MASUR–VEECH VOLUMES OF QUADRATIC DIFFERENTIALS AND THEIR ASYMPTOTICS

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Dedicated to the memory of Boris Anatol'evich Dubrovin

Abstract. Based on the Chen–Möller–Sauvaget formula, we apply the theory of integrable systems to derive three equations for the generating series of the Masur–Veech volumes $\mathrm{Vol}\,\mathcal{Q}_{g,n}$ associated with the principal strata of the moduli spaces of quadratic differentials, and propose refinements of the conjectural formulas given in [12, 4] for the large genus asymptotics of $\mathrm{Vol}\,\mathcal{Q}_{g,n}$ and of the associated area Siegel–Veech constants.

Contents

1. Statements of the results

Let $\mathcal{M}_{g,n}$ denote the moduli space of complex algebraic curves of genus g with n distinct marked points, and $\mathcal{Q}_{g,n}$ the moduli space of pairs $(\mathcal{C}, q)$, where $\mathcal{C} \in \mathcal{M}_{g,n}$ is a smooth algebraic curve and q is a meromorphic quadratic differential on $\mathcal{C}$ with only simple poles at the marked points. This moduli space of quadratic differentials $\mathcal{Q}_{g,n}$ is endowed with the canonical symplectic structure. The induced volume element on $\mathcal{Q}_{g,n}$ is called the Masur–Veech (MV) volume element. Denote by $\mathrm{Vol}\,\mathcal{Q}_{g,n}$ the volume of $\mathcal{Q}_{g,n}$; see e.g. [12] for its meaning. Recently, Chen–Möller–Sauvaget [8] proved that the volumes $\mathrm{Vol}\,\mathcal{Q}_{g,n}$ with $2g-2+n>0$ can be expressed in terms of linear Hodge integrals as follows:

$$\mathrm{Vol}\,\mathcal{Q}_{g,n} \;=\; 2^{2g+1}\frac{\pi^{6g-6+2n}(4g-4+n)!}{(6g-7+2n)!}\,\sum_{j=0}^{g}\int_{\overline{\mathcal{M}}_{g,3g-3+2n-j}}\frac{\lambda_j\psi_{n+1}^2\cdots\psi_{3g-3+2n-j}^2}{(3g-3+n-j)!}\,, \qquad (1)$$

where $\overline{\mathcal{M}}_{g,k}$ denotes the Deligne–Mumford compactification of $\mathcal{M}_{g,k}$, ψ_i denotes the first Chern class of the i_{th} tautological line bundle on $\overline{\mathcal{M}}_{g,k}$, and λ_j denotes the j_{th} Chern class of the rank g Hodge bundle $\mathbb{E}_{g,k}$ on $\overline{\mathcal{M}}_{g,k}$. The goal of the present paper is to study the numbers $\mathrm{Vol}\,\mathcal{Q}_{g,n}$ by using the Chen–Möller–Sauvaget (CMS) formula.

For $g, n \geq 0$, we define

$$a_{g,n} \;=\; \begin{cases} \sum_{j=0}^{g} \frac{1}{(3g-3+n-j)!} \int_{\overline{\mathcal{M}}_{g,3g-3+2n-j}} \lambda_j \psi_{n+1}^2 \cdots \psi_{3g-3+2n-j}^2 \,, & 2g-2+n>0\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{2}$$

Note that the $a_{g,n}$ are rational numbers, and differ from $\operatorname{Vol}\mathcal{Q}_{g,n}$ only by some simple factors. Define a generating series $\mathcal{H}(x,\epsilon)$ for the numbers $a_{g,n}$, called the MV free energy, by

$$\mathcal{H}(x,\epsilon) \;:=\; \sum_{g,n\geq 0} \epsilon^{2g-2} \frac{x^n}{n!} a_{g,n}\,. \tag{3}$$

The first result of this paper is then given by the following theorem.

Theorem 1. *The series $\mathcal{H}(x,\epsilon)$ satisfies the following two equations:*

$$\big[\partial_x(\mathcal{H}_+ \,-\, \mathcal{H}_-)\big]^2 \;+\; \partial_x^2\big(\mathcal{H}_+ + \mathcal{H}_-\big) \;=\; \frac{2x}{\epsilon^2}\,, \tag{4}$$

$$\left(\epsilon\partial_\epsilon + \frac{1}{2}x\partial_x - \frac{\epsilon^2}{24}\partial_x^3\right)\big(\mathcal{H}_+ - \mathcal{H}_-\big) \;+\; \frac{\epsilon^2}{12}\big[\partial_x(\mathcal{H}_+ - \mathcal{H}_-)\big]^3 \;=\; 0\,, \tag{5}$$

where $\mathcal{H}_\pm := \mathcal{H}\big(x \pm \frac{i\epsilon}{2}, \epsilon\big)$.

A statement equivalent to equation (4) is given by the following corollary.

Corollary 1. *For all $g \geq 0$ and $n \geq 2$, the numbers $a_{g,n}$ can be uniquely determined by the following recursion relation*

$$\begin{aligned} a_{g,q+2} \;=\; & \frac{q!}{2} \sum_{\substack{g_1+g_2+j_1+j_2=g \\ n_1+n_2=q+4+2(j_1+j_2)}} \frac{(-1)^{j_1+j_2} a_{g_1,n_1} a_{g_2,n_2}}{4^{j_1+j_2}(2j_1+1)!(2j_2+1)!(n_1-2j_1-2)!(n_2-2j_2-2)!} \\ & - \sum_{j=1}^{g} \frac{(-1)^j a_{g-j,q+2j+2}}{4^j(2j)!} \;+\; \delta_{q,1}\delta_{g,0} \end{aligned} \tag{6}$$

along with the boundary condition $a_{0,2} = 0$ (cf. (2)), where $q \geq 0$.

Another corollary of Theorem 1 is the following non-linear differential equation for the series $\mathcal{H}$.

Corollary 2. *The series $\mathcal{H} = \mathcal{H}(x,\epsilon)$ satisfies the following equation:*

$$\epsilon\partial_\epsilon\partial_x(\mathcal{H}) \;+\; x\partial_x^2(\mathcal{H}) \;+\; \frac{1}{2}\partial_x(\mathcal{H}) \;-\; \frac{\epsilon^2}{4}\big[\partial_x^2(\mathcal{H})\big]^2 \;-\; \frac{\epsilon^2}{24}\partial_x^4(\mathcal{H}) \;=\; 0\,. \tag{7}$$

The proof will be given in Section 3. We also show there that equation (7) implies a recursion given by Kazarian in [28] for the Hodge integrals

$$\frac{(5g-3-j)(5g-5-j)}{(3g-3-j)!} \int_{\overline{\mathcal{M}}_{g,3g-3-j}} \lambda_j \psi_1^2 \cdots \psi_{3g-3-j}^2\,, \quad 0 \leq j \leq g\,.$$

A third corollary of Theorem 1 (which apart from the boundary conditions is in fact equivalent to equation (7)) is the following recursion for the numbers $a_{g,n}$.

Corollary 3. *For all $g \geq 0$ and $n \geq 1$, the numbers $a_{g,n}$ are given recursively by*

$$a_{g,n} \;=\; \frac{1}{2} \sum_{\substack{g_1,g_2\geq 0\\ g_1+g_2=g}} \sum_{\substack{n_i\geq 2,(g_i,n_i)\neq(0,3),i=1,2\\ n_1+n_2=n+3}} \binom{n-1}{n_1-2} \frac{a_{g_1,n_1}a_{g_2,n_2}}{4g-4+n} \;+\; \frac{1}{12}\frac{a_{g-1,n+3}}{4g-4+n} \tag{8}$$

if $2g-2+n>0$, $(g,n)\notin\{(0,3),\,(0,4)\}$, $a_{0,3}=a_{0,4}=1$ and $a_{0,1}=a_{0,2}=a_{-1,n}=0$.

