z) \;=\; \frac{1}{\pi i}\tilde\psi(C)(z)\,. \tag{12.4}$$

Like in §7.1 this does not depend on the choice of the cycle C, on the choice of the lift $\tilde\psi$, or on the choice of ψ in its cohomology class, and satisfies $u_\psi(\delta z) = u_\psi(z)$, and leads to $u_\psi = u_{[\psi]} \in \mathcal{E}_s^\Gamma$.

Suppose that ψ is the cocycle q in (12.1) representing $\mathbf{q}u$, with $u \in \mathcal{E}_s^\Gamma$. We can take a lift $\tilde\psi(b) \in C^2(\mathfrak{H})$ of $q(b)$, for $b \in B$, equal to the value of the integral in (12.1) outside $N_\varepsilon(b)$. If ε is sufficiently small, then there is a non-empty open set $V \subset \mathring{\mathfrak{F}}_Y$ not intersecting the ε-neighborhood of any $e \in X_1^{\mathcal{T}}$. With Theorem 1.1 and Proposition 9.8 we obtain for $z \in V$:

$$u_\psi(z) \;=\; \frac{1}{\pi i}\int_{\partial\mathfrak{F}_Y} [u, q_s(\,\cdot\,, z)] \;=\; u(z)\,.$$

By analyticity $u_\psi = u$ everywhere on $\mathfrak{H}$.

If $s \neq \frac12$, we have $\mathbf{q}\mathcal{E}_s^\Gamma \subset H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\mathrm{exc}})$, from Theorem 9.20. Thus, $\psi \mapsto u_\psi$ induces a one-sided inverse of $\mathbf{q}$. For $s = \frac12$, we have $u_q = u$ only for those $u \in \mathcal{E}_{1/2}^\Gamma$ for which $\mathbf{q}u \in H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_{1/2}^\omega, \mathcal{W}_{1/2}^{\omega^*,\mathrm{exc}})$. This includes the Maass cusp forms in $\mathsf{Maass}_{1/2}^0(\Gamma)$.

We summarize:

Proposition 12.2. *There is a linear map $\psi \mapsto u_\psi$ from $Z^1(F_\cdot^{\mathcal{T}}; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\mathrm{exc}})$ to $\mathcal{E}_s^\Gamma$. The invariant eigenfunction u_ψ depends only on the parabolic cohomology class $[\psi] \in H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\mathrm{exc}})$ and can be given on each compact set in $\mathfrak{H}$ by a (finite) $\mathbb{C}$-linear combination of translates of $\psi(e)$, where e runs through the $\mathbb{Q}[\Gamma]$-basis B of $F_1^{\mathcal{T}}$ in* (11.4)*. If $s \neq \frac12$ the induced map $\alpha_s^\omega : H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\mathrm{exc}}) \to \mathcal{E}_s^\Gamma$ satisfies $\alpha_s^\omega\, \mathbf{q} = \mathrm{Id}$ on $\mathcal{E}_s^\Gamma$.*

In §12.4 we will discuss the question under what conditions on ψ the invariant eigenfunction u_ψ is in $\mathsf{Maass}_s^0(\Gamma)$ or $\mathsf{Maass}_s^1(\Gamma)$.

Now we have generalized the approach in §7.1 leading to Theorem 7.2. To generalize §7.2, we use the diagram (9.17). It satisfies the exactness condition (11.15), as follows from Lemma 9.22. Proposition 11.9 implies that there is a long exact sequence associated to (9.17), from which we use the following part:

$$\begin{aligned}\longrightarrow & H^1_{\mathrm{par}}(\Gamma; \mathcal{G}^\omega_s, \mathcal{G}^{\omega^*,\mathrm{exc}}_s) \longrightarrow H^1_{\mathrm{par}}(\Gamma; \mathcal{W}^\omega_s, \mathcal{W}^{\omega^*,\mathrm{exc}}_s) \\ & \longrightarrow H^2_{\mathrm{par}}(\Gamma; \mathcal{N}^\omega, \mathcal{N}^{\omega^*,\mathrm{exc}}) \longrightarrow\end{aligned}$$

We choose a lift $\tilde\psi \in C^1(F^{\mathcal{T}}_{\cdot}; \mathcal{G}^\omega_s, \mathcal{G}^{\omega^*,\mathrm{exc}}_s)$ of $\psi \in Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{W}^\omega_s, \mathcal{W}^{\omega^*,\mathrm{exc}}_s)$ as above. Then $d\tilde\psi \in Z^2(F^{\mathcal{T}}_{\cdot}; \mathcal{N}^\omega, \mathcal{N}^{\omega^*,\mathrm{exc}})$. As in §7.2, we obtain

$$u_\psi = \frac{1}{\pi i} \mathrm{Av}_\Gamma(d\tilde\psi(\mathfrak{F})), \tag{12.5}$$

independent of the choice of the lift $\tilde\psi$. The support of $d\tilde\psi(\mathfrak{F}) \in \mathcal{N}^{\omega^*,\mathrm{exc}}$ is not compact, but meets only finitely many Γ-translates of $\mathfrak{F}$. So the sum defining $\mathrm{Av}_\Gamma(d\tilde\psi(\mathfrak{F}))$ is locally finite and converges absolutely. The representation of u_ψ as an average does not depend on the choice of the lift $\tilde\psi$ satisfying (12.3).

12.3. *Injectivity.* Starting from a cocycle $\psi \in Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{W}^\omega_s, \mathcal{W}^{\omega^*,\mathrm{exc}}_s)$ and a lift $\tilde\psi$ of ψ satisfying (12.3), we have constructed in two ways, (12.4) and (12.5), a Γ-invariant eigenfunction u_ψ, thus obtaining a linear map α^ω_s from $H^1_{\mathrm{par}}(\Gamma; \mathcal{W}^\omega_s, \mathcal{W}^{\omega^*,\mathrm{exc}}_s)$ to $\mathcal{E}^\Gamma_s$. Now we will prove:

Proposition 12.3. *Let $s \in \mathbb{C}$, $0 < \mathrm{Re}\, s < 1$. The map*

$$\alpha^\omega_s : H^1_{\mathrm{par}}(\Gamma; \mathcal{W}^\omega_s, \mathcal{W}^{\omega^*,\mathrm{exc}}_s) \longrightarrow \mathcal{E}^\Gamma_s$$

in Proposition 12.2 is injective.

With Proposition 12.2 and with the fact that the transverse Poisson map gives isomorphisms $\mathcal{V}^\omega_s \cong \mathcal{W}^\omega_s$ and $\mathcal{V}^{\omega^*,\mathrm{exc}}_s \cong \mathcal{W}^{\omega^*,\mathrm{exc}}_s$, this implies:

Proposition 12.4. *Let $s \in \mathbb{C}$, $0 < \mathrm{Re}\, s < 1$, $s \neq \frac{1}{2}$. Then*

$$H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^\omega_s, \mathcal{V}^{\omega^*,\mathrm{exc}}_s) \cong \mathcal{E}^\Gamma_s .$$

To prove Proposition 12.3 we use the following results, of which we postpone the proofs.

Proposition 12.5. *The cohomology group $H^1_{\mathrm{par}}(\Gamma; \mathcal{G}^\omega_s, \mathcal{G}^{\omega^*,\mathrm{exc}}_s)$ is zero.*

Lemma 12.6. *Suppose $c \in C^2(F^{\mathcal{T}}_{\cdot}; \mathcal{N}^\omega, \mathcal{N}^{\omega^*,\mathrm{exc}})$ satisfies the following conditions:*

i) *There exists $R > 0$ such that* $\mathrm{Supp}\, c(V_\kappa) \subset N_R(e_\kappa)$ *for all $\kappa \in \mathfrak{F}^{\mathrm{cu}}$.*
ii) $\mathrm{Av}_\Gamma(c(\mathfrak{F})) = 0$.

Then the class $[c] \in H^2_{\mathrm{par}}(\Gamma; \mathcal{N}^\omega, \mathcal{N}^{\omega^*,\mathrm{exc}})$ is the zero class.

Proof of Proposition 12.3. For a given cocycle $\psi \in Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{W}^\omega_s, \mathcal{W}^{\omega^*,\mathrm{exc}}_s)$ we suppose that $u_\psi = 0$. We have to show that $[\psi] = 0$ in $H^1_{\mathrm{par}}(\Gamma; \mathcal{W}^\omega_s, \mathcal{W}^{\omega^*,\mathrm{exc}}_s)$.

To obtain u_ψ from ψ we have chosen a lift $\tilde\psi \in C^1(F^{\mathcal{T}}_{\cdot}; \mathcal{G}^\omega_s, \mathcal{G}^{\omega^*,\mathrm{exc}}_s)$ satisfying (12.3). The assumption implies that $\mathrm{Av}_\Gamma(d\tilde\psi(\mathfrak{F})) = \pi i\, u_\psi = 0$. For each $\kappa \in \mathfrak{F}^{\mathrm{cu}}$ we have

$$d\tilde\psi(V_\kappa) = -\tilde\psi(f_\kappa) + \tilde\psi(e_\kappa)|(1 - \pi_\kappa)$$

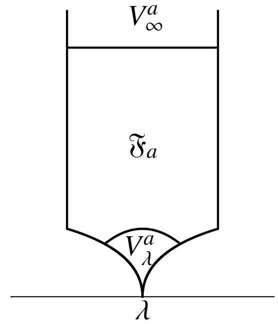
as an identity in $\mathcal{G}^{\omega^*,\mathrm{exc}}_s$. The singularities of $\tilde\psi(f_{\mathcal{K}})$ are contained in a compact subset of $\mathfrak{H}$, and those of $\tilde\psi(e_\kappa)$, and hence also of $\tilde\psi(e_\kappa)|(1 - \pi_\kappa)$, in a set $N_r(e_\kappa)$ as in (12.2). So there is an open neighborhood Ω of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\kappa\}$ in $\mathbb{P}^1_{\mathbb{C}}$ that is an excised neighborhood of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\kappa\}$, for which $d\tilde{\psi}(V_\kappa) \in \mathcal{E}_s(\Omega \cap \mathfrak{H})$. Since the cochain $\tilde{\psi}$ represents the cocycle ψ we have $d\tilde{\psi}(V_\kappa) = 0$ on $\Omega \cap \mathfrak{H}$. In particular $\operatorname{Supp} d\tilde{\psi}(V_\kappa) \subset N_R(e_\kappa)$ if R is sufficiently large. Now we can apply Lemma 12.6 to conclude that the class $[d\tilde{\psi}]$ in $H^2_{\mathrm{par}}(\Gamma; \mathcal{N}^\omega, \mathcal{N}^{\omega^*,\mathrm{exc}})$ is zero.

We use the part

$$H^1_{\mathrm{par}}(\Gamma; \mathcal{G}^\omega_s, \mathcal{G}^{\omega^*,\mathrm{exc}}_s) \longrightarrow H^1_{\mathrm{par}}(\Gamma; \mathcal{W}^\omega_s, \mathcal{W}^{\omega^*,\mathrm{exc}}_s) \xrightarrow{\delta} H^2_{\mathrm{par}}(\Gamma; \mathcal{N}^\omega, \mathcal{N}^{\omega^*,\mathrm{exc}})$$

of the long exact sequence associated to the diagram (9.17) by Proposition 11.9. Since $[d\tilde{\psi}] = \delta[\psi]$ we have $[\psi] = 0$ by Proposition 12.5. ■

Proof of Lemma 12.6. The cocycle c is determined by its values on the basis $\{\mathfrak{F}_Y\} \cup \{V_\kappa \,:\, \kappa \in \mathfrak{F}^{\mathrm{cu}}\}$ of $F^{\mathcal{T}}_2$.



The support of $c(\mathfrak{F}_Y)$ is a compact subset of $\mathfrak{H}$. Condition i) in the lemma ensures that we can find a large $a \geq Y$ such that $\operatorname{Supp} c(\mathfrak{F}_Y) \subset \Gamma\,\mathfrak{F}_a$ and that $\operatorname{Supp} c(V_\kappa) \subset \Gamma(\mathfrak{F}_a \cup V^a_\kappa)$ for all $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. In particular $\operatorname{Supp} c(V_\kappa)$ does not intersect ΓV^a_λ for some $\lambda \neq \kappa$, $\lambda \in \mathfrak{F}^{\mathrm{cu}}$. Condition ii) implies that $\mathrm{Av}_\Gamma(c(V_\kappa)) = 0$ on $\Gamma \mathring{V}^a_\kappa$.

We take $\alpha \in C^\infty_c(\mathbb{R})$ such that $\sum_{n\in\mathbb{Z}} \alpha(x+n) = 1$ for all $x \in \mathbb{R}$, and $\beta \in C^\infty(0,\infty)$ equal to 0 on $(0,a)$ and equal to 1 on $(a+\varepsilon,\infty)$ for some $\varepsilon > 0$. We define for each $\kappa \in \mathfrak{F}^{\mathrm{cu}}$ the function $\chi_\kappa \in \mathcal{N}^{\omega^*,\mathrm{exc}}$ by $\chi_\kappa(g_\kappa z) = \alpha(x)\beta(y)$. Let $\hat{c}$ be given by $\hat{c}(\mathfrak{F}_Y) = c(\mathfrak{F}_Y)$ and for each $\kappa \in \mathfrak{F}^{\mathrm{cu}}$

$$\hat{c}(V_\kappa) \;=\; c(V_\kappa) - \sum_{n\in\mathbb{Z}} (c(V_\kappa)\cdot(\chi_\kappa|\pi^n_\kappa))|(1-\pi^{-n}_\kappa) \;=\; \chi_\kappa \sum_{n\in\mathbb{Z}} c(V_\kappa)|\pi^{-n}_\kappa\,.$$

The cocycles $\hat{c}$ and c are in the same cohomology class, and $\mathrm{Av}_\Gamma(\hat{c}(\mathfrak{F})) = \mathrm{Av}_\Gamma(c(\mathfrak{F}))$. Since the support of $c(V_\kappa)$ is contained in $\Gamma_\kappa(V^a_\kappa \cup \mathfrak{F}_a)$ we have for $z \in \mathring{V}^{a+\varepsilon}_k$

$$\mathrm{Av}_\Gamma(c(V_\kappa))(z) \;=\; \chi_\kappa(z) \sum_{n\in\mathbb{Z}} c(V_\kappa)|\pi^{-n}_\kappa(z) \;=\; \hat{c}(V_\kappa)\,,$$

and on the other hand $\mathrm{Av}_\Gamma(c(V_\kappa))(z) = \mathrm{Av}_\Gamma(c(\mathfrak{F}_Y)) = 0$. Since $\varepsilon > 0$ was arbitrary we conclude that $\hat{c}(V_\kappa)$ has compact support for each $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. Hence $\hat{c}(\mathfrak{F})$ has compact support, and is an element of $\mathcal{N}^\omega$.

The proof of Proposition 7.3 works for $\hat{c}(\mathfrak{F})$, although the support of the function χ constructed there is not compact and intersects infinitely many fundamental domains. Thus we see that $\hat{c}(\mathfrak{F}) \in \sum \mathcal{N}^\omega|(1-\gamma)$. The cocycle $\hat{c}$ is in the same class as the cocycle $\tilde{c}$ given by $\tilde{c}(\mathfrak{F}_Y) = \hat{c}(\mathfrak{F})$ (which is in $\sum_\gamma \mathcal{N}^\omega|(1-\gamma)$) and $\tilde{c}(v_\kappa) = 0$ for $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. The cocycle $\tilde{c}$ is a coboundary. ■

Proof of Proposition 12.5. Let $\psi \in Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{G}^\omega_s, \mathcal{G}^{\omega^*,\mathrm{exc}}_s)$, where $\mathcal{T}$ is a tesselation of type **Fd**. The presence of cusps gives us directly $\psi_{P,Q} = \psi(e_{P,Q}) \in \mathcal{G}^{\omega^*,\mathrm{exc}}_s$ for all $P, Q \in X^{\mathcal{T}}_0$, including the cusps. There is no need of an extension to hyperbolic fixed points as in Lemma 7.5. For $P, Q \in X^{\mathcal{T},Y}_0 = X^{\mathcal{T}}_0 \cap \mathfrak{H}$, we have $\psi_{P,Q} \in \mathcal{G}^\omega_s$.

The cocycle ψ is determined by its values on the $\mathbb{Q}[\Gamma]$-basis $B = E \cup \{f_\kappa\} \cup \{e_\kappa\}$ in (11.4). For interior edges we have $\psi(e), \psi(f_\kappa) \in \mathcal{G}_s^\omega$. Since ψ is a cocycle, we have $\psi(e_\kappa) \,|\, (1-\pi_\kappa) = \psi(f_\kappa)$. This implies that (12.3) holds automatically for the cocycle ψ. Thus, we know that there is a number $R > 0$ such that $\mathsf{Sing}\,\psi(x) \subset N_R(x)$ for all $x \in X_1^{\mathcal{T}}$, with $N_R(\cdot)$ as defined in §12.2.

Let cusps $\xi, \eta \in C$ be given, and let $Z \subset \mathfrak{H}$ be compact. There is a chain $p = \sum_j \varepsilon_j e_j \in \mathbb{Z}[X_1^{\mathcal{T}}]$, representing a path from ξ to η, with $\varepsilon_j \in \{1, -1\}$, $e_j \in X_1^{\mathcal{T}}$ such that $N_R(e_j) \cap Z = \emptyset$ for all j. For $z \in Z$ we have

$$\psi_{\xi,\eta}(z) \;=\; \psi(p)(z) \;=\; \sum_j \varepsilon_j\, \psi(e_j)(z)\,.$$

So $\psi_{\xi,\eta} \in \mathcal{E}_s(\mathring{Z})$. The path can be adapted to any compact $Z \subset \mathfrak{H}$, and hence $\psi_{\xi,\eta} \in \mathcal{E}_s$ for each pair of cusps.

Let $P \in X_0^{\mathcal{T},Y}$. From $\psi_{P,\xi} \,|\, (1-\pi_\xi) = \psi_{P,\pi_\xi^{-1}P} \in \mathcal{G}_s^\omega$ it follows that $\overline{\mathsf{Sing}\,(\psi_{P,\xi})} \cap \partial\mathfrak{H} \subset \{\xi\}$. Let $\eta \in C$, $\eta \neq \xi$. In the relation $\psi_{\xi,\eta} = \psi_{P,\eta} - \psi_{P,\xi}$ singularities near ξ of $\psi_{P,\xi}$ cannot be canceled by singularities of $\psi_{P,\eta}$. So $\mathsf{Sing}\,(\psi_{P,\xi})$ is a compact set in $\mathfrak{H}$.

We apply Lemma 9.23 with $\hat{q} = \psi_{P,\xi} \,|\, g_\xi$. Since $\psi_{P,\xi} \,|\, (1-\pi_\xi) = \psi_{P,\pi_\xi^{-1}P} \in \mathcal{G}_s^\omega$, condition b) is satisfied as well. So there is $p_\xi \in \mathcal{G}_s^{\omega^*,\mathrm{exc}} \cap \mathcal{E}_s^{\pi_\xi}$ such that $\psi_{P,\xi} - p_\xi \in \mathcal{G}_s^\omega$. For $\eta = \gamma^{-1}\xi$, $\gamma \in \Gamma$, we take $p_\eta = p_\xi \,|\, \gamma$. The π_ξ-invariance of p_ξ implies that this does not depend on the choice of γ such that $\eta = \gamma^{-1}\xi$.

Let $h \in C^0(F_\cdot^{\mathcal{T}}; \mathcal{G}_s^\omega, \mathcal{G}_s^{\omega^*,\mathrm{exc}})$ be given by $h(Q) = 0$ if $Q \in X_0^{\mathcal{T}} \cap \mathfrak{H}$, and by $h(\xi) = p_\xi$ for $\xi \in C$. So h takes values in $\mathcal{E}_s$. We go over to the cocycle $\hat{\psi} = \psi - dh$ in the cohomology class of ψ. Then $\hat{\psi}_{\xi,\eta} \in \mathcal{E}_s$ for all $\xi, \eta \in C$, and $\hat{\psi}_{P,\xi} = \psi_{P,\xi} - p_\xi \in \mathcal{G}_s^\omega$. So $\hat{\psi}_{\xi,\eta} = \hat{\psi}_{P,\eta} - \hat{\psi}_{P,\xi} \in \mathcal{E}_s \cap \mathcal{G}_s^\omega = \{0\}$, as follows from (3.3). Taking the base point in C we obtain a group cocycle corresponding to $\hat{\psi}$ that is zero. ■

12.4. *Restriction to subspaces.* The map $\alpha_s^\omega : H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\mathrm{exc}}) \to \mathcal{E}_s^\Gamma$ has been constructed in §12.2 for all s with $0 < \operatorname{Re} s < 1$. Under the additional condition $s \neq \frac{1}{2}$ it is an isomorphism. Now we turn to its restriction to subspaces of $H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\mathrm{exc}})$.

Proposition 12.7. *Let $s \in \mathbb{C}$, $0 < \operatorname{Re} s < 1$. The following linear maps are isomorphisms:*

$$H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\infty}) \xrightarrow{\mathsf{P}_s^\dagger} H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\infty}) \xrightarrow{\alpha_s^\omega} \mathsf{Maass}_s^0(\Gamma)\,. \tag{12.6}$$

Under the additional condition $s \neq \frac{1}{2}$, the following linear maps are isomorphisms as well:

$$H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{simple}}) \xrightarrow{\mathsf{P}_s^\dagger} H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^\omega, \mathcal{W}_s^{\omega^*,\mathrm{simple}}) \xrightarrow{\alpha_s^\omega} \mathsf{Maass}_s^1(\Gamma)\,. \tag{12.7}$$

Proof. We consider two cases: (a) $W = \mathcal{W}_s^{\omega^*,\infty}$, $s = \frac{1}{2}$ allowed; (b) $W = \mathcal{W}_s^{\omega^*,\mathrm{simple}}$, $s \neq \frac{1}{2}$. Consider a group cocycle $\psi \in Z^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^\omega, W)$. Proposition 12.2 implies that $[\psi] = \mathbf{q} u_\psi$.

Consider a cusp $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. Propositions 9.11 and 9.15 imply that there is a unique eigenfunction v_κ such that the class $\mathbf{q} v_\kappa \in H^1(\Gamma_\kappa; \mathcal{W}_s^\omega, W)$ is represented by the restriction of ψ to Γ_κ, *i.e.*, $\psi_{\pi_\kappa} = v_\kappa$. We have $v_\kappa \in (\mathcal{K}_s^0)^{\uparrow}$ in case (a), and $v_\kappa \in \mathcal{K}_s^{\uparrow}$ in case (b). Proposition 5.1 shows that $v_\kappa = u_\psi$. Since this holds for all cusps $\kappa \in \mathfrak{F}^{\mathrm{cu}}$, we conclude that $u_\psi \in \mathsf{Maass}_s^0(\Gamma)$ in case (a), and $u_\psi \in \mathsf{Maass}_s^1(\Gamma)$ in case (b). Proposition 10.3 shows that α_s^ω gives isomorphisms in (12.6) and (12.7). The proof is completed by the fact that the transverse Poisson map gives isomorphisms $\mathcal{V}_s^\omega \to \mathcal{W}_s^\omega$ and $\mathcal{V}_s^{\omega^*,\mathrm{exc}} \to \mathcal{W}_s^{\omega^*,\mathrm{exc}}$. ■

13. Parabolic cohomology and mixed parabolic cohomology. In this section we shall prove the isomorphism

$$H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\infty}) \;\cong\; H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*,\infty}) \tag{13.1a}$$

in Theorem B, and also, under the assumption $s \neq \frac{1}{2}$, the isomorphisms

$$H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{simple}}) \;\cong\; H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*,\mathrm{simple}})\,, \tag{13.1b}$$

$$H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{exc}}) \;\cong\; H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^0,\mathrm{exc}})\,. \tag{13.1c}$$

Together with the previous section, this will complete the proof of Theorem C.

We recall that $\mathcal{V}_s^{\omega^0,\mathrm{exc}}$ consists of the $f \in \mathcal{V}_s^{\omega^*,\mathrm{exc}}$ for which the set of singularities $\mathsf{BdSing}\,(f)$ is contained in the set C of cusps of Γ. In Proposition 13.7 we shall give an example that shows that there are Γ for which we cannot replace $\mathcal{V}_s^{\omega^0,\mathrm{exc}}$ by $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ in (13.1c). The proofs will show that (13.1c) still holds if we replace $\mathcal{V}_s^{\omega^0,\mathrm{exc}}$ by the Γ-module of those $f \in \mathcal{V}_s^{\omega^*,\mathrm{exc}}$ for which $\mathsf{BdSing}\,(f)$ does not contain hyperbolic fixed points.

13.1. *Space of singularities.* The $\mathbb{Q}[G]$-module $\mathcal{S}_s = \mathcal{V}_s^{\omega^*}/\mathcal{V}_s^\omega$ is the space of singularities of semi-analytic vectors in the principal series. For $\xi \in \partial\mathfrak{H}$ we denote by $\mathcal{S}_{s,\xi}$ the subspace represented by elements of $\mathcal{V}_s^\omega[\xi]$. For each $g \in G$ the map $f \mapsto f|g$ induces an isomorphism

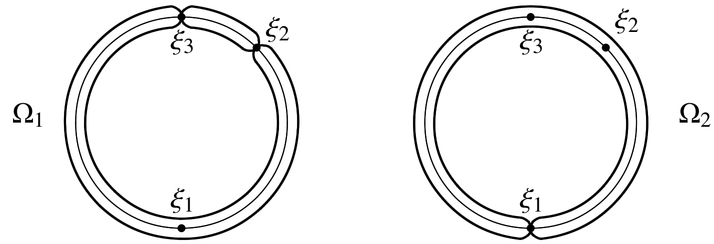
$$\mathcal{S}_{s,\xi} \longrightarrow \mathcal{S}_{s,\xi}|g \;=\; \mathcal{S}_{s,g^{-1}\xi}\,.$$

Clearly, $\mathcal{S}_s$ contains the direct sum of all $\mathcal{S}_{s,\xi}$ with $\xi \in \mathbb{P}^1_{\mathbb{R}} = \partial\mathfrak{H}$. We note that $\mathcal{S}_{s,\xi}$ is a subspace of $\mathcal{S}_s$, not a stalk of a sheaf.

Proposition 13.1. *The space $\mathcal{S}_s$ is equal to* $\bigoplus_{\xi\in\mathbb{P}^1_{\mathbb{R}}} \mathcal{S}_{s,\xi}$.

Proof. Suppose that $f \in \mathcal{V}_s^{\omega^*}$ represents an element of $\mathcal{S}_s$. We write $\mathsf{BdSing}\,(f) = \{\xi_1,\ldots,\xi_n\}$. We identify f (in the circle model) with a holomorphic function $f \in \mathcal{O}(\Omega)$, where $\Omega \subset \mathbb{C}$ is an open set such that $\mathbb{S}^1 \smallsetminus \Omega = \{\xi_1,\ldots,\xi_n\}$. We choose open sets Ω_1 and Ω_2 such that

$$\begin{aligned} &\Omega \;=\; \Omega_1 \cap \Omega_2\,,\\ &\xi_1 \in \Omega_1\,,\ \xi_2,\ldots,\xi_n \notin \Omega_1\,, \qquad \xi_1 \notin \Omega_2\,,\ \xi_2,\ldots,\xi_n \in \Omega_2\,. \end{aligned}$$



A theorem in complex function theory (e.g., [13], Proposition 1.4.5) gives the existence of $f_1 \in \mathcal{O}(\Omega_1)$, $f_2 \in \mathcal{O}(\Omega_2)$, such that $f = f_1 + f_2$ on Ω. (Apply Theorem 1.45 in [13] to the covering $\{\Omega_1, \Omega_2\}$ of $\Omega_1 \cup \Omega_2$, and put $g_{1,2} = -g_{2,1} = f$ on $\Omega_1 \cap \Omega_2$. The theorem gives $g_j \in \mathcal{O}(\Omega_j)$ such that $g_{1,2} = g_1 - g_2$ on $\Omega_1 \cap \Omega_2$.) Thus we have $\mathsf{BdSing}\,(f_1) \subset \{\xi_1\}$ and $\mathsf{BdSing}\,(f_2) \subset \{\xi_2, \ldots, \xi_n\}$. Repeating this construction gives $f + \mathcal{V}_s^\omega$ as an element of $\bigoplus_{\xi \in \mathbb{P}^1_{\mathbb{R}}} \mathcal{S}_{s,\xi}$. ■

• *Exact sequence.* For any $\mathbb{Q}[\Gamma]$-module W with $\mathcal{V}_s^\omega \subset W \subset \mathcal{V}_s^{\omega^*}$, we conclude from (11.17) that the following sequence is exact

$$H^0(\Gamma; W/\mathcal{V}_s^\omega) \longrightarrow H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, W) \longrightarrow H^1_{\mathrm{par}}(\Gamma; W) \longrightarrow H^1(\Gamma; W/\mathcal{V}_s^\omega)\,. \tag{13.2}$$

Since all Γ-orbits in $\partial\mathfrak{H}$ are infinite, we have $(W/\mathcal{V}_s^\omega)^\Gamma \subset \mathcal{S}_s^\Gamma = \{0\}$. Hence the natural map $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, W) \to H^1_{\mathrm{par}}(\Gamma; W)$ is injective. If the image of $H^1_{\mathrm{par}}(\Gamma; W)$ in $H^1(\Gamma; W/\mathcal{V}_s^\omega)$ is zero, then this map is surjective as well.

It seems unnatural that we go from parabolic cohomology to standard cohomology. The following lemma makes this step more explicit in the description of cohomology based on a tesselation $\mathcal{T}$ of type **Fd**, discussed in §11.1.

