Higher Kronecker "limit" formulas for real quadratic fields

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Abstract

For every integer $k \geq 2$ we introduce an analytic function of a positive real variable and give a universal formula expressing the values $\zeta(\mathcal{B}, k)$ of the zeta functions of narrow ideal classes in real quadratic fields in terms of this function and its derivatives up to order $k-1$ evaluated at reduced real quadratic irrationalities associated to $\mathcal{B}$. We show that our functions satisfy functional equations and use these to deduce explicit formulas for the rational numbers $\zeta(\mathcal{B}, 1-k)$. We also give an interpretation of our formula for $\zeta(\mathcal{B}, k)$ in terms of cohomology groups of $\mathrm{SL}(2, \mathbb{Z})$ with analytic coefficients, describe a "twisted" extension of the main formula that allows one to treat zeta values of zeta functions of ray classes rather than just ideal classes. Finally, we use our formulas to compute some zeta-values numerically and test that they are expressible as combinations of higher polylogarithm functons evaluated at algebraic arguments.

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Introduction

Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant D and $\mathcal{B} \in Cl^+(K)$ be a narrow ideal class. The ζ-function of $\mathcal{B}$ is defined for $\mathrm{Re}(s) > 1$ as

$$\zeta(\mathcal{B}, s) = \sum_{\mathfrak{a} \in \mathcal{B}} \frac{1}{N(\mathfrak{a})^s},$$

where the summation is over all integral ideals in the class $\mathcal{B}$. This function can be continued to a meromorphic function on the whole of $\mathbb{C}$ with its only singularity at $s = 1$, where it has a simple pole with residue $D^{-\frac{1}{2}} \log(\varepsilon)$. Here ε is the smallest totally positive unit of K with the property $\varepsilon > 1$. The Kronecker limit formula (hereafter abbreviated as KLF) for real quadratic fields is an expression for the 0th Laurent coefficient of $\zeta(\mathcal{B}, s)$ at $s = 1$. (The original KLF, of course, was for imaginary quadratic fields.) Such a formula was given in [13]: there is an analytic function $P(x, y)$ of $x > y > 0$ such that for all narrow ideal classes in all real quadratic fields

$$\lim_{s \to 1} \left(D^{s/2} \zeta(\mathcal{B}, s) - \frac{\log(\varepsilon)}{s-1} \right) = \sum_{w \in \mathrm{Red}(\mathcal{B})} P(w, w'). \tag{1}$$

Here $\mathrm{Red}(\mathcal{B})$ is the set of larger roots $w = \frac{-B+\sqrt{D}}{2A}$ of all reduced quadratic forms $Q(X,Y) = AX^2 + BXY + CY^2$ $(A, C > 0,\ A + B + C < 0)$ of discriminant D which belong to the class $\mathcal{B}$. Recall that narrow ideal classes correspond to $\mathrm{PSL}(2,\mathbb{Z})$-orbits on the set of integer quadratic forms: if $\mathfrak{b} = \mathbb{Z}w_1 + \mathbb{Z}w_2 \in \mathcal{B}$ with $\frac{w_1 w_2' - w_2 w_1'}{\sqrt{D}} > 0$, then the quadratic form $Q(X,Y) = \frac{N(Xw_1+Yw_2)}{N(\mathfrak{b})}$ is in the corresponding orbit. The set $\mathrm{Red}(\mathcal{B})$ is obviously finite and every $w \in \mathrm{Red}(\mathcal{B})$ satisfies $w > 1$, $1 > w' > 0$. Sometimes we identify $\mathrm{Red}(\mathcal{B})$ with the set of reduced forms themselves and write $Q \in \mathrm{Red}(\mathcal{B})$ for such a form. We denote $l(\mathcal{B}) = \#\,\mathrm{Red}(\mathcal{B})$ in the sequel.

The function $P(x,y)$ mentioned above is defined as

$$P(x,y) = \mathcal{F}(x) - \mathcal{F}(y) + \mathrm{Li}_2\Big(\frac{y}{x}\Big) - \frac{\pi^2}{6} + \Big(\log\frac{x}{y}\Big)\Big(\gamma - \frac{1}{2}\log(x-y) + \frac{1}{4}\log\frac{x}{y}\Big)\,,$$

where γ is Euler's constant, $\mathrm{Li}_2(z) = \sum\limits_{n=1}^{\infty}\dfrac{z^n}{n^2}$ $(|z| < 1)$ is the dilogarithm function, and

$$\mathcal{F}(x) = \sum_{p=1}^{\infty} \frac{\psi(px) - \log(px)}{p}\,, \tag{2}$$

with $\psi(x) = \Gamma'(x)/\Gamma(x)$ (digamma function). The sum defining $\mathcal{F}$ converges since $\psi(x) = \log x + \mathrm{O}(1/x)$ as $x \to \infty$, and defines an analytic function on $\mathbb{R}_+$ that satisfies the functional equations

$$\begin{aligned} \mathcal{F}(x) + \mathcal{F}\left(\frac{1}{x}\right) &= -\frac{\pi^2}{6}\left(x + \frac{1}{x}\right) + \frac{1}{2}\log^2 x + C\,, \\ \mathcal{F}(x) - \mathcal{F}(x-1) + \mathcal{F}\left(\frac{x-1}{x}\right) &= -\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{\pi^2}{6} + \frac{1}{2}\,C \qquad (x > 1), \end{aligned} \tag{3}$$

with the real constant $C \approx 1.45738783\cdots$ given explicitly in terms of the derivative of $\zeta(s) - (s-1)^{-1}$ at $s = 1$. In [13] these functional equations were used to deduce Meyer's theorem from the Kronecker limit formula (1), namely

$$\lim_{s\to 1}\big(\zeta(\mathcal{B},s) - \zeta(\mathcal{B}^*,s)\big) = \frac{\zeta(2)}{\sqrt{D}}\,\big(l(\mathcal{B}) - l(\mathcal{B}^*)\big)\,, \tag{4}$$

where $\mathcal{B}^*$ is the class of any ideal $\alpha\mathfrak{b}$ with any $\mathfrak{b} \in \mathcal{B}$ and $\alpha \in K$, $\alpha\alpha' < 0$.