The recursion relations (6) or (8) both give rapid (polynomial-time) algorithms for computing $a_{g,n}$ for $n\geq 2$ or $n \geq 1$, respectively. The first few values $a_{g,n}$ are given by the following table.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$g=0$	0	0	0	1	1	3	15
$g=1$	0	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{11}{24}$	$\frac{21}{8}$	$\frac{163}{8}$	$\frac{1595}{8}$
$g=2$	$\frac{1}{96}$	$\frac{29}{640}$	$\frac{337}{1152}$	$\frac{319}{128}$	$\frac{10109}{384}$	$\frac{42445}{128}$	$\frac{620641}{128}$
$g=3$	$\frac{575}{21504}$	$\frac{20555}{82944}$	$\frac{77633}{27648}$	$\frac{1038595}{27648}$	$\frac{16011391}{27648}$	$\frac{31040465}{3072}$	$\frac{201498115}{1024}$
$g=4$	$\frac{2106241}{7962624}$	$\frac{1103729}{294912}$	$\frac{160909109}{2654208}$	$\frac{14674841399}{13271040}$	$\frac{99177888029}{4423680}$	$\frac{442442475179}{884736}$	$\frac{10765584400823}{884736}$

TABLE 1. The numbers $a_{g,n}$ with $0 \leq g \leq 4$ and $0 \leq n \leq 6$.

The following proposition describes the property of $\operatorname{Vol}\mathcal{Q}_{g,n}$, which will enable us to determine also $a_{g,0}$ and $a_{g,1}$ from (4), and $a_{g,0}$ from (5) or (7).

Proposition 1 ([6, 5, 8])**.** *The following properties of the MV volumes hold:*

$$\operatorname{Vol}\mathcal{Q}_{0,n} \;=\; \frac{\pi^{2n-6}}{2^{n-5}}\,, \qquad \forall\, n \geq 3\,; \tag{9}$$

$$\operatorname{Vol}\mathcal{Q}_{1,n} \;=\; \frac{\pi^{2n}}{3}\left(\frac{n!}{(2n-1)!!} \;+\; \frac{2n}{(2n-1)2^n}\right), \qquad \forall\, n\geq 1\,; \tag{10}$$

$$\operatorname{Vol}\mathcal{Q}_{g,n} \;=\; 2^{2g+1+n}\frac{\pi^{6g-6+2n}(4g-4+n)!}{(6g-7+2n)!}\sum_{j=0}^{g}\frac{\langle\lambda_j\tau_2^{3g-3-j}\rangle_g}{(3g-3-j)!}\left(\frac{5g-5-j}{2}\right)_n, \tag{11}$$

where $g\geq 2$, $n\geq 0$, $(b)_n := b(b+1)\cdots(b+n-1)$ denotes the increasing Pochhammer symbol, and we used Witten's notation: for a cohomology class $\gamma \in H^*(\overline{\mathcal{M}}_{g,n};\mathbb{C})$,

$$\langle \gamma\tau_{i_1}\cdots\tau_{i_n}\rangle_g \;:=\; \int_{\overline{\mathcal{M}}_{g,n}} \gamma\,\psi_1^{i_1}\cdots\psi_n^{i_n}\,, \quad i_1,\ldots,i_n\geq 0\,.$$

The explicit expression for $\operatorname{Vol}\mathcal{Q}_{0,n}$, $n\geq 3$ was conjectured by Kontsevich, and was proved by Athreya-Eskin-Zorich in [6]. The formula (10) was conjecturally given by Andersen et. al. [5], and the formula (11) is equivalent to the Conjecture 5.4 of [5] (to see the equivalence, cf. [8]). A proof of Proposition 1 was given in [8]. In this paper we give a different proof of this proposition based on the following lemma.

Lemma 1. *Let $T=\sqrt{1-2x}$. Define the power series $\mathcal{H}_g(x)$, $g\geq 0$ by*

$$\mathcal{H}(x,\epsilon) \;=:\; \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{H}_g(x)\,. \tag{12}$$

Then we have

$$\mathcal{H}_0(x) \;=\; \frac{1}{40} - \frac{T^2}{12} + \frac{T^4}{8} - \frac{T^5}{15}\,, \tag{13}$$

$$\mathcal{H}_1(x) \;=\; \frac{1}{24}\log\frac{1}{T} + \frac{1}{24}(1-T)\,, \tag{14}$$

$$\mathcal{H}_2(x) \;=\; \frac{7}{1440}\,\frac{1}{T^5} + \frac{5}{1152}\,\frac{1}{T^4} + \frac{7}{5760}\,\frac{1}{T^3}\,. \tag{15}$$

In general, we have the following expression for $\mathcal{H}_g(x)$:

$$\mathcal{H}_g(x) \;=\; \sum_{j=0}^{g} \frac{\langle \lambda_j \tau_2^{3g-3-j}\rangle_g}{(3g-3-j)!}\,\frac{1}{T^{5g-5-j}}\,, \quad g\geq 2\,. \tag{16}$$

We give in Section 2 a proof of Lemma 1 by using the CMS formula (1) and the Dubrovin-Zhang formalism [18, 15, 16] on Hodge integrals. Substituting the expansion (12) into (7) we find

$$x\mathcal{H}_g'' + \Bigl(2g-\frac{3}{2}\Bigr)\mathcal{H}_g' - \frac{1}{4}\sum_{\substack{g_1,g_2\geq 0\\ g_1+g_2=g}} \mathcal{H}_{g_1}''\mathcal{H}_{g_2}'' - \frac{1}{24}\mathcal{H}_{g-1}'''' \;=\; 0\,. \tag{17}$$

Here, prime, "$'$" denotes d/dx. It turns out that this formula together with Lemma 1 determines $\mathcal{H}_g$, $g\geq 0$, and therefore the $a_{g,n}$, uniquely for all $g,n\geq 0$.

Recently, Aggarwal, Delecroix, Goujard, Zograf and Zorich [4] proposed a conjectural formula for the large g leading asymptotics of $\operatorname{Vol}\mathcal{Q}_{g,n}$ (the conjectural formula was given originally in [12] for $n=0$). The ADGZZ conjecture was very recently proved in [3]. Our next result is a refinement of the ADGZZ conjecture to the following more precise asymptotic statement.

Conjecture 1. *For any fixed $n\geq 0$, we have the asymptotic formula:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} \;\sim\; \frac{2^{12g+4n-10}}{3^{4g+n-4}\pi}\sum_{k=0}^{\infty}\frac{m_k(n)}{g^k}\,, \quad g\to\infty\,, \tag{18}$$

where each $m_k(n)$ is a polynomial in n with coefficients in $\mathbb{Q}[\pi^2]$, with the first four values (with $M = -\pi^2/144$ for convenience) given by

$$\begin{aligned}
m_0(n) &= 1\,, \quad m_1(n) \;=\; M\,,\\
m_2(n) &= \frac{M}{24}\,n^3 - \frac{3M}{8}\,n^2 + \frac{4M - 27M^2}{6}\,n + \frac{M + 19M^2}{2}\,,\\
m_3(n) &= -\frac{8M + 27M^2}{288}n^4 + \frac{17M + 65M^2}{48}n^3 - \frac{860M + 1890M^2 - 14256M^3}{576}n^2\\
&\quad + \frac{104M - 373M^2 - 6156M^3}{48}n\\
&\quad - \frac{55M - 3615M^2 - 28650M^3 + 126846M^4}{180}\,.
\end{aligned}$$

The asymptotic formula (18) with $\sum_{k=0}^{\infty} m_k(n)/g^k$ replaced by 1 is the ADGZZ conjecture. We refer to [1, 2, 9, 10, 21, 33, 34] for the analogues of the ADGZZ conjecture and Conjecture 1 (cf. also Conjecture 2 in Section 4 below) for the MV volumes and for the related area Siegel–Veech constants associated with the moduli spaces of abelian differentials, and the proofs of these analogues via different approaches. Conjecture 1 can also be stated in terms of the numbers $a_{g,n}$ defined in (2) as

$$a_{g,n} \;\sim\; \frac{(6g-7+2n)!}{(4g-4+n)!}\,\frac{2^{10g+4n-11}}{3^{4g+n-4}\pi^{6g-5+2n}}\,\sum_{k=0}^{\infty}\frac{m_k(n)}{g^k}\,,\quad g\to\infty\,. \tag{19}$$

Conjecture 1, like the related Conjecture 2 which will be stated in Section 4 below, is completely empirical. Specifically, we computed the values of $a_{g,n}$ numerically for $g \le 100$ and a number of small values of n, then interpolated by the numerical method explained in [36], [25, Section 5] and elsewhere to get an asymptotic power series in $1/g$ with coefficients known to high precision, and then used polynomial interpolation and the LLL (Lenstra–Lenstra–Lovasz) method to recognize the coefficients as polynomials in n with coefficients in $\mathbb{Q}[\pi^2]$.