Recall that $X_1^{\mathcal{T}}$ has three kind of edges: the edges going to a cusp, which are the Γ-translates of finitely many e_κ ($\kappa \in \mathfrak{F}^{\mathrm{cu}}$), the edges in $\partial\mathfrak{H}_Y$, which are the Γ-translates of finitely many $f_\kappa \in X_1^{\mathcal{T},Y}$ ($\kappa \in \mathfrak{F}^{\mathrm{cu}}$), and the interior edges in $X_1^{\mathcal{T},Y}$ of the form $\gamma\, e$ with $\gamma \in \Gamma$, $e \in E$.

Lemma 13.2. *Let $c_1 \in Z^1(F_\cdot^{\mathcal{T}}; W)$. There is a cocycle c in the same cohomology class such that $c(e) = 0$ for all edges e going to a cusp and for all edges contained in $\partial\mathfrak{H}_Y$.*

Proof. For each $\kappa \in \mathfrak{F}^{\mathrm{cu}}$, the edge e_κ goes from a point $P_\kappa \in \partial\mathfrak{H}_Y$ to κ. Define $f \in \mathrm{Map}(X_0^{\mathcal{T}}, W)^\Gamma$ by $f(P_\kappa) = c_1(e_\kappa)$ for all $\kappa \in \mathfrak{F}^{\mathrm{cu}}$, and $f = 0$ on all other Γ-orbits in $X_0^{\mathcal{T}}$. Take $c = c_1 - df$, then $c(e_\kappa) = 0$ and $c(f_\kappa) = c(e_\kappa)\,|\,(1 - \pi_\kappa) = 0$ for all $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. ■

The new cocycle c is effectively a 1-cocycle on $F_\cdot^{\mathcal{T},Y}$, and this resolution computes the standard group cohomology, as we have discussed in §11.2. Actually, the condition that $c(f_\kappa) = 0$ for all $\kappa \in \mathfrak{F}^{\mathrm{cu}}$ for some c in a cohomology class can be used to characterize $H^1_{\mathrm{par}}(\Gamma; W)$ inside $H^1(\Gamma; W)$.

Definition 13.3. We call a $\mathbb{Q}[\Gamma]$-module W such that $\mathcal{V}_s^\omega \subset W \subset \mathcal{V}_s^{\omega^*}$ *locally defined* if the image $\mathcal{S}_s^W$ of W in $\mathcal{S}_s$ has the form

$$\mathcal{S}_s^W = \bigoplus_{\xi \in \mathbb{P}^1_{\mathbb{R}}} \mathcal{S}_{s,\xi}^W\,,$$

where $\mathcal{S}_{s,\xi}^W = \mathcal{S}_s^W \cap \mathcal{S}_{s,\xi}$.

Since W is a $\mathbb{Q}[\Gamma]$-module, the local summands satisfy $\mathcal{S}^W_{s,\xi}|\gamma = \mathcal{S}^W_{s,\gamma^{-1}\xi}$ for all $\xi \in \mathbb{P}^1_{\mathbb{R}}$. The subspaces $\mathcal{V}_s^{\omega^*,\infty}$, $\mathcal{V}_s^{\omega^*,\text{simple}}$, $\mathcal{V}_s^{\omega^0,\text{exc}}$ and $\mathcal{V}_s^{\omega^*,\text{exc}}$ of $\mathcal{V}_s^{\omega^*}$ are all locally defined.

If W is a locally defined $\mathbb{Q}[\Gamma]$-module between $\mathcal{V}_s^{\omega}$ and $\mathcal{V}_s^{\omega^*}$, we have

$$H^1(\Gamma; \mathcal{S}_s^W) \cong \bigoplus_{x \in \Gamma\backslash\mathbb{P}^1_{\mathbb{R}}} H^1(\Gamma; \mathcal{S}_s^W(x)), \tag{13.3}$$

where $\mathcal{S}_s^W(x)$ is the $\mathbb{Q}[\Gamma]$-module $\bigoplus_{\xi\in x} \mathcal{S}^W_{s,\xi}$. So for the bijectivity of the natural map $H^1_{\text{par}}(\Gamma; \mathcal{V}_s^{\omega}, W) \to H^1_{\text{par}}(\Gamma; W)$ it suffices to show that the image of

$$H^1_{\text{par}}(\Gamma; W) \longrightarrow H^1(\Gamma; \mathcal{S}_s^W(x))$$

is zero for all Γ-orbits $x \in \Gamma\backslash\mathbb{P}^1_{\mathbb{R}}$.

The main result of this subsection is the following proposition:

Proposition 13.4. *Let W be a locally defined $\mathbb{Q}[\Gamma]$-module between $\mathcal{V}_s^{\omega}$ and $\mathcal{V}_s^{\omega^*}$. Let $\xi_0 \in \mathbb{P}^1_{\mathbb{R}}$. If ξ_0 is a hyperbolic fixed point fixed by $\eta \in \Gamma$, $\eta \neq 1$, we assume that the map*

$$1 - \eta : \mathcal{S}^W_{s,\xi_0} \longrightarrow \mathcal{S}^W_{s,\xi_0} \tag{13.4}$$

is injective. Then the image of $H^1_{\text{par}}(\Gamma; W)$ in $H^1(\Gamma; \mathcal{S}_s^W(\Gamma\xi_0))$ vanishes.

This implies that $H^1_{\text{par}}(\Gamma; \mathcal{V}_s^{\omega}, W) \cong H^1_{\text{par}}(\Gamma; W)$ for all locally defined W between $\mathcal{V}_s^{\omega}$ and $\mathcal{V}_s^{\omega^*}$ for which the map in (13.4) is injective for all hyperbolic fixed points of Γ.

Proof. The proof is long. Starting from $c \in Z^1(F^{\mathcal{T},Y}_{\cdot}; \mathcal{S}_s^W(\Gamma\xi_0))$ representing a class in the image of $H^1_{\text{par}}(\Gamma; W)$, we will show that c is a coboundary, separating the cases where Γ_{ξ_0} is trivial, hyperbolic or parabolic.

We can assume by Lemma 13.2 that $c(f_\kappa) = 0$ for each $\kappa \in \mathfrak{F}^{\text{cu}}$, and hence $c(f) = 0$ for any edge $f \in X_1^{\mathcal{T},Y}$ with support contained in $\partial\mathfrak{H}_Y$.

Let $\xi_0 \in \mathbb{P}^1_{\mathbb{R}}$. For $h \in \mathcal{S}_s^W(\Gamma\xi_0)$ and $\xi \in \Gamma\xi_0$ we denote by h_ξ the component of h in the summand $\mathcal{S}^W_{s,\xi}$. We have

$$(h|\gamma)_\xi \;=\; h_{\gamma\xi}|\gamma\,. \tag{13.5}$$

We put for $\xi \in \Gamma\xi_0$:

$$D(\xi) \;=\; \{e \in X_1^{\mathcal{T},Y} \;:\; c(e)_\xi \neq 0\}\,. \tag{13.6}$$

Lemma 13.5. *For each $\xi \in \Gamma\xi_0$ the set $D(\xi)$ consists of finitely many Γ_ξ-orbits.*

Proof. From (13.5):

$$\gamma e \in D(\xi) \Longleftrightarrow c(\gamma e)_\xi = c(e)_{\gamma^{-1}\xi}|\gamma^{-1} \neq 0 \Longleftrightarrow c(e)_{\gamma^{-1}\xi} \neq 0 \Longleftrightarrow e \in D(\gamma^{-1}\xi)\,.$$

This implies that $\Gamma_\xi D(\xi) = D(\xi)$.

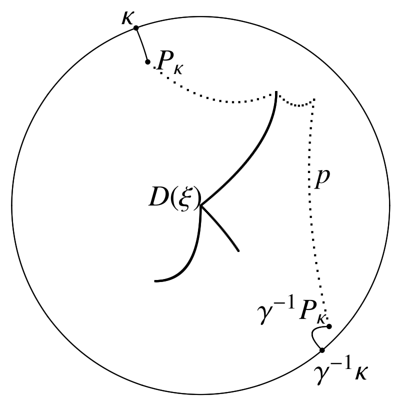
For each interior edge $e \in X_1^{\mathcal{T},Y}$, the set $\{\xi \in \Gamma\xi_0 \;:\; c(e)_\xi \neq 0\}$ is finite, since $\mathsf{BdSing}\,(f)$ is finite for each $f \in \mathcal{V}_s^{\omega^*}$. We use that $X_1^{\mathcal{T},Y} = \bigsqcup_{e\in E} \Gamma\, e \sqcup \bigsqcup_{\kappa\in\mathfrak{F}^{\mathrm{c}}} \Gamma\, f_\kappa$, where $E \cup \{f_\kappa\}$ is the finite $\mathbb{Q}[\Gamma]$-basis of $F_1^{\mathcal{T},Y}$ mentioned in §11.1. Since $c(f_\kappa) = 0$, we have

$$D(\xi) \;=\; \bigcup_{e\in E} \{\gamma e \;:\; \gamma \in \Gamma\,,\; c(e)_{\gamma^{-1}\xi} \neq 0\}\,,$$

which consists of finitely many Γ_ξ-orbits. ∎

- *Case* $\Gamma_{\xi_0} = \{1\}$.

Let $\kappa \in \mathfrak{F}^{\rm cu}$, and $\gamma \in \Gamma$. If $\gamma \in \Gamma_\kappa$, then $c(\gamma^{-1}P_\kappa, P_\kappa) = 0$, since there is a path from $\gamma^{-1}P_\kappa$ to P_κ along the horocycle of P_κ. If $\gamma \notin \Gamma_\kappa$, then $\gamma^{-1}P_\kappa$ is on another horocycle. For each $\xi \in \Gamma\xi_0$ the set $D(\xi)$ is finite. Hence there is a path p from $\gamma^{-1}P_\kappa$ to P_κ along edges of $X_1^{\mathcal{T},Y}$ none of which is in $D(\xi)$. (It does not matter if the path goes through an end point of an edge in $D(\xi)$.) Hence $c(\gamma^{-1}P_\kappa, P_\kappa)_\xi = c(p)_\xi = 0$. This holds for all $\xi \in \Gamma\xi_0$, so $c(\gamma^{-1}P_\kappa, P_\kappa) = 0$.

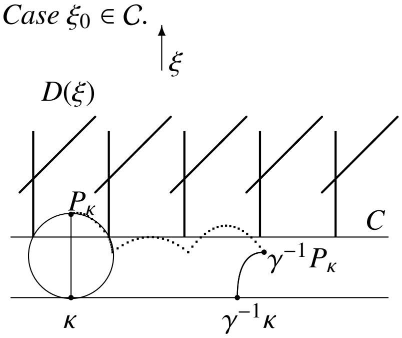


Take P_κ as the base point for the transition to group cocycles. This leads to the cocycle ψ satisfying

$$\psi_\gamma \;=\; c(\gamma^{-1}P_\kappa, P_\kappa) = 0 \qquad (\gamma \in \Gamma)\,.$$

This shows that c represents the trivial cohomology class.

- *Case* $\xi_0 \in C$.



We take $\kappa = \xi_0$, and use P_κ as the base point. As before, we have $c(\gamma^{-1}P_\kappa, P_\kappa) = 0$ if $\gamma \in \Gamma_\kappa$. Let $\gamma \in \Gamma \setminus \Gamma_\kappa$. For each $\xi \in \Gamma\kappa$ the edges in the Γ_ξ-invariant set $D(\xi)$ are all contained in a horocyclic disk C at ξ. If $\xi \neq \kappa$ and $\xi \neq \gamma^{-1}\kappa$, a path from $\gamma^{-1}P_\kappa$ to P_κ along edges of $X_1^{\mathcal{T},Y}$ may be forced to pass through C, but if so, the pieces of the path that are inside C can be chosen along edges in $\partial\mathfrak{H}_Y$, on which c vanishes. Thus we conclude that $c(\gamma^{-1}P_\kappa, P_\kappa)_\xi = 0$ for $\xi \notin \{\kappa, \gamma^{-1}\kappa\}$.

The group cocycle $\psi_\gamma = c(\gamma^{-1}P_\kappa, P_\kappa)$ is of the form

$$\psi_\gamma \;=\; (\psi_\gamma)_{\gamma^{-1}\kappa} + (\psi_\gamma)_\kappa \in \mathcal{S}^W_{s,\gamma^{-1}\kappa} \oplus \mathcal{S}^W_{s,\kappa}\,. \tag{13.7}$$

Let $\gamma, \delta \in \Gamma$ such that $\delta \notin \Gamma_\kappa$ and $\gamma\delta \notin \Gamma_\kappa$. Writing out the cocycle relation $\psi_{\gamma\delta} = \psi_\gamma\,|\,\delta + \psi_\delta$, we find for the components in $\mathcal{S}^W_{s,\kappa}$:

$$(\psi_{\gamma\delta})_\kappa \;=\; (\psi_\delta)_\kappa\,. \tag{13.8}$$

This implies that there is $b_\kappa \in \mathcal{S}^W_{s,\kappa}$ such that $(\psi_\gamma)_\kappa = b_\kappa$ for all $\gamma \in \Gamma \setminus \Gamma_\kappa$. For such γ:

$$(\psi_{\gamma^{-1}})_{\gamma\kappa} + b_\kappa \;=\; \psi_{\gamma^{-1}} \;=\; -\psi_\gamma\,|\,\gamma^{-1} \;=\; -(\psi_\gamma)_{\gamma^{-1}\kappa}\,|\,\gamma^{-1} - b_\kappa\,|\,\gamma^{-1}\,.$$

Hence

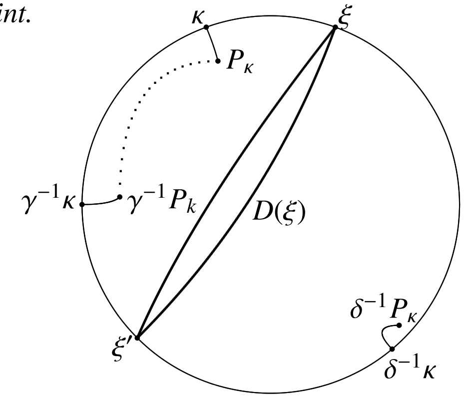
$$(\psi_\gamma)_{\gamma^{-1}\kappa} \;=\; -(\psi_{\gamma^{-1}})_{\gamma\kappa}\,|\,\gamma - b_\kappa\,|\,\gamma - b_\kappa \overset{(13.5)}{=} -(\psi_{\gamma^{-1}}\,|\,\gamma)_\kappa - b_\kappa\,|\,\gamma - (\psi_\gamma)_\kappa \;=\; -b_\kappa\,|\,\gamma\,.$$

We arrive at

$$\psi_\gamma = \begin{cases} 0 & \text{if } \gamma \in \Gamma_\kappa\,, \\ b_\kappa \,|\, (1-\gamma) & \text{if } \gamma \in \Gamma \smallsetminus \Gamma_\kappa\,. \end{cases} \tag{13.9}$$

For $\delta \in \Gamma_\kappa$, $\gamma \in \Gamma \smallsetminus \Gamma_\kappa$, the cocycle relation implies $\psi_{\delta\gamma} = \psi_\gamma$. Hence $b_\kappa \,|\, (1-\delta\gamma) = b_\kappa \,|\, (1-\gamma)$, so $b_\kappa \in (\mathcal{S}^W_{s,\kappa})^{\Gamma_\kappa}$. Thus we have $\psi_\gamma = b_\kappa \,|\, (1-\gamma)$ for all $\gamma \in \Gamma$, and c represents the trivial cohomology class.

- *Case where ξ_0 is a hyperbolic fixed point.*



We now suppose that ξ_0 is fixed by a hyperbolic $\eta \in \Gamma$. We fix $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. Proceeding as in the previous cases we find $c(\gamma^{-1}P_\kappa, P_\kappa) = 0$ if $\gamma \in \Gamma_\kappa$.

Let $\gamma \in \Gamma \smallsetminus \Gamma_\kappa$ and consider $\xi \in \Gamma\xi_0$. Let $\eta \in \Gamma$ be a generator of Γ_ξ; say ξ is the attracting fixed point $\omega(\eta)$ of η. Then η also generates $\Gamma_{\xi'}$, where $\xi' = \alpha(\eta)$ is the repelling fixed point.

The set $D(\xi)$ consists of finitely many Γ_ξ-orbits. If κ and $\gamma^{-1}\kappa$ are in the same cyclic interval $(\xi,\xi')_c \subset \mathbb{P}^1_\mathbb{R}$ or $(\xi',\xi)_c \subset \mathbb{P}^1_\mathbb{R}$, then we can find a path from $\gamma^{-1}P_\kappa$ to P_κ not containing edges in $D(\xi)$, and hence $c(\gamma^{-1}P_\kappa, P_\kappa)_\xi = 0$. If $\delta^{-1}\kappa$ and κ are separated by ξ and ξ' in $\mathbb{P}^1_\mathbb{R}$, then $D(\xi)$ may form a barrier between $\delta^{-1}P_\kappa$ and P_κ. But now we have $c(P_\kappa, \eta^{-1}P_\kappa)_\xi = 0$ and $c(\delta^{-1}P_\kappa, \eta^{-1}\delta^{-1}P_\kappa)_\xi = 0$, since κ and $\eta^{-1}\kappa$ are not separated by ξ and ξ', and similarly for $\delta^{-1}\kappa$ and $\eta^{-1}\delta^{-1}\kappa$. Thus we have

$$\big(c(\delta^{-1}P_\kappa, P_\kappa) \,|\, (1-\eta)\big)_\xi = c(\delta^{-1}P_\kappa, \eta^{-1}\delta^{-1}P_\kappa)_\xi - c(P_\kappa, \eta^{-1}P_\kappa)_\xi = 0\,.$$

The injectivity of the map in (13.4) implies by conjugation that $1-\eta$ is injective on $\mathcal{S}^W_{s,\xi}$, and hence $c(\delta^{-1}P_\kappa, P_\kappa)_\xi = 0$. We proceed as in the case $\Gamma_{\xi_0} = \{1\}$.

This completes the proof of Proposition 13.4. ■

Proposition 13.6. *For the spaces* $\mathcal{V}_s^{\omega^*,\infty}$, $\mathcal{V}_s^{\omega^*,\mathrm{simple}}$ *and* $\mathcal{V}_s^{\omega^0,\mathrm{exc}}$ *the map* (13.4) *in Proposition 13.4 is injective for all hyperbolic fixed points.*

Proof. Suppose after conjugation that $\eta = \left[\begin{smallmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{smallmatrix}\right]$ with $0 < t < 1$, fixing 0. If $f \in \mathcal{V}_s^{\omega^*,\infty}$ then f has an asymptotic behavior $f(x) \sim \sum_{n=0}^\infty a_n\,x^n$ as $x \to 0$ for some $a_n \in \mathbb{C}$. Then $f \,|\, (1-\eta)\,(x) = f(x) - t^s f(tx) \sim \sum_{n=0}^\infty (1-t^{n+s})\,a_n\,x^n$. If $f \,|\, (1-\eta) \in \mathcal{V}_s^\omega$, then $(1-t^{n+s})\,a_n = \mathrm{O}(C^n)$ for some $C > 0$, and consequently a_n is also $\mathrm{O}(C^n)$, since $1-t^{n+s}$ tends to 1 for n large, so $f \in \mathcal{V}_s^\omega$. This shows that $\mathcal{V}_s^{\omega^*,\infty}$ satisfies the condition.

If $f \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$, then $f(x) \sim \sum_{n=-1}^\infty a_n x^n$. Now the assumption that $f \,|\, (1-\eta)$ is analytic at $x = 0$ implies $(a - t^{s-1})a_{-1} = 0$, and hence $a_{-1} = 0$ since $\operatorname{Re} s \neq 1$. So f is analytic at 0 as before.

Finally, of course, the condition is vacuous for $\mathcal{V}_s^{\omega^0,\mathrm{exc}}$. ■

For the space $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ the map in (13.4) is not injective. This one sees by considering $\eta = \left[\begin{smallmatrix} t & 0 \\ 0 & 1/t \end{smallmatrix}\right]$ with $t > 0$. The function given by

$$\varphi(z) = z^{-s} \quad \text{for } \operatorname{Re} z > 0\,, \qquad \varphi(z) = 0 \quad \text{for } \operatorname{Re} z < 0 \tag{13.10}$$

determines a non-zero η-invariant element of $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$ (line model), with singularities 0 and ∞. However, of course, the subspace of those $f \in \mathcal{V}_s^{\omega^*,\mathrm{exc}}$ for which $\mathsf{BdSing}\,(f)$ does not contain hyperbolic fixed points also satisfies the conditions of Proposition 13.4.

• *Counterexample.* To show that the injectivity of the map in (13.4) is necessary, we give a counterexample, based on the *commutator subgroup* $\Gamma_c = [\Gamma_1, \Gamma_1]$ of the modular group. It is a subgroup of Γ_1 of index 6. It is free on the hyperbolic generators $C = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $D = \left[\begin{smallmatrix} 2 & -1 \\ -1 & 1 \end{smallmatrix}\right]$. It has one cuspidal orbit $\mathbb{P}^1_{\mathbb{Q}}$, and $(\Gamma_c)_\infty$ is generated by $T^6 = CDC^{-1}D^{-1}$. See [15], Chap. XI, §3E, on p. 362.

Proposition 13.7. *Denote* $\phi = \frac{1+\sqrt{5}}{2}$. *Then* $H^1_{\mathrm{par}}(\Gamma_c; \mathcal{V}_s^{\omega^*,\mathrm{exc}})$ *has non-zero image in* $H^1(\Gamma_c; \mathcal{S}_s^W(\Gamma_c\phi))$, *where* W *denotes* $\mathcal{V}_s^{\omega^*,\mathrm{exc}}$.

Proof. The element D is conjugate to $\eta = \left[\begin{smallmatrix} \phi^2 & 0 \\ 0 & \phi^{-2} \end{smallmatrix}\right]$, and fixes $-\phi$ and ϕ^{-1}. The example in (13.10) shows that we can find $f \in W^D$ with $\mathsf{BdSing}\,(f) = \{-\phi, \phi^{-1}\}$.

Define $\psi \in Z^1(\Gamma_c; W)$ in the group model of cohomology by $\psi_C = f$ and $\psi_D = 0$. This determines a parabolic cocycle:

$$\psi_{T^6} = \left(\psi_C|_{2s}(D-1) + \psi_D|_{2s}(1-C)\right)\Big|_{2s} C^{-1}D^{-1} = f|_{2s}(D-1)C^{-1}D^{-1} + 0 = 0\,.$$

Suppose that the image of $[\psi]$ in $H^1(\Gamma_c; \mathcal{S}_s^W(\Gamma\phi)$ is zero. Then the image of ψ is of the form dF with $F \in \mathcal{S}_s^W(\Gamma\phi)$. From $F\,|\,(D-1) = 0$ it follows that $F = F_{-\phi} + F_{\phi^{-1}} \in \mathcal{S}^W_{s,-\phi} \oplus \mathcal{S}^W_{s,\phi^{-1}}$, since $-\phi$ and ϕ^{-1} are the fixed points of D. Further, $f + \mathcal{V}_s^\omega = F\,|\,(C-1) = (F\,|\,C)_{C^{-1}(-\phi)} + (F\,|\,C)_{C^{-1}\phi^{-1}} - F_{-\phi} - F_{\phi^{-1}}$. The points $C^{-1}(-\phi)$, $C^{-1}\phi^{-1}$, $-\phi$ and ϕ^{-1} are all different. Since $\mathsf{BdSing}\,(f) = \{-\phi, \phi^{-1}\}$, we conclude that $0 = (F\,|\,C)_{C^{-1}(-\phi)} = F_{-\phi}\,|\,C$ and $0 = (F\,|\,C)_{C^{-1}\phi^{-1}} = F_{\phi^{-1}}\,|\,C$. Hence $F = F_{-\phi} + F_{\phi^{-1}}$ vanishes, a contradiction. ∎

13.2. *Recapitulation of the proof of Theorem C.* Let $s \neq \frac{1}{2}$. The injective map

$$\mathbf{r} : \mathcal{E}_s^\Gamma \longrightarrow H^1(\Gamma; \mathcal{V}_s^\omega)$$

in Proposition 5.1 has image in $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^0,\mathrm{exc}}) \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{exc}})$, according to Proposition 10.3. Proposition 12.4 shows that it is an isomorphism. The space $\mathcal{V}_s^{\omega^0,\mathrm{exc}}$ is locally defined and satisfies the conditions of Proposition 13.4 according to Proposition 13.6. So $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^0,\mathrm{exc}}) \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^0,\mathrm{exc}})$ by Proposition 13.4.

14. Period functions and periodlike functions for the full modular group. We return to the modular group $\Gamma_1 = \mathrm{PSL}_2(\mathbb{Z})$, which was the sole discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ considered in the earlier paper [19]. We give a cohomological interpretation of the period functions and the periodlike functions considered there. We show that the cohomology group $H^1(\Gamma_1; \mathcal{V}_s^\omega)$ is larger than the image $\mathbf{r}\mathcal{E}_s^{\Gamma_1}$, and end by describing briefly the generalization of Γ-invariant eigenfunctions corresponding to the classes in this larger cohomology group: the quantum Maass forms.

14.1. *Periodlike functions and cocycles.* The space $\mathsf{FE}_s(\mathbb{C}')$ of *periodlike functions* on $\mathbb{C}' = \mathbb{C} \smallsetminus (-\infty, 0]$ is defined, in Chap. III of [19], as the space of functions $\psi : \mathbb{C}' \to \mathbb{C}$ that satisfy the three term equation on $\mathbb{C}'$:

$$\psi(\tau) \;=\; \psi(\tau+1) + (\tau+1)^{-2s}\,\psi\Bigl(\frac{\tau}{\tau+1}\Bigr). \tag{14.1}$$

The subspace of holomorphic functions in $\mathsf{FE}_s(\mathbb{C}')$ is denoted $\mathsf{FE}_s(\mathbb{C}')_\omega$. Similarly, the space of functions on $\mathbb{R}_+ = (0,\infty)$ that satisfy (14.1) is denoted $\mathsf{FE}_s(0,\infty)$, with subspaces $\mathsf{FE}_s(0,\infty)_\omega \subset \mathsf{FE}_s(0,\infty)_\infty \subset \mathsf{FE}_s(0,\infty)_p$ of real analytic, respectively smooth, respectively p times differentiable periodlike functions.

The main theorem in [19] shows that $\mathsf{Maass}^0_s(\Gamma_1)$ is isomorphic to the space of *period functions* $\mathsf{FE}_s(\mathbb{C}')^0_\omega$, characterized inside $\mathsf{FE}_s(\mathbb{C}')_\omega$ by the growth conditions

$$\psi(x) \;=\; \begin{cases} \mathrm{O}(1) & (x \downarrow 0)\,,\\ \mathrm{O}(x^{-2s}) & (x \to \infty)\,. \end{cases} \tag{14.2}$$

These growth conditions also define $\mathsf{FE}_s(0,\infty)^0_\infty$ inside $\mathsf{FE}_s(0,\infty)_\infty$, and $\mathsf{FE}_s(0,\infty)^0_p$ inside $\mathsf{FE}_s(0,\infty)_p$. (These last notations and the next are not used in [19].) In Section 3, Chap. IV of [19] a discussion of eigenfunctions of the transfer operator leads to the less strict condition that there are $c_0, c_\infty \in \mathbb{C}$ such that

$$\psi(x) \;=\; \begin{cases} c_0\, x^{-1} + \mathrm{O}(1) & (x \downarrow 0)\,,\\ c_\infty\, x^{1-2s} + \mathrm{O}(x^{-2s}) & (x \to \infty)\,. \end{cases} \tag{14.3}$$

We use this condition to define $\mathsf{FE}_s(0,\infty)^1_\omega \subset \mathsf{FE}_s(0,\infty)_\omega$. In Section 3, Chapter III of [19] we see that general elements of $\mathsf{FE}_s(\mathbb{R}_+)_\infty$ have in their asymptotic behavior at ∞ an additional term $Q_\infty(x)$, and at 0 a term $x^{-2s}\,Q_0(1/x)$, with periodic functions Q_∞ and Q_0. So being in $\mathsf{FE}_s(\mathbb{R}_+)^1_\omega$ is a strong condition, almost as strong as being in $\mathsf{FE}_s(\mathbb{R}_+)^0_\omega$.

Periodlike functions are related to cocycles. Suppose that $\psi \in \mathsf{FE}_s(\mathbb{R}_+)_\omega$. Define ψ on $(-\infty, 0)$ by $\psi(x) = -|x|^{-2s}\,\psi(-1/x)$. One verifies that

$$\psi(x) \;=\; \psi(x+1) + |x+1|^{-2s}\,\psi\Bigl(\frac{x}{x+1}\Bigr) \tag{14.4}$$

for all $x \in \mathbb{R} \smallsetminus \{-1, 0\}$ by separately considering the cases $-1 < x < 0$ and $x < -1$. This extended function ψ satisfies in $\mathcal{V}^{\omega^*}_s$

$$\psi|_{2s}\, S \;=\; -\psi_S\,, \qquad \psi \;=\; \psi_{2s}|\,(T + T')\,, \tag{14.5}$$

with $T' = \left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$. These relations are equivalent to the parabolic cocycle relations

$$\psi|_{2s}\,(1+S) \;=\; 0\,, \qquad \psi|_{2s}\,(1 + ST + STST) \;=\; 0\,. \tag{14.6}$$

Hence $\psi \in \mathsf{FE}_s(\mathbb{R}_+)_\omega$ determines a parabolic cocycle $c \in Z^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}^{\omega^*}_s)$ given on the generators S and T of Γ_1 by $c_T = 0$ and $c_S = \psi$. Conversely, a parabolic cocycle $c \in Z^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}^{\omega^*}_s)$ such that $c_T = 0$ and such that the set of singularities $\mathsf{BdSing}\,(c_S)$ is contained in $\{0, \infty\}$ determines a periodlike function in $\mathsf{FE}_s(\mathbb{R}_+)_\omega$ by restriction to $(0,\infty)$.

