

# Information thermodynamics of financial markets: the Glosten-Milgrom model

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## Abstract

The Glosten-Milgrom model describes a single asset market, where informed traders interact with a market maker, in the presence of noise traders. We derive an analogy between this financial model and a Szilárd information engine by *i)* showing that the optimal work extraction protocol in the latter coincides with the pricing strategy of the market maker in the former and *ii)* defining a market analogue of the physical temperature from the analysis of the distribution of market orders. Then we show that the expected gain of informed traders is bounded above by the product of this market temperature with the amount of information that informed traders have, in exact analogy with the corresponding formula for the maximal expected amount of work that can be extracted from a cycle of the information engine. This suggests that recent ideas from information thermodynamics may shed light on financial markets, and lead to generalised inequalities, in the spirit of the extended second law of thermodynamics.

Information plays a key role in the working of financial markets. News about the performance of a company drives the activity of traders in the market, and as a result, the corresponding stock price adjusts in such a way as to reflect this information. The standard argument in economics [1] maintains that prices adjust to the discounted expected value of the future stream of payoffs that a share of that stock will deliver. In this ideal picture, (equilibrium) prices satisfy the efficient market hypothesis [2, 3], i.e. they reflect all available information, and no arbitrage (i.e risk-less gain) should be possible. This has been the dominating paradigm in finance. Several theoretical models have been proposed to show how information aggregation occurs, under specific market mechanisms [4, 5, 6, 7, 8], and how markets become informationally efficient [9]. Empirical studies suggest that the efficient market hypothesis [10, 11, 12] is typically satisfied in real markets, in spite of some anomalies [13].

This would suggest that information theory should play a major role in the theory of financial markets. But it does not. This is reminiscent of the state of affairs in statistical physics a few decades ago. Only very recent advances in stochastic thermodynamics [14, 15, 16] have led to recognise the relation between the physics of thermodynamic transformations and information theory [17]. There are good reasons to expect that a similar connection with the theory of financial

markets also exists. Indeed, the no-arbitrage hypothesis and the second law of thermodynamics both have the conceptual flavour of a no-free-lunch statement: the first maintains that no risk-less gain can be extracted from financial trading, the second asserts that no work can be extracted from a cyclic transformation of a thermodynamic system at constant temperature. The second law of thermodynamics refers to the average work over many cycles. Yet, work can be extracted in a single realisation of the engine's cycle with some probability [15]. Likewise, we know that statistical arbitrages do exist in real markets, but they're rare and they do not persist in the long run [13, 8]. Also, work can be extracted from a system, if one can acquire some information on its microscopic state, through a measurement. Likewise, an informed trader can extract a positive gain from trading. Furthermore, the no-arbitrage hypothesis in finance is formalised within the theory of martingales, which has recently found several applications in stochastic thermodynamics (see e.g. [18, 19]). This suggests a common conceptual basis for finance and stochastic thermodynamics.

In order to explore this connection, this paper focuses on the Glosten-Milgrom (GM) model [5], which provides a simple setting for a detailed analysis. This describes a single asset market where a population of informed and non-informed traders interact with a market maker, who sets the prices.

We derive a general analogy between the GM model and the Szilárd information engine [20, 21, 22], which is a prototype model in information thermodynamics. Within this analogy, we identify a “market temperature”  $T$  (Eq. (19)), which is the analogue in the GM model of the temperature of the heat bath in the Szilárd engine. We also show that the optimal pricing policy of the market maker in the GM model corresponds to the optimal work extraction protocol in the Szilárd engine. The sequence of trades in the financial system corresponds to a measurement process in the physics analogy. The generalised second law of thermodynamics [23] then states that, in the physical system, the maximal expected value of the work  $W$  that can be extracted from a cycle of the engine equals  $T$  times the amount of information  $H[Y]$  provided by the measurement, i.e.,

$$\mathbb{E}[W] \leq TH[Y]. \quad (1)$$

In the financial system  $H[Y]$  quantifies the amount of information that informed traders have on the value of the asset. In exact analogy with Eq. (1), we prove that the expected value of the gain  $G$  of informed traders in the GM model is bounded above by the product of the market temperature  $T$  and  $H[Y]$ , i.e.,

$$\mathbb{E}[G] \leq TH[Y]. \quad (2)$$

This is the main result of this paper. We also show that the inequality (2) becomes an asymptotic equality in the limit when noise traders dominate the market, which corresponds to the limit  $T \rightarrow \infty$ .

The similarity between inequalities (1) and (2) puts the analogy between thermodynamics and finance discussed above on firmer grounds, suggesting a common conceptual basis for both.

The inequality (2) is reminiscent of the classical result [24] relating the growth rate of the capital to the information that investor has on the odds, in a scheme of lotteries. The connection between this result and stochastic thermodynamics has been explored by several authors [25, 26, 27, 28]. In particular, Vinkler *et al.* [26] derive a physical analogy of the gambler's problem which is similar to the one we shall discuss below for the GM model. In this analogy, the heat bath has no direct counterpart, so the temperature is arbitrary. As we shall see, the analogy of the GM extends further, because we identify  $T$  in terms of the parameters of the model. Taken together, these results suggests a broader range of validity of inequalities such as (2), that bound the gain that informed traders can achieve.

In the rest of the paper, we first define the GM model in Section 1. In Section 2 we discuss its connection with a Szilárd information engine and we derive the inequality (1). There we also discuss the gain of informed traders and present the main result (Eq. (2)). The analytic proofs of the inequality (2) are presented in Section 3, together with a discussion of the analytic properties of the expected gain. We conclude with a general discussion of our findings and of their possible extensions in Section 4.

We should perhaps add a word about the authorship of the various parts of this paper, since they are quite disparate in both subject matter and style. The material relating to financial markets, stochastic thermodynamics and information theory (Sections 1, 2 and 4) was written by the first two authors only, but with the asymptotic statement (32) and the inequality (2) originally being conjectural. Analytic proofs of these two results were then found by the third author, and Section 3, which contains these proofs and some other related material, is due to him.

## 1 The Glosten-Milgrom model

We consider a simplified version of the GM model [5], that describes a population of traders, who buy and sell a stock from a dealer, whom we shall call the *market maker*. The stock has a value  $Y$  which is either one, with probability  $P(Y = 1) = p$ , or zero (with probability  $1 - p$ ). The value of  $Y$  is known to a fraction  $\nu$  of the traders – the informed traders – and is unknown to both the market maker and the remaining fraction  $(1 - \nu)$  of the traders – the noise traders. The amount of information that informed traders have is quantified by the entropy<sup>1</sup>

$$H[Y] = \mathbb{E}[-\log P(Y)] = -p \log p - (1 - p) \log(1 - p). \quad (3)$$

Trading occurs sequentially at discrete times,  $t = 0, 1, 2, \dots$ . At each time, one trader is randomly drawn from the population and submits an order, either to buy ( $X_t = 1$ ) or to sell ( $X_t = 0$ ) one unit of the stock. An informed trader will buy the stock if  $Y = 1$  and will sell it if  $Y = 0$ . A noise trader will buy or sell with equal probability. The probability that the market maker will receive a buy ( $X_t = 1$ ) or a sell ( $X_t = 0$ ) order can be written in the compact form

$$P(X_t = x|Y = y) = \frac{1 - \nu}{2} + \nu \delta_{x,y}. \quad (4)$$

The market maker doesn't know the value of  $Y$ , nor whether she<sup>2</sup> is dealing with an informed or an uninformed trader. Yet she knows the probabilities  $p$  and  $\nu$ . Before observing the next order  $X_{t+1}$ , the market maker announces an ask price  $a_{t+1}$ , which is the price at which she will sell, and

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<sup>1</sup>We shall use capital letters for random variables and the corresponding lowercase letter for their realised values. At time  $t$ , the sequence of realised transactions up to that time are known, whereas the transactions for later times are unknown. So we shall use  $x_\tau$  for  $\tau \leq t$  and  $X_\tau$  for  $\tau > t$ . The expected value of a random variable  $X$  with distribution  $P(X = x) = p(x)$  is denoted as  $\mathbb{E}[X] = \sum_x x p(x)$ . The entropy of  $X$  is given by the standard formula  $H[X] = -\sum_x p(x) \log p(x) = -\mathbb{E}[\log p(X)]$ . Likewise the mutual information between random variables  $X$  and  $Y$  is given by  $I(X, Y) = \sum_{x,y} p(x, y) \log \frac{p(x, y)}{p(x)p(y)}$ , in terms of the joint distribution  $p(x, y)$  and the marginals  $p(x), p(y)$ . The mutual information is also expressed in terms of the Kullback-Leibler divergence  $I(X, Y) = D_{KL}(P(X, Y) || P(X)P(Y)) = \mathbb{E}\left[\log \frac{P(X, Y)}{P(X)P(Y)}\right]$ . We shall use  $D_{KL}(P(Y|x) || P(Y))$  to indicate the Kullback-Leibler divergence between the distribution of  $Y$  conditional to  $X = x$  and the unconditional distribution. We measure information in nats, using natural logarithms.