Remark 1. It would be interesting to investigate the following generating series:

$$C_n(\epsilon) \;:=\; \sum_{g\ge 0}\epsilon^{2g-2}a_{g,n}\,,\quad n\ge 0\,. \tag{20}$$

In other words, $\mathcal{H}(x,\epsilon) = \sum_{n\ge0}\frac{x^n}{n!}C_n(\epsilon)$. Equation (7) then implies the following relations for $C_n(\epsilon)$:

$$C_{n+4} \;=\; \frac{24}{\epsilon}C'_{n+1} \;+\; 12\frac{2n+1}{\epsilon^2}C_{n+1} \;-\; 6n!\sum_{n_1+n_2=n}\frac{C_{n_1+2}C_{n_2+2}}{n_1!\,n_2!}\,,\quad n\ge 0\,. \tag{21}$$

Similarly, equation (5) implies relations for the analogue of $C_n(\epsilon)$ for $\mathcal{H}_+ - \mathcal{H}_-$. Understanding of $C_n(\epsilon)$ or its analogue might be useful for proving the above Conjecture 1.

The paper is organized as follows. In Section 2, we review the Dubrovin-Zhang theory and give a proof of Lemma 1. In Section 3, we prove Theorem 1. In Section 4, we extend a conjectural formula for the large genus asymptotics of the area Siegel–Veech constants.

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2. The Hodge free energy

In this section we first give a short review of the Dubrovin-Zhang approach to Hodge integrals [15, 7, 18, 16], and then specialize our discussions to linear Hodge integrals and prove Lemma 1. Recall that the genus g Hodge free energy $\mathcal{H}_g(\mathbf{t};\mathbf{s})$ is defined by

$$\mathcal{H}_g(\mathbf{t};\mathbf{s}) \;=\; \sum_{k\geq 0}\sum_{i_1,\dots,i_k\geq 0}\frac{t_{i_1}\cdots t_{i_k}}{k!}\int_{\overline{\mathcal{M}}_{g,k}}\Omega_{g,k}(\mathbf{s})\,\psi_1^{i_1}\cdots\psi_k^{i_k}\,, \tag{22}$$

$$\Omega_{g,k}(\mathbf{s}) \;:=\; \exp\biggl(\sum_{j\geq 0}s_{2j-1}\,\mathrm{ch}_{2j-1}(\mathbb{E}_{g,k})\biggr)\,. \tag{23}$$

Here $g\geq 0$, $\mathbf{t}=(t_0,t_1,\dots)$, $\mathbf{s}=(s_1,s_3,\dots)$, $t_0,t_1,t_2,\dots,\ s_1,s_3,\dots$ are indeterminates, and $\mathrm{ch}_1,\mathrm{ch}_3,\mathrm{ch}_5,\dots$ denote components of the Chern character of $\mathbb{E}_{g,k}$. Define the total Hodge free energy $\mathcal{H}$ by

$$\mathcal{H} \;=\; \mathcal{H}(\mathbf{t};\mathbf{s};\epsilon) \;=\; \sum_{g\geq 0}\mathcal{H}_g(\mathbf{t};\mathbf{s})\,\epsilon^{2g-2}\,.$$

Let $v\in\mathbb{C}[[\mathbf{t}]]$ be the unique power series solution to the following equation:

$$\sum_{i\geq 0}\frac{t_i}{i!}v^i \;=\; v\,. \tag{24}$$

It is well known that this unique power series $v=v(\mathbf{t})$ has the explicit expression

$$v(\mathbf{t}) \;=\; \sum_{k\geq 1}\frac{1}{k}\sum_{\substack{p_1,\dots,p_k\geq 0\\ p_1+\dots+p_k=k-1}}\frac{t_{p_1}}{p_1!}\cdots\frac{t_{p_k}}{p_k!}\,. \tag{25}$$

Denote

$$v_m(\mathbf{t}) \;=\; \partial_{t_0}^m(v(\mathbf{t}))\,,\quad m\geq 0\,. \tag{26}$$

Theorem A ([15]) *The genus 0 and 1 Hodge free energies have the expressions*

$$\mathcal{H}_0(\mathbf{t};\mathbf{s}) \;=\; \frac{v(\mathbf{t})^3}{6} \;-\; \sum_{i\geq 0}t_i\frac{v(\mathbf{t})^{i+2}}{i!(i+2)} \;+\; \frac{1}{2}\sum_{i,j\geq 0}t_it_j\frac{v(\mathbf{t})^{i+j+1}}{(i+j+1)i!j!}\,, \tag{27}$$

$$\mathcal{H}_1(\mathbf{t};\mathbf{s}) \;=\; \frac{1}{24}\log v_1(\mathbf{t}) \;+\; \frac{s_1}{24}v(\mathbf{t})\,. \tag{28}$$

For $g\geq 2$, there exist elements

$$H_g(z_1,\dots,z_{3g-2};s_1,s_3,\dots,s_{2g-1})\in\mathbb{C}\bigl[z_1,\dots,z_{3g-2},z_1^{-1};s_1,s_3,\dots,s_{2g-1}\bigr]$$

satisfing the conditions

$$\sum_{m=1}^{3g-2} m\, z_m \frac{\partial H_g}{\partial z_m} \;=\; (2g-2)\, H_g\,, \tag{29}$$

$$\sum_{m=2}^{3g-2} (m-1)\, z_m \frac{\partial H_g}{\partial z_m} \;+\; \sum_{j=1}^{g} (2j-1)\, s_{2j-1} \frac{\partial H_g}{\partial s_{2j-1}} \;=\; (3g-3)\, H_g\,, \tag{30}$$

such that

$$\mathcal{H}_g(\mathbf{t};\mathbf{s}) \;=\; H_g\big(v_1(\mathbf{t}),\dots,v_{3g-2}(\mathbf{t}); s_1, s_3, \dots, s_{2g-1}\big)\,. \tag{31}$$

This theorem was proved in [15]; see also [16] for a straightforward proof.

Define

$$u \;=\; u(\mathbf{t};\mathbf{s};\epsilon) \;:=\; \epsilon^2 \frac{\partial^2 \mathcal{H}(\mathbf{t};\mathbf{s};\epsilon)}{\partial t_0^2}\,, \tag{32}$$

then according to [15], u satisfies an integrable hierarchy of tau-symmetric Hamiltonian evolutionary PDEs, called the Hodge hierarchy, which is a deformation of the KdV hierarchy [35, 29] and has the form

$$\frac{\partial u}{\partial t_k} \;=\; P \frac{\delta \bar{h}_k}{\delta u(x)}\,, \quad k \geq 0. \tag{33}$$

Here $P = \partial_x \;+\; \cdots$ is a Hamiltonian operator, $\bar{h}_k$, $k \geq 0$ are Hamiltonians.

In [17] Theorem A was applied under a particular specialization of $\mathbf{t}, \mathbf{s}$, which gives the classical Hurwitz numbers according to the ELSV formula. In this paper, we consider a *different* specialization. Firstly, we specialize $\mathbf{s}$ to $\mathbf{s} = \mathbf{s}^*$ as follows:

$$s^*_{2k-1} \;:=\; (2k-2)!\, s^{2k-1}, \quad k \geq 1\,. \tag{34}$$

Denote by $\Lambda_{g,k}(s) := \sum_{j=0}^{g} \lambda_j\, s^j$ the Chern polynomial of $\mathbb{E}_{g,k}$. Applying the relationship between the Chern classes and the Chern character, and using Mumford's relations [32]

$$\mathrm{ch}_{2m}(\mathbb{E}_{g,k}) \;=\; 0\,, \quad m \geq 1\,,$$

we obtain $\Omega_{g,k}(\mathbf{s} = \mathbf{s}^*) = \Lambda_{g,k}(s)$. So we have

$$\mathcal{H}_g(\mathbf{t};\mathbf{s}^*) \;=\; \sum_{n\geq 0} \sum_{i_1,\dots,i_n \geq 0} \frac{t_{i_1}\cdots t_{i_n}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n}(s)\, \psi_1^{i_1} \cdots \psi_n^{i_n}\,. \tag{35}$$

Secondly, we specialize $\mathbf{t}$ to $\mathbf{t} = \mathbf{t}^*$ given by

$$t_0^* \;=\; x\,, \quad t_1^* \;=\; 0\,, \quad t_2^* \;=\; 1\,, \quad t_i^* \;=\; 0 \; (i \geq 3)\,. \tag{36}$$

Substituting (36) into (35) we arrive at

$$\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*) \;=\; \sum_{n_0 \geq 0} \frac{x^{n_0}}{n_0!} \sum_{j=0}^{g} s^j \frac{\langle \lambda_j\, \tau_0^{n_0}\, \tau_2^{3g-3+n_0-j} \rangle_g}{(3g-3+n_0-j)!}\,. \tag{37}$$

From the definition of $a_{g,n}$ given in (2), it follows that the MV free energy is a specialized linear Hodge free energy. More precisely, we have the following lemma.