Proposition 14.1. *The cohomology group $H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}^{\omega^0,\mathrm{exc}}_s)$ is isomorphic to the space* $\mathsf{FE}_s(\mathbb{C}')_\omega / \{h|_{2s}\,(1-S)\;:\; h \in O(\mathbb{C})^T\}$.

Proof. For a given $\psi \in \mathsf{FE}_s(\mathbb{C}')_\omega$ we define $c \in Z^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{\omega^0,\mathrm{exc}})$ by

$$c_T = 0, \qquad c_S(\tau) = \begin{cases} \psi(\tau) & \text{if } \operatorname{Re}\tau > 0, \\ -(-\tau)^{-2s}\,\psi(-1/\tau) & \text{if } \operatorname{Re}\tau < 0. \end{cases} \tag{14.7}$$

If $\psi = h|_{2s}(1-S)$ with $h \in \mathcal{O}(\mathbb{C})^T$, then $c_S = h|_{2s}(1-S)$. If $c = df$, $f \in \mathcal{V}_s^{\omega^*,\mathrm{exc}}$, then $c_S = f\,|\,(S-1)$, and from $c_T = f\,|\,(T-1) = 0$ we conclude that $f\,|\,T = f$, first on an excised neighborhood of $\mathbb{R} = \mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\infty\}$ and then on $\mathbb{C}$. Thus, we obtain a map from $\mathsf{FE}_s(\mathbb{C}')_\omega$ to $H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{\omega^0,\mathrm{exc}})$ with kernel $\{h\,|\,(1-S) \,:\, h \in \mathcal{O}(\mathbb{C})^T\}$.

Conversely, a given cohomology class in $H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{\omega^0,\mathrm{exc}})$ can be represented by a cocycle such that $c_T = 0$. In view of Theorem C there is $\eta \in \mathcal{V}_s^{\omega^0,\mathrm{exc}}$ such that the equivalent cocycle $c - d\eta$ has values in $\mathcal{V}_s^\omega$. From $\eta\,|\,(1-T) \in \mathcal{V}_s^\omega$ it follows that $\mathsf{BdSing}\,(\eta) \subset \{\infty\}$ and $\mathsf{BdSing}\,(c_S) \subset \{0,\infty\}$.

Restriction of c_S to $(0,\infty)$ gives $\psi \in \mathsf{FE}_s(\mathbb{R}_+)_\omega$. Moreover, ψ has a holomorphic extension to a right half-plane. The second step in the bootstrap procedure in §4, Chap. III of [19] can be applied, to see that ψ extends to $\mathbb{C}'$ and provides an element of $\mathsf{FE}_s(\mathbb{C}')_\omega$. ■

Proposition 14.2.

$$H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{\omega^*,\infty}) \cong \mathsf{FE}_s(\mathbb{R}_+)^0_\omega, \tag{14.8a}$$

$$H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{\omega^*,\mathrm{simple}}) \cong \mathsf{FE}_s(\mathbb{R}_+)^1_\omega \qquad (s \neq \tfrac{1}{2}), \tag{14.8b}$$

$$H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{\infty}) \cong \mathsf{FE}_s(\mathbb{R}_+)^0_\infty, \tag{14.8c}$$

$$H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{p}) \cong \mathsf{FE}_s(\mathbb{R}_+)^0_p \qquad (p \in \mathbb{N},\ p \geq 2). \tag{14.8d}$$

Proof. Each cohomology class in one of these four cohomology groups contains a unique cocycle such that $c_T = 0$. The uniqueness follows from Propositions 4.5 and 9.13. In cases (14.8a) and (14.8b) we conclude that $\mathsf{BdSing}\,(c_S) \subset \{0,\infty\}$ in the same way as in the proof of Proposition 14.1. Restriction of c_S to $(0,\infty)$ gives us $\psi \in \mathsf{FE}_s(\mathbb{R}_+)_\omega$ in cases (14.8a) and (14.8b), $\psi \in \mathsf{FE}_s(\mathbb{R}_+)_\infty$ in case (14.8c), and $\psi \in \mathsf{FE}_s(\mathbb{R}_+)_p$ in case (14.8d). In cases (14.8a) and (14.8c), the fact that $c_S \in \mathcal{V}_s^\infty$ implies that ψ satisfies (14.2). In case (14.8b), we get (14.3) from Definition 9.12.

Conversely, starting from a periodlike function on $(0,\infty)$, we construct a corresponding cocycle in each of the four cases. The hardest point is the behavior at 0 and ∞. We have the estimate (14.2) or (14.3), and want to derive the appropriate asymptotic behavior. We use averaging operators similar to the transfer operator discussed in §3 of Chap IV in [19]. From the three term relation $c_S|_{2s}(1-T) = c_S|_{2s}T'$ on $\mathbb{R} \smallsetminus \{-1,0\}$ we conclude that $c_S = c_S|_{2s}T'|_{2s}\mathrm{Av}_T^+$ on $(0,\infty)$ and $c_S = c_S|_{2s}T'|_{2s}\mathrm{Av}_T^-$ on $(-\infty,-1)$. The asymptotic formula (4.11) implies that c_S has the desired behavior near ∞ in each of the three cases, and also near 0 since $c_S|_{2s}S = -c_S$. ■

14.2. *Reconstruction.* A periodlike function $\psi \in \mathsf{FE}_s(\mathbb{C}')_\omega$ determines a cohomology class in $H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{\omega^0,\mathrm{exc}})$ (Proposition 14.1), which in turn determines an invariant eigenfunction $u \in \mathcal{E}_s^{\Gamma_1}$, provided $s \neq \frac{1}{2}$ (Theorem C). We want to construct u directly from the periodlike function ψ.

For a period function $\psi \in \mathsf{FE}_s(\mathbb{C}')^0_\omega$ we need not use cohomology. Proposition 2 in §2, Chap. I of [19] shows how to associate to ψ a function $f \in \mathcal{O}(\mathbb{C} \smallsetminus \mathbb{R})^T$ which in turn determines ψ. The Fourier expansion of f gives the Fourier coefficients $B_n(u)$ of u, hence determines u explicitly. This also works if $\psi \in \mathsf{FE}_s(\mathbb{C})^1_\omega$. For general periodlike functions ψ there still is a holomorphic 1-periodic function f, and its Fourier coefficients still give the $B_n(u)$, but the coefficients $A_n(u)$ cannot be read off from it directly. (They are hidden in the behavior of f near points of $\mathbb{Q}$.) In this case we will use the theory developed in these notes instead.

To a given $\psi \in \mathsf{FE}_s(\mathbb{C}')_\omega$ we have associated in the proof of Proposition 14.1 an explicit cocycle $c \in Z^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{\omega^0,\mathrm{exc}})$ by (14.7). To apply the method in §12 directly, we have to find $h \in \mathcal{V}_s^{\omega,\mathrm{exc}}[\infty]$ such that

$$c - dh \in Z^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^0,\mathrm{exc}}) \subset Z^1(\Gamma_1; \mathcal{V}_s^{\omega}) .$$

The existence of such a function h follows from (13.1c). The proof of Proposition 13.1 shows that the construction of a suitable h is not explicit. It seems better to apply the method in §12 directly to the cocycle $\hat{\psi} \in H^1_{\mathrm{par}}(\Gamma_1; \mathcal{W}_s^{\omega^0,\mathrm{exc}})$ given by $\hat{\psi}_\gamma = \mathsf{P}_s^\dagger c_\gamma$. So $\hat{\psi}_T = 0$ and $\hat{\psi}_S = \mathsf{P}_s^\dagger c_S \in \mathcal{W}_s^{\omega^0,\mathrm{exc}} \cap \mathcal{W}_s^{\omega}[0,\infty]$.

The first step is to determine a cocycle corresponding to $\hat{\psi}$ in a model of cohomology based on a tesselation. We use the fundamental domain

$$\mathfrak{F}_1 \;=\; \{z \in \mathfrak{H} \;:\; 0 \le \mathrm{Re}\, z \le 1\,,\ |z| \ge 1\,,\ |z-1| \ge 1\}$$

in Figure 6, which differs from the standard fundamental domain in Figure 4 on p. 67. Here we work with parabolic cohomology, not with mixed parabolic cohomology. So

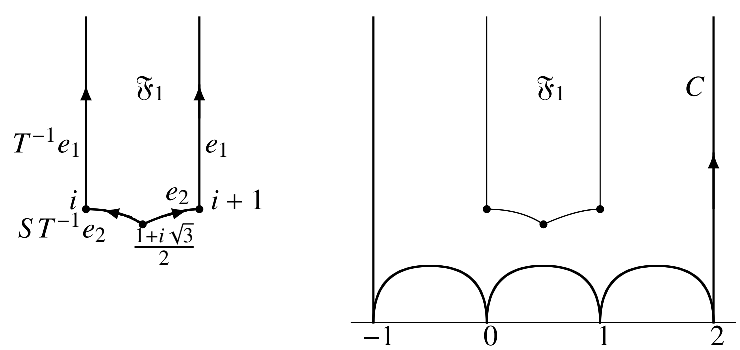


FIGURE 6. Fundamental domain $\mathfrak{F}_1$ for the modular group, and a 1-cycle around it.

there is no need to give the neighborhood of cusps a special treatment. In particular, we do not need an edge f_∞. We use the tesselation $\mathcal{T}$ obtained from all Γ_1-translates of $\mathfrak{F}_1$. In Figure 6 we have indicated a $\mathbb{Q}[\Gamma_1]$-basis $\{e_1, e_2\}$ of $\mathbb{Q}[X_1^{\mathcal{T}}]$. The tesselation $\mathcal{T}$ is not exactly a tesselation of type **Fd** as discussed in §11.1.

To find a cocycle $c \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{W}_s^{\omega^0,\mathrm{exc}})$ corresponding to $\hat{\psi}$ we write $c = df$, with the following $f \in \mathrm{Map}(X_0^{\mathcal{T}}; \mathcal{W}_s^{\omega^*,\mathrm{exc}})$. We put, with $\rho = \frac{1+i\sqrt{3}}{2}$,

$$f(\infty) \;=\; 0\,, \quad f(\rho) \;=\; \frac{1}{3}\,\hat{\psi}_S\,|\,(1 - ST^{-1})\,, \quad f(i) \;=\; \frac{1}{2}\,\hat{\psi}_S\,, \tag{14.9}$$

and check that f satisfies $f(P)\,|\,(1-\delta) = \hat\psi_\delta$ if $\delta P = P$ for $P \in \{\infty, \rho, i\}$, $\delta \in \Gamma$. Next we extend f to $X_1^{\mathcal{T}}$ by $f(\gamma^{-1}P) = f(P)\,|\,\gamma + \hat\psi_\gamma$ for all $\gamma \in \Gamma$. Taking ∞ as the base point we see that $c = df$ corresponds to the cohomology class $[\hat\psi]$.

On the $\mathbb{Q}[\Gamma_1]$-basis $\{e_1, e_2\}$ of $F_1^{\mathcal{T}}$:

$$c(e_1) \;=\; \frac{1}{2}\,\hat\psi_S\,|\,T^{-1}\,, \quad c(e_2) \;=\; -\frac{1}{2}\,\hat\psi_S\,|\,T^{-1} + \frac{1}{3}\,\hat\psi_S\,|\,(1-ST^{-1})\,. \tag{14.10}$$

To represent the corresponding invariant eigenfunction u on $\mathfrak{F}$, we take the following 1-cycle C around $\mathfrak{F}$

$$C \;=\; e_{2,\infty} + e_{\infty,-1} + e_{-1,0} + e_{0,1} + e_{1,2}\,, \tag{14.11}$$

which is indicated in Figure 6. It turns out that

$$\begin{aligned} e_{0,\infty} &= (1-S)T^{-1}e_1\,,\\ C &= (T^2 - T^{-1} - STS + ST^{-1}S + T^2ST)e_{0,\infty}\\ &= (T^2 - T^{-1} - STS + ST^{-1}S + T^2ST)(1-S)T^{-1}e_1\,. \end{aligned}$$

Application of the transverse Poisson transformation to the function in (14.7) gives a representative $\hat\psi_S \in \mathcal{G}_s^{\omega^*,\mathrm{exc}}$ of $\hat\psi_S \in \mathcal{W}_s^{\omega^*,\mathrm{exc}}$ with $\mathsf{Sing}\,(\hat\psi_S) \subset i\,(0,\infty)$. We have $c(e_{0,\infty}) = c(e_1)\,|\,T(1-S) = \hat\psi_S$, and hence $\mathsf{Sing}\,(c(e_{0,\infty}) \subset i\,(0,\infty)$. This implies that the singularities of $\psi(C)$ are contained in the support of C. Thus, for $z \in \mathfrak{F}_1$ we have

$$\begin{aligned} u(z) &= \frac{1}{\pi i}\psi(C) \;=\; \frac{1}{\pi i}\psi(e_{0,\infty})\,|\,(T^{-2} - T - ST^{-1}S + STS + T^{-1}ST^{-2})\\ &= \frac{1}{\pi i}\left(\hat\psi_S(z-2) - \hat\psi_S(z+1) - \hat\psi_S\Big(\frac{z}{z+1}\Big) + \hat\psi_S\Big(\frac{z}{1-z}\Big) + \hat\psi_S\Big(\frac{1-z}{z-2}\Big)\right). \end{aligned} \tag{14.12}$$

Each of these values of $\hat\psi_S(z_1)$ can be expressed by a transverse Poisson integral from $\bar z_1$ to z_1 of the original periodlike function ψ. See (3.4).

14.3. *The image of the invariant eigenfunctions in the first cohomology group.* In the previous section we have shown that $\mathbf{r}\mathcal{E}_s^\Gamma = H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{exc}})$ for $s \neq \frac{1}{2}$. We now show that for $\Gamma = \Gamma_1$ the image is not the total first cohomology group with analytic coefficients:

Proposition 14.3. *Let $s \neq \frac{1}{2}$. The inclusion $H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{exc}}) \subset H^1(\Gamma_1; \mathcal{V}_s^\omega)$ is not an equality.*

Proof. We determine a cohomology class $[\psi] \in H^1(\Gamma_1; \mathcal{V}_s^\omega)$ by taking in the line model

$$\psi_{TS} \;=\; 0\,, \qquad \psi_T \;=\; a \in \mathcal{V}_s^\omega \quad \text{with } a(\tau) \;=\; \frac{-4i\tau}{(1+\tau^2)^{s+1}}\,. \tag{14.13}$$

(See the introduction of §10.2 for the relations.) The sum

$$\mathrm{Av}_T(\psi_T)(\tau) \;=\; \sum_{n\in\mathbb{Z}} \frac{-4i\,(\tau+n)}{\big(1+(\tau+n)^2\big)^{s+1}} \tag{14.14}$$

converges without regularization, and has singularities in the points of $\pm i + \mathbb{Z}$. So $\mathrm{Av}_T(\psi_T)$ does not extend as a holomorphic 1-periodic function on $\mathbb{C}$, which it should according to Proposition 8.2 if $[\psi]$ were in $H^1_{\mathrm{par}}(\Gamma_1; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{exc}}) = \mathbf{r}\,\mathcal{E}_s^{\Gamma_1}$. ∎

14.4. *Quantum Maass forms.* Let $s \neq \frac{1}{2}$. Now we have the following inclusions and isomorphisms for the modular group. (We have suppressed the symbol Γ_1.)

$$\begin{tikzcd} \mathsf{Maass}_s^0 \arrow[r, hook] \arrow[d, leftrightarrow, "\cong"] & \mathsf{Maass}_s^1 \arrow[r, hook, "\neq"] \arrow[d, leftrightarrow, "\cong"] & \mathcal{E}_s^{\Gamma_1} \arrow[r, dotted, no head] \arrow[d, leftrightarrow, "\cong"] & ? \arrow[d, dotted, no head] \\ H^1_{\mathrm{par}}(\mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\infty}) \arrow[r, hook] & H^1_{\mathrm{par}}(\mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{simple}}) \arrow[r, hook, "\neq"] & H^1_{\mathrm{par}}(\mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{exc}}) \arrow[r, hook, "\neq"] & H^1(\mathcal{V}_s^\omega) \end{tikzcd} \tag{14.15}$$

The space of quantum Maass forms may be put at the position of the question mark. Informally, we can characterize quantum Maass forms for Γ_1 as functions on $\mathbb{Q}$ that satisfy a modular transformation property modulo simpler functions.

We first discuss the quantum Maass form associated to a cusp form $u \in \mathsf{Maass}_s^0(\Gamma_1)$. In the main theorem in [19] we associate to u among other objects a periodic holomorphic function f_u on $\mathbb{C} \smallsetminus \mathbb{R}$ given by

$$f_u(\tau) \;=\; \begin{cases} \sum_{n>0} n^{s-\frac{1}{2}} A_n(u)\, e^{2\pi i n\tau} & \text{if } \operatorname{Im}\tau > 0\,, \\ -\sum_{n<0} |n|^{s-\frac{1}{2}} A_n(u) e^{2\pi i n\tau} & \text{if } \operatorname{Im}\tau < 0\,, \end{cases} \tag{14.16}$$

with the Fourier coefficients $A_n(u)$ from the expansion (8.1). In [19], Chap. I, §4, the function f_u is expressed in terms of the L-functions of u by inverse Mellin transformation. By moving the line of integration in this representation we obtain an asymptotic expansion for $f_u(iy)$ as $y \to 0$. This expansion is the same for $y \downarrow 0$ and $y \uparrow 0$. So f_u has a smooth continuation through 0. Instead of approaching 0 vertically, we can let τ tend to 0 along a geodesic half-line in $\mathfrak{H}$ or in $\mathfrak{H}^-$. One may show that there exists $f_u(0) \in \mathbb{C}$ such that

$$f_u(\tau) \;=\; f_u(0) + \mathrm{o}(1) \qquad \text{as } \tau \overset{\mathrm{ga}}{\longrightarrow} 0\,, \tag{14.17}$$

where $\overset{\mathrm{ga}}{\to}$ (geodesic approach) indicates uniformity on sectors in the upper or lower half plane bounded by geodesic half-lines.

The periodicity of f_u and the formula $f_u(\tau) - \tau^{-2s} f_u(-1/\tau) = \psi_u(\tau)$, where ψ_u is the period function associated to u, leads to a unique extension of f_u to $\mathbb{Q}$ that satisfies

$$f_u(\tau) \;=\; f_u(\xi) + \mathrm{o}(1) \qquad \text{as } \tau \overset{\mathrm{ga}}{\longrightarrow} \xi \qquad\qquad \text{for each } \xi \in \mathbb{Q}\,, \tag{14.18}$$

$$f_u|_{2s}\,\gamma(\xi) \;=\; f_u(\xi) - c_\gamma(\xi) \quad \text{for almost all } \xi \in \mathbb{Q} \quad \text{for all } \gamma \in \Gamma_1\,, \tag{14.19}$$

where $\gamma \mapsto c_\gamma$ is the $\mathcal{V}_s^{\omega^*,\infty}$-valued group cocycle determined by $c_S = \psi_u$ on $(0,\infty)$.

The isomorphism (13.1a) implies that there exists $\eta \in \mathcal{V}_s^{\omega^*,\infty}$ such that $\tilde{c}_\gamma = c_\gamma + \eta|_{2s}(\gamma - 1)$, $\gamma \in \Gamma_1$, is a $\mathcal{V}_s^\omega$-valued cocycle on Γ_1. Replacing f_u by $\tilde{f}_u = f_u + \eta$, we obtain the relation $\tilde{f}_u|_{2s}\,\gamma = \tilde{f}_u - \psi_\gamma$ on $\mathbb{Q}$, with $\psi \in Z^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\infty})$. If we add to $\tilde{f}_u$ an element of $\mathcal{V}_s^\omega$, then nothing essential changes. We say that $\tilde{f}_u$ represents the *quantum Maass form* associated to $u \in \mathsf{Maass}_s^0(\Gamma_1)$.

It is argued in [3] that to define quantum Maass forms for other invariant eigenfunctions, we should work not with functions $\mathbb{Q} \to \mathbb{C}$, but with systems of expansions, giving for each $\xi \in \mathbb{Q}$ a short asymptotic expansion

$$p(\xi,\tau) \;=\; \frac{d_\xi}{\tau - \xi} + c_\xi + \mathrm{o}(1) \qquad (\tau \overset{\mathrm{ga}}{\longrightarrow} \infty)\,. \tag{14.20}$$

In the case of $u \in \mathsf{Maass}_s^0(\Gamma_1)$, the function $\tilde{f}_u$ gives an system where $d_\xi = 0$ for all $\xi \in \mathbb{Q}$. Each $\varphi \in \mathcal{V}_s^\omega$ (line model) defines an uninteresting example with $d_\xi = 0$ and $c_\xi = \varphi(\xi)$ for all cusps ξ.

The group Γ_1 acts on the space $\mathcal{R}_s$ of all expansions as in (14.20) by

$$p|_{2s}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\xi,\tau) \;=\; \big((c\tau+d)^2\big)^{-s} p(\gamma\xi,\gamma\tau) + \mathrm{o}(1) \qquad (\tau \overset{\mathrm{ga}}{\longrightarrow} \xi)\,, \tag{14.21}$$

with $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_1$. We define the Γ_1-module $\mathcal{Q}_s$ as the quotient in the exact sequence

$$0 \longrightarrow \mathcal{V}_s^\omega \longrightarrow \mathcal{R}_s \longrightarrow \mathcal{Q}_s \longrightarrow 0 \tag{14.22}$$

and we call the elements of $\mathcal{Q}_s^{\Gamma_1}/\mathcal{R}_s^{\Gamma_1}$ *quantum Maass forms*, with the notation

$$\mathsf{qMaass}_s(\Gamma_1) = \mathcal{Q}_s^{\Gamma_1} \,/\, \mathcal{R}_s^{\Gamma_1}\,. \tag{14.23}$$

In this way, we ignore elements of $\mathcal{V}_s^\omega$ and systems in $\mathcal{R}_s$ that are exactly Γ_1-invariant. One can show that there is an injection $\mathsf{qMaass}_s(\Gamma_1) \to H^1(\Gamma;\mathcal{V}_s^\omega)$. Theorems 2 and 4 in [3] give for $s \neq \frac{1}{2}$ the following commuting diagram:

$$\begin{tikzcd} \mathcal{E}_s^{\Gamma_1} \arrow[d, hook] \arrow[dr, hook, "\mathbf{r}"] & \\ \mathsf{qMaass}_s(\Gamma_1) \arrow[r, "\cong"] & H^1(\Gamma_1;\mathcal{V}_s^\omega) \end{tikzcd} \tag{14.24}$$

Thus, $\mathsf{qMaass}_s(\Gamma_1)$ can take the place of the question mark in the diagram (14.15). Proposition 3 in [3] relates the vanishing of $A_0(u)$, the coefficient of y^{1-s} in the Fourier expansion of $u \in \mathcal{E}_s^{\Gamma_1}$, to the vanishing of the d_ξ in (14.20).

We expect that these results can be extended to all cofinite discrete Γ with cusps.

15. Maass forms and holomorphic functions. In §8.2 we have associated to $u \in \mathcal{E}_s^T$ the holomorphic function $\beta(u) \in \mathcal{O}(\mathbb{C})^T$ given by

$$\beta(u)(\zeta) \;=\; \sum_{n\in\mathbb{Z}} B_n(u)\, e^{2\pi i n\zeta}\,,$$

based on the coefficients $B_n(u)$ in the Fourier expansion (8.1). To have a well defined coefficient $B_0(u)$, we assume $s \neq \frac{1}{2}$. For a Γ-invariant function $u \in \mathcal{E}_s^\Gamma$, where Γ is a group with cusps, $\beta(u\,|\,g_\kappa) \in \mathcal{E}_s^T$ for each cusp $\kappa \in C$. In the introduction of §10 we have chosen the g_κ such that $\beta(u\,|\,g_\kappa)$ depends only on the class of κ in $\Gamma\backslash C$. Thus we are led to define

$$\mathbf{j} : \mathcal{E}_s^\Gamma \longrightarrow \bigoplus_{\kappa\in\Gamma\backslash C} \mathcal{O}(\mathbb{C})^T\,, \quad \mathbf{j}u = \big(\beta(u\,|\,g_\kappa)\big)_\kappa\,. \tag{15.1}$$

The kernel of $\mathbf{j}$ is, by definition, the space $\mathsf{Maass}_s^1(\Gamma)$ introduced in §10.1. To get information on the cokernel, we start with $u, v \in \mathcal{E}_s^\Gamma$ and integrate the Green's form $\{u,v\}$ in (1.9) over the boundary of a truncated fundamental domain $\mathfrak{F}_Y$. Since $\{u,v\}$ is closed, this integral vanishes. All edges of $\mathfrak{F}_Y$, except for the edges f_κ near the cusps $\kappa \in \mathfrak{F}^{\mathrm{cu}}$, occur in Γ-equivalent pairs for which the integrals of $\{u,v\}$ cancel. Hence

$$\sum_{\kappa\in\mathfrak{F}^{\mathrm{cu}}} \int_{f_\kappa} \{u,v\} \;=\; 0\,.$$

Inserting the Fourier expansions and working with Wronskians, we obtain the following relation, valid for all $u, v \in \mathcal{E}_s^\Gamma$:

$$\begin{aligned} 0 \;=\; \sum_{\kappa \in \mathfrak{F}^{\mathrm{cu}}} \Big(&(2s-1) \big(A_0(u\,|\,g_\kappa)\, B_0(v\,|\,g_\kappa) - B_0(u\,|\,g_\kappa)\, A_0(v\,|\,g_\kappa) \big) \\ &+ \pi^{\frac12 - s}\, \Gamma\Big(s+\frac12\Big) \sum_{n \neq 0} |n|^{\frac12 - s} \big(A_n(u\,|\,g_\kappa)\, B_{-n}(v\,|\,g_\kappa) - B_n(u\,|\,g_\kappa)\, A_{-n}(v\,|\,g_\kappa) \big) \Big). \end{aligned} \tag{15.2}$$

This is the so-called *Maass-Selberg relation*. See, e.g., §3 in Chap. IV of [20].

In particular, if $v \in \mathsf{Maass}_s^1(\Gamma)$, then

$$\begin{aligned} 0 \;=\; \sum_{\kappa \in \mathfrak{F}^{\mathrm{cu}}} \Big(&(2s-1)\, B_0(u\,|\,g_\kappa)\, A_0(v\,|\,g_\kappa) \\ &+ \pi^{\frac12 - s}\, \Gamma\Big(s+\frac12\Big) \sum_{n \neq 0} |n|^{\frac12 - s}\, B_n(u\,|\,g_\kappa)\, A_{-n}(v\,|\,g_\kappa) \Big). \end{aligned}$$

The right hand side of this expression makes sense if we replace the $B_m(u\,|\,g_\kappa)$ by the coefficients b_m^κ of an arbitrary element

$$(b_\kappa)_\kappa \;\in\; \bigoplus_{\kappa \in \Gamma\backslash C} \mathcal{O}(\mathbb{C})^T\,, \qquad b_\kappa(\zeta) \;=\; \sum_{m \in \mathbb{Z}} b_m^\kappa\, e^{2\pi i m \zeta}\,.$$

Note that the convergence of these series implies that

$$b_n^\kappa \;\ll_A\; e^{-A|n|} \qquad \text{for } n \in \mathbb{Z}\,,\ \kappa \in \Gamma\backslash C\,,\ \text{for each } A > 0\,. \tag{15.3}$$

Thus, we have a linear map $\mathbf{m}$ from $\bigoplus_{\kappa \in \Gamma\backslash C} \mathcal{O}(\mathbb{C})^T$ to the dual space $\mathsf{Maass}_s^1(\Gamma)^\vee$.

Theorem 15.1. *Let* $0 < \mathrm{Re}\, s < 1$, $s \neq \frac12$.

i) *The following sequence is exact:*

$$0 \longrightarrow \mathsf{Maass}_s^1(\Gamma) \longrightarrow \mathcal{E}_s^\Gamma \xrightarrow{\ \mathbf{j}\ } \bigoplus_{\kappa \in \Gamma\backslash C} \mathcal{O}(\mathbb{C})^T \xrightarrow{\ \mathbf{m}\ } \mathsf{Maass}_s^1(\Gamma)^\vee \longrightarrow 0\,. \tag{15.4}$$

ii) *Every $u \in \mathcal{E}_s^\Gamma$ is the specialization of a family $(u_{s'})_{s' \in U}$ of elements $u_{s'} \in \mathcal{E}_{s'}^\Gamma$ depending holomorphically on a parameter s' on a neighborhood U of s.*

Remark 1. Part ii) of the theorem will be used in Chapter VI, when we will study the relation between Γ-invariant eigenfunctions and distribution-valued cohomology.
Remark 2. The proof will show that the restriction on the spectral parameter s is nonessential.
Remark 3. This theorem is essentially known if almost all b_n^κ vanish. Since the result is peripheral to the main themes of these notes, we will only sketch the proof.

Proof. The surjectivity of the map $\mathbf{m}$ is clear since $\mathsf{Maass}_s^1(\Gamma)$ has finite dimension and already the restriction of $\mathbf{m}$ to $\bigoplus_\kappa \mathbb{C}[q]$, with $q = e^{2\pi i \zeta}$, is surjective. The Maass-Selberg relation shows that the image of $\mathbf{j}$ is contained in the kernel of $\mathbf{m}$. The main point is to show that $\mathrm{Im}\,\mathbf{j}$ is equal to $\mathrm{Ker}\,\mathbf{m}$. We will sketch how this follows from the spectral theory of automorphic forms. One may use [14] as a general reference.