<sup>2</sup>We follow Osborne's suggestion (see preface in [29]) on the gender of players in game theory, referring to the market maker as female and to traders as males.

a bid price  $b_{t+1}$ , which is the price at which she will buy. These prices are set in order to ensure that her profit is zero, on average. This condition is standard in competitive markets. A market maker with negative expected profit will be exploited by traders, and if the expected profit is positive, she will be outcompeted by other market makers. Since trading is a zero-sum game, this condition arises from a minimax principle: traders submit their orders to the market maker that make minimal profit (i.e. minimal loss to them). Given this, market makers will try to maximise their profits. The profit of the market maker for selling (buying) the stock at price  $a_{t+1}$  at time  $t + 1$  is  $a_{t+1} - Y$  (respectively  $Y - b_{t+1}$ ). The equations for  $a_{t+1}$  and  $b_{t+1}$  are a consequence of the zero expected profit condition, where the expectation on  $Y$  is taken conditional on the information that the dealer has up to time  $t$ . This leads to

$$a_{t+1} = \mathbb{E}[Y|x_{\leq t}, X_{t+1} = 1] \quad (5)$$

$$b_{t+1} = \mathbb{E}[Y|x_{\leq t}, X_{t+1} = 0], \quad (6)$$

where  $x_{\leq t} = (x_0, x_1, \dots, x_t)$  is the observed history of transactions up to time  $t$ . The price of the realised transaction will be  $p_{t+1} = a_{t+1}$  if  $X_{t+1} = 1$  and  $p_{t+1} = b_{t+1}$  otherwise.

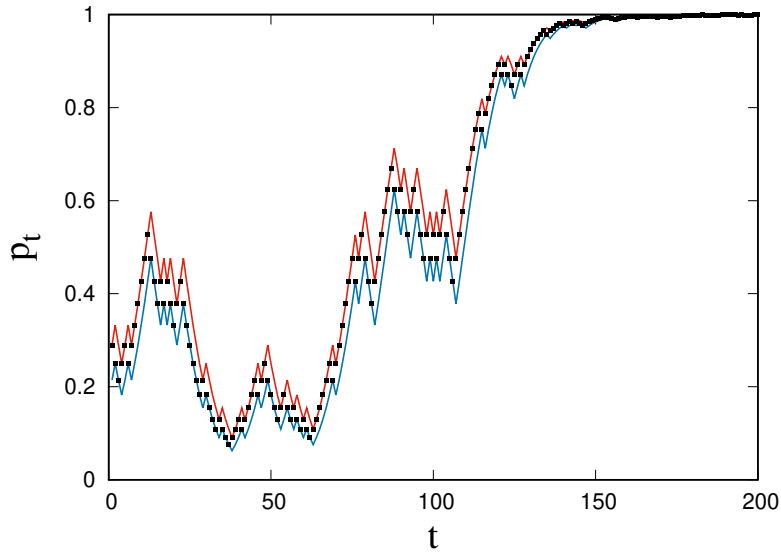


Figure 1: Sample trajectories of the price  $p_t$  ( $\blacksquare$ ) for  $Y = 1$ ,  $p = 1/4$  and  $\nu = 0.1$ . The values of  $a_t$  and  $b_t$  are shown as full red and blue lines, respectively.

As time goes on, the dealer will acquire more and more information on the true value  $Y$  of the stock, from the sequence of trades  $x_{\leq t}$ . As a consequence, both bid and ask prices will converge to  $Y$  as  $t \rightarrow \infty$ . In order to see this, let us derive explicit expressions for  $a_{t+1}$  and  $b_{t+1}$ . We start by using Bayes's formula to evaluate Eq. (5):

$$a_{t+1} = P(Y = 1|x_{\leq t}, X_{t+1} = 1) \quad (7)$$

$$= \frac{P(x_{\leq t}, X_{t+1} = 1|Y = 1)P(Y = 1)}{P(x_{\leq t}, X_{t+1} = 1)} \quad (8)$$

$$= \left[ 1 + \frac{1-p}{p} \left( \frac{1+\nu}{1-\nu} \right)^{t-2n_t-1} \right]^{-1}. \quad (9)$$

Here we used  $P(Y = 1) = p$  and the fact that, according to Eq. (4),  $x_t$  are independently drawn conditional on  $Y$ . Hence

$$P(x_{\leq t}, x_{t+1} = 1 | Y = 1) = \left(\frac{1+\nu}{2}\right)^{n_t+1} \left(\frac{1-\nu}{2}\right)^{t-n_t}, \quad n_t = \sum_{t'=0}^t x_{t'}.$$

As a consequence  $a_t$  depends only on the number  $n_t$  of buy trades up to  $t$ , not on the order in which they occurred. Similarly, we find

$$b_{t+1} = \left[1 + \frac{1-p}{p} \left(\frac{1+\nu}{1-\nu}\right)^{t-2n_t+1}\right]^{-1}. \quad (10)$$

The dealer will always sell at a price  $a_t$  higher than the one ( $b_t$ ) at which she buys. The difference  $a_t - b_t \geq 0$  is called the bid-ask spread in finance [8].

Notice that  $n_t$  follows a binomial distribution

$$P(n_t | Y) = \binom{t}{n_t} \left(\frac{1-\nu}{2} + \nu Y\right)^{n_t} \left(\frac{1+\nu}{2} - \nu Y\right)^{t-n_t}, \quad (11)$$

that depends on  $Y$ . The sequence  $x_{\leq t}$  of trades will reveal which of the two distributions  $P(n_t | Y = 0)$  or  $P(n_t | Y = 1)$  is realised.

If  $Y = 1$ , the number of buy transactions will almost surely grow like  $n_t \simeq t(1+\nu)/2$ . Likewise, if  $Y = 0$  we find  $n_t \simeq t(1-\nu)/2$ . Eqs. (9,10) then show that, almost surely,<sup>3</sup>

$$\lim_{t \rightarrow \infty} a_t = \lim_{t \rightarrow \infty} b_t = Y. \quad (12)$$

This implies that the price will ultimately aggregate all information conveyed by the activity of informed traders, and converge to the true value of the stock. Fig. 1 show sample trajectories for  $a_t$  and  $b_t$  in both cases  $Y = 0$  and  $Y = 1$ .

The realised price  $p_t$  at time  $t$ , i.e. the price at which the transaction occurs, can be written as

$$p_t = \mathbb{E}[Y | x_{\leq t}]. \quad (13)$$

This coincides with  $a_t$  if  $x_t = 1$  and with  $b_t$  if the last trader was a seller ( $x_t = 0$ ). The price satisfies the martingale property<sup>4</sup>  $\mathbb{E}[p_{t+1} | x_{\leq t}] = p_t$  [5].

## 2 The Information Thermodynamics of the GM model

The protocol that the market maker follows generates a stochastic price process. This process is driven by the information  $x_t$  on the type of orders that the market maker receives, that leads her to discover the true value of  $Y$ . In this section we map this process to a Szilárd engine and we show that optimal work extraction coincides with the optimal policy that the market maker implements.