Lemma 2. *For any $g \geq 0$, the following identities hold:*

$$\mathcal{H}_g(x) \;=\; \mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*)|_{s=1}\,, \tag{38}$$

where $\mathcal{H}_g(x)$ is the $g^{\rm th}$ part of the MV free energy (12)*. Equivalently, we have*

$$\mathrm{Vol}\,\mathcal{Q}_{g,n} \;=\; 2^{2g+1}\frac{\pi^{6g-6+2n}(4g-4+n)!}{(6g-7+2n)!}\,\partial_x^n\big(\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*)\big)\big|_{x=0,s=1}\,. \tag{39}$$

Let us now apply Theorem A to the computation of $\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*)$, which, due to (39), gives rise to $\mathrm{Vol}\,\mathcal{Q}_{g,n}$. Substituting (36) into (24) we find that $v = v(\mathbf{t}^*)$ satisfies the following quadratic equation

$$x \,+\, \frac{v^2}{2} \;=\; v\,. \tag{40}$$

By solving this and observing that the power series v starts with x, we obtain

$$v(\mathbf{t}^*) = 1 - \sqrt{1-2x}\,.$$

Denote

$$T \;:=\; \sqrt{1-2x}\,. \tag{41}$$

Then by noticing $\partial_x \;=\; -\frac{1}{T}\partial_T$ we find

$$v_m(\mathbf{t}^*) \;=\; \frac{(2m-3)!!}{T^{2m-1}} \,+\, \delta_{m,0}\,, \quad m\geq 0\,. \tag{42}$$

Lemma 3. *The power series $\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*)$ of x,t are given explicitly for $g=0,1,2$ by*

$$\mathcal{H}_0(\mathbf{t}^*;\mathbf{s}^*) \;=\; \frac{1}{40} - \frac{T^2}{12} + \frac{T^4}{8} - \frac{T^5}{15}\,, \tag{43}$$

$$\mathcal{H}_1(\mathbf{t}^*;\mathbf{s}^*) \;=\; \frac{1}{24}\log\frac{1}{T} + \frac{s}{24}\,(1-T)\,, \tag{44}$$

$$\mathcal{H}_2(\mathbf{t}^*;\mathbf{s}^*) \;=\; \frac{7}{1440}\,\frac{1}{T^5} + \frac{5}{1152}\,\frac{s}{T^4} + \frac{7}{5760}\,\frac{s^2}{T^3}\,. \tag{45}$$

In general, for $g\geq 2$, $\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*)$ has the following expression:

$$\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*) \;=\; \sum_{j=0}^{g}\frac{\langle\lambda_j\tau_2^{3g-3-j}\rangle_g}{(3g-3-j)!}\,\frac{s^j}{T^{5g-5-j}}\,, \quad g\geq 2\,. \tag{46}$$

Proof. By substituting (42) into (27) and (28), we arrive at the formulas for $\mathcal{H}_0(\mathbf{t}^*;\mathbf{s}^*)$ and $\mathcal{H}_1(\mathbf{t}^*;\mathbf{s}^*)$, respectively. The formula for $\mathcal{H}_2(\mathbf{t}^*;\mathbf{s}^*)$ can be obtained by using the algorithm of [15] with $v_m(\mathbf{t}^*)$ given by (42). To show the validity of the formula for $\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*)$, $g\geq 2$, we first observe that, according to (31), (42) and the homogeneity conditions (29), (30), the function $\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*)$ can be written in the form

$$\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*) \;=\; \sum_{j=0}^{g}\frac{C_{g,j}s^j}{T^{5g-5-j}}\,, \quad g\geq 2\,, \tag{47}$$

where $C_{g,j}\in\mathbb{Q}$. Therefore,

$$\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*)|_{x=0} \;=\; \sum_{j=0}^{g} C_{g,j}s^j\,, \quad g\geq 2\,.$$

On the other hand, it follows from (37) that

$$\mathcal{H}_g(\mathbf{t}^*;\mathbf{s}^*)|_{x=0} \;=\; \sum_{j=0}^{g} \frac{\langle \lambda_j \tau_2^{3g-3-j}\rangle_g}{(3g-3-j)!} s^j \,.$$

By comparing the coefficients of s^j in the two formulas given above we arrive at

$$C_{g,j} \;=\; \frac{\langle \lambda_j \tau_2^{3g-3-j}\rangle_g}{(3g-3-j)!}\,, \quad j=0,\dots,g\,, \tag{48}$$

where $g\geq 2$. The lemma is proved. □

Proof of Lemma 1. By putting $s=1$ in Lemma 3, we arrive at the result of Lemma 1. □

Now let us give a proof of Proposition 1 based on Lemma 1.

Proof of Proposition 1. By using (13) and the fact that $\frac{d}{dx}=-\frac{1}{T}\frac{d}{dT}$ we have

$$\mathcal{H}_0'(x) \;=\; \frac{1}{6}-\frac{T^2}{2}+\frac{T^3}{3}\,, \quad \mathcal{H}_0''(x) \;=\; v(\mathbf{t}^*)\,, \tag{49}$$

$$\frac{d^n\mathcal{H}_0(x)}{dx^n} \;=\; v_{n-2}(\mathbf{t}^*) \;=\; \frac{(2n-7)!!}{T^{2n-5}}\,, \quad n\geq 3\,. \tag{50}$$

Therefore, $\frac{d^n\mathcal{H}_0(x)}{dx^n}\big|_{x=0}=(2n-7)!!\,\delta_{n\geq 3}$. Due to the definition (3) and the CMS formula this gives (9). Similarly, by using (14) we obtain

$$\frac{d^n\mathcal{H}_1(x)}{dx^n} \;=\; \frac{\delta_{n\geq 1}}{24}\frac{2^{n-1}(n-1)!}{T^{2n}} \;+\; \frac{\delta_{n,0}}{24}\log\frac{1}{T} \;+\; \frac{1}{24}\frac{(2n-3)!!}{T^{2n-1}} \;+\; \frac{\delta_{n,0}}{24}\,, \tag{51}$$

from which we arrive at (10). Finally, by using (16) we have for $g\geq 2$,

$$\frac{d^n\mathcal{H}_g(x)}{dx^n} \;=\; \sum_{j=0}^{g}\frac{\langle \lambda_j \tau_2^{3g-3-j}\rangle_g}{(3g-3-j)!}\frac{\prod_{i=0}^{n-1}(5g-5-j+2i)}{T^{5g-5-j+2n}}\,, \tag{52}$$

which yields formula (11). Proposition 1 is proved. □

Remark 2. The explicit expressions of the numbers $\langle\lambda_g\tau_2^{2g-3}\rangle_g$ that appear in (11) of Proposition 1 are given by the following λ_g-conjecture proven in [22, 19]:

$$\frac{\langle\lambda_g\tau_2^{2g-3}\rangle_g}{(2g-3)!} \;=\; \frac{2^{2g-1}-1}{2^{2g-1}}(4g-7)!!\,\frac{|B_{2g}|}{(2g)!}\,, \qquad g\geq 2\,, \tag{53}$$

where B_k denotes the $k_{\rm th}$-Bernoulli number. The number $\langle\tau_2^{3g-3}\rangle_g$ for $g\geq 2$ has the expression [26]:

$$\frac{\langle\tau_2^{3g-3}\rangle_g}{(3g-3)!} \;=\; \frac{24^{-g}c_g}{(5g-3)(5g-5)}\,, \tag{54}$$

where c_g are given by the recursion

$$c_g \;=\; 50\,(g-1)^2\,c_{g-1}+\frac{1}{2}\sum_{h=2}^{g-2}c_h\,c_{g-h}, \quad g\geq 3 \tag{55}$$

together with $c_0=-1, c_1=2, c_2=98$.