We use Eisenstein series and Poincaré series. These converge absolutely only for $\mathrm{Re}\, s > 1$. This forces us to consider also values of s outside the strip $0 < \mathrm{Re}\, s < 1$.

For $\mathrm{Re}\, s > 1$, $\kappa \in C$, the Eisenstein series

$$E_s^\kappa(z) \;=\; \sum_{\gamma\in\Gamma_\kappa\backslash\Gamma} (\mathrm{Im}\, g_\kappa^{-1}\gamma z)^s \tag{15.5}$$

converges absolutely and defines an element of $\mathcal{E}_s^\Gamma$. It depends holomorphically on s, and has a meromorphic continuation to $s \in \mathbb{C}$ as a family of elements of $\mathcal{E}_s^\Gamma$. The singularities in the region $\mathrm{Re}\, s \geq \frac{1}{2}$, $s \neq \frac{1}{2}$ are of first order and occur at $s = 1$ and possibly at finitely many points in $(\frac{1}{2}, 1)$. The latter singularities are absent for congruence subgroups of Γ_1. If E_s^κ has a singularity at $s_0 \in (\frac{1}{2}, 1)$ then $\mathrm{Res}_{s_0} E_s^\kappa \in \mathsf{Maass}_{s_0}^1(\Gamma)$ and for all $u \in \mathsf{Maass}_{s_0}^1(\Gamma)$:

$$\int_{\Gamma\backslash\mathfrak{H}} u \operatorname*{Res}_{s_0} E_s^\kappa \, d\mu \;=\; A_0^\kappa(u)\,. \tag{15.6}$$

Suppose that the collection (b_n^κ) satisfies (15.3). The series

$$F_s^\kappa(z) \;=\; \sum_{n\in\mathbb{Z}} b_n^\kappa \, i_{s,2\pi n}(g_\kappa^{-1} z)$$

converges absolutely for all $s \in \mathbb{C}$ and defines a holomorphic family of elements of $\mathcal{E}_s^{\pi_\kappa} \cap \mathcal{W}_s^\omega[\kappa]$. The Poincaré series

$$P_s(z) \;=\; \sum_{\kappa\in\Gamma\backslash C} \sum_{\gamma\in\Gamma_\kappa\backslash\Gamma} F_s^\kappa(\gamma z) \tag{15.7}$$

converges absolutely for $\mathrm{Re}\, s > 1$. It defines a holomorphic family of elements of $\mathcal{E}_s^\Gamma$, and

$$B_m(P_s \,|\, g_\kappa) \;=\; b_m^\kappa \qquad (m \in \mathbb{Z},\ \kappa \in \mathfrak{F}^{\mathrm{cu}})\,. \tag{15.8}$$

We consider the following families of Γ-invariant functions, only the first of which has values in $\mathcal{E}_s^\Gamma$:

$$E_s \;=\; \sum_{\kappa\in\mathfrak{F}^{\mathrm{cu}}} b_0^\kappa \, E_s^\kappa\,, \qquad \tilde{P}_s(z) \;=\; \sum_{\kappa\in\mathfrak{F}^{\mathrm{cu}}} \sum_{\gamma\in\Gamma_\kappa\backslash\Gamma} \tilde{F}_s^\kappa(\gamma z)\,,$$

where

$$\tilde{F}_s^\kappa(z) \;=\; \sum_{n\neq 0} b_n^\kappa (\mathrm{Im}\, g_\kappa^{-1} z)^s \, e^{2\pi i n \mathrm{Re}\,(g_\kappa^{-1} z)}\,.$$

This defines $\tilde{P}_s \in C^\infty(\Gamma\backslash\mathfrak{H})$ for $\mathrm{Re}\, s > 1$. The difference $P_s - E_s - \tilde{P}_s$ is given by a series converging absolutely for $\mathrm{Re}\, s > 0$. Compared with P_s, the advantage of $\tilde{P}_s$ is its square integrability. Its decay at the cusps implies that for any Maass form $u \in \mathsf{Maass}_{s_1}(\Gamma)$, $0 < \mathrm{Re}\, s_1 < 1$, $s_1 \neq \frac{1}{2}$, the integral $\int_{\Gamma\backslash\mathfrak{H}} \tilde{P}_s \, u \, d\mu$ converges. Its value can be explicitly computed:

$$\int_{\Gamma\backslash\mathfrak{H}} \tilde{P}_s \, u \, d\mu \;=\; \sum_{k\in\mathfrak{F}^{\mathrm{cu}}} \sum_{n\neq 0} \frac{(\pi|n|)^{\frac{1}{2}-s}}{4} \Gamma\Big(\frac{s+s_1-1}{2}\Big) \Gamma\Big(\frac{s-s_1}{2}\Big) A_{-n}(u \,|\, g_\kappa)\, b_n^\kappa\,. \tag{15.9}$$

This quantity occurs in the expansion of $\tilde{P}_s$ in the spectral decomposition of the Laplace operator in $L^2(\Gamma\backslash\mathfrak{H})$. This expansion converges absolutely for $\mathrm{Re}\, s > 1$. On checks that the convergence is even better in the region $0 < \mathrm{Re}\, s \leq 1$, except for the terms that have singularities in this region. See, e.g., the reasoning in the proof of Satz 6.2 in [21]. For $0 < \mathrm{Re}\, s < 1$, $s \neq \frac{1}{2}$, these singularities have at most first order and occur at values for which there are square integrable elements of $\mathcal{E}_s^\Gamma$ and at values at which an Eisenstein series has a pole, in other words, at values of s for which $\mathsf{Maass}^1_s(\Gamma) \neq \{0\}$. This means that $P_s = E_s + \tilde{P}_s + \left(P_s - E_s - \tilde{P}_s\right)$ has a meromorphic continuation to $\operatorname{Re} s > 0$, with singularities of at most order one at the same points, and for such a point s_0

$$\operatorname*{Res}_{s_0} P_s = \operatorname*{Res}_{s_0} E_s + \operatorname*{Res}_{s_0} \tilde{P}_s\,.$$

By analytic continuation the equality $(\Delta - \lambda_s)P_s = 0$ goes through where P_s is holomorphic. (Work first in distribution sense.) Furthermore,

$$B_n(P_s\,|\,g_\kappa) \;=\; b_n^\kappa \qquad (\kappa \in \mathfrak{C}^{\mathrm{cu}},\ n \in \mathbb{Z})\,. \tag{15.10}$$

At points s for which $\mathsf{Maass}^1_s(\Gamma) = \{0\}$, we thus have $P_s \in \mathcal{E}_s^\Gamma$ with prescribed Fourier coefficients $B_m(P_s\,|\,g_\kappa)$. This implies that the sequence (15.4) is exact for these s, and that all elements of $\mathcal{E}_s^\Gamma$ occur in holomorphic families on this region in the parameter space.

It remains to consider $s_0 \neq \frac{1}{2}$, $0 < \operatorname{Re} s_0 < 1$ for which $\mathsf{Maass}^1_{s_0} \neq \{0\}$. We first take $\operatorname{Re} s_0 \geq \frac{1}{2}$. Then all elements of $\mathsf{Maass}^1_{s_0}$ are square integrable, and $\mathsf{Maass}^1_{s_0}(\Gamma) \neq \{0\}$ can occur only for $s_0 \in \frac{1}{2} + i\mathbb{R}$ or $s_0 \in \left(\frac{1}{2}, 1\right)$. Hence the space $\mathsf{Maass}^1_{s_0}(\Gamma)$ is invariant under complex conjugation. For $u \in \mathsf{Maass}^1_{s_0}(\Gamma)$ we have from (15.6) and (15.9):

$$\int_{\Gamma\backslash\mathfrak{H}} u \operatorname*{Res}_{s_0} P_s\,d\mu \;=\; \sum_{k\in\mathfrak{C}^{\mathrm{cu}}} b_0^\kappa\,A_0(u\,|\,g_\kappa) + \sum_{\kappa\in\mathfrak{C}^{\mathrm{cu}}}\sum_{n\neq 0} \frac{(\pi|n|)^{\frac{1}{2}-s_0}}{2}\,\Gamma\Big(s_0 - \frac{1}{2}\Big)\,b_n^\kappa\,A_{-n}(u\,|\,g_\kappa)\,.$$

This implies that the finite dimensional space $\mathsf{Maass}^1_{s_0}(\Gamma)$ is spanned by residues of finitely many Poincaré series P_s, for choices of the b_n^κ such that almost all of them are zero. Hence elements of $\mathsf{Maass}^1_{s_0}(\Gamma)$ occur as values of holomorphic families $s \mapsto (s - s_0)\,P_s$.

Now suppose that the b_n^κ are chosen such that $\mathbf{m}(h) = 0$ for $h = (h_\kappa)_\kappa$, $h_\kappa(\zeta) = \sum_n b_n^\kappa\,e^{2\pi i n\zeta}$. Then $\operatorname{Res}_{s_0} P_s \in \mathsf{Maass}^1_{s_0}(\Gamma)$ is orthogonal to all $\bar{u} \in \mathsf{Maass}^1_{s_0}(\Gamma)$. In other words, $\operatorname{Res}_{s_0} P_s = 0$, and $s \mapsto P_s$ is holomorphic at $s = s_0$ and $P_{s_0} \in \mathcal{E}_{s_0}^\Gamma$ satisfies $\mathbf{j}P_{s_0} = h$. Thus, the sequence (15.4) is exact for the value s_0 of the spectral parameter as well, and all elements of $\mathcal{E}_{s_0}^\Gamma$ occur as the value of a holomorphic family of Γ-invariant λ_s-eigenfunctions of Δ. This finishes the case $\frac{1}{2} \leq \operatorname{Re} s_0 < 1$.

Since $\mathcal{E}_s^\Gamma = \mathcal{E}_{1-s}^\Gamma$, all elements of $\mathcal{E}_s^\Gamma$ with $0 < \operatorname{Re} s < \frac{1}{2}$, $s \neq \frac{1}{2}$, occur as values of holomorphic families as well.

We are left with the kernel of $\mathbf{m}$ for s_0 with $0 < \operatorname{Re} s_0 < \frac{1}{2}$. Suppose that the collection (b_n^κ) satisfies (15.3) and that for all $u \in \mathsf{Maass}^1_{s_0}(\Gamma)$

$$\sum_{\kappa\in\mathfrak{C}^{\mathrm{cu}}} (2s_0 - 1)\,b_0^\kappa\,A_0(u\,|\,g_\kappa) + \pi^{\frac{1}{2}-s}\,\Gamma\Big(s + \frac{1}{2}\Big)\sum_{n\neq 0} |n|^{\frac{1}{2}-s}\,b_n^\kappa\,A_{-n}(u\,|\,g_\kappa) \;=\; 0\,. \tag{15.11}$$

As in the case $\operatorname{Re} s_0 \geq \frac{1}{2}$, it follows that $\operatorname{Res}_{s_0} P_s$ is orthogonal to $\mathsf{Maass}^0_{s_0}(\Gamma)$ which is contained in $\mathsf{Maass}^1_{s_0}(\Gamma)$. If $\operatorname{Res}_{s_0} P_s \neq 0$ it is known that then it is a linear combination of residues at s_0 of Eisenstein series, and that $A_0(\operatorname{Res}_{s_0} P_s\,|\,g_\mu) \neq 0$ for some $\mu \in \mathfrak{C}^{\mathrm{cu}}$. We apply the Maass-Selberg relation (15.2) to E_s^λ and the Poincaré series $P_s^{\kappa,n}$, which is the Poincaré series with $b_n^\kappa = 1$ and all other $b_{n'}^{\kappa'}$ equal to zero. This gives

$$2\,A_0(P_s^{\kappa,n}\,|\,g_\lambda) \;=\; (\pi|n|)^{\frac{1}{2}-s}\,\Gamma\Big(s - \frac{1}{2}\Big)\,A_{-n}(E_s^\lambda\,|\,g_\kappa)\,.$$

Application of (15.2) to E_s^κ and E_s^μ gives $A_0(E_s^\kappa \,|\, g_\mu) = A_0(E_s^\mu \,|\, g_\kappa)$, which is the symmetry of the scattering matrix. Hence we have

$$\begin{aligned} A_0(\operatorname*{Res}_{s_0} P_s \,|\, g_\mu) &= \sum_\kappa b_0^\kappa \operatorname*{Res}_{s_0} A_0(E_s^\kappa \,|\, g_\mu) + \sum_\kappa \sum_{n\neq 0} b_n^\kappa \operatorname*{Res}_{s_0} A_0(P_s^{\kappa,n} \,|\, g_\mu) \\ &= \sum_\kappa b_0^\kappa A_0(\operatorname*{Res}_{s_0} E_s^\mu \,|\, g_\kappa) + \sum_\kappa \sum_{n\neq 0} \frac{(\pi|n|)^{\frac{1}{2}-s_0}}{2} \Gamma\Big(s_0 - \frac{1}{2}\Big) A_{-n}(\operatorname*{Res}_{s_0} E_s^\mu \,|\, g_\kappa) \\ &= 0\,, \end{aligned}$$

by assumption (15.11) on the b_n^κ. Hence $\operatorname{Res}_{s_0} P_s = 0$, and $P_{s_0} \in \mathcal{E}_{s_0}^\Gamma$ has the prescribed Fourier coefficients $B_n(P_{s_0} \,|\, g_\kappa) = b_n^\kappa$. This implies that the sequence (15.4) is exact for $s = s_0$.

Part ii) is a consequence of part i). For instance, if s does not belong to the (discrete) set when $\mathsf{Maass}_s^1(\Gamma)$ is non-zero, then we can take $u_{s'}$ to be the preimage of any holomorphic variation of $\mathbf{j}u \in \bigoplus_{\kappa\in\Gamma\backslash C} O(\mathbb{C})^T$, e.g., the constant family $s' \mapsto (\beta(u \,|\, g_\kappa))_\kappa$.

In the proof of part i) we have seen that for any $(\sum_n b_n^\kappa e^{2\pi i n\zeta})_\kappa$ in the kernel of $\mathbf{m}$ at s there is a family of Poincaré series $s' \mapsto P_{s'}$ such that $\mathbf{j}P_s$ or $\mathbf{j}\operatorname{Res}_{s'=s} P_{s'}$ is equal to $(\sum_n b_n^\kappa e^{2\pi i n\zeta})_\kappa$. ∎

Chapter V. Maass forms and differentiable cohomology

In the chapters II–IV we studied the relation between invariant eigenfunctions and (semi)-analytic cohomology, and proved Theorem C and most of the statements in Theorems A and B. In this chapter we turn to smooth and differentiable cohomology. We will give a proof of the isomorphisms $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*,\infty}) \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\infty) \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^p)$, $p \geq 3$, in Theorem B. Actually, we will show that

$$\mathsf{Maass}_s^0(\Gamma) \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\infty)\,, \tag{V.1}$$

and

$$\mathsf{Maass}_s^0(\Gamma) \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^p) \qquad (p \in \mathbb{N}\,,\ p \geq 3)\,, \tag{V.2}$$

which together with the previously proved results (in particular, (12.6) and (13.1a)) proves the remaining isomorphisms in Theorem B.

The isomorphism (V.2) will be established in §16 by a method analogous to that used in §12.2, with adaptions to the differentiable context. In §17 this leads to Theorem 17.1, which gives the isomorphism (V.1). A consequence is that the space of modular Maass cusp forms is in bijective correspondence with the space of smooth period functions. This result, Theorem 17.2, extends the main result in [19].

Most of the proofs in this chapter work for general cofinite discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$. At the end of Section 16 we use separate approaches for cocompact groups and groups with cusps. The cocompact case is the harder one. As in § 7.3 in Chapter II we need for cocompact groups to extend cocycles to hyperbolic fixed points.

The isomorphy of $\mathcal{E}_s^\Gamma$ and $H^1(\Gamma; \mathcal{V}_s^\infty)$ in the cocompact case is already known, though with a quite different proof, from the work of Bunke and Olbrich [6].

16. Differentiable parabolic cohomology. In this section we relate the cohomology group $H^1_{\mathrm{par}}(\Gamma;\mathcal{V}_s^p)$, $p\in\mathbb{N}$, $p\geq 2$, to the space of Maass cusp forms $\mathsf{Maass}_s^0(\Gamma)$. The space $\mathcal{V}_s^p$ was defined in §2.1. Recall that in Chapters II and IV the map from parabolic cohomology to cusp forms was constructed, not with cohomology with values in the space $\mathcal{V}_s^\omega$ of analytic functions on $\partial\mathbb{H}$, but in the isomorphic space $\mathcal{W}_s^\omega$ of boundary germs of λ_s-eigenfunctions of the Laplace operator Δ. Here we work with the corresponding space $\mathcal{W}_s^p$ of boundary jets, as defined in §3.3.

The construction of Maass forms from cocycles in §7.1 and §12.2 used locally finite sums. To generalize this, in §16.1, to differentiable cocycles we will need infinite sums. The resulting convergence questions require some geometric considerations, carried out in §16.2. The proof for discrete groups with cusps is completed in §16.3, where we prove that if $p\geq 3$ the map that we have given is injective with image in the cusp forms. (The surjectivity will be an easy consequence of that in the analytic case.) The injectivity in the cocompact case is handled in §16.4.

16.1. *Construction of a Maass form from a given cocycle.* We start with a cocycle $\psi\in Z^1(F_\cdot^{\mathcal{T}};\mathcal{W}_s^p)$ with $p\in\mathbb{N}$, $p\geq 2$, and use a tesselation $\mathcal{T}$ of type **Fd**. (See §6.2 and §11.1.) We choose a Γ-equivariant lift $\tilde\psi\in C^1(F_\cdot^{\mathcal{T}},\mathcal{G}_s^p)$ of ψ, corresponding to the exact sequence

$$0\longrightarrow \mathcal{N}_s^p\longrightarrow \mathcal{G}_s^p\longrightarrow \mathcal{W}_s^p\longrightarrow 0\,, \tag{16.1}$$

with the space $\mathcal{G}_s^p$ of representatives and the space $\mathcal{N}_s^p$ of functions with decay of order p at the boundary, as introduced in §3.3. Then $d\tilde\psi\in C^2(F_\cdot^{\mathcal{T}},\mathcal{N}_s^p)$ and we put

$$u_\psi(z)\;=\;\frac{1}{\pi i}\sum_{\gamma\in\Gamma} d\tilde\psi(\mathfrak{F})(\gamma z)\;=\;\mathrm{Av}_\Gamma d\tilde\psi(\mathfrak{F})(z)\qquad (z\in\mathbb{H})\,. \tag{16.2}$$

Proposition 16.1. *The series in* (16.2) *converges absolutely and uniformly on compact sets in $\mathfrak{H}$. The sum it defines belongs to $\mathcal{E}_s^\Gamma$, and does not depend on the choice of the lift $\tilde\psi$ or on the choice of the representative ψ in the cohomology class* $[\psi]\in H^1_{\mathrm{par}}(\Gamma;\mathcal{W}_s^p)$.

If $Z\in\mathbb{Z}[X_1^{\mathcal{T}}]$ is a cycle consisting only of edges that are Γ-translates of edges in $\partial\mathfrak{F}$ that describes a simple positively oriented closed curve, then

$$u_\psi\;=\;\frac{1}{\pi i}\Bigl(\tilde\psi(Z)+\sum_{\gamma\in\Gamma,\,\gamma^{-1}\mathfrak{F}\text{ outside }Z} d\tilde\psi(\mathfrak{F})|\gamma\Bigr)\,. \tag{16.3}$$

Proof. The function $h=d\tilde\psi(\mathfrak{F})$ is in $\mathcal{N}_s^p$ and hence satisfies $h(z)=o\bigl((\frac{y}{|z+i|^2})^{s+p}\bigr)$ as $z\to\partial\mathfrak{H}$. Since $\frac{y}{|z+i|^2}=\bigl(2+2\cosh \mathrm{d}(z,i)\bigr)^{-1}$ (Table 1 in §1.1), we have $h(z)=o(e^{-(s+p)\,\mathrm{d}(z,i)})$ as $z\to\partial\mathfrak{H}$.

Let $K\subset\mathbb{H}$ be a compact set. For $z\in K$ the number of $\gamma\in\Gamma$ such that $\mathrm{d}(\gamma z,i)\leq R$ is at most $\mathrm{O}_K(e^R)$ since the area of a hyperbolic circle with large radius R is asymptotic to $4\pi e^R$. We get

$$\sum_{\gamma\in\Gamma}\bigl|h(\gamma z)\bigr|\;\ll_K\;\sum_{R=1}^\infty e^R e^{-(s+p)R}\;<\;\infty\,.$$

This proves the first statement of the proposition.

Our next observation, which will be used repeatedly, is that we have

$$\tilde{\psi}(Z) = \sum_{\gamma\in\Gamma,\ \gamma^{-1}\mathfrak{F}\text{ inside }Z} h|\gamma$$

for any cycle Z as in the proposition, since $h = d\tilde{\psi}$. This gives the expression (16.3) for u_ψ. It also follows that u_ψ is the limit of $\frac{1}{\pi i}\tilde{\psi}(Z_R)$ for any sequence of cycles $\{Z_R)\}_{R\in\mathbb{N}}$ approaching the boundary, for instance those given in Lemma 16.3 below, where Z_R has distance at least R to i and consists of $\mathrm{O}(e^R)$ edges. This observation is useful both to prove that u_ψ is independent of the choice of the lift $\tilde{\psi}$ and that it is an eigenfunction. For the former we observe that changing $\tilde{\psi}$ to $\tilde{\psi} + \chi$ with $\chi \in C^1(F^{\mathcal{T}}_{\cdot}; \mathcal{N}^p_s)$ each edge e in Z_R contributes at most $o(e^{-(s+p)R})$ to $\chi(Z_R)$, which gives $\chi(Z_R) = \mathrm{O}(e^R)\, o(e^{-(s+p)R})$. Hence $\lim_{R\to\infty}\chi(Z_R) = 0$.

The definition of $\mathcal{W}^s_p$ implies that $(\Delta-\lambda_s)\tilde{\psi} \in C^1(F^{\mathcal{T}}_{\cdot}; \mathcal{N}^p_s)$. It follows that with $\{Z_R\}$ as above we have $(\Delta - \lambda_s)\,\tilde{\psi}(Z_R) = \mathrm{O}(e^R\, e^{-(s+p)R}) = o(1)$, this estimate being uniform on compact sets. Hence we have with a test function $\theta \in C^\infty_c(\mathfrak{H})$

$$\begin{aligned}
\langle\theta, (\Delta - \lambda_s)u_\psi\rangle &= \int_{\mathfrak{H}} ((\Delta - \lambda_s)\theta)\, u_\psi\, d\mu \qquad \text{(since } \Delta - \lambda_s \text{ is self-adjoint)}\\
&= \frac{1}{\pi i}\lim_{R\to\infty}\int_{\mathfrak{H}} ((\Delta - \lambda_s)\theta)\, \tilde{\psi}(Z_R)\, d\mu \qquad \text{(since } \theta \text{ is compactly supported)}\\
&= \frac{1}{\pi i}\lim_{R\to\infty}\int_{\mathfrak{H}} \theta\, ((\Delta - \lambda_s)\tilde{\psi}(Z_R))\, d\mu \qquad \text{(again since } \Delta - \lambda_s \text{ is self-adjoint)}\\
&= \lim_{R\to\infty}\mathrm{O}\Big(\int_{\mathfrak{H}} |\theta|\, d\mu \sup_{z\in\operatorname{Supp}\theta}\big|(\Delta - \lambda_s)\tilde{\psi}(Z_R)(z)\big|\Big)\\
&= 0 \qquad \text{(by the estimate (3.10a))}\,.
\end{aligned}$$

So $(\Delta - \lambda_s)u_\psi = 0$ weakly. By elliptic regularity, $(\Delta - \lambda_s)u_\psi = 0$ holds also at the level of functions. This shows that $u_\psi \in \mathcal{E}_s$, and the Γ-invariance is obvious.

Finally, we add to ψ a coboundary df with $f \in C^0(F^{\mathcal{T}}_{\cdot}; \mathcal{W}^p_s)$. We can lift f to $\tilde{f} \in C^0(F^{\mathcal{T}}_{\cdot}; \mathcal{G}^p_s)$, and change the lift $\tilde{\psi}$ of ψ to the lift $\tilde{\psi} + d\tilde{f}$ of ψ_f. This changes $d\tilde{\psi}(\mathfrak{F})$ by $dd f(\mathfrak{F}) = 0$, and does not influence the definition of u_ψ. ■

Thus we have defined a map $\alpha^p_s : [\psi] \mapsto u_\psi$ from $H^1_{\mathrm{par}}(\Gamma; \mathcal{W}^p_s) \to \mathcal{E}^\Gamma_s$. If $q \in \mathbb{N}$, $q \geq p$, then $\psi \in Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{W}^q_s)$ also determines an element in $Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{W}^p_s)$. The construction of u_ψ shows that $\alpha^q_s[\psi] = \alpha^p_s[\psi]$. For $\psi \in Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{W}^\omega_s)$ the sequence $\tilde{\psi}(Z_R)(z)$ stabilizes, uniformly for z in compact sets, and this shows that u_ψ coincides with u_ψ defined in §7.1 and §12.2. Thus $\alpha^p_s[\psi] = \alpha^\omega_s[\psi]$ on $H^1_{\mathrm{par}}(\Gamma; \mathcal{W}^\omega_s)$, with α^ω_s as in Theorem 7.2 and Proposition 12.2. This implies that in the differentiable case we also have $u_{\mathbf{q}u} = u$ for $u \in \mathcal{E}^\Gamma_s$, where $\mathbf{q} : \mathcal{E}^\Gamma_s \to H^1(\Gamma; \mathcal{W}^\omega_s)$ is the map constructed in §5.2, in the version given after (6.10c). Summarizing, we have:

Proposition 16.2. *For each $p, q \in \mathbb{N}$, $q \geq p \geq 2$ there are linear maps α_s^p and α_s^q induced by $\psi \mapsto u_\psi$ such that the following diagram commutes:*

$$\begin{tikzcd}
& \mathsf{Maass}^0_s(\Gamma) \arrow[r, hook] & \mathcal{E}^\Gamma_s & \\
H^1_{\mathrm{par}}(\mathcal{W}^\omega_s, \mathcal{W}^{\omega^0,\infty}_s) \arrow[ur, "\cong", "\alpha^\omega_s"'] \arrow[r, "\cong"'] & H^1_{\mathrm{par}}(\mathcal{W}^{\omega^0,\infty}_s) \arrow[r] & H_{\mathrm{par}}(\mathcal{W}^q_s) \arrow[u, "\alpha^q_s"] \arrow[r] & H_{\mathrm{par}}(\mathcal{W}^p_s) \arrow[ul, "\alpha^p_s"']
\end{tikzcd}$$

We have suppressed Γ in the notation for cohomology groups. See (13.1a) for the isomorphism $H^1_{\mathrm{par}}(\Gamma; \mathcal{W}^\omega_s, \mathcal{W}^{\omega^0,\infty}_s) \to H^1_{\mathrm{par}}(\Gamma; \mathcal{W}^{\omega^0,\infty}_s)$.

It remains to be shown that the linear map α_s^p is injective, and, if Γ has cusps, that its image is in the space of cusp forms. Before turning to that question, in §16.3 and §16.4, we prove the geometrical result that we used in the proof of Proposition 16.1.

16.2. *Geometrical lemmas.* The result that we used is the following:

Lemma 16.3. *Let $R > 0$. There exists a cycle $Z = Z_R \in \mathbb{Z}[X_1^{\mathcal{T}}]$ consisting of $\mathrm{O}(e^R)$ edges contained in $\Gamma\partial\mathfrak{F}$ and lying outside the open hyperbolic disk with center i and radius R, with winding number 1 around i.*

Remark. The circumference of a hyperbolic cycle with radius R is approximately $2\pi\, e^R$ as $R \to \infty$. The lemma says that the conditions on Z do not force it to have substantially more edges that is to be expected from the length of a hyperbolic circle.

The proof of Lemma 16.3 will be very simple if Γ has no cusps, but if Γ has cusps and R is large, the path corresponding to Z will always have to go through some cusps, and in that case we will need a bound for the number of times that the curve Z is forced to go through a cusp. We first estimate this quantity.

We recall that for groups with cusps the fundamental domain has a decomposition $\mathfrak{F} = \mathfrak{F}_Y \cup \bigcup_{\kappa \in \mathfrak{F}^{\mathrm{cu}}} V_\kappa$ where $\mathfrak{F}_Y$ is compact and where V_κ is the closure of the intersection of $\mathfrak{F}$ with the open horocyclic disk $D_Y(\kappa)$ in (11.1). The boundary of $D_Y(\kappa)$ is the horocycle $H_Y(\kappa)$.

Lemma 16.4. *The number of horocycles $\gamma^{-1} H_Y(\kappa)$, $\gamma \in \Gamma$, $\kappa \in \mathfrak{F}^{\mathrm{cu}}$, that intersect the hyperbolic circle C_R around i with radius R is $\mathrm{O}(e^R)$ as $R \to \infty$. The number of cuspidal triangles $\gamma^{-1} V_\kappa$, $\gamma \in \Gamma$, $\kappa \in \mathfrak{F}^{\mathrm{cu}}$, that intersect the circle C_R is also $\mathrm{O}(e^R)$ as $R \to \infty$.*

Proof. $\mathfrak{F}^{\mathrm{cu}}$ is finite, so we may restrict ourselves to considering one cusp $\kappa \in \mathfrak{F}^{\mathrm{cu}}$, which we conjugate to ∞.