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<sup>3</sup>A more refined analysis shows that  $\alpha_t = \log(1/a_t - 1)$  performs a random walk with drift  $\nu(1 - 2Y) \log \frac{1+\nu}{1-\nu}$ . Hence for  $Y = 1$ ,  $\alpha_t \rightarrow -\infty$  and  $\alpha_t \rightarrow \infty$  for  $Y = 0$ , as  $t \rightarrow \infty$ . Therefore  $a_t = (1 + e^{\alpha_t})^{-1} \rightarrow Y$  as  $t \rightarrow \infty$ . Similarly  $b_t$  can be expressed in terms of the same random walk process  $\alpha_t$ , but with different initial conditions.

<sup>4</sup>The price  $p_{t+1} = \mathbb{E}[Y | x_{\leq t}, X_{t+1}]$  at time  $t$  is a random variable, because it depends on the realisation  $X_{t+1}$  of the next trade.  $\mathbb{E}[p_{t+1} | x_{\leq t}] = \mathbb{E}[\mathbb{E}[Y | x_{\leq t}, X_{t+1}]]$  is the expected value of this random variable on  $X_{t+1}$ . By the property of expectations, this equals  $\mathbb{E}[Y | x_{\leq t}] = p_t$ .

## 2.1 The Szilárd engine

The Szilárd engine is the simplest realisation of the Maxwell’s demon idea, that the second law of thermodynamics can be violated if some information on the microscopic state of a system is available. In its original form, it consists of a box of unit lateral length in each direction, containing a single point-like particle. The box is in contact with a heat bath at temperature  $T$ . At a certain time  $t$ , a measurement  $Y$  is taken that reveals whether the particle is in the left ( $Y = 0$ ) or the right ( $Y = 1$ ) side of the box. This allows the observer to extract work from this system, using the following protocol. A wall is inserted without friction and instantaneously right after the measurement. The measurement reveals on which side the particle is confined. Hence the observer knows on which side the particle will push the wall. The force exerted on the wall by the particle can be exploited to do work, e.g. to lift a weight. There are several details in this idealised system that have been discussed at length elsewhere [20, 21, 22]. In the present context, let it suffice to say that in the limit of a quasi-static process (i.e. when the wall moves infinitely slowly), the work that can be extracted can be computed considering the expansion of an ideal gas composed of a single particle. This is characterised by an equation of state  $PV = Nk_B T$ , where  $N$  is the number of particles and  $k_B$  is Boltzmann constant, that we shall take equal to one ( $k_B = 1$ ) in what follows. After the wall is inserted, the pressure on the side containing the particle is  $P = T/V$ , since  $N = 1$ . On the other side the pressure is zero, because  $N = 0$ . In an isothermal expansion, the work done for a change  $dV$  in the volume is  $dW = PdV$ . Integrating this from the initial volume  $V_i = 1/2$  to the final volume  $V_f = 1$  yields,

$$W = \int_{1/2}^1 \frac{dV}{V} = T \log 2. \quad (14)$$

This protocol leaves the system in the same state as the one before the measurement. So the whole process constitutes a cycle whose outcome is to extract  $\log 2$  units of work from the system, in apparent<sup>5</sup> violation of the second law of thermodynamics. It has recently been realised [23, 17] that, measuring a quantity  $M$  during an isothermal cyclic transformation, allows an observer to extract, on average, an amount of work

$$\mathbb{E}[W] \leq TI(Y, M) \quad (15)$$

that cannot exceed  $T$  times the mutual information  $I(Y, M)$  between the the measurement  $M$  and the microscopic state  $Y$  of the system. Eq. (15) generalises the second law of thermodynamics, for processes that use information acquired by a measurement system. Note that  $I(Y, M) \leq H[Y]$ , with equality holding when the measurement  $M$  fully determines the state  $Y$  of the system.

## 2.2 An analogy with the Szilárd engine

The relation of the GM model to stochastic thermodynamics is sketched in Fig. 2. The variable  $Y$  describes the microscopic state of a particle in a box of lateral size  $L_x = 1$  at constant temperature  $T$ . At time  $t = 0$ , a wall is introduced in the box, separating it in two parts. If  $Y = 0$  the particle is in the left side of the box, whereas if  $Y = 1$  the particle is on the right of the partition. At each consecutive transaction  $t = 1, 2, \dots$ , the market maker receives a noisy signal  $x_t$ . Using this, she can operate a feedback protocol by moving the wall, in order to extract work<sup>6</sup>  $W$ . As we shall see,

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<sup>5</sup>Later Landauer [30] showed that, considering the cost of storing the measurement in a memory, no violation of the second law of thermodynamics occurs.

<sup>6</sup>We adopt the convention that extracted work (i.e. work done by the system) is positive and work done on the system is negative.

the optimal work extraction protocol coincides with the market maker pricing strategy in the GM model.

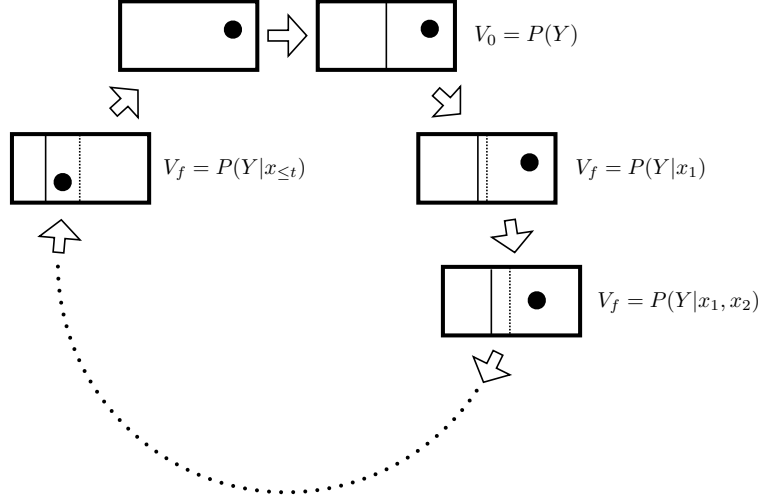


Figure 2: Sequential expansion in a Szilárd engine. Initially the particle is in equilibrium in the box. At  $t = 0$ , a wall is inserted at position  $1 - p$ , and the particle is either on the left ( $Y = 0$ ) or on the right ( $Y = 1$ ) of the wall. At  $t = 1$ , a noisy measurement  $x_1$  is made of the position of the particle and the wall is moved from the initial position (dashed line) to  $P(Y = 0|x_1)$ , therefore extracting work  $W_1$ . After the second measurement  $x_2$  the wall is moved to  $P(Y = 0|x_1, x_2)$ . The process continues for  $t$  steps, when the wall is moved to  $P(Y = 0|x_1, \dots, x_t)$ . Then the wall is removed and the system returns to the initial equilibrium state.