Proposition 1 and formula (54) imply immediately the following corollary.

Corollary 4. *For any fixed $g \geq 0$, the following asymptotic formula is true:*

$$\mathrm{Vol}\, \mathcal{Q}_{g,n} \sim \kappa_g \frac{n^{\frac{g}{2}} \pi^{2n}}{2^n} \quad (n \to \infty), \tag{56}$$

where

$$\kappa_g = \frac{64\, \pi^{6g - \frac{11}{2}}}{384^g\, \Gamma(\frac{5g-1}{2})}\, c_g\,, \tag{57}$$

and c_g are defined by (55).

The reader may notice that certain universality found in [17] about asymptotics of enumerations related to $\overline{\mathcal{M}}_{g,n}$ reappears in (56), (57). The first few κ_g are given by $\kappa_0 = 32/\pi^6$, $\kappa_1 = \pi^{\frac{1}{2}}/3$, $\kappa_2 = 7\pi^6/1080$, $\kappa_3 = 245\pi^{25/2}/7962624$.

3. Relations for the MV volumes

The goal of this section is to prove Theorem 1 and Corollary 2.

Proof of Theorem 1. It was shown by Buryak [7] that the Hodge hierarchy associated with $\Lambda(s)$ is normal Miura equivalent [18, 15] to the intermediate long wave (ILW) hierarchy. To be precise, define $\tilde{u} = \tilde{u}(\mathbf{t}; s; \epsilon)$ by

$$\tilde{u}(\mathbf{t}; s; \epsilon) := \sum_{g=0}^{\infty} \epsilon^{2g} \frac{(-1)^g s^g}{2^{2g}(2g+1)!} \frac{\partial^{2g} u}{\partial t_0^{2g}}, \tag{58}$$

where u is defined in (32) with the specialization $\mathbf{s} = \mathbf{s}^*$; then $\tilde{u}$ satisfies [7] the ILW hierarchy, which has the first two flows

$$\tilde{u}_{t_1} = \tilde{u} \frac{\partial \tilde{u}}{\partial t_0} + \sum_{g \geq 1} \frac{|B_{2g}|}{(2g)!} \epsilon^{2g} s^{g-1} \frac{\partial^{2g+1} \tilde{u}}{\partial t_0^{2g+1}}\,, \tag{59}$$

$$\begin{aligned}\tilde{u}_{t_2} = {}& \frac{1}{2} \tilde{u}^2 \frac{\partial \tilde{u}}{\partial t_0} + \sum_{g \geq 1} \frac{|B_{2g}|}{(2g)!} \epsilon^{2g} \frac{s^{g-1}}{4} \left(2\partial_{t_0} \left(\tilde{u} \frac{\partial^{2g} \tilde{u}}{\partial t_0^{2g}} \right) + \frac{\partial^{2g+1}(\tilde{u}^2)}{\partial t_0^{2g+1}} \right) \\ & + \sum_{g \geq 2} \frac{|B_{2g}|}{(2g)!} (g+1) \epsilon^{2g} s^{g-2} \frac{\partial^{2g+1} \tilde{u}}{\partial t_0^{2g+1}}\,. \end{aligned} \tag{60}$$

Let us now do the specialization (36) with $s = 1$, and denote the series $u(\mathbf{t}^*; \mathbf{s}^*; \epsilon)|_{s=1}$, $\tilde{u}(\mathbf{t}^*; s; \epsilon)|_{s=1}$ by $u = u(x, \epsilon)$, $\tilde{u} = \tilde{u}(x, \epsilon)$, respectively. Then $u(x, \epsilon) = \epsilon^2 \partial_x^2(\mathcal{H}(x, \epsilon))$, and from (58) it follows that $\tilde{u}(x, \epsilon)$ and $u(x, \epsilon)$ are related by

$$\tilde{u} = \sum_{g=0}^{\infty} \epsilon^{2g} \frac{(-1)^g}{2^{2g}(2g+1)!} \frac{\partial^{2g} u}{\partial x^{2g}}\,. \tag{61}$$

Proposition 2. *The series $\tilde{u} = \tilde{u}(x, \epsilon)$ satisfies the following non-linear equation:*

$$x + \frac{\tilde{u}^2}{2} + \sum_{g=1}^{\infty} \epsilon^{2g} \frac{|B_{2g}|}{(2g)!} \frac{\partial^{2g} \tilde{u}}{\partial x^{2g}} = \tilde{u}\,. \tag{62}$$

Proof. Recall that the Hodge partition function $Z = Z(\mathbf{t};\mathbf{s};\epsilon) := e^{\mathcal{H}(\mathbf{t};\mathbf{s};\epsilon)}$ satisfies the string equation (cf. e.g. [15, 16]), that is,

$$\sum_{i=0} t_{i+1}\frac{\partial Z}{\partial t_i} \;+\; \frac{t_0^2}{2\epsilon^2}Z \;+\; \frac{s_1}{24}Z \;=\; \frac{\partial Z}{\partial t_0}\,. \tag{63}$$

Dividing both sides of (63) by Z and differentiating with respect to x we obtain

$$\sum_{i=0}^{\infty} t_{i+1}\frac{\partial^2\mathcal{H}(\mathbf{t};\mathbf{s};\epsilon)}{\partial t_i\partial x} \;+\; \frac{x}{\epsilon^2} \;=\; \frac{\partial^2\mathcal{H}(\mathbf{t};\mathbf{s};\epsilon)}{\partial x^2}\,. \tag{64}$$

We recall that

$$\epsilon^2\frac{\partial^2\mathcal{H}(\mathbf{t};\mathbf{s};\epsilon)}{\partial t_i\partial x} \;=\; \Omega_{i,0}\big(u(\mathbf{t};\mathbf{s};\epsilon), u_x(\mathbf{t};\mathbf{s};\epsilon),\dots\big)\,,\quad i\geq 0\,, \tag{65}$$

where $\Omega_{i,0}$ are certain differential polynomials [7, 15] of u. Then by using the Miura transformation (58) we obtain

$$\sum_{i=0}^{\infty} t_{i+1}\widetilde{\Omega}_{i,0}\big(\tilde u(\mathbf{t};\mathbf{s};\epsilon), \tilde u_x(\mathbf{t};\mathbf{s};\epsilon),\dots\big) \;+\; x \;=\; \tilde u(\mathbf{t};\mathbf{s};\epsilon)\,. \tag{66}$$

Here $\widetilde{\Omega}_{i,0}$, $i\geq 0$ are differential polynomials of $\tilde u$. Buryak [7] showed that the Miura transformation (58) transforms the Hamiltonian structure P of the linear Hodge hierarchy to ∂_x, in particular, $\tilde u(\mathbf{t};\mathbf{s};\epsilon)$ satisfies the Hamiltonian system

$$\frac{\partial \tilde u}{\partial t_1} \;=\; \partial_x\frac{\delta\tilde{\bar h}_1}{\delta\tilde u(x)}\,, \tag{67}$$

where

$$\tilde{\bar h}_1 \;=\; \int\bigg(\frac{\tilde u^3}{6} \;+\; \sum_{g=1}^{\infty}\frac{|B_{2g}|}{2(2g)!}\tilde u\tilde u_{2g}\bigg)dx\,.$$

Therefore, according to [15] we know that

$$\widetilde{\Omega}_{1,0} \;=\; \frac{\delta\tilde{\bar h}_1}{\delta\tilde u(x)} \;=\; \frac{\tilde u^2}{2} \;+\; \sum_{g=1}^{\infty}\epsilon^{2g}\frac{|B_{2g}|}{(2g)!}\frac{\partial^{2g}\tilde u}{\partial x^{2g}}\,. \tag{68}$$

Thus equation (66) with the specialization $s=1$ leads to (62). The proposition is proved. $\square$

We are in a position of proving equation (4). Indeed, observe that

$$\sum_{g\geq 1}\epsilon^{2g}\frac{|B_{2g}|}{(2g)!}\partial_x^{2g} \;=\; 1 \;-\; \frac{i}{2}\epsilon\,\partial_x \;-\; \frac{i\epsilon\partial_x}{e^{i\epsilon\partial_x}-1}, \tag{69}$$

so it follows from (62) that

$$x \;+\; \frac{\tilde u^2}{2} \;-\; \frac{i}{2}\epsilon\,\partial_x(\tilde u) \;-\; \frac{i\epsilon\partial_x}{e^{i\epsilon\partial_x}-1}(\tilde u) \;=\; 0\,. \tag{70}$$

By using the fact that $\tilde u = -i\epsilon\partial_x(\mathcal{H}_+ - \mathcal{H}_-)$ we arrive at equation (4).