In the counting of the horocycles, γ runs over $\Gamma_\infty \backslash \Gamma$. We write $\gamma^{-1} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The horocycle is determined by the first column $\begin{pmatrix} a \\ c \end{pmatrix}$ of γ^{-1}. The maximum of

$$x \mapsto \frac{\mathrm{Im}\,(\gamma^{-1}(iY + x))}{|\gamma^{-1}(iY + x) + i|^2} = \frac{Y}{(ax + b - cY)^2 + (aY + cx + d)^2}$$

occurs for $x = \frac{-ab-cd}{a^2+c^2}$, and has value $\frac{(a^2+c^2)\,Y}{\left(1+(a^2+c^2)\,Y\right)^2} \leq \frac{1}{(a^2+c^2)\,Y}$. Since $\frac{y}{|z+i|^2} \sim e^{-R}$ on C_R, the number of horocycles intersecting C_R is bounded by $N(e^R/Y)$, where

$$N(B) \;:=\; \#\Big\{ \gamma \in \Gamma_\infty \backslash \Gamma \;:\; \gamma \notin \Gamma_\infty\,,\; a^2 + c^2 \leq B \Big\}\,. \tag{16.4}$$

Lemma 2.10 in [14], applied with $z = i$ and $g_{\mathfrak{a}} = 1$, implies that $N(B) = \mathrm{O}(B)$ as $B \to \infty$. This gives the first statement in the lemma.

For a fixed horocycle $\gamma^{-1}H_Y(\kappa)$, there may be many cuspidal triangles $\gamma^{-1}V_\infty$ intersecting C_R. Their number differs by no more than 2 from the number of intervals $[-\frac{1}{2}+n, \frac{1}{2}+n]$, $n \in \mathbb{Z}$, containing an element x with $\frac{\mathrm{Im}\,(\gamma^{-1}(iY+x)}{|\gamma^{-1}(iY+x)+i|^2} \geq e^{-R}$. If such x occur, the equation $(ax+b-cY)^2 + (aY+cx+d)^2 = Ye^R$ has solutions x_1 and x_2. These solutions satisfy

$$(x_2-x_1)^2 = \frac{4}{a^2+c^2}\,(e^R-2)\,Y - 4Y^2 - \frac{4}{(a^2+c^2)^2} \leq \frac{4Y(1-2e^{-R})}{(a^2+c^2)e^{-R}} \ll \frac{Ye^R}{a^2+c^2}\,.$$

So the number of sectors is bounded by

$$2 + |x_2 - x_1| \ll \frac{\sqrt{Ye^R}}{\sqrt{a^2+c^2}}\,.$$

For $\gamma \in \Gamma_\infty$ this gives $\mathrm{O}\left(\sqrt{Ye^R}\right)$ sectors. The number of other sectors to be counted is estimated by

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma,\,\gamma\notin\Gamma_\infty,\\ a^2+c^2\leq e^R/Y}} \frac{\sqrt{Ye^R}}{\sqrt{a^2+c^2}} \ll \sum_{l=0}^{\infty} \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma,\,\gamma\notin\Gamma_\infty,\\ 2^{-l-1}e^R/Y\leq a^2+c^2\leq 2^{-l}e^R/Y}} \frac{\sqrt{Ye^R}}{2^{-(l+1)/2}\sqrt{e^R/Y}}$$
$$\ll Y\sum_{l=0}^{\infty} 2^{(l+1)/2}\,N(2^{-l}\,e^R/Y) \ll e^R\,,$$

where in the last line we have again used $N(B) = \mathrm{O}(B)$. ■

Proof of Lemma 16.3. Let $\Gamma(R) = \{\gamma \in \Gamma \;:\; \gamma^{-1}\mathfrak{F} \cap D_R \neq \emptyset\}$, where D_R denotes the hyperbolic disk around i with radius R. We put $A_R = \bigcup_{\gamma\in\Gamma(R)} \gamma^{-1}\mathfrak{F}$, and take $Z = \partial A_R$. Since $\partial\mathfrak{F}$ has finitely many edges, Z consists of at most $\mathrm{O}(\#\Gamma(R))$ edges. Each edge occurring in Z (with non-zero factor) has a distance at least R to i. The curve Z encircles D_R once in the positive direction. To complete the proof we show that $\#\Gamma(R) = \mathrm{O}(e^R)$ as $R \to \infty$.

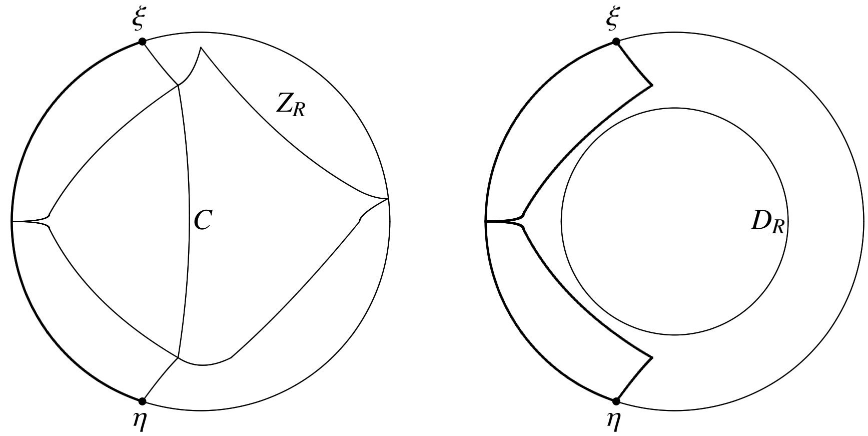
We use that $\mathfrak{F}$ is the union of a compact set $\mathfrak{F}_Y$ and finitely many cuspidal triangles V_κ, with $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. Lemma 16.4 estimates the number of $\gamma \in \Gamma$ such that $\gamma^{-1}V_\kappa$ intersects D_R for some $\kappa \in \mathfrak{F}^{\mathrm{cu}}$ by $\mathrm{O}(e^R)$. To count the number of $\gamma^{-1}\mathfrak{F}_Y$ intersecting D_R we note that the distance between any two points of $\gamma^{-1}\mathfrak{F}_Y$ is bounded by some number t, independently of $\gamma \in \Gamma$. Hence if $\gamma^{-1}\mathfrak{F}_Y$ intersects D_R, then $\gamma^{-1}\mathfrak{F}_Y$ is contained in D_{R+t}. This leads to an estimate by

$$\frac{\mathrm{area}(D_{R+t})}{\mathrm{area}(\mathfrak{F}_Y)} = \mathrm{O}(e^{R+t}) = \mathrm{O}(e^R)\,. \qquad ■$$

In the next subsection we will use the following modification of Lemma 16.3:

Lemma 16.5. *Suppose that ξ and η are cusps of Γ. Let $R > 0$, and denote by D_R the open hyperbolic disk around i with radius R. There exists a chain $A = A_R \in \mathbb{Z}[X_1^{\mathcal{T}}]$ consisting of $\mathrm{O}_{\xi,\eta}(e^R)$ edges in $\Gamma\partial\mathfrak{F}$ that describes a path from ξ to η in the region $\mathfrak{H} \smallsetminus D_R$ and is homotopic in $\bar{\mathfrak{H}} \smallsetminus D_R$ to the (oriented) arc in $\mathbb{P}^1_{\mathbb{R}}$ from ξ to η.*

Proof. Let $C \in \mathbb{Z}[X_1^{\mathcal{T}}]$ denote a path from ξ to η along edges of the tesselation. It consists of $\mathrm{O}_{\xi,\eta}(1)$ edges. Let Z_R be a cycle as in Lemma 16.3, consisting of $\mathrm{O}(e^R)$ edges and encircling D_R once.



The cycle Z_R and the direct path intersect each other in points of $X_0^{\mathcal{T}}$. As in the sketch on the right, this leads to a path with the desired properties, with $\mathrm{O}_{\xi,\eta}(1) + \mathrm{O}(e^R) + \mathrm{O}_{\xi,\eta}(1)$ edges, going from ξ along C to the first intersection point, then counterclockwise along Z_R to the last intersection point, and then along C to η. ∎

16.3. *Maass cusp forms associated to differentiable parabolic cohomology classes.* In the cocompact case, $\mathsf{Maass}_s^0(\Gamma) = \mathcal{E}_s^\Gamma$, and parabolic cohomology coincides with standard cohomology. In this case it only remains to show that the maps α_s^p are injective. If Γ has cusps, we need to show not only that α_s^p is injective, but also that its image is in the space of Maass cusp forms $\mathsf{Maass}_s^0(\Gamma)$. Surprisingly, the presence of cusps actually helps in proving the injectivity.

The cocompact case will be discussed in the next subsection. Here we suppose that Γ has cusps. We use a tesselation $\mathcal{T}$ of type **Fd**. (See §11.1.)

Let $u = u_\psi \in \mathcal{E}_s^\Gamma$ be the invariant eigenfunction associated to the cocycle $\psi \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{W}_p^s)$ via a lift $\tilde\psi \in C^1(F_\cdot^{\mathcal{T}}; \mathcal{G}_s^p)$. We use the following variant of the formula (16.3) in Proposition 16.1: if ξ and η are distinct cusps of Γ, we have the splitting

$$u = u_{\xi,\eta} + u_{\eta,\xi} \tag{16.5}$$

with

$$\begin{aligned} u_{\xi,\eta} &= \frac{1}{\pi i}\Big(\tilde\psi(A) + \sum_{\gamma\in\Gamma,\,\gamma^{-1}\mathfrak{F}\text{ to the right of }A} d\tilde\psi(\mathfrak{F})|\gamma\Big),\\ u_{\eta,\xi} &= \frac{1}{\pi i}\Big(-\tilde\psi(A) + \sum_{\gamma\in\Gamma,\,\gamma^{-1}\mathfrak{F}\text{ to the left of }A} d\tilde\psi(\mathfrak{F})|\gamma\Big), \end{aligned} \tag{16.6}$$

where the chain $A \in Z[X_1^{\mathcal{T}}]$ represents a path from ξ to η without self-intersections, following edges contained in $\Gamma\,\partial\mathfrak{F}$. The words "to the left" and "to the right" of A depend on the orientation of A: "to the left of A" is equivalent to "to the right of the opposite path $-A$." The definitions of $u_{\xi,\eta}$ and $u_{\eta,\xi}$ do not depend on the choice of path from ξ to η for the same reason that the right hand side of (16.3) was independent of the chosen path A.

Proposition 16.6. *For cusps ξ, η, θ satisfying $\xi < \eta < \theta < \xi$ for the cyclic order of $\partial\mathbb{H}$, and $\gamma \in \Gamma$ we have*

$$u_{\xi,\theta} = u_{\xi,\eta} + u_{\eta,\theta}\,, \tag{16.7a}$$

$$u_{\gamma^{-1}\xi,\gamma^{-1}\eta} = u_{\xi,\eta}|\gamma\,, \tag{16.7b}$$

$$u_{\xi,\eta} \in \mathcal{E}_s\,. \tag{16.7c}$$

Proof. The statements in (16.7a) and (16.7b) are clear from the definitions. For (16.7c) we proceed as in the proof that $u \in \mathcal{E}_s$ in Proposition 16.1, now with a sequence of paths A_R from ξ to η as in Lemma 16.5. ■

The following lemma implies that $u_{\xi,\eta}$ is relatively small near the arc from η to ξ in $\partial\mathfrak{H}$.

Lemma 16.7. *Let ψ, $\tilde{\psi}$ and $u_{\xi,\eta}$ be as above. We denote by $g = g_{\xi,\eta}$ the geodesic from ξ to η. For any choice of path A from ξ to η we have*

$$\begin{aligned} \sum_{\gamma\in\Gamma,\,\gamma^{-1}\mathfrak{F}\text{ to the right of }A} d\tilde{\psi}(\mathfrak{F})(\gamma z) &= \mathrm{O}\Big(\Big(\frac{y}{|z+i|^2}\Big)^{s+p-2}\Big) \\ \textit{and}\quad u_{\xi,\eta}(z) &= \mathrm{O}\Big(\Big(\frac{y}{|z+i|^2}\Big)^{s}\Big). \end{aligned} \tag{16.8}$$

In both estimates $z \to \partial\mathfrak{H}$ through the region to the left of g or on g.

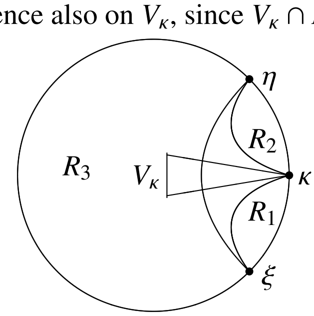
We first apply this result, postponing its proof.

Proposition 16.8. *Let $p \geq 2$. The function u_ψ is an element of $\mathsf{Maass}^0_s(\Gamma)$ for all $\psi \in Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{W}^p_s)$.*

Proof. Since $0 < \operatorname{Re} s < 1$, it suffices to show that u_ψ is bounded on the cuspidal sectors $V_\kappa \subset \mathfrak{F}$ for any $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. (This follows from the Fourier expansion at the cusp κ: see §10.1 and equation (8.1), and use the asymptotic behavior as $y \to \infty$ of the special functions in the expansion.) Take $\xi, \eta \in C$ with κ between ξ and η for the positive (counterclockwise) orientation of $\partial\mathbb{H}$ and such that V_κ is between the geodesic $g_{\kappa,\eta}$ from κ to η and the geodesic $g_{\xi,\kappa}$ from ξ to κ. Let R_1, R_2, R_3 denote the regions to the right of the geodesics $g_{\xi,\kappa}$, $g_{\kappa,\eta}$, and $g_{\eta,\xi}$, respectively, as in the picture below. We have

$$u = u_{\xi,\kappa} + u_{\kappa,\eta} + u_{\eta,\xi}\,. \tag{16.9}$$

By Lemma 16.7, $u_{\xi,\kappa}$ is bounded on the complement of R_1 (and hence also on V_κ), $u_{\kappa,\eta}$ is bounded on the complement of R_2 (and hence also on V_κ), and $u_{\xi,\eta}$ is bounded on the complement of R_3 (and hence also on V_κ, since $V_\kappa \cap R_3$ is compact). ■



Proposition 16.9. *The map* $[\psi] \mapsto u_\psi$ *from* $H^1_{\mathrm{par}}(\Gamma; \mathcal{W}^p_s)$ *to* $\mathcal{E}^\Gamma_s$ *is injective for* $p \geq 3$.

Proof. Suppose that $u_\psi = 0$. Let A be a path from the cusp ξ to the cusp η as in the definition of $u_{\xi,\eta}$ in (16.6). Lemma 16.7 shows that

$$\pi i\, u_{\xi,\eta}(z) \;=\; \tilde{\psi}(A)(z) + \mathrm{O}\Bigl(\Bigl(\frac{y}{|z+i|^2}\Bigr)^{s+p-2}\Bigr) \tag{16.10}$$

for z on or on the left of the geodesic from ξ to η, and also that $\pi i\, u_{\xi,\eta}(z) = -\pi i\, u_{\eta,\xi}(z)$ satisfies (16.10) on the right of that geodesic. Hence (16.10) holds for all $z \in \mathfrak{H}$.

Since $\tilde{\psi}(A) \in \mathcal{G}^p_s$, there is a continuous function B on a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$ such that

$$\tilde{\psi}(A)(z) \;=\; \Bigl(\frac{y}{|z+i|^2}\Bigr)^s B(z) + \mathrm{O}\Bigl(\Bigl(\frac{y}{|z+i|^2}\Bigr)^{s+1}\Bigr) \qquad (z \to \partial\mathfrak{H})\,.$$

With (16.10), and with the assumption $p \geq 3$, we get the same estimate for $\pi i\, u_{\xi,\eta}(z)$. Lemma 4.4 in [4] tells us that elements of $\mathcal{E}_s$ satisfying an estimate of this type vanish. We conclude that $u_{\xi,\eta} = 0$ for all cusps ξ and η.

Now we have by (16.10) that $\tilde{\psi}(A) \in \mathcal{N}^{p-2}_s$, so $\psi(A)$ is 0 in $\mathcal{W}^{p-2}_s$. Then $\psi(A) = 0$ because $\mathcal{W}^p_s \to \mathcal{W}^{p-2}_s$ is injective (since $\mathcal{W}^p_s \cong \mathcal{V}^p_s \subset \mathcal{V}^{p-2}_s \cong \mathcal{W}^{p-2}_s$).

Take a cusp ξ as base point. The group cocycle $\gamma \mapsto \psi(C_{\gamma^{-1}\xi,\xi})$ with $C_{\gamma^{-1}\xi,\xi}$ a path in $\mathbb{Z}[X^{\mathcal{T}}_1]$ from $\gamma^{-1}\xi$ to ξ is zero. Hence the class $[\psi] \in H^1_{\mathrm{par}}(\mathcal{G}, m; \mathcal{W}^p_s)$ is the trivial cohomology class. ■

We summarize the results in the following proposition:

Proposition 16.10. *Suppose that* Γ *has cusps. For all* $p, q \in \mathbb{N}$, $q \geq p \geq 3$, *the following diagram is commutative and all arrows are isomorphisms.*

$$\begin{tikzcd}
 & \mathrm{Maass}^0_s(\Gamma) \arrow[dl, shift left, "\mathbf{q}"] & & \\
H^1_{\mathrm{par}}(\mathcal{W}^\omega_s, \mathcal{W}^{\omega^0,\infty}_s) \arrow[ur, shift left, "\alpha^\omega_s"] \arrow[r] & H^1_{\mathrm{par}}(\mathcal{W}^{\omega^0,\infty}_s) \arrow[r] & H_{\mathrm{par}}(\mathcal{W}^q_s) \arrow[ul, "\alpha^q_s"'] \arrow[r] & H_{\mathrm{par}}(\mathcal{W}^p_s) \arrow[ull, "\alpha^p_s"']
\end{tikzcd}$$

It remains to prove the estimate (16.8).

Proof of Lemma 16.7. Since $\tilde{\psi}(C) \in \mathcal{G}^p_s$, the second estimate in (16.8) follows from the first.

By conjugation and symmetry it suffices to consider $\xi = \infty$, $\eta = 0$. Then $g = i\,\mathbb{R}_+$. We can assume that $i \in \mathfrak{F}$. Changing A means adding an element of $\mathcal{N}^p_s$ to the sum, and does not influence the estimate. So we can assume that A runs from ∞ to 0 through the left half of the upper half-plane. It now suffices to estimate for $z \in \mathfrak{H}$ with $\mathrm{Re}\, z \geq 0$ the sum

$$\sum_\gamma h(\gamma z) \;\ll\; \sum_\gamma \Bigl(\frac{\mathrm{Im}\,\gamma z}{|\gamma z + i|^2}\Bigr)^{s+p},$$

where $\gamma \in \Gamma$ is such that $\gamma^{-1}\mathfrak{F}$ is in the left half of $\mathfrak{H}$ and where $h = d\tilde{\psi}(\mathfrak{F}) \in \mathcal{N}^p_s$.

For any $\gamma = \bigl[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\bigr]$ in the sum we have $\mathrm{Re}\,\gamma^{-1} i < 0$, so $ab + cd > 0$. Moreover, ∞ is a cusp of Γ, so $\mathrm{Im}\,\gamma^{-1} i$ stays under a bound B_∞, depending only on the group Γ and the cusp ∞. Hence $a^2 + c^2 \geq \frac{1}{B_\infty}$. For $\operatorname{Re} z \geq 0$, $|z| \geq 1$:

$$\begin{aligned} \frac{\operatorname{Im} \gamma z}{|\gamma z + i|^2} \Big/ \frac{y}{|z+i|^2} \;&=\; \frac{|z|^2 + 2y + 1}{(a^2+c^2)|z|^2 + 2(ab+cd)x + 2y + b^2 + d^2} \\ &\leq\; \frac{|z|^2 + 2|z| + 1}{(a^2+c^2)|z|^2} \;\leq\; 4B_\infty\,. \end{aligned} \tag{16.11a}$$

We apply the reflection $z \mapsto 1/\bar{z}$ to conclude that there is a bound $B_0 > 0$ such that for $\operatorname{Re} z \geq 0$, $|z| \leq 1$:

$$\frac{\operatorname{Im} \gamma z}{|\gamma z + i|^2} \Big/ \frac{y}{|z+i|^2} \;\leq\; 4B_0\,. \tag{16.11b}$$

In the proof of Lemma 16.3 we saw that the number of Γ-translates of $\mathfrak{F}$ that intersect the closed disk D_k around i with radius k is $\mathrm{O}(e^k)$ as $k \to \infty$. We use this to estimate the size of the set F_k of γ occurring in the sum such that $\gamma^{-1}\mathfrak{F}$ intersects D_k but does not intersect D_{k-1}. We use the estimates in (16.11) for the sum up to $k = k_0 - 1$, with k_0 to be chosen later.

$$\sum_{k=0}^{k_0-1} \sum_{\gamma \in F_k} h(\gamma z) \;\ll\; \sum_{k=0}^{k_0-1} e^k \Big(\frac{y}{|z+i|^2}\Big)^{s+p} \;\ll\; e^{k_0} \Big(\frac{y}{|z+i|^2}\Big)^{s+p}.$$

For the tail of the series, we employ another estimate. Applying the triangle inequality $\mathrm{d}(\gamma z, i) + \mathrm{d}(z,i) \geq \mathrm{d}(\gamma^{-1} i, i)$, and using $\frac{y}{|z+i|^2} \sim e^{-\mathrm{d}(z,i)}$ as $z \to \partial\mathfrak{H}$, we find

$$\frac{\operatorname{Im} \gamma z}{|\gamma z + i|^2} \, \frac{y}{|z+i|^2} \;\sim\; e^{-\mathrm{d}(\gamma z, i) - \mathrm{d}(z,i)} \;\leq\; e^{-\mathrm{d}(\gamma^{-1} i, i)} \;\ll\; \frac{\operatorname{Im} \gamma^{-1} i}{|\gamma^{-1} i + i|^2}\,,$$

and if $\gamma^{-1}\mathfrak{F}$ has distance at least $k-1$ to i, then

$$\frac{\operatorname{Im} \gamma^{-1} i}{|\gamma^{-1} i + i|^2} \;\ll\; e^{-k}\,.$$

This leads to

$$\begin{aligned} \sum_{k \geq k_0} \sum_{\gamma \in F_k} h(\gamma z) \;&\ll\; \sum_{k \geq k_0} e^k \Big(\frac{\operatorname{Im} \gamma^{-1} i}{|\gamma^{-1} i + i|^2}\Big)^{s+p} \Big(\frac{y}{|z+i|^2}\Big)^{-s-p} \\ &\ll\; \sum_{k \geq k_0} e^{(1-s-p)k} \Big(\frac{y}{|z+i|^2}\Big)^{-s-p} \;\ll\; e^{(1-s-p)k_0} \Big(\frac{y}{|z+i|^2}\Big)^{-s-p}. \end{aligned}$$

Combining both estimates we obtain for the total sum:

$$\sum_{\gamma} h(\gamma z) \;\ll\; e^{k_0 - (\operatorname{Re} s + p)\mathrm{d}(z,i)} + e^{(1 - \operatorname{Re} s - p)k_0 + (\operatorname{Re} s + p)\mathrm{d}(z,i)}\,.$$

We take $k_0 = [2\mathrm{d}(z,i)]$, and obtain

$$\ll\; e^{(2 - \operatorname{Re} s - p)\mathrm{d}(z,i)}\,,$$

which gives the desired estimate of the sum. ∎

16.4. *Injectivity of the map from differentiable cohomology to invariant eigenfunctions, cocompact case.* We can also prove the injectivity of the map from $\mathcal{W}_s^p$-valued cohomology classes to invariant eigenfunctions in the cocompact case, now even for $p \geq 2$, but the proof in the absence of cusps is harder, because the chain A used in (16.6) must be replaced by an infinite path. We will show:

Proposition 16.11. *Let Γ be cocompact, and let $p, q \in \mathbb{N}$, $q \geq p \geq 2$. Then the following diagram is commutative and all arrows are isomorphisms.*

$$\begin{tikzcd}
 & \mathcal{E}_s^\Gamma \arrow[dl, shift left, "\mathbf{q}"] & \\
H^1(\mathcal{W}_s^\omega) \arrow[ur, shift left, "\alpha_s^\omega"] \arrow[r] & H(\mathcal{W}_s^q) \arrow[u, "\alpha_s^q"] \arrow[r] & H(\mathcal{W}_s^p) \arrow[ul, "\alpha_s^p"']
\end{tikzcd}$$

Only the injectivity of $[\psi] \mapsto u_\psi$ from $H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^p)$ to $\mathcal{E}_s^\Gamma$ remains to be proved. In the non-cocompact case we used the decomposition (16.5) to write the eigenform u_ψ as a sum of two pieces associated to paths between cusps. Now there are no edges in $X_1^{\mathcal{T}}$ that go to points of the boundary $\partial\mathbb{H}$ and we will use paths between *hyperbolic* points instead. To do this we will extend a lift $\tilde{\psi} \in C^1(F_{\cdot}^{\mathcal{T}}; \mathcal{G}_s^p)$ of ψ to a function defined on infinite paths going from a point $P \in X_0^{\mathcal{T}} \subset \mathbb{H}$ to a hyperbolic fixed point on $\partial\mathbb{H}$, like we did in §7.3 in the analytic case, using a one-sided average. However, because our knowledge of the behavior of the extension near the hyperbolic fixed point is incomplete, we are forced to perform some complicated estimates.

The proof will consist of five steps, of which steps a) and b) were not needed for groups with cusps, and step d) requires more work than in the previous subsection.

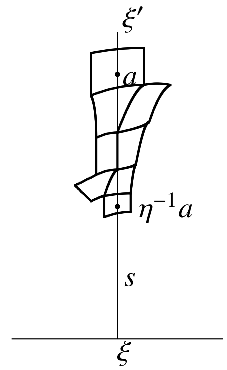
a) Choice of paths from points of $X_0^{\mathcal{T}}$ to hyperbolic fixed points.
b) Extension of $\tilde{\psi}$ to these paths.
c) Definition of $u_{\tilde{\psi}}(\xi_1, \xi_2)$ for hyperbolic fixed points ξ_1 and ξ_2.
d) Proof that if $u_\psi = 0$ then $u_{\tilde{\psi}}(\xi_1, \xi_2) = 0$.
e) Proof that if $u_\psi = 0$ then $[\psi]$ is the trivial cohomology class.

• *Step a).* We assume that the tesselation $\mathcal{T}$ of type **Fd** is based on a Dirichlet fundamental domain $\mathfrak{F}$. We choose once and for all a Γ-orbit H of hyperbolic fixed points of Γ, and consider only hyperbolic points belonging to this orbit. For each $\xi \in H$ we denote by $\eta_\xi \in \Gamma$ the generator of Γ_ξ that has ξ as its repelling fixed point.

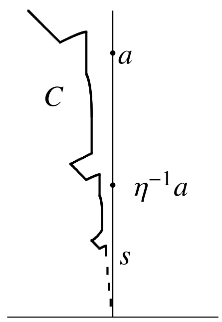
In the proof of Proposition 7.4 we extended a cocycle on $X_0^{\mathcal{T}} \times X_0^{\mathcal{T}}$ to a cocycle on $\left(X_0^{\mathcal{T}} \cup H\right) \times \left(X_0^{\mathcal{T}} \cup H\right)$. Here we work with a cochain $\tilde{\psi}$, and have to deal with actual paths between points of $X_0^{\mathcal{T}} \cup H$. We choose for each $\xi \in H$ and each $P \in X_0^{\mathcal{T}} \subset \mathbb{H}$ an infinite chain $p(P, \xi)$ of elements of $X_1^{\mathcal{T}}$, describing a path from P to ξ. We require:

(i) Γ-*equivariance*, *i.e.*, $\gamma p(P, \xi) = p(\gamma P, \gamma \xi)$ for all $\gamma \in \Gamma$,
(ii) $p(P, \xi)$ has no self-intersection,
(iii) for $P, Q \in X_0^{\mathcal{T}}$ and $\xi \in H$ the difference $p(Q, \xi) - p(P, \xi)$ is finite, *i.e.*, it is in $\mathbb{Z}[X_1^{\mathcal{T}}]$

To see that such a choice is possible we start with one $\xi \in H$, and abbreviate $\eta_\xi = \eta$. Let s be the geodesic connecting ξ with the other fixed point ξ' of η. (In the sketch on the right in the upper half-plane model we place ξ at $0 \in \mathbb{P}^1_{\mathbb{R}}$ and ξ' at $\infty \in \mathbb{P}^1_{\mathbb{R}}$.) We choose a point a on s and cover the segment of s between a and $\eta^{-1}a$ with the finitely many translates $\gamma\mathfrak{F}$ such that the closure $\overline{\gamma\mathfrak{F}}$ contains the segment between these two points.



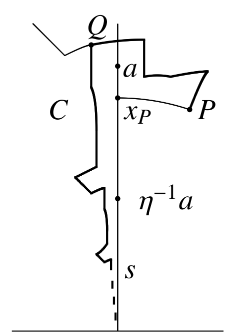
Next we take the infinite cover of s consisting of all translates by η^n, $n \in \mathbb{Z}$, of the cover of the segment between a and $\eta^{-1}a$.



The union of all these translates is a connected simply connected region in $\mathbb{H}$. The boundary of this region consists of two components. We take the component on the right of the geodesic with respect to the direction from ξ' to ξ.
This component can be described as an infinite chain of the form $C = \sum_{i=-\infty}^{\infty} \varepsilon_i \, x_i$ with $x_i \in X_1^{\mathcal{T}}$ and $\varepsilon_i \in \{1, -1\}$. The chain is η-invariant: there exists $n \geq 1$ such that $\eta x_i = x_{i+n}$. It has no self-intersections.