We first address the issue of identifying the temperature  $T$ . The box containing the particle interacts with the system generating the signals  $x_t$ . Both have to be at the same temperature  $T$ . The signals  $x_t$  are drawn from the distribution Eq. (4) independently, conditional on  $Y$ . The signals  $x_{\leq t}$  can be described as a physical system of  $t$  non-interacting particles, in equilibrium at temperature  $T$ . Each particle can be either in the “right” state  $x_t = Y$ , or in the “wrong” state  $x_t = 1 - Y$  and the energy of configuration  $x_{\leq t}$  is the sum

$$E\{x_{\leq t}|Y\} = \sum_{\tau \leq t} [\epsilon_w \delta_{x_\tau, 1-Y} + \epsilon_r \delta_{x_\tau, Y}] = (t - m_t)\epsilon_w + m_t\epsilon_r \quad (16)$$

of the energies of the individual particles, which equal  $\epsilon_w$  or  $\epsilon_r$  depending on whether  $x_\tau \neq Y$  or  $x_\tau = Y$ , respectively. In Eq. (16)  $m_t = \sum_{\tau \leq t} \delta_{x_\tau, Y}$  is the number of particles in the “right” state ( $x_\tau = Y$ ). In order to define the temperature  $T$ , we rewrite the probability of a micro-state  $x_{\leq t}$  as a Gibbs-Boltzmann distribution

$$P(x_{\leq t}|Y) = \left(\frac{1+\nu}{2}\right)^{m_t} \left(\frac{1-\nu}{2}\right)^{t-m_t} = \frac{1}{Z} e^{-E\{x_{\leq t}|Y\}/T}, \quad (17)$$

where  $Z$  is the normalisation constant. The thermodynamics of the system of signals  $x_{\leq t}$  is defined by the free energy

$$\mathcal{F} = -T \log Z = \mathbb{E}[E] - TH[X_{\leq t}|Y], \quad (18)$$

where

$$H[X_{\leq t}|Y] = -\mathbb{E}[\log P(X_{\leq t}|Y)] = t \left[ -\frac{1+\nu}{2} \log \frac{1+\nu}{2} - \frac{1-\nu}{2} \log \frac{1-\nu}{2} \right]$$

is the entropy of the sequence  $X_{\leq t}$ , conditional on  $Y$ . We require that all thermodynamic effects related to variations in  $\nu$  should be ascribed to variations in the entropic term, i.e. that the average energy

$$\mathbb{E}[E] = t \left( \frac{1+\nu}{2} \epsilon_r + \frac{1-\nu}{2} \epsilon_w \right)$$

should be independent of  $\nu$ <sup>7</sup>. Without loss of generality, we set the zero of the energy so that  $\mathbb{E}[E] = 0$ . This allows us to express  $\epsilon_r$  in terms of  $\epsilon_w$  and it yields  $Z = e^{H[X_{\leq t}|Y]}$  (see Eq. 18). These relations, combined with the expression for  $H[X_{\leq t}|Y]$ , turn Eq. (17) into an identity for arbitrary values of  $t$  and  $m_t$ , provided that  $T$  satisfies the equation

$$\frac{\epsilon_w}{T} = \frac{1+\nu}{2} \log \frac{1+\nu}{1-\nu}.$$

Here  $\epsilon_w$  remains as the only scale of the energy. Without loss of generality, we set  $\epsilon_w = 1$ , which is equivalent to measuring  $T$  and  $W$  in units of  $\epsilon_w$ . With this choice, the temperature takes the value

$$T = \left( \frac{1+\nu}{2} \log \frac{1+\nu}{1-\nu} \right)^{-1}. \quad (19)$$

This definition of the temperature in Eq. (19) coincides with the micro-canonical one

$$\frac{\partial S}{\partial E} = \frac{1}{T}, \quad (20)$$

in the thermodynamic limit  $t \rightarrow \infty$ , as it should be, by the equivalence of ensembles. The entropy  $S$  in Eq. (20) is the logarithm of the number of configurations with a certain energy  $E$ , i.e.

$$S = \log \binom{t}{m_t} \simeq t \left[ -\frac{m_t}{t} \log \frac{m_t}{t} - \left(1 - \frac{m_t}{t}\right) \log \left(1 - \frac{m_t}{t}\right) \right]$$

where the last relation holds in the limit  $t \rightarrow \infty$ .  $S$  as a function of  $E$  is obtained using Eq. (16) to express  $m_t$  in terms of  $E$ , i.e.  $m_t = \frac{t\epsilon_w - E}{\epsilon_w - \epsilon_r}$ . Its derivative evaluated at the typical value of the energy ( $E = \mathbb{E}[E] = 0$ ), with  $\epsilon_w = 1$ , yields Eq. (20), with  $T$  given by Eq. (19).

The market temperature  $T$  depends only on  $\nu$ , i.e. on the fraction of informed traders. It is a decreasing function of  $\nu$ . It diverges when  $\nu \rightarrow 0$ , i.e. when the market is dominated by noise traders. In the absence of noise traders (i.e. when  $\nu \rightarrow 1$ ) the temperature  $T$  tends to zero.

Let us now show that the protocol of optimal work extraction in a Szilárd engine operated at temperature  $T$  coincides with the market maker strategy. Our derivation follows steps similar to Ref. [26]. The following work extraction protocol is operated by an agent that observes the sequence of signals  $x_1, x_2, \dots$  that the market maker receives. We identify this agent with the market maker for simplicity, but that is not necessary. The main aim of the present section is to derive an upper bound on the work that these signals allows an observer to extract from a Szilárd engine.

Let us discuss the first step  $t = 1$ . Let the wall divide the box in two partitions of volumes  $V_0(0)$  on the left and  $V_0(1) = 1 - V_0(0)$  on the right of the wall. In this way,  $V_0(Y)$  is the volume

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<sup>7</sup>A variation of  $\nu$  would correspond to a thermodynamic transformation where some work  $W$  is done on the system. By the first law of thermodynamics  $\Delta \mathbb{E}[E] = W - Q$ , the work is related to the change in the internal energy and to the heat  $Q$  absorbed by the system. Assuming that  $\mathbb{E}[E]$  is independent of  $\nu$  implies that all work done on the system by changing  $\nu$  is turned into heat  $Q$ , which then results in a change of the temperature  $T$ .



of the partition containing the particle. The distribution of the particle's position in the box is uniform, so the probability that the particle is initially found in  $Y$  is equal to  $V_0(Y)$ . Equivalence with the GM model implies  $V_0(1) = P(Y = 1) = p$ .

The market maker then observes the value of  $x_1$  and she can move the partition quasi-statically. Let  $V_f^{(0)}(Y|x_1)$  be the volume of the partition containing the particle after this change. The work extracted for  $t = 1$  is

$$W_1(Y) = T \log \frac{V_f^{(0)}(Y|x_1)}{V_0(Y)}. \quad (21)$$

Note that  $W_1$  is a random variable, because it depends on the (unknown) position of the particle  $Y$ . The expected value of  $W_1$ , conditional on  $X_1 = x_1$  is

$$\mathbb{E}[W_1|X_1 = x_1] = T \sum_y P(y|x_1) \log \frac{V_f^{(0)}(y|x_1)}{V_0(y)},$$

which is maximal for

$$V_f^{(0)}(y|x_1) = P(y|x_1).$$

So the maximum value of the expected work extracted at  $t = 1$  is

$$\max \mathbb{E}[W_1|X_1 = x_1] = T I(Y, x_1),$$

where  $I(Y, x_1) = D_{KL}(P(Y|x_1)||P(Y))$  is the information gained on  $Y$  from the measurement  $x_1$ . The expected value of  $I(Y, x_1)$  on  $x_1$  is the mutual information  $I(Y, X_1) = \sum_{y, x_1} p(y, x_1) \log \frac{p(y|x_1)}{p(y)}$ .

After the first step of quasi-static expansion, without removing the wall, we find ourselves in a situation very similar to the initial one. The new initial volume of the partition containing the particle is  $V_0^{(1)}(Y) = V_f^{(0)}(Y|x_1) = P(Y|x_1)$ , which is the probability that the particle is in partition  $Y$  of the box, given the information  $x_1$ . At  $t = 2$  the market maker receives the signal  $x_2$ , and performs a quasi-static expansion to volume  $V_f^{(1)}(Y|x_2)$ . Maximising the expected work that can be extracted, we find  $V_f^{(1)}(Y|x_2) = P(Y|x_1, x_2)$ . The maximum value of the expected work done for  $t = 2$  is

$$\max \mathbb{E}[W_2|x_1, x_2] = T \sum_y P(y|x_1, x_2) \log \frac{P(y|x_1, x_2)}{P(y|x_1)} = T I(Y, x_2|x_1).$$