We will now prove equation (5). We first switch on the t_2-dependence and denote it by t in the specialization (36). More precisely, we consider

$$\mathcal{H} \;=\; \mathcal{H}(x,t,\epsilon) \;:=\; \sum_{g,n\geq 0}\sum_{j=0}^{g} \frac{\langle \lambda_j \tau_0^n \tau_2^{3g-3+n-j}\rangle}{(3g-3+n-j)!}\,\epsilon^{2g-2}\frac{x^n}{n!}t^{3g-3+n-j}\,, \tag{71}$$

and denote $\mathcal{H}_\pm := \mathcal{H}\big(x\pm\frac{i\epsilon}{2},t,\epsilon\big)$. Then by using equation (59) and an argument like the one we used above to derive equation (4), we find that $\mathcal{H}$ satisfies the following equation:

$$t\frac{\epsilon^2}{2}\Big[\partial_x(\mathcal{H}_+\,-\,\mathcal{H}_-)\Big]^2 \;+\; t\frac{\epsilon^2}{2}\partial_x^2\big(\mathcal{H}_+ + \mathcal{H}_-\big) \;-\; (1-t)\,i\,\epsilon\,\partial_x(\mathcal{H}_+\,-\,\mathcal{H}_-) \;=\; x\,. \tag{72}$$

Then by using equations (60) and (72) we obtain the following equation for $\mathcal{H}$:

$$\begin{aligned}-\,i\,\epsilon\,\partial_t\,(\mathcal{H}_+\,-\,\mathcal{H}_-) \;=\; &\frac{1}{6}\tilde{u}^3 + \frac{3}{4}\tilde{u}^2 + \tilde{u} - \frac{i\epsilon}{2}\tilde{u}\tilde{u}_x - \frac{3i\epsilon}{4}\tilde{u}_x - \frac{\epsilon^2}{6}\tilde{u}_{xx} \;+\; \frac{x}{2t} - \frac{i\epsilon}{4t}\\ &-\frac{1+2t}{2t}\epsilon^2\partial_x^2(\mathcal{H}_-) + \frac{i\epsilon^3}{4}\partial_x^3(\mathcal{H}_-) - \frac{\epsilon^2}{2}\tilde{u}\,\partial_x^2(\mathcal{H}_-)\,.\end{aligned} \tag{73}$$

Here we recall that $\tilde{u} = -i\epsilon\partial_x(\mathcal{H}_+ - \mathcal{H}_-)$, and we also used Theorem A to get the constant in x term $-i\epsilon/4t$. It is not difficult to deduce from Theorem A the following homogeneity property for $\mathcal{H}$:

$$t\frac{\partial\mathcal{H}}{\partial t} \;+\; \Big(x-\frac{1}{t}\Big)\frac{\partial\mathcal{H}}{\partial x} \;+\; \epsilon\frac{\partial\mathcal{H}}{\partial\epsilon} \;=\; -\frac{1}{24} - \frac{1}{24t} - \frac{x^2}{2\epsilon^2 t}\,. \tag{74}$$

From the above equations (72)–(74) we arrive at equation (5). The theorem is proved. $\square$

Let us proceed to prove Corollary 2.

Proof of Corollary 2. Differentiating equation (5) with respect to x we obtain

$$\Big(\epsilon\partial_x\partial_\epsilon + \frac{1}{2}\partial_x + \frac{1}{2}x\partial_x^2 - \frac{\epsilon^2}{24}\partial_x^4\Big)\big(\mathcal{H}_+ - \mathcal{H}_-\big) \;+\; \frac{\epsilon^2}{4}\big[\partial_x(\mathcal{H}_+ - \mathcal{H}_-)\big]^2\big[\partial_x^2(\mathcal{H}_+ - \mathcal{H}_-)\big] \;=\; 0\,,$$

so from equation (4) it follows that

$$\Big(\epsilon\partial_\epsilon + \frac{1}{2} + x\partial_x - \frac{\epsilon^2}{24}\partial_x^3\Big)\circ\partial_x\big(\mathcal{H}_+ - \mathcal{H}_-\big) \;-\; \frac{\epsilon^2}{4}\Big[\big(\partial_x^2(\mathcal{H}_+)\big)^2 - \big(\partial_x^2(\mathcal{H}_-)\big)^2\Big] \;=\; 0\,.$$

Observing that $\big[x\partial_x, e^{\pm i\epsilon\partial_x/2}\big] = \mp\frac{i\epsilon}{2}e^{\pm i\epsilon\partial_x/2}\partial_x$ one can simplify this equation and find

$$\Big(e^{\frac{i\epsilon\partial_x}{2}} - e^{-\frac{i\epsilon\partial_x}{2}}\Big)\left[\Big(\epsilon\partial_\epsilon + \frac{1}{2} + x\partial_x - \frac{\epsilon^2}{24}\partial_x^3\Big)\circ\partial_x\big(\mathcal{H}\big) \;-\; \frac{\epsilon^2}{4}\big[\partial_x^2(\mathcal{H})\big]^2\right] \;=\; 0\,. \tag{75}$$

Since the operator $(e^{\frac{i\epsilon\partial_x}{2}} - e^{-\frac{i\epsilon\partial_x}{2}})/\partial_x$ is invertible on power series of x, we find that equation (75) is equivalent to

$$\partial_x\left[\Big(\epsilon\partial_\epsilon + \frac{1}{2} + x\partial_x - \frac{\epsilon^2}{24}\partial_x^3\Big)\circ\partial_x\big(\mathcal{H}\big) \;-\; \frac{\epsilon^2}{4}\big[\partial_x^2(\mathcal{H})\big]^2\right] \;=\; 0\,. \tag{76}$$

It follows that

$$\left(\epsilon\partial_\epsilon+\frac{1}{2}+x\partial_x-\frac{\epsilon^2}{24}\partial_x^3\right)\circ\partial_x\big(\mathcal{H}\big)\;-\;\frac{\epsilon^2}{4}\big[\partial_x^2(\mathcal{H})\big]^2\;=\;C(\epsilon)\,,$$

where $C(\epsilon)=\sum_{g\geq 0}\epsilon^{2g-2}C_g$ with C_g being constants. It remains to show that C_g all vanish. Indeed, for $g=0$ and $g=1$, this can be verified directly with the explicit expressions of $\mathcal{H}_0$ and $\mathcal{H}_1$ given in Lemma 1. For $g\geq 2$, by using Lemma 1 and the fact that $\partial_x=-\frac{1}{T}\partial_T$ we arrive at $C_g=0$. The corollary is proved. □

Let us now show that Corollary 2 implies Kazarian's recursion on the linear Hodge integrals $\frac{(5g-3-j)(5g-5-j)}{(3g-3-j)!}\int_{\overline{\mathcal{M}}_{g,3g-3-j}}\lambda_j\psi_1^2\cdots\psi_{3g-3-j}^2$. Indeed, differentiating (7) with respect to x we find that the series $u=\epsilon^2\partial_x^2(\mathcal{H})$ satisfies the equation

$$2\epsilon u_\epsilon\;+\;2xu_x\;-\;u\;=\;\partial_x\Big(\frac{1}{2}u^2\Big)\;+\;\frac{1}{12}\epsilon^2u_{xxx}\,. \tag{77}$$