We use this chain to form infinite paths from each $P \in X_0^{\mathcal{T}}$ to ξ. Through each $P \in X_0^{\mathcal{T}}$ there is a unique geodesic g_P that intersects s orthogonally. We denote the intersection point by x_P. If P happens to be a point of s we take $x_P = P$.

First we consider those points $P \in X_0^{\mathcal{T}}$ for which x_p is between a and $\eta^{-1}a$, or is equal to a. For each of these points P we choose a chain $r \in \mathbb{Z}[X_1^{\mathcal{T}}]$ describing a path from P to a point Q in $C \cap X_0^{\mathcal{T}}$. This can be done in such a way that the path corresponding to r intersects the path given by C only in Q. Then we choose $p(P, \xi)$ as the sum of r and the part of the chain C describing a path from Q to ξ.



The path $x(P, \xi)$ that we have constructed has no self-intersection. If $P, R \in X_0^{\mathcal{T}}$ satisfy $x_P, x_R \in [a, \eta^{-1}a)$ then $p(R, \xi)$ and $p(P, \xi)$ form the same path, except for an initial part. So requirement (iii) is satisfied. Thus we have completed the choice of paths going to the fixed $\xi \in H$.

he condition on the intersection points x_P implies that the set $H \times X_0^{\mathcal{T}}$ is freely generated by the (ξ, P) for which we have defined $p(P, \xi)$. We put $p(\Gamma P, \gamma\xi) = \gamma p(P, \xi)$ for $\gamma \in \Gamma$ to define p on $H \times X_0^{\mathcal{T}}$. This choice clearly satisfies requirements (i) and (ii). For requirement (iii) we observe that $p(\eta^{-n}P, \xi)$ and $p(P, \xi)$ differ only in an initial part.

In the sequel we assume that $p(P,\xi)$ for $\xi \in H$ and $P \in X_0^{\mathcal{T}}$ has been chosen such that requirements (i)–(iii) hold.

• *Step b).* The lift $\tilde\psi \in C^1(F_{\boldsymbol{\cdot}}^{\mathcal{T}}; \mathcal{G}_s^p)$ of the given cocycle $\psi \in Z^1(F_{\boldsymbol{\cdot}}^{\mathcal{T}}; \mathcal{W}_s^p)$ is defined on paths in $\mathbb{Z}[X_1^{\mathcal{T}}]$. For $P \in X_0^{\mathcal{T}}$ and $\xi \in H$ we have arranged that $p(P,\xi) - \eta_\xi^{-1} p(P,\xi) \in \mathbb{Z}[X_1^{\mathcal{T}}]$. So $\tilde\psi(p(P,\xi) - \eta_\xi^{-1} p(P,\xi))$ is an element of $\mathcal{G}_s^p$. We use the one-sided average $\mathrm{Av}_{\eta_\xi}^+$ discussed in §4.1 to define:

$$F_{\tilde\psi}(P,\xi) \;=\; \mathrm{Av}_{\eta_\xi}^+ \tilde\psi((1-\eta_\zeta^{-1})p(P,\xi)) \;\in\; \mathcal{G}_s^p(\partial\mathbb{H} \smallsetminus \{\xi\})\,. \tag{16.12}$$

The dependence of $F_{\tilde\psi}(P,\xi)$ on the choice of the path $p(P,\xi)$ is not visible in the notation. We think of $F_{\tilde\psi}(P,\xi)$ as $\tilde\psi$ evaluated on the infinite path $p(P,\xi)$.

Near points of $\partial\mathbb{H} \smallsetminus \{\xi\}$ we have good information on the behavior of $F_{\tilde\psi}(P,\xi)$. It is locally (in the disk model) of the form $w \mapsto (1-|w|^2)^s \cdot (\text{analytic})$, as defined in §3.3. The next lemma gives information concerning the behavior near the point ξ. We formulate it in the upper half-plane model, with ξ at position zero, and use polar coordinates $\mathfrak{H} \ni z = \rho e^{i\varphi}$ $(\rho > 0, \varphi \in (0,\pi))$.

Lemma 16.12. *Let $g \in G$ such that $g \cdot 0 = \xi$, in the upper half-plane model, and $g\eta_\xi g^{-1} = \left[\begin{smallmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{smallmatrix}\right]$ with $t > 1$. Then we have, uniformly for $0 < \varphi < \pi$:*

$$\begin{aligned} F_{\tilde\psi}(P,\xi)(g \cdot \rho e^{i\phi}) \;&\ll\; 1 && (\rho \downarrow 0)\,,\\ \partial_\rho F_{\tilde\psi}(P,\xi)(g \cdot \rho e^{i\phi}) \;&\ll\; \rho^{-1} && (\rho \downarrow 0)\,. \end{aligned}$$

Proof. This is a statement concerning $H(\rho e^{i\phi}) = \sum_{n\geq 0} h(t^n x)$ for some $h \in \mathcal{G}_s^p$. There are $R_\infty > R_0 > 0$ such that

$$h(\rho e^{i\phi}) \;=\; \begin{cases} \rho^s (\sin\phi)^s a_0(\rho\cos\phi, \rho\sin\phi) & \text{for } 0 < \rho \leq R_0\,,\\ \mathrm{O}(1) & \text{for } R_0 \leq \rho \leq R_\infty\,,\\ \rho^{-s}(\sin\phi)^s a_\infty(-\rho^{-1}\cos\phi, \rho^{-1}\sin\phi) & \text{for } \rho \geq R_\infty\,, \end{cases}$$

with C^p-functions a_0 and a_∞ on a neighborhood of 0 in $\mathbb{R}^2$ containing a disk of radius R_0, respectively R_∞^{-1}. In the intermediate region we also have $\partial_\rho h(\rho e^{i\phi}) = \mathrm{O}(1)$.

For $\rho < R_0$ we split up the sum at $B = -\frac{\log(\rho/R_0)}{\log t}$ and $A = -\frac{\log(\rho/R_\infty)}{\log t}$. The sum over $0 \leq n < B$ contributes

$$\begin{aligned} \text{to } H(\rho e^{i\phi}) : \quad & \sum_{0\leq n<B} t^{ns}\rho^s (\sin\phi)^s a_0(t^n\rho\cos\phi, t^n\rho\sin\phi) \;\ll\; \rho^s(\sin\phi)^s t^{Bs}\\ & \ll\; \rho^s (\sin\phi)^s \rho^{-s} \;\ll\; 1\,,\\ \text{to } \partial_\rho H(\rho e^{i\phi}) : \quad \ll\; & \sum_{0\leq n<B} \rho^s (\sin\phi)^s \left(t^{ns}\rho^{-1} + t^{n(s+1)}\right) \;\ll\; \rho^{-1}\,. \end{aligned}$$

The sum over $n \geq A$ contributes

$$\begin{aligned} \text{to } H(\rho e^{i\phi}) : \quad & \sum_{n\geq A} t^{-ns}\rho^{-s}(\sin\phi)^s a_\infty(-t^{-n}\rho^{-1}\cos\phi, t^{-n}\rho^{-1}\sin\phi)\\ & \ll\; \rho^s\rho^{-s} \;\ll\; 1\,,\\ \text{to } \partial_\rho H(\rho e^{i\phi}) : \quad & \sum_{n\geq A} t^{-n(s+1)}\rho^{-s}(\sin\phi)^s \left(s\rho^{-1}\mathrm{O}(1) + \rho^{-2}\mathrm{O}(1)\right) \;\ll\; \rho^{-1}\,. \end{aligned}$$

The region $B \le n < A$ contributes $\mathrm{O}(A-B) = \mathrm{O}(1)$ to $H(\rho e^{i\phi})$, and to $\partial_\rho H(\rho e^{i\phi})$

$$\sum_{B\le n<A} t^n \mathrm{O}(1) \ll t^A \ll \rho^{-1}.$$

On this region we do not obtain a factor $\rho^s(\sin\varphi)^s$. So we have to be content with $\mathrm{O}(1)$ and $\mathrm{O}(\rho^{-1})$ as the final estimates. ■

Let $p(\cdot,\cdot)$ and $\hat p(\cdot,\cdot)$ denote two choices of paths, both satisfying the requirements. Then $\hat p(P,\xi) - p(P,\xi)$ can be written as an infinite sum $\sum_{\gamma\in\Gamma} m_\gamma\,(\gamma^{-1}\mathfrak{F})$ of elements of $X_2^{\mathcal{T}}$, with $m_\gamma = \pm 1$ if $\gamma^{-1}\mathfrak{F}$ is between both paths, with the choice of the sign depending on the winding number of $\hat p(P,\xi) - p(P,\xi)$ around $\gamma^{-1}\mathfrak{F}$, and $m_\gamma = 0$ for all other $\gamma^{-1}\mathfrak{F}$. For the influence of the choice of the path on $F_{\tilde\psi}(P,\xi)$ we would like to get an estimate of the sum

$$\sum_\gamma m_\gamma\, d\tilde\psi(\gamma^{-1}\mathfrak{F}). \tag{16.13}$$

For later use we formulate this more generally. We need estimates for absolutely converging sums over γ such that $\gamma^{-1}\mathfrak{F}$ is contained in a set X. Near pieces of the boundary away from the closure of X in $\mathbb{P}^1_{\mathbb{C}}$ the estimates are better. In the proof of Lemma 16.16 we will need also an estimate for derivatives of the sum.

For convenience we use in the following lemma the disk model $\mathbb{H} = \mathbb{D}$, with coordinates $w = re^{i\theta}$, $0 \le r < 1$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Lemma 16.13. *Let X be a union of Γ-translates of $\mathfrak{F}$. Denote by $\bar X$ the closure of X in $\mathbb{D}\cup\partial\mathbb{D}$. Let $h \in \mathcal{N}_s^p$, $p = 2,3,\ldots$, let $\{\varepsilon_\gamma : \Gamma\}$ be a bounded set of complex numbers, and put*

$$H = \sum_{\gamma\in\Gamma,\ \gamma^{-1}\mathfrak{F}\subset X} \varepsilon_\gamma\, h|\gamma.$$

Then for any $w \in \mathbb{D}$ with hyperbolic distance $\mathrm{d}(w,X)$ to X at least equal to $R \ge 0$

$$H(w) \ll e^{-(s+p-1)R}, \quad (1-r^2)\,\partial_r H(re^{i\theta}),\ (1-r^2)\,\partial_\theta H(re^{i\theta}) \ll e^{-(s+p-1)R}. \tag{16.14}$$

The differential operators operators $(1-r^2)\,\partial_r$ and $r^{-1}(1-r^2)\,\partial_\theta$ are natural in the coordinates r and θ, since they are Γ-equivariant up to a factor of absolute value one.

Proof. In fact, we will prove the estimates (16.14) for the functions obtained by replacing all terms in the defining sums by their absolute values.

Near the boundary, $h(re^{i\theta}) = (1-r^2)^s\, a(re^{i\theta})$ with a C^p-function a on a neighborhood of the boundary, vanishing up to order p on the boundary. Hence $h(re^{i\theta}) = \mathrm{O}((1-r^2)^{s+p})$ and $\partial_r h(re^{i\theta})$ and $\partial_\theta h(re^{i\theta})$ are $\mathrm{O}((1-r^2)^{s+p-1})$. Since h is a C^2-function, we can use these estimate everywhere on $\mathbb{D}$. To estimate $\partial_r \sum h\circ\gamma$ and $\partial_\theta \sum h\circ\gamma$ we use $\sum\left|\frac{d\gamma w}{dw}\, h_r(\gamma w)\right|$ and $\sum\left|\frac{d\gamma w}{dw}\, h_\theta(\gamma w)\right|$, and note that $\left|\frac{d\gamma w}{dw}\right| = \frac{1-|\gamma w|^2}{1-|w|^2}$. (We use that h_r and h_θ are linear combinations of h_w and $h_{\bar w}$ with bounded coefficients.) So for H and for its derivatives we have to deal with sums of the type

$$\sum_{\gamma\in\Gamma,\,\gamma^{-1}\mathfrak{F}\subset X} (1-|\gamma w|^2)^{s+q}, \tag{16.15}$$

with $q = p$ or $q = p-1$, and have to add a factor $(1-r^2)$ for the derivatives.

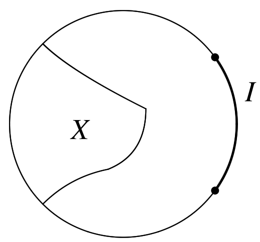
We can assume that $0 \in \mathfrak{F}$. For all $\gamma \in \Gamma$

$$1 - |\gamma w|^2 \ll e^{-\mathrm{d}(\gamma w, 0)} = e^{-\mathrm{d}(w, \gamma^{-1} 0)} \leq e^{-\mathrm{d}(w, \gamma^{-1}\mathfrak{F})}.$$

An area consideration shows that the number of $\gamma \in \Gamma$ such that $\mathrm{d}(w, \gamma^{-1}\mathfrak{F}) \leq R$ is $\mathrm{O}(e^R)$. Hence the sum in (16.15) satisfies

$$\ll \sum_{l \geq 1} e^{\mathrm{d}(w,X)+l} e^{-(s+q)(l+\mathrm{d}(w,X))} \ll e^{-(s+q-1)\mathrm{d}(w,X)}.$$

This gives (16.14). ■



Lemma 16.13 has as a direct consequence that if I is a closed cyclic interval in $\mathbb{S}^1 = \partial\mathbb{D}$ contained in the open set $\partial\mathbb{D} \smallsetminus \bar{X}$, then as $r \uparrow 1$, uniform in $e^{i\theta} \in I$:

$$\begin{aligned} H(re^{i\theta}) &\ll (1-r^2)^{s+p-1}, \\ \partial_r H(re^{i\theta}) &\ll (1-r^2)^{s+p-2}. \end{aligned} \tag{16.16}$$

To see this we simply observe that $e^{-\mathrm{d}(w,X)} = \mathrm{O}((1-r^2)$ uniformly, because the nearest point of X to any point near I lies in a fixed compact subset of X.

If we have two choices $\hat{p}(P,\xi)$ and $p(P,\xi)$ for the path from P to ξ we get for the difference of the corresponding values $\hat{F}_{\tilde{\psi}}(P,\xi)$ and $F_{\tilde{\psi}}(P,\xi)$ the expression

$$\sum_{\gamma \in \Gamma} m_\gamma \, d\tilde{\psi}(\gamma^{-1}\mathfrak{F}) = \sum_{\gamma \in \Gamma} m_\gamma \, d\tilde{\psi}(\mathfrak{F})|\gamma$$

with m_γ as in (16.13). We have $d\tilde{\psi}(\mathfrak{F}) \in \mathcal{N}_s^p$, and the estimate (16.16) shows that this sum is estimated by $\mathrm{O}((1-r^2)^{s+p-1})$ near points of $\partial\mathbb{D} \setminus \{\xi\} =: I$. The restriction morphism $\rho_s : \mathcal{W}_s^p(I) \to \mathcal{V}_s^p(I)$ in §3.3 is given by $\rho_s f(\xi) = \lim_{r\uparrow 1}((1-r^2))^{-s} f(re^{i\theta}))$ on the class in $\mathcal{W}_s^p(I)$ represented by $f \in \mathcal{G}_s^p(I)$. Thus, we obtain the following result:

Lemma 16.14. *The element $\rho_s F_{\tilde{\psi}}(P,\xi) \in \mathcal{V}_s^p(\partial\mathbb{D} \smallsetminus \{\xi\})$ does not depend on the choice of $p(P,\xi)$, and satisfies*

$$\rho_s F_{\tilde{\psi}}(\gamma^{-1}P, \gamma^{-1}\xi) = \bigl(\rho_s F_{\tilde{\psi}}(P,\xi)\bigr)|\gamma \qquad (\gamma \in \Gamma), \tag{16.17}$$

and

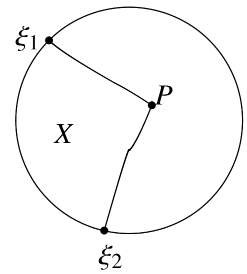
$$\rho_s F_{\tilde{\psi}}(P_1,\xi) = \rho_s F_{\tilde{\psi}}(P,\xi) + \psi(r) \tag{16.18}$$

for $P_1, P \in X_0^{\mathcal{T}}$, $\xi \in H$ and $r \in \mathbb{Z}[X_1^{\mathcal{T}}]$ corresponding to a path from P_1 to P.

Thus we get a cocycle c on $(X_0^{\mathcal{T}} \cup H) \times (X_0^{\mathcal{T}} \cup H)$ with values in a Γ-module containing $\mathcal{V}_s^p$ in which singularities at points in H are allowed. The restriction of c to $X_0^{\mathcal{T}} \times X_0^{\mathcal{T}}$ is related to $\psi \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{W}_s^p)$ by $c(P,Q) = \rho_s \psi(p)$, where $p \in \mathbb{Z}[X_1^{\mathcal{T}}]$ is a path from P to Q. For $P \in X_0^{\mathcal{T}}$ and $\xi \in H$ we have $c(P,\xi) = \rho_s F_{\tilde{\psi}}(P,\xi)$.

• *Step c).* Let $\xi_1, \xi_2 \in H$, and choose $P \in X_0^{\mathcal{T}}$. Motivated by (16.6) we might consider

$$F_{\tilde\psi}(P,\xi_2) - F_{\tilde\psi}(P,\xi_1) + \sum_{\gamma\in\Gamma,\, \gamma^{-1}\mathfrak{F}\subset X} d\tilde\psi(\mathfrak{F})|\gamma\,,$$



where X is in the region indicated on the left. However, this sketch is misleading, because the paths $p(P,\xi_1)$ and $p(P,\xi_2)$ may intersect each other.

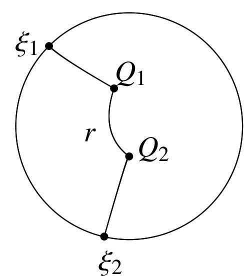
We need to work with a sum $\sum_\gamma m_\gamma\, d\tilde\psi(\mathfrak{F})|\gamma$, where m_γ is the winding number around $\gamma^{-1}\mathfrak{F}$ of the closed path $-p(P,\xi_2) + p(P,\xi_1)$ from ξ_2 via P to ξ_1 and then back to ξ_2 along the positively oriented arc of $\partial\mathbb{D}$ from ξ_1 to ξ_2. (In the sketch above, where $p(P,\xi_1)$ and $p(P,\xi_2)$ do not intersect each other except in P, we have $m_\gamma = 1$ if $\gamma^{-1}\mathfrak{F}\subset X$ and $m_\gamma = 0$ otherwise.) We define

$$u_{\tilde\psi}(\xi_1,\xi_2) \;=\; \frac{1}{\pi i}\Big(F_{\tilde\psi}(P,\xi_2) - F_{\tilde\psi}(P,\xi_1) + \sum_{\gamma\in\Gamma} m_\gamma\, d\tilde\psi(\mathfrak{F})|\gamma\Big). \tag{16.19}$$

There is a more general representation

$$u_{\tilde\psi}(\xi_1,\xi_2) \;=\; \frac{1}{\pi i}\Big(\tilde\psi(A) + \sum_{\gamma\in\Gamma} m_\gamma\, d\tilde\psi(\mathfrak{F})|\gamma\Big), \tag{16.20}$$

where A is any infinite path given by an infinite chain of the form $-p(Q_1,\xi_1) + r + p(Q_2,\xi_2)$, where $Q_1, Q_2 \in X_0^{\mathcal{T}}$, where $r\in\mathbb{Z}[X_1^{\mathcal{T}}]$ corresponds to a path from Q_1 to Q_2, and where m_γ is the winding number around $\gamma^{-1}\mathfrak{F}$ of the closed path consisting of $-A$ and the positively oriented arc in $\partial\mathbb{D}$ from ξ_1 to ξ_2.



Lemma 16.15. *We have $u_{\tilde\psi}(\xi_1,\xi_2)\in\mathcal{E}_s$ and*

$$u_{\tilde\psi}(\xi_1,\xi_2) + u_{\tilde\psi}(\xi_2,\xi_1) \;=\; u_\psi\,. \tag{16.21}$$

Proof. The second assertion follows directly from (16.2) and (16.20).

To see that $u_{\tilde\psi}(\xi_1,\xi_2)$ is in $\mathcal{E}_s$ we want to apply the same reasoning as in Proposition 16.1. We need a sequence A_l of infinite paths from ξ_1 to ξ_2 that have distance at least l to a fixed point P_0. To construct such a sequence we consider first the sequences $(\eta_1^{-n}P_0)$ and $(\eta_2^{-n}P_0)$ tending to ξ_1 and ξ_2, respectively. Put $p_n = \eta_1^{-n}p(P_0,\xi_1) - \eta_1^{-n-1}p(P_0,\xi_2)$. This is a finite path from $\eta_1^{-n}P_0$ to $\eta_1^{-n-1}P_0$, and $F_{\tilde\psi}(P_0,\xi_1)$ is equal to $\sum_{n\ge0}\tilde\psi(p_0)|\eta_1^n$. (The infinite chain $\sum_{n\ge0}p_n$ describes an infinite path from P_0 to ξ_1, that will in general be different from $p(P_0,\xi_1)$.) For each n the maximum of the distance between two points of p_n does not depend on n. The sequence of finite paths $q_n = \eta_2^{-n}p(P_0,\xi_2) - \eta_2^{-n-1}p(P_0,\xi_2)$ has similar properties.

For each $l\ge1$ there is by Lemma 16.3 a cycle Z_l going around $\mathbb{D}$ once in the positive direction in the region with hyperbolic distance at least l to P_0. There are points $P_l\in X_0^{\mathcal{T}}$ in the intersection of Z_l with $\sum_{n\ge0}p_n$, and $Q_l\in X_0^{\mathcal{T}}$ in the intersection of Z_l with the infinite path $\sum_{n\ge0}q_n$. Take n_1 such that P_l is on the finite path p_{n_1}, and

n_2 such that Q_l is on the path a_{n_2}. We form the chain A_l as the sum of the following chains:

			from	to
a_1	$=$	$-\sum_{n\geq n_1} p_n$	ξ_1	$\eta_1^{-n_1} P_0\,,$
a_2		along edges of p_{n_1}	$\eta_1^{-n_1} P_0$	$P_l\,,$
a_3		along edges of C_l	P_l	$Q_l\,,$
a_4		along edges of q_{n_2}	Q_l	$\eta_2^{-n_2} P_0\,,$
a_5	$=$	$\sum_{n\geq n_2} q_n$	$\eta_2^{-n_2} P_0$	$\xi_2\,.$

To apply the method used in the proof of Proposition 16.1 we estimate $(\Delta-\lambda_s)\tilde{\psi}(a_j)$ for each of these paths. We have $(\Delta-\lambda_s)\tilde{\psi}(a_3) = \mathrm{O}(e^l)\,\mathrm{O}(e^{-(s+p)l}) = o(1)$. The paths a_2 and a_4 consist of finitely many edges, and have distance at least $l-\mathrm{O}(1)$ to P_0. This leads to $(\Delta-\lambda_s)\tilde{\psi}(a_i) = o(1)$ for $i = 2,4$. The path q_{n_2} has distance at least $l-\mathrm{O}(1)$ to P_0, and there exists a factor $\alpha > 0$ such that $\eta_2^{-n} q_{n_2}$ has distance at least $\alpha n + l - \mathrm{O}(1)$ to P_0. Hence $(\Delta-\lambda_s)\tilde{\psi}(a_5) \ll \sum_{n\geq 0} \mathrm{O}(e^{-(s+p)(\alpha n+l)}) = o(1)$, and similarly for a_1.

Now we can proceed as in the proof of Proposition 16.1. ■

- *Step d).* We now show:

Lemma 16.16. *If $u_\psi = 0$, then $u_{\tilde{\psi}}(\xi_1,\xi_2) = 0$ for all $\xi_1,\xi_2 \in H$.*

Proof. Since $u_\psi = 0$, equations (16.21) and (16.19) give

$$\begin{aligned}\pi i\, u_{\tilde{\psi}}(\xi_1,\xi_2) &= F_{\tilde{\psi}}(P,\xi_2) - F_{\tilde{\psi}}(P,\xi_1) + S_{\tilde{\psi}}(P,\xi_1,\xi_2) \\ &= -\pi i\, u_{\tilde{\psi}}(\xi_2,\xi_1) \;=\; -F_{\tilde{\psi}}(P,\xi_1) + F_{\tilde{\psi}}(P,\xi_2) - S_{\tilde{\psi}}(P,\xi_2,\xi_1)\,,\end{aligned} \tag{16.22}$$

where

$$S_{\tilde{\psi}}(P,\xi,\eta) \;=\; \sum_{\gamma\in\Gamma} m_\gamma\, d\tilde{\psi}(\mathfrak{F})|\gamma\,, \tag{16.23}$$

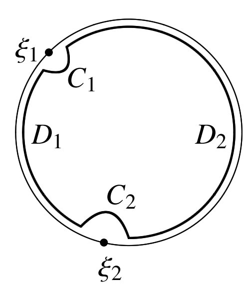
with m_γ the winding number around $\gamma^{-1}\mathfrak{F}$ of the path consisting of $-p(P,\eta)$, $p(P,\xi)$ and the positively oriented arc in $\partial\mathbb{D}$ from ξ to η.

Let $w_1 \in \mathbb{D}$. By Lemma 16.15 and (1.9) we can use Theorem 1.1 to write

$$u_{\tilde{\psi}}(\xi_1,\xi_2)(w_1) = -2\pi \int_C \{u_{\tilde{\psi}}(\xi_1,\xi_2), v\}\,, \tag{16.24}$$

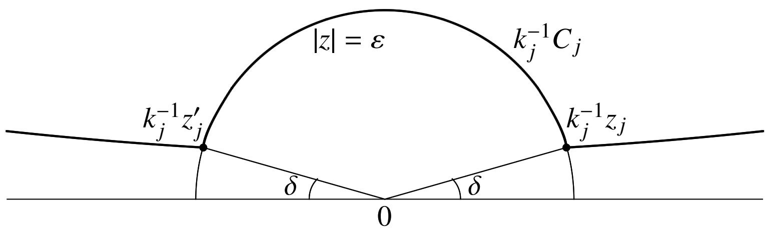
where $v(w) = q_s(w,w_1)$ and C is a curve encircling w_1 once in the positive direction.

We will choose the path C consisting of four pieces, two being small arcs C_1 and C_2 near ξ_1 and ξ_2, and two being arcs D_1 and D_2 between ξ_1 and ξ_2. (See diagram.)



Near ξ_j it is convenient to use the upper half-plane model, writing $\xi_j = k_j \cdot 0$, with $k_j \in \mathrm{PSO}(2) \subset G = \mathrm{PSL}_2(\mathbb{R})$.

We choose the C_j depending on two small positive parameters ε and δ, as indicated in the following sketch of $k_j^{-1}\,C_j$ in the upper half-plane:



The points z_j and z_j', where the arc C_j meets D_1 and D_2, correspond to $w_j = \frac{z_j - i}{z_j + i}$ and $w_j' = \frac{z_j' - i}{z_j' + i}$ in the disk model. The absolute value $b := |w_j| = |w_j'|$ (near 1) is related to ε and δ (near 0) by an explicit formula (namely $1 - b^2 = \frac{4\varepsilon \sin\delta}{1 + 2\varepsilon \sin\delta + \varepsilon^2}$). The arcs D_j are of the form $w = be^{i\theta}$ where θ runs through an interval I_j in $\mathbb{R}/2\pi\mathbb{Z}$. We must show that each of the four contributions to (16.24) tends to zero as $\delta \to 0$ and then $\varepsilon \to 0$.

We begin with C_j. In the upper half-plane model we use for z near ξ_j the coordinate $\rho e^{i\varphi} = k_j^{-1} z \in \mathfrak{H}$, for which

$$\{u, v\} \;=\; \frac{1}{\rho}\,(vu_\varphi - uv_\varphi)\,d\rho + \rho\,(uv_\rho - vu_\rho)\,d\varphi\,,$$

and hence simply $\{u, v\} = \varepsilon\,(uv_\rho - vu_\rho))\,d\varphi$ on C_j, where $\rho = \varepsilon$ is constant.

We have $F_{\tilde\psi}(P, \xi_j) = \mathrm{O}(1)$ and $\partial_\rho F_{\tilde\psi}(P, \xi_j) = \mathrm{O}(\varepsilon^{-1})$ according to Lemma 16.12. The corresponding term for $F_{\tilde\psi}(P, \xi_{j'})$, with $j' \neq j$, is smaller, since $F_{\tilde\psi}(\xi_1, \xi_2) \in \mathcal{G}_s^p(J)$ for an interval J containing ξ_j; the derivative is $\mathrm{O}(\rho^{s-1})$. Lemma 16.13 and its consequence (16.16) applied to $S_{\tilde\psi}(P, \xi_1, \xi_2) = -S_{\tilde\psi}(P, \xi_2, \xi_1)$ show that $S_{\tilde\psi}(P, \xi_1, \xi_2)$ is bounded near ξ_j, and that near ξ_j the derivatives with respect to r and θ are estimated by $\mathrm{O}((1 - r^2)^{-1}) = \mathrm{O}(\varepsilon^{-1})$. The derivative with respect to ρ can be expressed in these derivatives with bounded coefficients. Finally, the functions v and v_ρ for $v = q_s(\,\cdot\,, w_1) \in \mathcal{W}_s^p$ are $\mathrm{O}(\varepsilon^s)$ and $\mathrm{O}(\varepsilon^{s-1})$ respectively. All this leads to the following estimate of the integral over C_j:

$$\int_{\pi-\delta}^{\delta} \varepsilon\Big(\mathrm{O}(1)\,\mathrm{O}(\varepsilon^{1-s}) + \mathrm{O}(\varepsilon^s)\,\mathrm{O}(\varepsilon^{-1})\Big)\,d\phi \;=\; \mathrm{O}(\varepsilon^s)\,.$$

This estimate is uniform in $\delta \in (0, \pi/2)$.