This argument can be extended for all  $t \geq 1$ . Given all the information  $x_{\leq t-1} = (x_1, \dots, x_{t-1})$  received before  $t$ , the optimal policy implies that  $V_0^{(t-1)}(y) = P(y|x_{\leq t-1})$ . The work extracted is

$$W_t(Y) = T \log \frac{V_f^{(t-1)}(Y|x_t)}{P(Y|x_{\leq t-1})},$$

whose expected value is maximal for  $V_f^{(t-1)}(y|x_t) = P(y|x_{\leq t})$ . The expected value of the work extracted at time  $t$  is  $\max \mathbb{E}[W_t|x_{\leq t}] = T I(Y, x_t|x_{\leq t-1})$ . The total work extracted up to time  $t$  following this protocol is

$$W_{\leq t}(Y) = W_1(Y) + W_2(Y) + \dots, W_t(Y) \quad (22)$$

$$= T \log \frac{P(Y|x_{\leq t})}{P(Y)}. \quad (23)$$

so its expected value over  $Y$  is  $\max \mathbb{E}[W_{\leq t} | x_{\leq t}] = I(Y, x_{\leq t})$ . Taking the expected value over the realisation of the trading process  $x_{\leq t}$ , we find

$$\max \mathbb{E}[W_{\leq t}] = T I(Y, X_{\leq t}) \leq T H[Y], \quad (24)$$

where the last inequality holds because  $I(Y, X_{\leq t}) = H[Y] - H[Y | X_{\leq t}]$  and  $H[Y | X_{\leq t}] \geq 0$ . If the wall is removed at time  $t$ , the system reverts back to the original equilibrium state, closing the cycle. The inequality (24) holds as an equality in the limit  $t \rightarrow \infty$ , i.e.

$$\boxed{\lim_{t \rightarrow \infty} \max \mathbb{E}[W_{\leq t}] = T H[Y],} \quad (25)$$

because  $H[Y | X_{\leq t}] \rightarrow 0$  as  $t \rightarrow \infty$ . Indeed,  $P(y | x_{\leq t})$  converges to a singleton in either  $y = 0$  or  $y = 1$ , for all typical realisations of  $x_{\leq t}$ .

Finally, notice that the price

$$p_t = V_0^{(t)}(1 | x_{\leq t}) \quad (26)$$

is given by the volume of the right partitions, along the process. Therefore the feedback protocol that extracts the maximal amount of work coincides with the optimal behaviour of the market maker.

## 2.3 The Gain of Informed Traders

In this section we discuss the profits and losses of different market participants. As discussed in Section 1, the market-maker sets the prices in order to have zero expected profit at each time step. Since the market is a zero sum game, the expected total gain of informed traders equals the expected total loss of noise traders. Therefore it is sufficient to focus attention on the gain of informed traders, which is what we shall do in what follows.

Let  $G_t$  denote the total expected gain of informed traders up to time  $t$ . This satisfies the recursion relation

$$G_{t+1} = U_{t+1}[b_{t+1}(1 - Y) + (1 - a_{t+1})Y] + G_t, \quad (27)$$

where  $U_t$  is a random variable that takes value  $U_t = 1$  if the trader at time  $t$  is informed and  $U_t = 0$  otherwise. Indeed, when  $Y = 0$  an informed trader will gain  $b_{t+1}$  by selling the stock, whereas if  $Y = 1$  he will buy at price  $a_{t+1}$  realising a gain  $1 - a_{t+1}$ . We are interested in the total asymptotic gain, defined as

$$G = \lim_{t \rightarrow \infty} G_t = \sum_{t=0}^{\infty} U_{t+1}[b_{t+1}(1 - Y) + (1 - a_{t+1})Y], \quad (28)$$

where the last equality derives from the recursion relation (27) and the initial condition  $G_0 = 0$ . The expected value of  $G$  can be computed taking conditional expectations

$$\mathbb{E}[G] = p \mathbb{E}[G | Y = 1] + (1 - p) \mathbb{E}[G | Y = 0]. \quad (29)$$

Taking the expectation of Eq. (28) conditional to  $Y = 0$ , we find

$$\begin{aligned} \mathbb{E}[G | Y = 0] &= \nu \sum_{t=0}^{\infty} \mathbb{E}[b_{t+1}] \\ &= \nu \sum_{t=0}^{\infty} \sum_{k=0}^t \binom{t}{k} \frac{\left(\frac{1-\nu}{2}\right)^k \left(\frac{1+\nu}{2}\right)^{t-k}}{1 + \frac{1-p}{p} \left(\frac{1+\nu}{1-\nu}\right)^{t-2k+1}}. \end{aligned} \quad (30)$$

In the first line we used the fact that the variables  $U_t$  are independent and identically distributed, with  $\mathbb{E}[U_{t+1}] = \nu$ . Eq. (30) follows from the expression Eq. (10) for  $b_{t+1}$  and Eq. (11) for the binomial distribution over which the expectation is taken. The expectation, conditional to  $Y = 1$  is computed in the same way, substituting  $b_{t+1}$  with  $1 - a_{t+1}$ :

$$\begin{aligned}\mathbb{E}[G|Y = 1] &= \nu \sum_{t=0}^{\infty} \sum_{k=0}^t \binom{t}{k} \left(\frac{1+\nu}{2}\right)^k \left(\frac{1-\nu}{2}\right)^{t-k} \left(1 - \frac{1}{1 + \frac{1-p}{p} \left(\frac{1+\nu}{1-\nu}\right)^{t-2k-1}}\right) \\ &= \nu \sum_{t=0}^{\infty} \sum_{k=0}^t \binom{t}{k} \frac{\left(\frac{1-\nu}{2}\right)^k \left(\frac{1+\nu}{2}\right)^{t-k}}{1 + \frac{p}{1-p} \left(\frac{1+\nu}{1-\nu}\right)^{t-2k+1}}.\end{aligned}\quad (31)$$

Here we have used the change of index  $k \rightarrow t - k$  in the last step. Notice that  $\mathbb{E}[G|Y = 1]$  in Eq. (31) is equal to Eq. (30) with  $p$  replaced by  $1 - p$ . This implies that  $\mathbb{E}[G]$  is invariant under the transformation  $p \rightarrow 1 - p$ , as it should because the GM model enjoys the same invariance.

From equations (30) and (31) it is clear that both  $\mathbb{E}[G|Y = 0]$  and  $\mathbb{E}[G|Y = 1]$ , and hence also their sum  $\mathbb{E}[G]$ , are decreasing functions of  $\nu$ . Also note that  $\mathbb{E}[G] \rightarrow 0$  as  $\nu \rightarrow 1$ , as it should be, since if all traders are informed, the market maker can guess the value of  $Y$  from the first transaction.