Denote

$$u(x,\epsilon)=:\sum_{g\geq 0}\epsilon^{2g}u^{[g]}(x)\,. \tag{78}$$

Then we can write (77) equivalently as follows:

$$\Big(4g-1+2x\frac{d}{dx}\Big)\big(u^{[g]}\big)\;=\;\frac{1}{2}\frac{d}{dx}\bigg(\sum_{g_1+g_2=g}u^{[g_1]}u^{[g_2]}\bigg)\;+\;\frac{1}{12}\frac{d^3}{dx^3}\big(u^{[g-1]}\big)\,,\quad g\geq 0\,. \tag{79}$$

To proceed we note that it follows easily from Lemma 1 that $u^{[g]}(x)$ has the expression

$$u^{[0]}=1-T\,,\quad u^{[1]}\;=\;\frac{1}{12}\frac{1}{T^4}\;+\;\frac{1}{24}\frac{1}{T^3}\,, \tag{80}$$

$$u^{[g]}\;=\;\sum_{j=0}^{g}\frac{\langle\lambda_j\tau_2^{3g-3-j}\rangle_g}{(3g-3-j)!}\frac{\prod_{i=0}^{1}(5g-5-j+2i)}{T^{5g-1-j}}\,,\quad g\geq 2\,. \tag{81}$$

Thus using the fact that $\frac{d}{dx}=-\frac{1}{T}\frac{d}{dT}=:D_T$ we find that (79) is equivalent to

$$\big(4g-1+(1-T^2)D_T\big)\big(u^{[g]}\big)\;=\;\frac{1}{2}D_T\bigg(\sum_{\substack{g_1,g_2\geq 0\\ g_1+g_2=g}}u^{[g_1]}u^{[g_2]}\bigg)\;+\;\frac{1}{12}D_T^3\big(u^{[g-1]}\big)\,. \tag{82}$$

Substituting (80), (81) into (82) we find

$$\begin{aligned} c_{g,j}\;=\;&\frac{g+1-k}{5g-2-j}\,c_{g,j-1}\;+\;\frac{(5g-6-j)(5g-4-j)}{12}\,c_{g-1,j}\\ &+\;\frac{1}{2}\sum_{\substack{g_1,g_2\geq 1,j_1,j_2\geq 0\\ g_1+g_2=g,j_1+j_2=j}}c_{g_1,j_1}c_{g_2,j_2}\,,\qquad g\geq 1,\,0\leq j\leq g\,, \end{aligned} \tag{83}$$

where the numbers $c_{g,j}$ are defined by

$$c_{g,j}\;:=\;\frac{\langle\lambda_j\tau_2^{3g-3-j}\rangle_g}{(3g-3-j)!}\prod_{i=0}^{1}(5g-5-j+2i)\,. \tag{84}$$

The recursion relations (83) for $c_{g,j}$ were obtained by Kazarian [28] from the KP hierarchy [27] satisfied by the linear Hodge integrals.

It is not clear at the moment whether Corollary 2 and Lemma 1 imply Theorem 1.

We end this section with two remarks on the computational aspects. Firstly, as a consequence of equation (4) and Lemma 1, the $u^{[g]}$ can be computed from the recursion

$$\begin{aligned} u^{[0]} &= 1-T\,,\\ u^{[g]} &= \frac{1}{2T}\sum_{\substack{0\le g_1,g_2\le g-1\\ g_1+g_2+j_1+j_2=g}}\Big(-\frac14\Big)^{j_1+j_2}\frac{D_T^{2j_1}\big(u^{[g_1]}\big)D_T^{2j_2}\big(u^{[g_2]}\big)}{(2j_1+1)!(2j_2+1)!}-\frac1T\sum_{j=1}^{g}\Big(-\frac14\Big)^j\frac{D_T^{2j}\big(u^{[g-j]}\big)}{(2j)!}\,, \end{aligned}$$

where $g\ge1$. Then one can further compute $\mathcal{H}_g$, $g\ge2$ from $u^{[g]}$ via

$$\mathcal{H}_g = \sum_{j=0}^{g}\frac{C_{g,j}}{T^{5g-5-j}}\,,\qquad C_{g,j} = \frac{\text{coefficient of } 1/T^{5g-1-j} \text{ in } u^{[g]}}{(5g-3-j)(5g-5-j)}\quad (0\le j\le g)\,. \tag{85}$$

Secondly, the series $\tilde u$ (see (61)) also presents good properties. Denote

$$\tilde u(x,\epsilon) =: \sum_{g\ge0}\epsilon^{2g}\tilde u^{[g]}(x)\,. \tag{86}$$

If then follows from (80), (81) that $\tilde u^{[g]}$ has the expression

$$\tilde u^{[0]} = 1-T\,,\quad \tilde u^{[1]} = \frac{1}{12T^4}\,,\quad \tilde u^{[g]} = \sum_{j=0}^{g}\frac{d_{g,j}}{T^{5g-1-j}}\ (g\ge2)\,, \tag{87}$$

where $d_{g,j}\in\mathbb{Q}$ are constants. In terms of intersection numbers we have for $g\ge2$,

$$\begin{aligned} \tilde u^{[g]} &= \sum_{g_1=0}^{g-2}\frac{(-1)^{g_1}}{2^{2g_1}(2g_1+1)!}\sum_{j=0}^{g-g_1}\frac{\langle\lambda_j\tau_2^{3g-3g_1-3-j}\rangle_{g-g_1}}{(3g-3g_1-3-j)!}\frac{\prod_{i=0}^{1+2g_1}(5g-5g_1-5-j+2i)}{T^{5g-g_1-1-j}}\\ &\quad+\frac{(-1)^{g-1}2^{2g}}{12}\frac{1}{T^{4g}}+\frac{(-1)^g(4g-3)!!}{2^{2g}(2g+1)!}\frac{5-2g}{6}\frac{1}{T^{4g-1}}\,. \end{aligned}$$

Substituting (86) into (62) we find that $\tilde u^{[g]}$, $g\ge0$ satisfy the following recursion

$$\tilde u^{[0]} = 1-T\,, \tag{88}$$

$$\tilde u^{[g]} = \frac{1}{2T}\sum_{g_1=1}^{g-1}\tilde u^{[g_1]}\tilde u^{[g-g_1]}+\frac1T\sum_{g_1=1}^{g}\frac{|B_{2g_1}|}{(2g_1)!}D_T^{2g_1}\big(\tilde u^{[g-g_1]}\big)\,,\quad g\ge1\,. \tag{89}$$

This recursion gives an algorithm for computing $\tilde u$. From (61) we know that

$$u = \tilde u + \sum_{g\ge1}\epsilon^{2g}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}D_T^{2g}(\tilde u)\,.$$

Therefore, for $g\ge0$,

$$u^{[g]} = \tilde u^{[g]} + \sum_{g_1=1}^{g}\frac{2^{2g_1-1}-1}{2^{2g_1-1}}\frac{|B_{2g_1}|}{(2g_1)!}D_T^{2g_1}\big(\tilde u^{[g-g_1]}\big)\,.$$

So this gives rise to another algorithm for computing the MV volumes. One could also use (5) to study $\tilde{u}$.

4. Asymptotics of the area Siegel–Veech constants

In this section we use Goujard's formula to compute the area Siegel–Veech (SV) constants associated with principal strata of moduli spaces of quadratic differetials. Indeed, according to Goujard [23] the area SV constants can be expressed explicitly in terms of the number $a_{g,n}$ as follows:

$$C_{\mathrm{area}}(\mathcal{Q}_{g,n}) \;=\; \frac{\pi^{-2}}{4a_{g,n}}\left(n(n-1)a_{g,n-1} + a_{g-1,n+2} + \sum_{\substack{g_1,g_2\geq 0,\, n_1,n_2\geq 1\\ g_1+g_2=g,\, n_1+n_2=n+2\\ 3g_i-3+n_i>0\,(i=1,2)}} \binom{n}{n_1-1} a_{g_1,n_1}a_{g_2,n_2}\right). \tag{90}$$

The result in this section is a refinement of the conjectural formula for the large g asymptotics of $C_{\mathrm{area}}(\mathcal{Q}_{g,n})$ given in [12, 4] to the following more precise asymptotic statement.