Now we turn to D_j. In the disk model with coordinate $w = re^{i\theta} \in \mathbb{D}$ we have

$$\{u, v\} \;=\; \frac{1}{r}\,(vu_\theta - uv_\theta)\,dr + r\,(uv_r - vu_r)\,d\theta\,,$$

and hence $\{u, v\} = b\,(uv_r - vu_r)\,d\theta$ on D_j, where $r = b$ is constant. Since I_j has length at most $2\pi = \mathrm{O}(1)$, the contribution of the integral over D_j is bounded by a multiple of the maximum of $|uv_r| + |vu_r|$ over the arc D_j.

We use that $F_{\tilde\psi}(P, \xi_j) \in \mathcal{G}_s^p(I)$ and that v represents an element of $\mathcal{W}_s^p$ to get from $F_{\tilde\psi}(P, \xi_j)(re^{i\theta}) = (1 - r^2)^s a(r, \theta)$ and $v(re^{i\theta}) = (1 - r^2)^s b(r, \theta)$ the contribution

$$r(1 - r^2)^{2s}\big(a(r, \theta) b_r(r, \theta) - a_r(r, \theta) b(r, \theta)\big)\,d\theta\,.$$

This contributes $\mathrm{O}_\varepsilon((1-b^2)^{2s})$ to the integral. (We note that the implicit constant depends on the interval, hence on ε.) By Lemma 16.13 the contribution of $S_{\tilde\psi}(P,\xi_1,\xi_2)$ or $-S_{\tilde\psi}(P,\xi_2,\xi_1)$ is $\mathrm{O}_\varepsilon\left((1-b^2)^{2s+p-2}\right)$. Since $p \geq 2$, the final estimate of the integral over D_j is $\mathrm{O}_\varepsilon((1-b^2)^{2s})$.

We have arrived at

$$u_{\tilde\psi}(\xi_1,\xi_2) \;=\; \mathrm{O}(\varepsilon^s) + \mathrm{O}_\varepsilon((1-b^2)^{2s}),$$

with the first term uniform in δ. Letting $\delta \downarrow 0$, and hence $b \uparrow 1$, we arrive at the bound $\mathrm{O}(\varepsilon^s)$ for all positive ε. Hence $u_{\tilde\psi}(\xi_1,\xi_2) = 0$. ■

• *Step e).* If $u_\psi = 0$, then $u_{\tilde\psi}(\xi_1,\xi_2) = u_{\tilde\psi}(\xi_2,\xi_1) = 0$ by step d). Lemma 16.16 and (16.22) give for $P \in X_0^{\mathcal{T}}$ and $\xi_1,\xi_2 \in H$:

$$F_{\tilde\psi}(P,\xi_2) - F_{\tilde\psi}(P,\xi_1) \;=\; S_{\tilde\psi}(P,\xi_2,\xi_1) \;=\; -S_{\tilde\psi}(P,\xi_1,\xi_2). \tag{16.25}$$

The cocycle c on $(X_0^{\mathcal{T}} \cup H) \times (X_0^{\mathcal{T}} \cup H)$ introduced after Lemma 16.14 satisfies $c(\xi_1,\xi_2) = \rho_s(F_{\tilde\psi}(P,\xi_2) - F_{\tilde\psi}(P,\xi_1))$. We have $F_{\tilde\psi}(P,\xi_j)(w) = A_j(w)\,(1-|w|^2)^s$ where A_j is p times differentiable on a neighborhood Ω_j of $\mathbb{S}^1 \setminus \{\xi_j\}$ in $\mathbb{P}^1_{\mathbb{C}}$. Estimate (16.16) shows that $S_{\tilde\psi}(P,\xi_1,\xi_2)$ is $o(1-|w|^2)^{p-1}$ near closed intervals in $(\xi_2,\xi_1)_c$ and, with reversed roles of ξ_1 and ξ_2, also near $(\xi_1,\xi_2)_c$. So $A_1 - A_2$ is in $C^{p-1}(\Omega_1 \cap \Omega_2)$. We glue A_1 and A_2 to obtain $f \in \mathcal{G}_s^{p-1}$ that represents $c(P,\xi_j)$ on $\mathbb{S}^1 \setminus \{\xi_j\}$ for $j = 1,2$. The sheaf $\mathcal{W}_s^{p-1}$ has no sections with support consisting of one point, so $F_{\tilde\psi}(P,\xi_1)$ and $F_{\tilde\psi}(P,\xi_2)$ are in $\mathcal{W}_s^{p-1}$ and and their difference is represented by an element of $\mathcal{N}_s^{p-1}$. The proof can now be completed in the same way as in the last part of the proof of Proposition 16.9.

17. Smooth parabolic cohomology. We combine the results on $\mathcal{V}_s^p$-valued cohomology of the previous section to show that $H^1_{\mathrm{par}}(\Gamma;\mathcal{V}_s^\infty) \cong \mathsf{Maass}_s^0(\Gamma)$. In this way, we complete the proofs of Theorems A and B, and conclude that the main theorem of [19] extends to three times differentiable period functions.

Theorem 17.1. *Let* $0 < \mathrm{Re}\, s < 1$. *Let* Γ *be a cofinite discrete subgroup of* $G = \mathrm{PSL}_2(\mathbb{R})$.

If Γ *is cocompact then* $\mathcal{E}_s^\Gamma$ *is isomorphic to* $H^1(\Gamma;\mathcal{V}_s^\infty)$ *and to* $H^1(\Gamma;\mathcal{V}_s^p)$ *for all* $p \in \mathbb{N}$, $p \geq 2$.

If Γ *has cusps, then* $\mathsf{Maass}_s^0(\Gamma)$ *is isomorphic to* $H^1_{\mathrm{par}}(\Gamma;\mathcal{V}_s^\infty)$ *and to* $H^1_{\mathrm{par}}(\Gamma;\mathcal{V}_s^p)$ *for all* $p \in \mathbb{N}$, $p \geq 3$.

Proof. With the restriction map ρ_s we obtain from Propositions 16.10 and 16.11, in the case that Γ has cusps and $p \geq 3$

$$\begin{tikzcd} \mathsf{Maass}_s^0(\Gamma) \arrow[r, "\cong"] & H^1_{\mathrm{par}}(\Gamma;\mathcal{V}_s^{\omega^0,\infty}) \arrow[r] \arrow[rr, bend left, "\cong"] & H^1_{\mathrm{par}}(\Gamma;\mathcal{V}_s^\infty) \arrow[r] & H^1_{\mathrm{par}}(\Gamma;\mathcal{V}_s^p) \end{tikzcd},$$

and if Γ is cocompact and $p \geq 2$

$$\begin{tikzcd} \mathcal{E}_s^\Gamma \arrow[r, "\cong"] & H^1(\Gamma;\mathcal{V}_s^\omega) \arrow[r] \arrow[rr, bend left, "\cong"] & H^1(\Gamma;\mathcal{V}_s^\infty) \arrow[r] & H^1(\Gamma;\mathcal{V}_s^p) \end{tikzcd},$$

where in both cases the composition is an isomorphism. So it suffices to show that the natural maps $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^\infty_s) \to H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^p_s)$ and $H^1(\Gamma; \mathcal{V}^\infty_s) \to H^1(\Gamma; \mathcal{V}^p_s)$ are injective for all p.

We show this in the standard model of cohomology. Suppose that for $\psi \in Z^1(\Gamma; \mathcal{V}^\infty_s)$ there exists $a_p \in \mathcal{V}^p_s$ such that $\psi_\gamma = a_p \,|\, (\gamma - 1)$ for all $\gamma \in \Gamma$. Since all arrows in the diagrams in Propositions 16.10 and 16.11 are isomorphisms, there are also $a_q \in \mathcal{V}^q_s$ such that $\psi_\gamma = a_q \,|\, (\gamma - 1)$ for all γ. We take a hyperbolic $\gamma \in \Gamma$, and conclude from Proposition 4.1 that all a_q coincide, and hence give elements of $\mathcal{V}^\infty_s$. ■

As a consequence of (14.8c) we obtain for the modular group the following extension of the main theorem in [19] on period functions and Maass cusp forms:

Theorem 17.2. *Let* $0 < \mathrm{Re}\, s < 1$, *and* $p \in \mathbb{N}$, $p \geq 3$.

$$\mathsf{Maass}^0_s(\Gamma_1) \;\cong\; \mathsf{FE}_s(\mathbb{R}_+)^0_\omega \;=\; \mathsf{FE}_s(\mathbb{R}_+)^0_\infty \;=\; \mathsf{FE}_s(\mathbb{R}_+)^0_p\,.$$

So thrice differentiable functions on $(0, \infty)$ that satisfy

$$f(x) \;=\; f(x+1) \;+\; (x+1)^{-2s}\, f\Big(\frac{x}{x+1}\Big)^s$$

and the estimates $\psi(x) = \mathrm{O}(1)$ as $x \downarrow 0$ and $\psi(x) = \mathrm{O}(x^{-2s})$ as $x \to \infty$ automatically are real analytic, and occur as the period function of a Maass cusp form.

17.1. *Recapitulation of the proof of Theorem A.* The definition in (5.5a) induces an injective (Proposition 5.1) map $\mathbf{r} : \mathcal{E}^\Gamma_s \to H^1(\Gamma; \mathcal{V}^\omega_s)$. The isomorphism $\mathcal{V}^\omega_s \cong \mathcal{W}^\omega_s$ (in §3.2) and Theorem 7.2 give a left inverse $\alpha^\omega_s \circ \mathsf{P}^\dagger_s$ of $\mathbf{r}$, with

$$\alpha : [\psi] \mapsto b(s)\, u_{[\psi]}\,,$$

with $b(s)$ as in (3.4d). The injectivity of $\alpha^\omega_s \circ \mathsf{P}^\dagger_s$ follows from Proposition 7.3, Proposition 7.4 and the exact sequence (7.6).

Proposition 16.2 extends α^ω_s to $\alpha^p_s : H^1(\Gamma; \mathcal{W}^p_s) \to \mathcal{E}^\Gamma_s$, in a compatible way for different values of p. Theorem 17.1 shows that these extensions are injective, and also determine an isomorphism $H^1(\Gamma; \mathcal{V}^\infty_s) \to \mathcal{E}^\Gamma_s$.

17.2. *Recapitulation of the proof of Theorem B.* The injective map $\mathbf{r}$ in Proposition 5.1 from $\mathsf{Maass}^0_s(\Gamma)$ to $H^1(\Gamma; \mathcal{V}^\omega_s)$ has its image in the subspace $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^\omega_s, \mathcal{V}^{\omega^*,\infty}_s)$, according to Proposition 10.3. Proposition 12.7 shows that

$$\mathbf{r} : \mathsf{Maass}^0_s(\Gamma) \longrightarrow H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^\omega_s, \mathcal{V}^{\omega^*,\infty}_s)$$

is an isomorphism. Propositions 13.4 and the fact that $\mathcal{V}^{\omega^*,\infty}_s$ is locally defined (Definition 13.3) imply that

$$H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^\omega_s, \mathcal{V}^{\omega^*,\infty}_s) \;\cong\; H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^{\omega^*,\infty}_s)\,.$$

Finally, Theorem 17.1 gives

$$\mathsf{Maass}^0_s(\Gamma) \;\cong\; H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^\infty_s)\,.$$

Chapter VI. Distribution cohomology and Petersson product

In the previous chapters we have considered the relation between automorphic forms and cohomology groups with values in principal series spaces consisting of functions, possibly with singularities. In this chapter we turn to cohomology with values in the spaces $\mathcal{V}_s^{-\infty}$ and $\mathcal{V}_s^{-\omega}$ of distribution and hyperfunction vectors in the principal series. For groups with cusps, $H^1(\Gamma;\mathcal{V}_s^{-\infty})$ is isomorphic to $\mathsf{Maass}_s^0(\Gamma)$, by a result of Bunke and Olbrich, [7].[3] The natural map from $H^1(\Gamma;\mathcal{V}_s^{-\infty})$ to $H^1(\Gamma;\mathcal{V}_s^{-\omega})$ turns out to be the zero map if Γ has cusps. This contrasts with the cocompact case, where Bunke and Olbrich, [6], have shown that $H^1(\Gamma;\mathcal{V}_s^{-\infty}) = H^1(\Gamma;\mathcal{V}_s^{-\omega})$.

The Petersson scalar product can be transformed into a bilinear form on the space $\mathsf{Maass}_s^0(\Gamma)$. The isomorphisms with cohomology groups transform this bilinear form into a duality between $H^1_{\mathrm{par}}(\Gamma;\mathcal{V}_s^{\infty})$ and $H^1(\Gamma;\mathcal{V}_{1-s}^{-\infty})$. We will show in Section 19 that this bilinear form coincides, up to a multiple, with that given by the cup product, which we have to adapt to parabolic cohomology if the discrete group has cusps.

18. Distribution cohomology. The obvious way to obtain a map from Maass forms to distribution cohomology uses the natural homomorphism associated to $\mathcal{V}_s^{\infty} \to \mathcal{V}_s^{-\infty}$. In §18.1 we see that this leads to the zero map. In §18.2 we discuss another map from Maass forms to distribution cohomology, which gives an isomorphism on cusps forms.

18.1. *Vanishing image in distribution and hyperfunction cohomology.* The injective map $\mathbf{r}$ in Proposition 5.1 can be followed by the natural map to $H^1(\Gamma;\mathcal{V}_s^{\infty})$. The composition is injective for cusp forms:

$$\begin{tikzcd}
\mathsf{Maass}_s^0(\Gamma) \arrow[r, "\mathbf{r}", "\cong"'] \arrow[d, hook] & H^1_{\mathrm{par}}(\Gamma;\mathcal{V}_s^{\omega},\mathcal{V}_s^{\omega^*,\infty}) \arrow[r, "\cong"'] \arrow[d, hook] & H^1_{\mathrm{par}}(\Gamma;\mathcal{V}_s^{\infty}) \arrow[d, hook] \\
\mathcal{E}_s^{\Gamma} \arrow[r, hook, "\mathbf{r}"] & H^1(\Gamma;\mathcal{V}_s^{\omega}) \arrow[r] & H^1(\Gamma;\mathcal{V}_s^{\infty})
\end{tikzcd}$$

The inclusions $\mathcal{V}_s^{\infty} \subset \mathcal{V}_s^{-\infty} \subset \mathcal{V}_s^{-\omega}$ give further natural homomorphisms

$$H^1(\Gamma;\mathcal{V}_s^{\infty}) \longrightarrow H^1(\Gamma;\mathcal{V}_s^{-\infty}) \longrightarrow H^1(\Gamma;\mathcal{V}_s^{-\omega})\,. \tag{18.1}$$

Proposition 18.1. *The image of* $\mathbf{r}\,\mathcal{E}_s^{\Gamma}$ *under the natural maps in* (18.1) *vanishes in* $H^1(\Gamma;\mathcal{V}_s^{-\omega})$*, and the image of* $\mathbf{r}\,\mathsf{Maass}_s(\Gamma)$ *vanishes in* $H^1(\Gamma;\mathcal{V}_s^{-\infty})$.

Proof. Let $u \in \mathcal{E}_s^{\Gamma}$. In §5.2 we have seen that the class of $\mathbf{r}u$ is represented by a $\mathcal{V}_s^{\omega}$-valued cocycle $\psi_\gamma = g|_{2s}(\gamma - 1)$, where $g \in \mathbf{H}$ is a representative of the hyperfunction $\mathsf{P}_s^{-1}u$. Hence the image of $\mathbf{r}u$ in $H^1(\Gamma;\mathcal{V}_s^{-\omega})$ is zero. If $u \in \mathsf{Maass}_s(\Gamma)$ then u has polynomial growth. Theorem 2.3 implies that $\mathsf{P}_s^{-1}u \in \mathcal{V}_s^{-\infty}$. Hence $\mathbf{r}u$ is zero in $H^1(\Gamma;\mathcal{V}_s^{-\infty})$. ∎

18.2. *From cusp forms to distribution cohomology.* In §2.4 of [4] we have considered two spaces of *second order eigenfunctions:*

$$\begin{aligned}
\mathcal{E}_s' &= \operatorname{Ker}\left((\Delta-\lambda_s)^2 : C^\infty(\mathbb{H}) \longrightarrow C^\infty(\mathbb{H})\right),\\
(\mathcal{E}_s^{-\infty})' &= \{f \in \mathcal{E}_s' \;:\; f \text{ has polynomial growth}\}\,.
\end{aligned} \tag{18.2}$$

[3] See the footnote in §7 on p. 36.

The space of eigenfunctions of polynomial growth is equal to $\mathcal{E}_s^{-\infty} = \mathsf{P}_s\mathcal{V}_s^{-\infty}$ (Theorem 2.3).

Propositions 2.6 and 2.7 in [4] show that the spaces of second order eigenfunctions fit into the following exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{E}_s & \to & \mathcal{E}'_s & \overset{\Delta-\lambda_s}{\to} & \mathcal{E}_s & \to & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \to & \mathcal{E}_s^{-\infty} & \to & (\mathcal{E}'_s)^{-\infty} & \overset{\Delta-\lambda_s}{\to} & \mathcal{E}_s^{-\infty} & \to & 0 \end{array} \tag{18.3}$$

The second of these exact sequences leads to a non-zero map from cusp forms to distribution cohomology. Indeed, define $\mathbf{b} : \mathsf{Maass}_s(\Gamma) \to H^1(\Gamma; \mathcal{V}_s^{-\infty})$ as the composition

$$\mathsf{Maass}_s(\Gamma) = (\mathcal{E}_s^{-\infty})^\Gamma = H^0(\Gamma; \mathcal{E}_s^{-\infty}) \longrightarrow H^1(\Gamma; \mathcal{E}_s^{-\infty}) \cong H^1(\Gamma; \mathcal{V}_s^{-\infty}),$$

of the connecting homomorphism in the long exact sequence associated to the second row in (18.3) and the isomorphism provided by the Poisson transformation.

If Γ has cusps and $u \in \mathsf{Maass}_{s_0}(\Gamma)$ is the value u_{s_0} of a holomorphic family $s \mapsto u_s \in \mathsf{Maass}_s(\Gamma)$, then a Γ-invariant lift $\tilde{u}_{s_0} \in (\mathcal{E}'_s)^{-\infty}$ can be obtained by differentiation with respect to s, as in the proofs of Propositions 2.6 and 2.7 in [4]. So $\mathbf{b}$ vanishes on Eisenstein series. Bunke and Olbrich have shown in [7], Proposition 8.1:

Proposition 18.2. *Let* $0 < \operatorname{Re} s < 1$. *The map* $\mathbf{b} : \mathsf{Maass}_s^0(\Gamma) \to H^1(\Gamma; \mathcal{V}_s^{-\infty})$ *is an isomorphism.*

Their proof uses an exact sequence

$$0 \longrightarrow \mathcal{E}_s^{-\infty} \longrightarrow C^{-\infty} \overset{\Delta-\lambda_s}{\longrightarrow} C^{-\infty} \longrightarrow 0,$$

where $C^{-\infty}$ consists of the $f \in C^\infty(\mathfrak{H})$ such that $\partial_z^n \partial_{\bar{z}}^m f$ has polynomial growth for all $n, m \in \mathbb{N}$.

In the cocompact case, this result amounts to $\mathcal{E}_s^\Gamma \cong H^1(\Gamma; \mathcal{V}_s^{-\infty})$. In [6], Bunke and Olbrich have shown that

$$\mathcal{E}_s^\Gamma \cong H^1(\Gamma; \mathcal{V}_s^{-\infty}) = H^1(\Gamma; \mathcal{V}_s^{-\omega}). \tag{18.4}$$

If Γ has cusps and $s \neq \frac{1}{2}$, then all elements $u \in \mathcal{E}_s$ occur in holomorphic families, as we have seen in part ii) of Theorem 15.1. Differentiation with respect to s gives for $u \in \mathcal{E}_s^\Gamma$ a Γ-invariant lift in $(\mathcal{E}'_s)^\Gamma$. Hence the image of $\mathbf{b}u$ in $H^1(\Gamma; \mathcal{V}_s^{-\omega})$ vanishes:

Proposition 18.3. *Let* $0 < \operatorname{Re} s < 1$, $s \neq \frac{1}{2}$. *For groups with cusps the composition*

$$\mathsf{Maass}_s(\Gamma) \overset{\mathbf{b}}{\longrightarrow} H^1(\Gamma; \mathcal{V}_s^{-\infty}) \longrightarrow H^1(\Gamma; \mathcal{V}_s^{-\omega})$$

is the zero map.

19. Duality. In this final section we give a cohomological description of the Petersson scalar product.

19.1. *Petersson scalar product.* The *Petersson scalar product* on the space of cusp forms $\mathsf{Maass}_s^0(\Gamma)$ is given by

$$(u, v) = \int_{\Gamma\backslash\mathfrak{H}} u\,\bar{v}\,d\mu.$$

It can be computed by integration over any measurable fundamental domain, for instance over $\mathfrak{F}$ as chosen in §6.2 in the cocompact case and §11.1 if Γ has cusps.

Instead of $(\cdot\,,\,\cdot)$ we use the *bilinear Petersson scalar product*

$$\langle u, v\rangle \;=\; \int_{\Gamma\backslash\mathfrak{H}} u\, v\; d\mu \qquad (u, v \in \mathsf{Maass}^0_s(\Gamma))\,. \tag{19.1}$$

Since $\mathsf{Maass}^0_s(\Gamma) \neq \{0\}$ only if $s \in (\frac{1}{2} + i\mathbb{R}) \cup (0,1)$, the space $\mathsf{Maass}^0_s(\Gamma)$ is invariant under conjugation. Hence $(u, v) = \langle u, \bar{v}\rangle$ for all $u, v \in \mathsf{Maass}^0_s(\Gamma)$ for all s with $0 < \operatorname{Re} s < 1$.

19.2. *Cup product.* The cup product in cohomology can be described with any augmentation preserving chain map $F_. \to F_. \otimes F_.$ of projective resolutions, called a diagonal approximation, which gives an isomorphism in cohomology. In the standard model of group cohomology one may use the diagonal approximation described in [1], Chap. V, §1. This leads to a linear map

$$\cup : H^i(\Gamma; V) \otimes H^j(\Gamma; W) \longrightarrow H^{i+j}(\Gamma; V \otimes W)\,.$$

For $i = j = 1$ this is induced on 1-cocycles by $(b \cup c)(x) = -(b \otimes c)(\Delta_2 x))$, where $\Delta_.$ denotes a diagonal approximation. For $i = j = 1$, the cup product ends up in $H^2(\Gamma; V \otimes W)$, which is isomorphic to $(V \otimes W)_\Gamma$ in the cocompact case. For groups with cusps, the second cohomology groups vanish, and we need parabolic cohomology to have a reasonable cup product. Applied in the cocompact case, we get back the usual cup product.

For our purpose it suffices to consider a resolution $F^{\mathcal{T}}_.$ based on a tesselation $\mathcal{T}$ of type **Mix**, as discussed in §6.2 and §11.1. The tensor product $G_. = F^{\mathcal{T}}_. \otimes F^{\mathcal{T}}_.$ with $G_i = \bigoplus_{a=0}^{i} F^{\mathcal{T}}_a \otimes F^{\mathcal{T}}_{i-a}$ gives a resolution of $\mathbb{Q}$. The boundary maps are determined by $\partial_{i+j} x \otimes y = (\partial_i x) \otimes y + (-1)^i x \otimes (\partial_j y)$ for $x \in F^{\mathcal{T}}_i$ and $y \in F^{\mathcal{T}}_j$. The augmentation is given by $\varepsilon(P \otimes Q) = 1$ for $P, Q \in X^{\mathcal{T}}_0$. With minimal sets $B_i \subset X^{\mathcal{T}}_i$ of generators of $F^{\mathcal{T}}_i$, the sets

$$\bigcup_{a=0}^{i} \left\{ x \otimes \gamma^{-1} y \;:\; x \in B_a,\, y \in B_{i-a},\, \gamma \in \Gamma_y\backslash\Gamma \right\}$$

generate G_i over $\mathbb{Q}[\Gamma]$, and form a basis if $i \geq 1$. For $\kappa \in C$:

$$G_0^{\pi_\kappa} \;=\; \mathbb{Q} \cdot \kappa \otimes \kappa\,.$$

These are the sole generators with an infinite isotropy subgroup. The conclusion is that $G_.$ is a parabolic resolution as defined in §11.3.

• *Explicit basis elements.* For the sequel we need an explicit description of generating elements of $F^{\mathcal{T}}_i$ for the tesselation $\mathcal{T}$ of type **Mix**. The Dirichlet fundamental domain $\mathfrak{F}$ underlying the tesselation has the following boundary:

$$\partial\mathfrak{F} \;=\; \sum_{e \in E} e \,|\, (1 - \gamma_e^{-1}) + \begin{cases} 0 & \text{(cocompact)}\,, \\ \sum_{\kappa \in \mathfrak{F}^{\mathrm{cu}}} e_\kappa \,|\, (1 - \pi_\kappa^{-1}) & \text{(not cocompact)}\,. \end{cases} \tag{19.2}$$

If Γ is cocompact, then $\mathfrak{F}^{\mathrm{cu}}$ is empty.

We introduce some notation, illustrated in Figure 7 for the modular group. Define vertices P_e and Q_e of $\mathfrak{F}_Y$ (or of $\mathfrak{F}$ if Γ is cocompact) such that $e = e(P_e, Q_e)$, and also vertices P_κ of $\mathfrak{F}_Y$ such that $e_\kappa = e(P_\kappa, \kappa)$ if Γ has cusps. Note that Q_e may occur as $P_{e'}$ for some $e' \in E$.

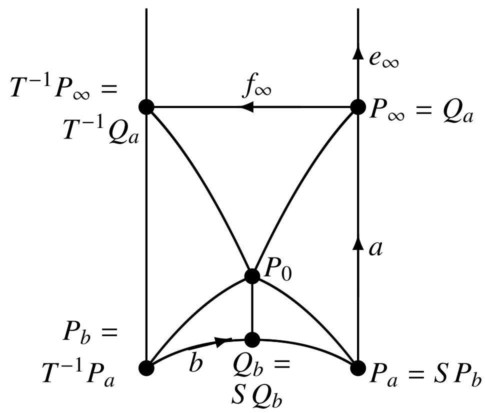


FIGURE 7. The points P_e and Q_e in the standard fundamental domain of the modular group.

So $F_0^{\mathcal{T}}$ is generated by P_0, the P_e and Q_e with $e \in E$, the P_κ and κ, with $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. We use the $\mathbb{Q}[\Gamma]$-basis of $F_1^{\mathcal{T}}$ consisting of the $e \in E$, the edges $e(P_0, R)$ where R runs through the set of vertices $\{P_e, Q_e\} \cup \{P_\kappa, \pi_\kappa^{-1} P_\kappa\}$, the $e_\kappa = e(P_\kappa, \kappa)$ and the $f_\kappa = e(P_\kappa, \pi_\kappa^{-1} P_\kappa)$ with $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. A $\mathbb{Q}[\Gamma]$-basis of $F_2^{\mathcal{T}}$ consists of the polygons in $X_2^{\mathcal{T}}$ contained in $\mathfrak{F}$. For the tesselation $\mathcal{T}$ of type **Mix** these polygons are triangles. Their sum represents the fundamental class. Denoting by $\Delta(A, B, C)$ the triangle in $X_2^{\mathcal{T}}$ with boundary $e_{A,B} + e_{B,C} + e_{C,A}$ we have:

$$\begin{aligned} (\mathfrak{F}) \;=\; & \sum_{e\in E}\Big(\Delta(P_0, P_e, Q_e) + \Delta(P_0, \gamma_e^{-1} Q_e, \gamma_e^{-1} P_e)\Big) \\ & + \sum_{\kappa\in\mathfrak{F}^{\mathrm{cu}}}\Big(\Delta(P_0, P_k, \pi_\kappa^{-1} P_\kappa) + V_\kappa\Big), \end{aligned} \tag{19.3}$$

where $V_\kappa = \Delta(\pi_\kappa^{-1} P_\kappa, P_\kappa, \kappa)$.

• *Explicit diagonal approximation.* The cup product in parabolic cohomology is induced by any augmentation preserving chain map $\delta_. : F_.^{\mathcal{T}} \to F_.^{\mathcal{T}} \otimes F_.^{\mathcal{T}}$. With the notations just introduced we indicate a special choice that will work well in connection with the Petersson scalar product.

In dimension 0 there is only one sensible choice:

$$\delta_0(P) \;=\; (P) \otimes (P) \quad (P \in X_0^{\mathcal{T}})\,. \tag{19.4}$$

This is continued $\mathbb{Q}[\Gamma]$-linearly, and gives an augmentation preserving map.

In dimension 1 we prescribe δ_1 on the basis discussed above. For each of the basis elements $e_{P,Q}$ we put

$$\delta_1 e_{P,Q} \;=\; P \otimes e_{P,Q} + e_{P,Q} \otimes Q\,. \tag{19.5}$$

After $\mathbb{Q}[\Gamma]$-linear extension, δ_1 turns out to be compatible with the boundary maps: $\partial_0\delta_1 = \delta_1\partial_1$.