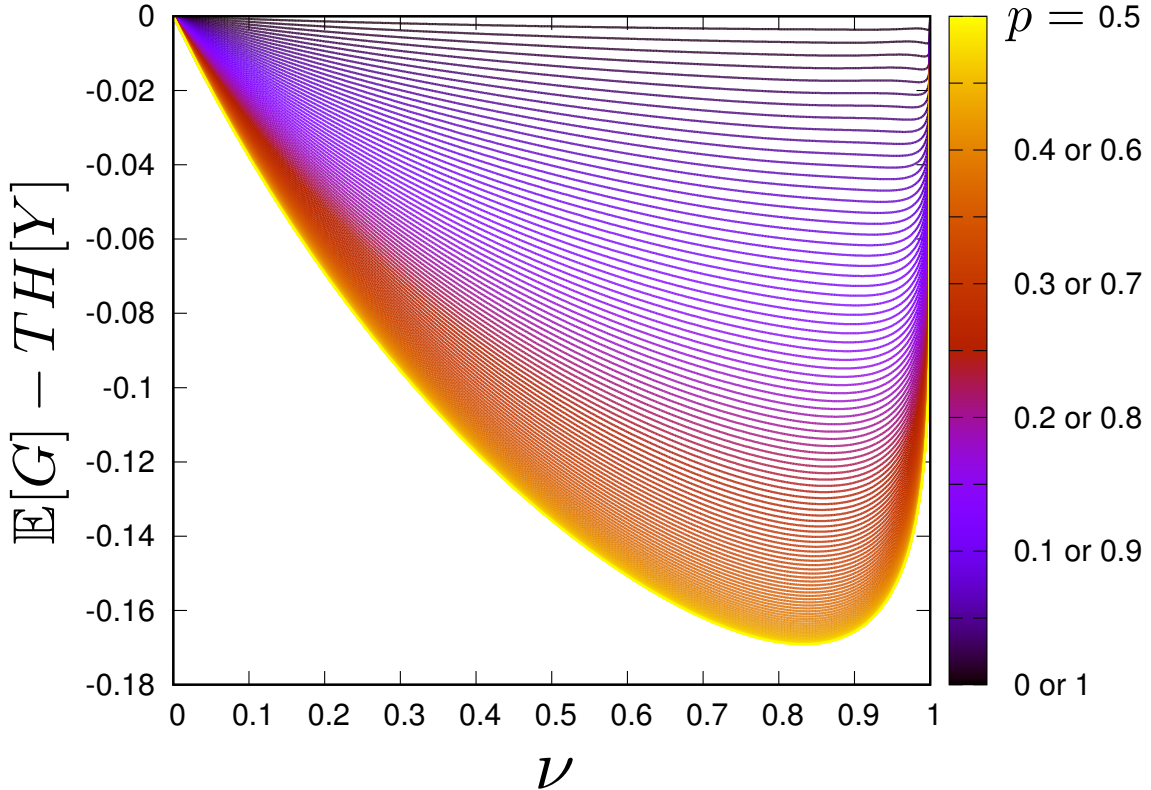


Figure 3: Numerical evaluation of the function  $\mathbb{E}[G] - TH[Y]$  as a function of  $\nu$ , for different values of  $p$ , on a scale defined by the color-code on the right.

The expression for  $\mathbb{E}[G]$  that we obtain by combining Eqs. (29), (30) and (31) has no evident analogue in the physical system. Yet numerical analysis gives strong indications in support of the

inequality (2), as shown in Fig. 3. This also suggests the limiting behavior

$$\lim_{\nu \rightarrow 0} \frac{\mathbb{E}[G]}{T} = \lim_{T \rightarrow \infty} \frac{\mathbb{E}[G]}{T} = H[Y] \quad (32)$$

as  $\nu \rightarrow 0$  or  $T \rightarrow \infty$  (recall that  $\nu$  and  $T$  are related by (19)). In the next section we will give proofs of the inequality (2) and of Eq. (32). In fact, in Section 3 we will derive a refinement of (32) to a complete asymptotic expression

$$\mathbb{E}[G] \simeq H[Y]T - \frac{5p(1-p)}{3T} + \frac{10p(1-p)}{3T^2} - \frac{2p(1-p)(173-3p+3p^2)}{45T^3} + \dots \quad (33)$$

for  $\mathbb{E}[G]$  as a Laurent series in  $1/T$  as  $T \rightarrow \infty$  (or  $\nu \rightarrow 0$ ), giving a quantitative version of the inequality (2) in the large temperature limit. The coefficients in the expansion (33) are all polynomials in  $p(1-p)$ , consistently with the invariance under the transformation  $p \rightarrow 1-p$  discussed above.

The inequality (2), which is the main result of this paper, gives an upper bound to the gains that informed traders can extract from their side information. Combined with the derivation in the previous section, it shows that the maximal profit that informed traders can gain cannot exceed the maximal work that can be extracted from the analogous physical system (the Szilárd information engine). This reveals a precise analogy between the generalized second law of thermodynamics (15) and a generalised efficient market hypothesis, for the GM model. The asymptotic statements (32) and (33), on the other hand, says that for  $\nu \rightarrow 0$  the approach of the price  $p_t$  to its asymptotic value  $Y$  becomes infinitely slow, resembling a quasi-static limit in thermodynamics. In this limit, the transactions of informed traders are well separated in time and the market has enough time to relax to an equilibrium dominated by noise traders, as in a reversible process. Eq. (32), hence, is consistent with the observation [31] that the generalised second law of thermodynamics holds as an equality for reversible feedback protocols.

### 3 Asymptotics and inequalities for the expected gain function

In this section we will study the properties of the functions  $\mathbb{E}[G|Y=0]$ ,  $\mathbb{E}[G|Y=1]$  and  $\mathbb{E}[G|Y=1]$  as defined in equations (30), (31) and (29), respectively, and in particular prove the asymptotic development (33) and inequality (2).

It will be convenient to replace  $\nu$  by an equivalent variable  $q$ , again between 0 and 1, by setting

$$q = \frac{1-\nu}{1+\nu}, \quad \nu = \frac{1-q}{1+q}. \quad (34)$$

We can then rewrite equations (31), (30) and (29) in terms of this new variable as

$$\mathbb{E}[G|Y=1] = G_q(p), \quad \mathbb{E}[G|Y=0] = G_q(1-p), \quad \mathbb{E}[G] = p G_q(p) + (1-p) G_q(1-p), \quad (35)$$

where  $G_q(p)$  is defined by

$$G_q(p) = \sum_{t=0}^{\infty} \frac{1-q}{(1+q)^{t+1}} \sum_{k=0}^t \binom{t}{k} \frac{q^k}{1 + \frac{p}{1-p} q^{2k-t-1}}. \quad (36)$$

It is therefore this function that we have to study.

The formula for  $G_q(p)$  can be simplified considerably. We first split up the sum into two parts according to whether  $k \leq t/2$  or  $k > t/2$ . Writing  $(t, k)$  as  $(2m + n, m)$  in the first case and as  $(2m + n, m + n)$  in the second leads to

$$G_q(p) = \sum_{n=0}^{\infty} \frac{P_n(q)}{1 + \frac{p}{1-p} q^{-n-1}} + \sum_{n=1}^{\infty} \frac{q^n P_n(q)}{1 + \frac{p}{1-p} q^{n-1}} \quad (37)$$

with  $P_n(q)$  defined by

$$P_n(q) = \frac{1-q}{(1+q)^{n+1}} \sum_{m=0}^{\infty} \binom{2m+n}{m} \frac{q^m}{(1+q)^{2m}}. \quad (38)$$

But  $P_n(q) = 1$  by a standard identity (that can be proved, from the binomial theorem for  $n = 0$  and then by induction on  $n$ ), so the formula for  $G_q(p)$  simplifies to

$$G_q(p) = \sum_{n=0}^{\infty} \frac{1}{1 + \frac{p}{1-p} q^{-n-1}} + \sum_{n=1}^{\infty} \frac{q^n}{1 + \frac{p}{1-p} q^{n-1}} = \sum_{n \in \mathbb{Z}} \frac{q^{\max(0, n)}}{1 + \frac{p}{1-p} q^{n-1}}.$$

This can be rewritten as

$$G_q(p) = F\left(\frac{1-p}{p}, q\right) + \frac{1-p}{p} q F\left(\frac{p}{1-p}, q\right) - \frac{(1-p)(1-q)}{2}, \quad (39)$$

where  $F(x, q)$  is the relatively simple function defined for all  $x$  and  $q$  in  $\mathbb{C}$  with  $|q| < 1$  by

$$F(x, q) = \frac{1}{2} \frac{x}{1+x} + \sum_{n=1}^{\infty} \frac{q^n x}{1+q^n x}. \quad (40)$$

Eq. (35) then gives

$$\mathbb{E}[G] = (1+q) \left[ p F\left(\frac{1-p}{p}, q\right) + (1-p) F\left(\frac{p}{1-p}, q\right) \right] - p(1-p)(1-q). \quad (41)$$

In the remainder of this section we will give a series of properties of the function  $F(x, q)$  and use them and Eq. (41) to prove (33) and (2).