Conjecture 2. *For any fixed $n \geq 0$, we have the asymptotic formula*

$$C_{\mathrm{area}}(\mathcal{Q}_{g,n}) \;\sim\; \sum_{k=0}^{\infty} \frac{C_k(n)}{g^k}\,, \qquad g\to\infty\,, \tag{91}$$

where each $C_k(n)$ is a polynomial with rational coefficients in n and $M = -\pi^2/144$, with the first four of them being

$$C_0(n) \;=\; \frac{1}{4}\,, \quad C_1(n) \;=\; \frac{1}{48}n^2 - \frac{3}{16}n + \frac{1-2M}{4}\,,$$

$$C_2(n) \;=\; -\frac{5+12M}{576}\,n^3 + \frac{59+180M}{576}\,n^2 - \frac{11+24M-72M^2}{32}\,n + \frac{23+15M-648M^2}{72}\,,$$

$$\begin{aligned} C_3(n) \;=\; & \frac{4+17M+54M^2}{1152}\,n^4 - \frac{179+978M+3564M^2}{3456}\,n^3 \\ & + \frac{929+5169M+13554M^2-42768M^3}{3456}\,n^2 \\ & - \frac{989+4851M-4428M^2-192456M^3}{1728}\,n \\ & + \frac{295+1165M-16140M^2-105300M^3+253692M^4}{720}\,. \end{aligned}$$

The asymptotic formula (91) with $\sum_{k=0}^{\infty} C_k(n)/g^k$ replaced by $1/4$ becomes the ADGZZ conjecture for the area SV constants. As we mentioned in the Introduction, the above Conjecture 2 is also not based on theoretical reasoning but on numerical computations. Very recently Aggarwal [3] proved the ADGZZ conjecture for the area SV constants by showing that the leading term asymptotics in (18) implies the leading term asymptotics in (91) with the knowledge of Goujard's formula (90). However, we do not know whether Conjecture 1 implies Conjecture 2 in the same way. This would be an interesting point to investigate next.

References

[1] Aggarwal, A., Large genus asymptotics for Siegel–Veech constants, Geom. Funct. Anal., **29** (2019), 1295–1324.

[2] Aggarwal, A., Large genus asymptotics for volumes of strata of abelian differentials, With an appendix by Zorich, A., arXiv:1804.05431.

[3] Aggarwal A., Large genus asymptotics for intersection numbers and principal strata volumes of quadratic differentials, arXiv:2004.05042.

[4] Aggarwal A., Delecroix, V., Goujard, É., Zograf, P., Zorich, A., Conjectural large genus asymptotics of Masur–Veech volumes and of area Siegel–Veech constants of strata of quadratic differentials, arXiv:1912.11702.

[5] Andersen, J.E., Borot, G., Charbonnier, S., Delecroix, V., Giacchetto, A., Lewański, D., Wheeler, C., Topological recursion for Masur–Veech volumes, arXiv:1905.10352.

[6] Athreya, J., Eskin, A., Zorich, A., Right-angled billiards and volumes of moduli spaces of quadratic differentials on $\mathbb{C}\mathrm{P}^1$, Ann. Scient. ENS, 4ème série, **49** (2016), 1307–1381.

[7] Buryak, A., Dubrovin-Zhang hierarchy for the Hodge integrals, Commun. Number Theory Phys., **9** (2015), 239–271.

[8] Chen, D., Möller, M., Sauvaget, A., Masur–Veech volumes and intersection theory: The principal strata of quadratic differentials, With an appendix by Borot, G., Giacchetto, A., Lewański, D., arXiv:1912.02267.

[9] Chen, D., Möller, M., Sauvaget, A., Zagier, D., Masur–Veech volumes and intersection theory on moduli spaces of abelian differentials, Invent. Math. (to appear), arXiv:1901.01785.

[10] Chen, D., Möller, M., Zagier, D., Quasimodularity and large genus limits of Siegel-Veech constants, J. Amer. Math. Soc., **31** (2018), 1059–1163.

[11] Delecroix, V., Goujard, E., Zograf, P., Zorich, A., Contribution of one-cylinder square-tiled surfaces to Masur–Veech volumes, arXiv:1903.10904.

[12] Delecroix, V., Goujard, E., Zograf, P., Zorich, A., Masur–Veech volumes, frequencies of simple closed geodesics and intersection numbers of moduli spaces of curves, arXiv:1908.08611.

[13] Deligne, P., Mumford, D., The irreducibility of the space of curves of given genus, Inst. Hautes Études Sci. Publ. Math., **45** (1969), 75–109.

[14] Dubrovin, B., Geometry of 2D topological field theories, In "Integrable Systems and Quantum Groups" (Montecatini Terme, 1993), Editors: Francaviglia, M., Greco, S., Springer Lecture Notes in Math. **1620**, 1996, 120–348.

[15] Dubrovin, B., Liu, S.-Q., Yang, D., Zhang, Y., Hodge integrals and tau-symmetric integrable hierarchies of Hamiltonian evolutionary PDEs, Adv. Math., **293** (2016), 382–435.

[16] Dubrovin, B., Yang, D., Remarks on intersection numbers and integrable hierarchies. I. Quasi-triviality, arXiv:1905.08106.

[17] Dubrovin, B., Yang, D., Zagier, D., Classical Hurwitz numbers and related combinatorics, Mosc. Math. J., **17** (2017), 601–633.

[18] Dubrovin, B., Zhang, Y., Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants, arXiv math/0108160.

[19] Ekedahl, T., Lando, S., Shapiro, M., Vainshtein, A., On Hurwitz numbers and Hodge integrals, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 1175–1180.

[20] Eskin, A., Kontsevich, M., Zorich, A., Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow, Publ. Math. IHES, **120** (2014), 207–333.

[21] Eskin, A., Zorich, A., Volumes of strata of abelian differentials and Siegel-Veech constants in large genera, Arnold Math. J., **1** (2015), 481–488.

[22] Faber, C., Pandharipande, R., Hodge integrals and Gromov-Witten theory, Invent. Math., **139** (2000), 173–199.

[23] Goujard, E., Siegel-Veech constants for strata of moduli spaces of quadratic differentials, Geom. Funct. Anal., **25** (2015), 1440–1492.

[24] Goujard, E., Volumes of strata of moduli spaces of quadratic differentials: getting explicit values, Ann. Inst. Fourier, Grenoble, **66** (2016), 2203–2251.
[25] Grünberg, D., Moree, P., Sequences of enumerative geometry: congruences and asymptotics, With an Appendix by Zagier, D., Experim. Math. **17** (2008), 409–426.
[26] Itzykson, C., Zuber, J.B., Combinatorics of the modular group II. The Kontsevich integrals, Internat. J. Modern Phys. A, **7** (1992), 5661–5705.
[27] Kazarian, M., KP hierarchy for Hodge integrals, Adv. Math., **221** (2009), 1–21.
[28] Kazarian, M., Recursion for Masur-Veech volumes of moduli spaces of quadratic differentials, arXiv:1912.10422.
[29] Kontsevich, M., Intersection theory on the moduli space of curves and the matrix Airy function, Comm. Math. Phys., **147** (1992), 1–23.
[30] Kontsevich, M., Zorich, A., Connected components of the moduli spaces of Abelian differentials with prescribed singularities, Invent. Math., **153** (2003), 631–678.
[31] Mirzakhani, M., Growth of the number of simple closed geodesics on hyperbolic surfaces, Ann. Math., **168** (2008), 97–125.
[32] Mumford, D., Towards an enumerative geometry of the moduli space of curves, In: Arithmetic and geometry (271–328), Birkhäuser Boston, 1983.
[33] Sauvaget, A., The large genus asymptotic expansion of Masur–Veech volumes, arXiv:1903.04454.
[34] Sauvaget, A., Volumes and Siegel-Veech constants of $\mathcal{H}(2g-2)$ and Hodge integrals, Geom. Funct. Anal. **28** (2018), 1756–1779.
[35] Witten, E., Two-dimensional gravity and intersection theory on moduli space, Surveys in differential geometry (Cambridge, MA, 1990), 243–320, Lehigh Univ., Bethlehem, PA, 1991.
[36] Zagier, D., Vassiliev invariants and a strange identity related to the Dedekind eta-function, Topology **40** (2001), 945–960.

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