For the basis elements of $F_2^{\mathcal{T}}$ we make the choice in Table 2, with $e \in E$, $\kappa \in \mathfrak{F}^{\mathrm{cu}}$. It is some work to check that this is compatible with ∂_2 and ∂_1. This choice has the

$\delta_2\Delta(P_0,P_e,Q_e)$	$=$	$P_0\otimes\Delta(P_0,P_e,Q_e)+e(P_0,P_e)\otimes e$ $+\Delta(P_0,P_e,Q_e)\otimes Q_e$
$\delta_2\Delta(P_0,\gamma_e^{-1}Q_e,\gamma_e^{-1}P_e)$	$=$	$P_0\otimes\Delta(P_0,\gamma_e^{-1}Q_e,\gamma_e^{-1}P_e)$ $-e(P_0,\gamma_e^{-1}P_e)\otimes e(\gamma_e^{-1}P_e,\gamma_e^{-1}Q_e)$ $+\Delta(P_0,\gamma_e^{-1}Q_e,\gamma_e^{-1}P_e)\otimes\gamma_e^{-1}Q_e$
$\delta_2\Delta(P_0,P_\kappa,\pi_\kappa^{-1}P_\kappa)$	$=$	$P_0\otimes\Delta(P_0,P_\kappa,\pi_\kappa^{-1}P_\kappa)+e(P_0,P_\kappa)\otimes f_\kappa$ $+\Delta(P_0,P_\kappa,\pi_\kappa^{-1}P_\kappa)\otimes\pi_\kappa^{-1}P_\kappa$
δ_2V_κ	$=$	$P_\kappa\otimes V_\kappa - f_\kappa\otimes\pi_\kappa^{-1}e_\kappa + V_\kappa\otimes\kappa$

TABLE 2. Basis elements of $F_2^{\mathcal{T}}$, with e running through E and κ through $\mathfrak{F}^{\mathrm{cu}}$.

special property

$$\delta_2F_2^{\mathcal{T}} \subset \left(F_0^{\mathcal{T}}\otimes F_2^{\mathcal{T}}\right)\oplus\left(F_1^{\mathcal{T},Y}\otimes F_1^{\mathcal{T}}\right)\oplus\left(F_2^{\mathcal{T}}\otimes F_1^{\mathcal{T}}\right), \tag{19.6}$$

where the first factor of the $(1,1)$-term is $F_1^{\mathcal{T},Y}$.

• *Cup product.* Let V and W be $\mathbb{Q}[\Gamma]$-modules. The cup product of cocycles $b \in Z^1(F_\cdot^{\mathcal{T},Y};V)$ and $c \in Z^1(F_\cdot^{\mathcal{T}};W)$ is computed as $(b\cup c)(x) = -(b\otimes c)(\delta_2x)$ for $x \in X_2^{\mathcal{T}}$. The tensor $b\otimes c$ sees only the component of x in $F_1^{\mathcal{T},Y}\otimes F_1^{\mathcal{T}}$. The result represents an element of $H^2_{\mathrm{par}}(\Gamma;V\otimes W)$, which does not depend on the choice of b and c in their cohomology classes. Thus, we have obtained

$$\cup : H^1(\Gamma;V)\otimes H^1_{\mathrm{par}}(\Gamma;W) \longrightarrow H^2_{\mathrm{par}}(\Gamma;V\otimes W). \tag{19.7}$$

By evaluation on the fundamental class we obtain an element of $H^2_{\mathrm{par}}(\Gamma;V\otimes W) \cong (V\otimes W)_\Gamma$ represented by:

$$\begin{aligned}(b\cup c)(\mathfrak{F}) \;=\; &\sum_{e\in E}\Big(-b(P_0,P_e)\otimes c(e) + \big(b(\gamma_eP_0,P_e)\otimes c(e)\big)\,|\,\gamma_e\Big)\\ &+\sum_{\kappa\in\mathfrak{F}^{\mathrm{cu}}}\Big(-b(P_0,P_\kappa)\otimes c(f_\kappa) + b(f_\kappa)\otimes c(\pi_\kappa^{-1}e_\kappa)\Big).\end{aligned} \tag{19.8}$$

• *Duality.* In the special case that there is a Γ-invariant bilinear form $\langle\cdot,\cdot\rangle : V\times W \to \mathbb{C}$ we have a linear map $(V\otimes W)_\Gamma \to \mathbb{C}$. Thus we have a linear form $H^1(\Gamma,V)\otimes H^1_{\mathrm{par}}(\Gamma;W) \to \mathbb{C}$ given by

$$[b]\otimes[c] \mapsto \Big\langle (b\cup c)([\Gamma\backslash\mathbb{H}])\Big\rangle,$$

where we denote by $\langle\cdot\rangle$ the linear form on $V\otimes W$ corresponding to the bilinear form $\langle\cdot,\cdot\rangle$. We will use this with $V = \mathcal{V}_{1-s}^{-\infty}$ and $W = \mathcal{V}_s^{\infty}$. See §2.1.

19.3. *Cohomological interpretation of the Petersson scalar product.*

Theorem 19.1. *For $0 < \operatorname{Re} s < 1$ and all cofinite $\Gamma \subset G$, the bilinear Petersson scalar product is given by the cup product of $\mathbf{b}u \in H^1(\Gamma; \mathcal{V}_{1-s}^{-\infty})$ and $\mathbf{r}v \in H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\infty})$ evaluated on the fundamental class:*

$$\langle u, v\rangle \;=\; \frac{-b(s)}{2i}\, \langle (\mathbf{b}u \cup \mathbf{r}v)([\Gamma\backslash\mathbb{H}])\rangle$$

for $u, v \in \mathsf{Maass}_s^0(\Gamma)$. See (3.4d) *for the gamma factor $b(s)$.*

Remarks. **1**. For discrete groups with cusps we understand the cup product in the parabolic sense of the previous subsection.

2. The choice of the spectral parameter s such that $\lambda_s = s(1-s)$ is the eigenvalue of Δ on u and v is not visible in the notations $\mathbf{b}$ and $\mathbf{r}$. Here it is important to use opposite choices for the spectral parameter for $\mathbf{b}u$ and $\mathbf{r}v$. For the Maass cusp forms the choice does not matter: $\mathsf{Maass}_{1-s}^0(\Gamma) = \mathsf{Maass}_s^0(\Gamma)$.

Proof. The proof takes the remainder of this subsection, and consists of three separate steps, which use several results from the previous chapters. For cocompact Γ we use $\mathfrak{F}^{\mathrm{cu}} = \emptyset$ throughout the proof.

- *Use of a distribution-valued cocycle.* First we use the description of cohomology with cocycles on the group Γ.

The map $\mathbf{b}$ in §18.2 gives rise to

$$\mathbf{b} \otimes \mathrm{id} : \mathsf{Maass}_{1-s}^0(\Gamma) \otimes \mathsf{Maass}_s^0(\Gamma) \longrightarrow H^1(\Gamma; \mathcal{E}_{1-s}^{-\infty}) \otimes \mathsf{Maass}_s^0(\Gamma)\,,$$

where we have used the identification $\mathcal{V}_{1-s}^{-\infty} \cong \mathcal{E}_{1-s}^{-\infty}$ by the Poisson transformation. We define a linear map $\mathbf{d} : Z^1(\Gamma; \mathcal{E}_{1-s}) \otimes \mathsf{Maass}_s^0(\Gamma) \to \mathbb{C}$ by

$$\mathbf{d}(b \otimes v) \;=\; 2i \sum_{e \in E'} \int_e [b_{\gamma_e^{-1}}, v]\,, \tag{19.9}$$

where $E' = E \cup \{e_\kappa \;:\; \kappa \in \mathfrak{F}^{\mathrm{cu}}\}$ and $\gamma_e \in \Gamma$ for $e \in E'$ such that $\partial\mathfrak{F} = \sum_{e\in E'} e\,|\,(1-\gamma_e)$. For $e = e_\kappa$, $\kappa \in \mathfrak{F}^{\mathrm{cu}}$, and hence $\gamma_e = \pi_\kappa$, the convergence of the integral is assured by the exponential decay of v and its derivatives, and the polynomial growth of $b_{\gamma^{-1}}$.

Lemma 19.2. *Definition* (19.9) *induces a linear map*

$$\mathbf{d} : H^1(\Gamma; \mathcal{E}_{1-s}^{-\infty}) \otimes \mathsf{Maass}_s^0(\Gamma) \longrightarrow \mathbb{C}$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Maass}_{1-s}^0(\Gamma) \otimes \mathsf{Maass}_s^0(\Gamma) \arrow[d, "\mathbf{b}\otimes\mathrm{id}"'] \arrow[dr, "\langle\cdot,\cdot\rangle"] & \\
H^1(\Gamma; \mathcal{E}_{1-s}^{-\infty}) \otimes \mathsf{Maass}_s^0(\Gamma) \arrow[r, "\mathbf{d}"] & \mathbb{C}
\end{tikzcd}$$

Proof. Let $b \in Z^1(\Gamma; \mathcal{E}_{1-s}^{-\infty})$ and $v \in \mathsf{Maass}_s^0(\Gamma)$. To see that $\mathbf{d}(b \otimes v)$ does not depend on the choice of b in its cohomology class, we look at $b = da \in B^1(\Gamma; \mathcal{E}_{1-s}^{-\infty})$.

$$\begin{aligned}\mathbf{d}(b \otimes v) \;&=\; 2i \sum_{e \in E'} \int_e \big[a \,|\, \gamma_e^{-1} - a, v\big] \;=\; 2i \sum_{e \in E'} \Big(\int_{\gamma_e^{-1} e} [a, v] - \int_e [a, v] \Big) \\ &=\; 2i \int_{\partial \mathfrak{F}} [a, v] \;=\; 0\,.\end{aligned}$$

Let $u, v \in \mathsf{Maass}_s^0(\Gamma) = \mathsf{Maass}_{1-s}^0(\Gamma)$. The exactness of the sequences in (18.3) implies that $\mathbf{b}u$ can be represented by a cocycle $\gamma \mapsto b_\gamma = \tilde{u} \,|\, (\gamma - 1)$ with $\tilde{u} \in (\mathcal{E}'_{1-s})^{-\infty}$ such that $(\Delta - \lambda_s)\tilde{u} = u$. Thus,

$$\mathbf{d}(\mathbf{b}u \otimes v) \;=\; \mathbf{d}(b \otimes v) \;=\; 2i \sum_{e \in E'} [\tilde{u} \,|\, (\gamma_e^{-1} - 1), v] \;=\; 2i \int_{\partial \mathfrak{F}} [\tilde{u}, v]\,.$$

Since $(\Delta - \lambda_s)\tilde{u}$ may be non-zero, the form $[\tilde{u}, v]$ is not closed. For cocompact Γ we obtain with (1.10c):

$$d(\mathbf{b}u \otimes v) \;=\; 2i \int_{\partial \mathfrak{F}} [\tilde{u}, v] \;=\; \int_{\mathfrak{F}} uv \, d\mu \;=\; \langle u, v \rangle\,.$$

If Γ has cusps, we replace $\mathfrak{F}$ by the truncated fundamental domain $\mathfrak{F}_a = \mathfrak{F} \cap \mathfrak{H}_a$ with a large. (See §11.1 for $\mathfrak{H}_a$.) The exponential decay of v and the polynomial growth of $\tilde{u}$ and their derivatives shows that for all large values of a

$$\mathbf{d}(\mathbf{b}u \otimes v) \;=\; 2i \int_{\partial \mathfrak{F}_a} [\tilde{u}, v] + \mathrm{o}(1) \;=\; \int_{\mathfrak{F}_a} uv \, d\mu + \mathrm{o}(1) \;=\; \langle u, v \rangle + \mathrm{o}(1)\,.$$

Taking the limit as $a \to \infty$ we obtain the desired equality. ■

• *Reformulation with a* 2*-cocycle.* We switch to the description of cohomology with a tesselation $\mathcal{T}$ of type **Mix**, and define for $b \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{E}_{1-s}^{-\infty})$ and $v \in \mathsf{Maass}_s^0(\Gamma)$:

$$\omega_{P,Q} \;=\; \big[b(P, Q), v\big] \qquad (P, Q \in X_0^{\mathcal{T}})\,. \tag{19.10}$$

The map $(P, Q) \mapsto \omega_{P,Q}$ defines a 1-cocycle on $X_0^{\mathcal{T}} \times X_0^{\mathcal{T}}$ with values in the smooth closed differential forms, satisfying $\omega_{\gamma^{-1}P, \gamma^{-1}Q} = \omega_{P,Q} \circ \gamma$ for $\gamma \in \Gamma$. For $P, Q, R \in X_0^{\mathcal{T}}$ we put

$$C(P, Q, R) \;=\; -2i \int_Q^R \omega_{P,Q}\,, \tag{19.11}$$

where the path of integration from Q to P follows edges in $X_1^{\mathcal{T}}$.

Lemma 19.3. *Equation* (19.11) *defines a cocycle* $C \in Z^2(F_\cdot^{\mathcal{T}}; \mathbb{C})$ *and induces a linear map*

$$\mathbf{a} : H^1(\Gamma; \mathcal{E}_{1-s}^{-\infty}) \otimes \mathsf{Maass}_s^0(\Gamma) \longrightarrow H^2_{\mathrm{par}}(\Gamma; \mathbb{C}) \tag{19.12}$$

such that the following diagram commutes:

$$\begin{tikzcd} H^1(\Gamma; \mathcal{E}_{1-s}^{-\infty}) \otimes \mathsf{Maass}_s^0(\Gamma) \arrow[rr, "\mathbf{d}"] \arrow[drr, "\mathbf{a}"'] & & \mathbb{C} \\ & & H^2_{\mathrm{par}}(\Gamma; \mathbb{C}) \arrow[u, "\textit{eval. on } (\mathfrak{F})"'] \end{tikzcd}$$

Proof. If P or Q is a cusp, the decay properties used in the proof of Lemma 19.2 ensure convergence here as well. The Γ-invariant $\mathbb{C}$-valued 2-cochain $\Delta(P,Q,R) \mapsto C(P,Q,R)$ in $C^2(F_\cdot^{\mathcal{T}};\mathbb{C})$, also called C, is automatically in $Z^2(F_\cdot^{\mathcal{T}};\mathbb{C})$, since $F_3^{\mathcal{T}} = \{0\}$. To see that it does not depend on the choice of b in its cohomology class, we suppose that $b = da$, with a Γ-equivariant a. Then $\omega_{P,Q} = \eta_P - \eta_Q$, with $\eta_P = [a(P), v]$ also Γ-equivariant, and

$$\frac{i}{2} C(P,Q,R) \;=\; \int_P^R \eta_P - \int_P^Q \eta_P - \int_Q^R \eta_Q \;=\; df(P,Q,R)\,, \tag{19.13}$$

where $f(P,Q) = -\int_P^Q \eta_P$. So $\mathbf{a} : [b] \otimes v \mapsto [C]$ can be extended to give a linear map $H^1(\Gamma;\mathcal{E}_{1-s}) \otimes \mathsf{Maass}_s^0(\Gamma) \to H^2(\Gamma;\mathbb{C})$.

We evaluate C on the representative of the fundamental class in (19.3):

$$\begin{aligned} C(\mathfrak{F}) \;=\; & \sum_{e\in E} \bigl(C(P_0,P_e,Q_e) - C(\gamma_e P_0, P_e, Q_e)\bigr) \\ & + \sum_{\kappa \in \mathfrak{F}^{\mathrm{cu}}} \bigl(C(P_0, P_\kappa, \pi_\kappa^{-1} P_\kappa) - C(P_\kappa, \pi_\kappa^{-1} P_\kappa, \kappa)\bigr)\,, \end{aligned} \tag{19.14}$$

where we have used the Γ-invariance of C.

Each term in the sum over $e \in E$ contributes:

$$-2i \int_{P_e}^{Q_e} (\omega_{P_0,P_e} - \omega_{\gamma_e P_0, P_e}) \;=\; 2i \int_e \omega_{\gamma_e P_0, P_0} \;=\; 2i \int_e [b(\gamma_e P_0, P_0), v]\,.$$

This is equal to the corresponding term in the definition of $\mathbf{d}(b \otimes v)$ in (19.9), provided we use P_0 as the base point in the description of b in the standard model of group cohomology.

The contribution of the terms for $\kappa \in \mathfrak{F}^{\mathrm{cu}}$ are also in accordance with (19.9), as one sees from the following slightly more complicated computation:

$$\begin{aligned} \frac{1}{2i}\Bigl(C(P_0, P_\kappa, \pi_\kappa^{-1} P_\kappa) \quad & - C(P_\kappa, \pi_\kappa^{-1} P_\kappa, \kappa)\Bigr) \\ = -\int_{P_\kappa}^{\pi_\kappa^{-1} P_\kappa} \omega_{P_0,P_\kappa} \quad & + \int_{\pi_\kappa^{-1} P_\kappa}^{\kappa} \omega_{P_\kappa, \pi_\kappa^{-1} P_\kappa} \\ = -\int_{P_\kappa}^{\kappa} \omega_{P_0,P_\kappa} + \int_{\pi_\kappa^{-1} P_\kappa}^{\kappa} \omega_{P_0,P_\kappa} \quad & + \int_{\pi_\kappa^{-1} P_\kappa}^{\kappa} (\omega_{P_0, \pi_\kappa^{-1} P_\kappa} - \omega_{P_0,P_\kappa}) \\ = -\int_{P_\kappa}^{\kappa} \omega_{P_0,P_\kappa} \quad & + \int_{\pi_\kappa^{-1} P_\kappa}^{\kappa} \omega_{P_0, \pi_\kappa^{-1} P_\kappa} \\ = \int_{e_\kappa} (-\omega_{P_0,P_\kappa} + \omega_{\pi_\kappa P_0, P_\kappa}) \quad & = \int_{e_\kappa} [b_{\pi_\kappa^{-1}}, v]\,. \quad \blacksquare \end{aligned}$$

- *Reformulation with cup product.* The map

$$\mathsf{Maass}_s^0(\Gamma) \xrightarrow{\mathbf{q}} H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^{\omega}, \mathcal{W}_s^{\omega^*,\infty}) \longrightarrow H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^{\omega^*,\infty})$$

induces a map

$$\begin{aligned} H^1(\Gamma; \mathcal{E}_{1-s}^{-\infty}) \otimes \mathsf{Maass}_s^0(\Gamma) & \xrightarrow{\mathrm{id}\otimes\mathbf{q}} H^1(\Gamma; \mathcal{E}_{1-s}^{-\infty}) \otimes H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^{\omega^*,\infty}) \\ & \xrightarrow{\cup} H^2_{\mathrm{par}}(\Gamma; \mathcal{E}_{1-s}^{-\infty} \otimes \mathcal{W}_s^{\omega^*,\infty})\,. \end{aligned}$$

Since $\mathcal{E}_{1-s}^{-\infty} \cong \mathcal{V}_{1-s}^{-\infty}$ and $\mathcal{W}_s^{\omega^*,\infty} \cong \mathcal{V}_s^{\omega^*,\infty} \subset \mathcal{V}_s^{\infty}$, there is a G-invariant $\mathbb{C}$-bilinear duality $\langle\,\cdot\,,\,\cdot\,\rangle : \mathcal{E}_{1-s}^{-\infty} \times \mathcal{W}_s^{\omega^*,\infty} \to \mathbb{C}$. On the subspace $\mathcal{E}_{1-s}^{-\infty} \times \mathcal{W}_s^{\omega}$ we have the description in (3.9):

$$\langle f, g\rangle \;=\; b(s)\beta(f,g) \;=\; \frac{b(s)}{\pi i}\int_C [f,g]\,,$$

with a suitable contour C. This leads to the following result:

Lemma 19.4. *Fix $f \in \mathcal{E}_{1-s}^{-\infty}$, and define $g \in \mathcal{W}_s^{\omega^*,\infty}$ by $g(z) \;=\; \int_p [v, q_s(\,\cdot\,,z)]$, where $v \in \mathsf{Maass}_s^0(\Gamma)$ and $p \in X_1^{\mathcal{T}}$. Then*

$$\langle f, g\rangle \;=\; b(s)\int_p [v,f]\,. \tag{19.15}$$

Proof. First we suppose that $p \in X_1^{\mathcal{T},Y}$. Then $\mathsf{Sing}\,(g) \subset \mathrm{Supp}\,(p)$ which is compact in $\mathfrak{H}$, and $g \in \mathcal{W}_s^{\omega}$. Let C be a positively oriented closed curve encircling p once. With (3.9) and Theorem 1.1:

$$\begin{aligned}\langle f, g\rangle \;&=\; b(s)\int_C [f,g] \;=\; \frac{b(s)}{\pi i}\int_{z\in C}\Big[f(z), \int_{z'\in p}\big[v(z'), q_s(z,z')\big]_{z'}\Big]\\ &=\; \frac{b(s)}{\pi i}\int_{z'\in p}\Big[v(z'), \int_{z\in C}\big[f(z), q_s(z,z')\big]_{z}\Big]_{z'} \;=\; b(s)\int_p [v,f]\,.\end{aligned}$$

The legitimicity of the interchange of the order of square brackets must (and can!) be checked. Thus, we have obtained (19.15) if $p \in X_1^{\mathcal{T},Y}$.

We still have to consider the case that $p = e(P,\kappa)$ with $P \in X_0^{\mathcal{T}} \cap \mathfrak{H}$ and $\kappa \in \mathcal{C}$. We approximate p by $p_Q = e(P,Q)$ with $Q \in \mathfrak{H}$ on p. Put, for $Q \in p$ (including $Q = \kappa$):

$$\begin{aligned} g_Q(z) \;&=\; \int_{e(P,Q)} [v, q_s(\,\cdot\,,z)]\,,\\ r_Q(\zeta) \;&=\; \int_{e(P,Q)} [v, R(\zeta;\,\cdot\,)^s]\,.\end{aligned}$$

With (3.6c) and Proposition 12.1 we have $\mathsf{P}_s^{\dagger} r_Q = b(s)^{-1}\, g_Q$ for all $Q \in p$. If $Q \neq \kappa$ then $r_Q \in \mathcal{V}_s^{\omega}$. Proposition 9.7 implies that $\lim_{Q\to\kappa} r_Q = r_\kappa$ in the topology of $\mathcal{V}_s^{\infty}$. Let $\beta \in \mathcal{V}_{1-s}^{-\infty}$ be such that $f = \mathsf{P}_{1-s}\beta$.

$$\begin{aligned}\langle f, g\rangle \;&=\; \langle \beta, b(s)\, r_\kappa\rangle \;=\; b(s)\lim_{Q\to\kappa}\langle\beta, r_Q\rangle \;=\; \lim_{Q\to\kappa}\langle f, g_Q\rangle\\ &=\; b(s)\lim_{Q\to\kappa}\int_{e(P,Q)} [v,f] \;=\; b(s)\int_{e(P,\kappa)} [v,f]\,.\end{aligned}$$

The last equality follows from the fact that $\int_{e(P,\kappa)} [f,v]$ converges absolutely. ■

Lemma 19.5. *The following diagram commutes:*

$$\begin{tikzcd} H^1(\Gamma;\mathcal{E}_{1-s}^{-\infty}) \otimes \mathsf{Maass}_s^0(\Gamma) \arrow[d, "\frac{-b(s)}{2i}\,\mathrm{id}\otimes\mathbf{q}"'] \arrow[drr, "\mathbf{a}"] & & \\ H^1(\Gamma;\mathcal{E}_{1-s}^{-\infty}) \otimes H^1_{\mathrm{par}}(\Gamma;\mathcal{W}_s^{\omega^*,\infty}) \arrow[r, "\cup"'] & H^2_{\mathrm{par}}(\Gamma;\mathcal{E}_{1-s}^{-\infty}\otimes\mathcal{W}_s^{\omega^*,\infty}) \arrow[r, "duality"'] & H^2_{\mathrm{par}}(\Gamma;\mathbb{C}) \end{tikzcd}$$

Proof. We start with $b \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{E}_{1-s}^{-\infty})$ representing a class in $H^1(\Gamma; \mathcal{E}_{1-s}^{-\infty})$, and with $v \in \mathsf{Maass}_s^0(\Gamma)$. The image $\mathbf{q}v \in H^1_{\mathrm{par}}(\Gamma; \mathcal{W}_s^{\omega^*,\infty})$ is represented by $q \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{W}_s^{\omega^*,\infty})$ given by

$$q(y)(z) \;=\; \int_y [v, q_s(\,\cdot\,, z)] \qquad (y \in X_1^{\mathcal{T}})\,.$$

Let $C_1 \in Z^2(F_\cdot^{\mathcal{T}}; \mathbb{C})$ be the cocycle obtained from the cup product

$$C_1(V) \;=\; \Big\langle (b \cup \mathbf{q}v)(V) \Big\rangle \qquad (V \in X_1^{\mathcal{T}})\,.$$

Our aim is to relate C_1 to $C \circ \delta_2$, where C is the cocycle in (19.11) representing $\mathbf{a}([b] \otimes v)$, and where $\delta_\cdot : F_\cdot^{\mathcal{T}} \to F_\cdot^{\mathcal{T}} \otimes F_\cdot^{\mathcal{T}}$ is as in §19.2.

It suffices to consider C_1 and $C \circ \delta_2$ on the generators in (19.3), given in the first column of Table 3. For each of these generators we have $(b \cup q)(\Delta) = -b(x) \otimes q(y)$

basis elt. Δ	$x \in X_1^{\mathcal{T},T}$	$y \in X_1^{\mathcal{T}}$	
$\Delta(P_0, P_e, Q_e)$	$e(P_0, P_e)$	e	$e \in E$
$\Delta(P_0, \gamma_e^{-1} Q_e, \gamma_e^{-1} P_e)$	$-e(P_0, \gamma_e^{-1} P_e)$	$\gamma_e^{-1} e(P_e, Q_e)$	$e \in E$
$\Delta(P_0, P_\kappa, \pi_\kappa^{-1} P_\kappa)$	$e(P_0, P_\kappa)$	f_κ	$\kappa \in \mathfrak{F}^{\mathrm{cu}}$
V_κ	$-f_\kappa$	$\pi_\kappa^{-1} e_\kappa$	$\kappa \in \mathfrak{F}^{\mathrm{cu}}$

TABLE 3. Basis elements of $F_2^{\mathcal{T}}$

with x and y as indicated in Table 3. Lemma 19.4 shows that

$$C_1(\Delta) \;=\; -\langle b(x), q(y) \rangle \;=\; -b(s) \int_y [v, b(x)]\,.$$

With (19.10) and (19.11) we have $C(\Delta) \;=\; -2i \int_y [b(x), v]$. In view of (1.10b):

$$\frac{i}{2} C(\Delta) - b(s)^{-1}\, C_1(\Delta) \;=\; \int_y d\big(v\, b(x)\big)\,.$$

A computation on the basis elements of $F_2^{\mathcal{T}}$ in Table 3 shows that

$$\int_y d(v\, b(x)) \;=\; (dF)(\Delta)$$

where $F \in \mathrm{Map}(F_1^{\mathcal{T}}; \mathbb{C})^\Gamma$ satisfies

$$\begin{aligned} F(e(P_0, P)) \;&=\; -v(P)\, b(e(P_0, P))(P) && \text{for all vertices } P \text{ of } \mathfrak{F}_Y\,,\\ F(e) \;&=\; -v(Q_e)\, b(e)(Q_e) && \text{for } e \in E\,,\\ F(f_\kappa) \;&=\; -v(P_\kappa)\, b(f_\kappa)(\pi_\kappa^{-1} P_\kappa) && \text{for } \kappa \in \mathfrak{F}^{\mathrm{cu}}\,. \end{aligned}$$

Hence $[C] = -2i\, b(s)^{-1}\, [C_1]$. (For $\Delta = V_\kappa$ with $\kappa \in \mathfrak{F}^{\mathrm{cu}}$ we use that $v(\kappa) = 0$, and that $F(e_\kappa)$ is unimportant since $F(\pi)\kappa^{-1} e_\kappa) = F(e_\kappa)|\pi_\kappa = F(e_\kappa)$.) ∎

• *Combination.* We combine the commuting diagrams in the Lemmas 19.2, 19.3 and 19.5:

$$\begin{tikzcd}
\mathsf{Maass}^0_{1-s}(\Gamma)\otimes\mathsf{Maass}^0_s(\Gamma) \arrow[d, "\mathbf{b}\otimes\mathrm{id}"'] \arrow[drr, "\langle\cdot,\cdot\rangle"] & & \\
H^1(\Gamma;\mathcal{E}^{-\infty}_{1-s})\otimes\mathsf{Maass}^0_s \arrow[rr, "\mathbf{d}"] \arrow[d, "\frac{-b(s)}{2i}\,\mathrm{id}\otimes\mathbf{q}"'] \arrow[drr, "\mathbf{a}"] & & \mathbb{C} \\
H^1(\Gamma;\mathcal{E}^{-\infty}_{1-s})\otimes H^1_{\mathrm{par}}(\Gamma;\mathcal{W}^{\omega^*,\infty}_s) \arrow[r, "\cup"'] & H^2_{\mathrm{par}}(\Gamma;\mathcal{E}^{-\infty}_{1-s}\otimes\mathcal{W}^{\omega^*,\infty}_s) \arrow[r, "\text{duality}"'] & H^2_{\mathrm{par}}(\Gamma;\mathbb{C}) \arrow[u, "\text{eval. on }(\mathfrak{F})"']
\end{tikzcd}$$

Since $\mathcal{V}^{-\infty}_{1-s}\cong\mathcal{E}^{-\infty}_{1-s}$ and $\mathcal{V}^{\infty}_s\supset\mathcal{V}^{\omega^*,\infty}_s\cong\mathcal{W}^{\omega^*,\infty}_s$, this completes the proof of Theorem 19.1. ■

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Notations

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