1. The simplest property of  $F(x, q)$  is that it satisfies the functional equation

$$F(x, q) - F(qx, q) = \frac{1}{2} \left( \frac{x}{1+x} + \frac{qx}{1+qx} \right), \quad (42)$$

as one sees by replacing  $n$  by  $n+1$  in (40). A consequence is that the antisymmetrized function

$$F_-(x, q) = F(x, q) - F(x^{-1}, q) + \frac{\log x}{\log q} \quad (43)$$

is invariant under  $x \mapsto qx$  and therefore has a Fourier expansion in  $\frac{\log x}{\log q}$ , and indeed by using the Poisson summation formula we find the rapidly convergent Fourier sine expansion

$$F_-(e^{uh}, e^{-h}) = \frac{2\pi}{h} \sum_{n=1}^{\infty} \frac{\sin(2\pi nu)}{\sinh(2\pi^2 n/h)} \quad (h > 0). \quad (44)$$

We omit the proof, since we will not use this formula. We also mention in passing that  $F_-$  can be expressed in closed form in terms of the Weierstrass zeta-function  $\zeta(z; \tau)$  (the function whose derivative is the Weierstrass  $\wp$ -function; we do not give the complete definition since it plays no further role in this paper) with  $q = e^{2\pi i \tau}$  and  $x = -e^{2\pi i z}$ , and that Eq. (44) can then also be obtained as a consequence of the transformation behavior of  $\zeta(z; \tau)$  under  $(z, \tau) \mapsto (z/\tau, -1/\tau)$ .

2. If we expand each term  $\frac{q^n x}{1+q^n x}$  ( $n \geq 0$ ) in (40) as a geometric series  $\sum_{r=1}^{\infty} (-1)^{r-1} q^{nr} x^r$  and then resum the resulting geometric series in  $n$ , we obtain the formula

$$F(x, q) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{2} \frac{1+q^r}{1-q^r} x^r \quad (45)$$

giving the Taylor expansion of  $F$  at  $x = 0$ . This formula is only valid for  $|x| < 1$ , but if we retain the first  $N$  terms of (40) and expand only the others as geometric series we obtain the more general hybrid series expansion

$$F(x, q) = \frac{1}{2} \frac{x}{1+x} + \sum_{n=1}^{N-1} \frac{q^n x}{1+q^n x} + \sum_{r=1}^{\infty} (-1)^{r-1} \frac{q^{Nr}}{1-q^r} x^r, \quad (46)$$

which is now valid whenever  $|x| < |q|^{-N}$ , and hence for any  $x \in \mathbb{C}$  if we take  $N$  large enough. This is also useful computationally (although the convergence of either (40) or (45) is already exponential and hence good enough in practice), since truncating the second sum in (46) after  $N$  terms gives an approximation of  $F(q, x)$  up to order  $q^{N^2}$  in only  $O(N)$  rather than  $O(N^2)$  steps.

3. We now consider the behavior of  $F(x, q)$  near  $q = 1$ . Set  $q = e^{-h}$  with  $h > 0$ . Replacing  $\frac{1}{2} \frac{1+q^r}{1-q^r} = \frac{1}{2} + \frac{1}{e^{rh}-1}$  in equation (45) by its Laurent expansion in terms of Bernoulli numbers, we get the asymptotic expansion

$$\begin{aligned} F(x, e^{-h}) &\sim \sum_{r=1}^{\infty} (-1)^{r-1} \left( \frac{1}{rh} + \sum_{n=2}^{\infty} \frac{B_n(rh)^{n-1}}{n!} \right) x^r \\ &= \frac{\log(1+x)}{h} + \sum_{n=2}^{\infty} \frac{B_n}{n!} \left( x \frac{d}{dx} \right)^{n-1} \left( \frac{x}{1+x} \right) h^{n-1} \\ &= \frac{\log(1+x)}{h} + \frac{x}{(1+x)^2} \frac{h}{12} - \frac{x-4x^2+x^3}{(1+x)^4} \frac{h^3}{720} + \dots \end{aligned} \quad (47)$$

as a Laurent series in  $h$ , in which the coefficient of  $h^{2k-1}$  for each  $k > 0$  is  $\frac{B_{2k}}{(2k)!}$  times a polynomial of degree  $k$  in  $\frac{x}{(1+x)^2}$  with integral coefficients. The fact that this series is odd implies that  $F_-(x, e^{-h})$  vanishes to all orders in  $h$  as  $h \searrow 0$ , but in fact we know from (44) that  $F_-(x, e^{-h}) = O(e^{-2\pi^2/h})$ .

4. Finally, we discuss upper bounds for  $F$ . The expansion (47) implies that the difference

$$F_0(x, q) := F(x, q) - \frac{1}{h} \log(1+x)$$

is bounded by a multiple of  $h$  as  $h \rightarrow 0$  or  $q \rightarrow 1$  with  $x$  fixed. The first part of the following proposition makes this upper bound explicit, the second part gives an explicit upper bound for  $F_0(x, q)$  that is independent of  $h$  (and hence stronger than the first bound when  $h$  is large), and the third part combines these two to give a uniform upper bound valid for all values of  $x$  and  $h$ . In each case the proof would also give a lower bound, but we do not write these out explicitly since they are not particularly relevant for the purposes of this paper.

**Proposition.** For  $x, h > 0$  and  $q = e^{-h}$  we have the upper bounds

$$F_0(x, q) < \frac{h}{12} \cdot \begin{cases} \frac{x}{(1+x)^2} & \text{if } x \leq 1, \\ \frac{1}{2} & \text{if } x \geq 1, \end{cases} \quad (48)$$

$$F_0(x, q) < \frac{1}{2} \frac{x}{1+x} - \frac{1}{h} \cdot \begin{cases} \log\left(\frac{1+x}{1+x\sqrt{q}}\right) & \text{if } x \leq q^{-1/2}, \\ \log\left(\frac{1+\sqrt{q}}{1+q}\right) & \text{if } x \geq q^{-1/2}, \end{cases} \quad (49)$$

$$F_0(x, q) < \frac{1}{2} \frac{x}{1+x} \frac{1-q}{1+q} \quad \text{for all } x \text{ and } h. \quad (50)$$

*Proof.* We first observe that the Laurent expansion (47) could also have been deduced directly from the definition (40), rather than from the alternative formula (45), by using the Euler-Maclaurin summation formula, and that the Euler-Maclaurin formula also has a finite form that gives an explicit upper bound for the truncation at any point. This finite form, obtained by  $K$ -fold integration by parts (see for instance Prop. 3 of [32] and its proof) says that for any smooth function  $f : [0, \infty) \rightarrow \mathbb{C}$  that is small at infinity and any integer  $K \geq 1$  we have

$$\frac{f(0)}{2} + \sum_{n=1}^{\infty} f(nh) = \frac{1}{h} \int_0^{\infty} f(t) dt - \sum_{k=2}^K \frac{B_k}{k!} f^{(k-1)}(0) h^{k-1} + \frac{(-h)^{K-1}}{K!} \int_0^{\infty} \overline{B_K(t/h)} f^{(K)}(t) dt$$

for all  $h > 0$ , where  $\overline{B_K(x)} = B_K(x - [x])$  is the periodic version of the  $K$ th Bernoulli polynomial. Applying this with  $K = 2$  and  $f(t) = f_x(t) := \frac{xe^{-t}}{1+xe^{-t}}$  gives

$$F_0(x, q) = \frac{h}{12} \frac{x}{(1+x)^2} - \frac{h}{2} \int_0^{\infty} \overline{B_2(t/h)} \frac{xe^{-t}(1-xe^{-t})}{(1+xe^{-t})^3} dt$$

and this implies our first bound (48) since  $|\overline{B_2(x)}| \leq \max_{0 \leq x \leq 1} |x^2 - x + \frac{1}{6}| = \frac{1}{6}$  and

$$\int_0^{\infty} \left| \frac{xe^{-t}(1-xe^{-t})}{(1+xe^{-t})^3} \right| dt = \begin{cases} \frac{x}{(1+x)^2} & \text{if } 0 \leq x \leq 1, \\ \frac{1}{2} - \frac{x}{(1+x)^2} & \text{if } x \geq 1. \end{cases}$$

If we took  $K = 1$  instead of  $K = 2$  we would get  $F_0(x, q) \leq \frac{1}{2} \frac{x}{1+x}$ , and in fact for this we would not need the Euler-Maclaurin formula at all but simply the observation that  $f_x(n) \leq \int_{n-1}^n f_x(t) dt$  for all  $n \geq 1$  because  $f_x$  is monotone decreasing. To obtain the stronger inequality (49), we use instead the observation that  $f(n) < \int_0^{1/2} (f(n+t) + f(n-t)) dt$  if  $f$  is convex on the interval  $[n - \frac{1}{2}, n + \frac{1}{2}]$ . For  $f = f_x$  this holds for all  $n \geq 1$  if  $x \leq q^{-1/2}$  and hence  $\sum_{n=1}^{\infty} f_x(n) \leq \int_{1/2}^{\infty} f_x(t) dt = \frac{1}{h} \log(1+x\sqrt{q})$ , which is equivalent to the first inequality in (49). If  $x \geq q^{-1/2}$ , then there is a unique integer  $N \geq 1$  for which  $q^{1/2} < xq^N \leq q^{-1/2}$ . Then  $f_x(t)$  is decreasing on  $(0, N)$  and convex on  $(N + \frac{1}{2}, \infty)$ , so we get instead

$$\sum_{n=1}^{\infty} f_x(n) - \int_0^{\infty} f_x(t) dt \leq - \int_N^{N+1/2} f_x(t) dt = \frac{1}{h} \log\left(\frac{1+xq^N}{1+xq^{N+1/2}}\right) \leq \frac{1}{h} \log\left(\frac{1+q^{1/2}}{1+q}\right),$$

proving the second inequality in (49) as well. Finally, the inequality (50) follows from (49) if  $x \leq q^{-1/2}$  because the difference

$$\delta_h(x) := \frac{1}{h} \log\left(\frac{1+x}{1+x\sqrt{q}}\right) - \frac{x}{1+x} \frac{q}{1+q}$$

between their right-hand sides is  $\geq 0$  for all  $x \geq 0$ , since  $\delta_h(0) = 0$  and

$$\begin{aligned} (1+x)^2 \delta'_h(x) &= \frac{1-\sqrt{q}}{h} \frac{1+x}{1+x\sqrt{q}} - \frac{q}{1+q} \\ &\geq \frac{1-\sqrt{q}}{h} - \frac{q}{1+q} = \frac{1}{1+e^h} \sum_{n=2}^{\infty} \left(1 - \frac{1+(-1)^n}{2^n}\right) \frac{h^{n-1}}{n!} \geq 0. \end{aligned}$$

If  $x \geq q^{-1/2}$  then the difference between the right-hand sides of (49) and (50) is bounded above by  $\frac{1}{1+q} - \frac{1}{h} \log\left(\frac{1+\sqrt{q}}{1+q}\right)$ , which is negative for  $h > 2.784$ , while the difference between the right-hand sides of (48) and (50) is bounded by  $\frac{x}{1+x} \left(h \frac{1+\sqrt{q}}{24} - \frac{1}{2} \frac{1-q}{1+q}\right)$ , which is negative for  $h < 11.969$ .  $\square$

We can now complete the proofs of the assertions in Section 2 of this paper by combining the formula (41) for  $\mathbb{E}[G]$  in terms of  $F(x, q)$  with the results 1.–4. With the changes of variables (34) and  $q = e^{-h}$  (or equivalently  $\nu = \tanh(h/2)$ ), the market temperature defined in (19) is given by

$$T = \frac{1+q}{h} = \frac{1+e^{-h}}{h} = \frac{2}{h} - 1 + \frac{h}{2} - \frac{h^2}{6} + \dots \quad (51)$$

so that  $q \rightarrow 1$  or  $h \rightarrow 0$  corresponds to large temperature, while the entropy  $H[Y]$  is given by (3). Substituting the expansion (47) into (41) therefore gives

$$\frac{\mathbb{E}[G]}{T} \sim H[Y] - \frac{5p(1-p)}{12} h^2 + \frac{p(1-p)(29+6p(1-p))}{720} h^4 + \dots,$$

which in view of (51) is equivalent to (33), but is somewhat simpler because it is an even power series  $h$  whereas (33) is not even or odd in powers of  $1/T$ . Finally, substituting (50) into (39) gives

$$(1+q)p F\left(\frac{1-p}{p}, q\right) < p \log(1/p) T + p(1-p) \frac{1-q}{2}.$$

Symmetrizing this with respect to  $p \leftrightarrow 1-p$  and substituting into (41) we get the inequality (2).

## 4 Discussion

This paper establishes an upper bound to the gain that informed traders can extract from their trading activity, in the context of the Glosten-Milgrom model [5]. The upper bound is derived from an analogy with a physical system. This offers the ground for applying the generalised second law of thermodynamics to financial systems, suggesting how the no-arbitrage hypothesis can be generalised in the presence of informed traders.

The key elements in the inequality are the amount of information  $H[Y]$  that informed traders have on the value  $Y$  of the asset, the market temperature  $T$  and the expected gain  $\mathbb{E}[G]$  of informed traders. The market temperature  $T$  measures the level of noise in the market. It increases with the fraction of noise traders. It vanishes when these are absent ( $\nu = 1$ ) and it diverges when the fraction of informed traders vanishes ( $\nu \rightarrow 0$ ). Therefore, (2) is very similar to the bound for work extraction in information engines. Interestingly, we find that the bound is attained asymptotically in the latter limit, i.e. when  $T \rightarrow \infty$ . This is consistent with the fact that, in this limit, the convergence of the price to the true value  $Y$  and the activity of informed traders is infinitely slow, as in a quasi-static process in physics. In the analogy with physics, gain extraction approximates



a reversible process because at infinite temperature thermalisation is infinitely fast. Interestingly, reversibility is the condition that allows maximal work extraction, as shown in [31].

A similar analogy has been drawn in Ref. [26] between the Szilárd box and gambling in a sequence of lotteries. In that case, the optimal work extraction protocol coincides with the optimal (betting) strategy, and the work extracted with the optimal rate of growth of the gambler’s gain. This suggests that the bound (2) has a wider validity than the framework of Ref. [26] or of the Glosten-Milgrom model, and it hints at a generalised second law of thermodynamics for financial markets. This is a very interesting avenue of further research<sup>8</sup>.

The present version of the GM model considers a population of an infinite number of informed traders who behave in a competitive fashion. The situation is very different from that of a single informed trader, who trades sequentially. The probability  $\nu$  that a trade is executed by an informed trader becomes, in this setting, the frequency with which the informed trader submits orders. The GM model becomes a repeated game of incomplete information [33] between the market maker and the informed trader. The informed trader can decide the frequency  $\nu$  of his orders so as to maximise his gain, taking into account that his trading activity reveals information on  $Y$  to the market maker. By taking an infinitesimally small value of  $\nu$ , the informed trader can access the regime where the inequality (2) holds asymptotically as an equality, and increase his gain by making the market temperature  $T(\nu)$  arbitrarily large. Note that the time needed to accumulate the gain also diverges in the limit  $\nu \rightarrow 0$ , corresponding to a slower impact of the activity of the informed trader on the price dynamics. An impatient trader would opt for a finite frequency while at the same time leaving a more significant impact on price’s dynamics. Further work in this direction may help shed light on the trade-offs between time-constraints and profits, which are at the basis of the theories of market impact [12].

As the GM model shows, a simple market mechanism can allow private information to be incorporated into prices. Ref. [34] argues that financial transformations, such as diversification or securitisation, degrade private information in the sense that a considerable fraction of the side information on financial returns is lost under aggregation. Extensions of the present results to multi-asset markets may shed light on the interplay between diversification and information aggregation.

We hope the present paper will stimulate further research in the direction of providing an information theoretic basis to finance.

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<sup>8</sup>After this manuscript first appeared, Pierre Carmier informed us that numerical results support the conjecture that the inequality Eq. (2) extends to finite times, if  $\mathbb{E}[G]$  is replaced by the expected gain up to time  $t$  and  $H[Y]$  by the mutual information  $I(X_{\leq t}, Y)$  between  $Y$  and the trading activity up to time  $t$ .

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