In the present paper we generalize formulas (1) and (4) to the higher zeta values $\zeta(\mathcal{B},2), \zeta(\mathcal{B},3), \ldots$. We begin by generalizing the function (2).

Definition 1. *For every integer $k > 2$ and real number $x > 0$ let*

$$\mathcal{F}_k(x) \;=\; \sum_{p=1}^{\infty} \frac{\psi(px)}{p^{k-1}}\,. \tag{5}$$

Here we do not need to subtract $\log(px)$ from $\psi(px)$ since the estimate $\psi(x) = \mathrm{O}(\log x)$ suffices for (absolute) convergence, and the functions $\mathcal{F}_k$ are analytic on $\mathbb{R}_+$. They already occurred incidentally in [17]. We also introduce a collection of differential operators $\mathcal{D}_n$ $(n = 0, 1, 2, \ldots)$ that turn differentiable functions of one variable into differentiable functions of two variables:

$$\begin{aligned}
(\mathcal{D}_0 F)(x,y) &= F(x) - F(y)\\
(\mathcal{D}_1 F)(x,y) &= F'(x) + F'(y) \;-\; 2\,\frac{F(x)-F(y)}{x-y}\\
(\mathcal{D}_2 F)(x,y) &= \frac{F^{(2)}(x) - F^{(2)}(y)}{2} \;-\; 3\,\frac{F'(x)+F'(y)}{x-y} \;+\; 6\,\frac{F(x)-F(y)}{(x-y)^2}\\
&\;\;\vdots
\end{aligned}$$

where the coefficients, given explicitly in Definition 2 below, are chosen so that $\mathcal{D}_n$ kills polynomials of degree $\le 2n$ and sends $F(x) = x^{2n+1}$ to $(x-y)^{n+1}$. Then we have

Theorem 1. *For any narrow ideal class $\mathcal{B} \in Cl^+(K)$ and integer $k \ge 2$,*

$$D^{k/2}\,\zeta(\mathcal{B},k) \;=\; \sum_{w\in \mathrm{Red}(\mathcal{B})} P_k(w,w') \tag{6}$$

where $P_k(x,y)$ is the function of two variables $x,y>0$ defined by

$$P_k(x,y) = \bigl(\mathcal{D}_{k-1}\,\mathcal{F}_{2k}\bigr)(x,y)\;. \tag{7}$$

We call this formula the higher Kronecker "limit" formula because it has a form similar to (1). The reason for the quotes is that for $k \ge 2$ no limit is involved, since the series defining $\zeta(\mathcal{B},k)$ converges.

For any differentiable function F on $\mathbb{R}$ one can also consider the two-variable function $\mathcal{D}_n\,F$ as a homogeneous function of degree $-n-1$ on the space $\mathcal{Q}^+$ of real quadratic forms with positive discriminant by setting

$$(\mathcal{D}_n F)(Q) \;=\; D^{-\frac{n+1}{2}}\,(\mathcal{D}_n F)\biggl(\frac{-B+\sqrt{D}}{2A},\,\frac{-B-\sqrt{D}}{2A}\biggr) \tag{8}$$

for $Q(X,Y) = AX^2 + BXY + CY^2$ with $B^2 - 4AC = D > 0$. (This is defined whenever F is smooth near the roots of Q.) Here $\sqrt{D}$ denotes the positive square root and $D^{-(n+1)/2}$ is defined as $(\sqrt{D})^{-n-1}$, but because $\mathcal{D}_n\,F$ is $(-1)^{n+1}$-symmetric the right-hand side of (8) is actually independent of the choice of square root. By Proposition 4 below we have the equivariance property

$$\mathcal{D}_n\bigl(F\big|_{-2n}g\bigr)(Q) \;=\; (\det g)^{2n+1}\,(\mathcal{D}_n\,F)(Q\circ g) \qquad \Bigl(\forall\, g\in \mathrm{GL}(2,\mathbb{R})\Bigr),$$

where $|_w$ for $w \in \mathbb{Z}$ is defined as usual by $\bigl(F|_w\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\bigr)(x) = (cx+d)^{-w}\,F\bigl(\frac{ax+b}{cx+d}\bigr)$. With this notation, Theorem 1 can be rewritten in the form

$$\zeta(\mathcal{B},k) \;=\; \sum_{Q\in\mathrm{Red}(\mathcal{B})} \bigl(\mathcal{D}_{k-1}\,\mathcal{F}_{2k}\bigr)(Q)\;.$$

In the first part of the paper we prove Theorem 1 and study the functions $\mathcal{F}_k$. In particular we show that they (and their derivatives) can be computed easily to high accuracy and satisfy functional equations analogous to (3). These functional equations turn out to be sufficient to deduce from our higher KLF explicit formulas, analogous to (4), for the numbers $\zeta(\mathcal{B},k)+(-1)^k\,\zeta(\mathcal{B}^*,k)\in\pi^{2k}D^{-1/2}\mathbb{Q}$, or equivalently, if one uses the functional equations for $\zeta(\mathcal{B},s)\pm\zeta(\mathcal{B}^*,s)$, for the numbers $\zeta(\mathcal{B},1-k)\in\mathbb{Q}$, where the rationality statement is the well-known theorem of Klingen and Siegel. One of the formulas we obtain was already proved in [14].

The second part of the paper is devoted to the interpretation of the higher KLF in terms of (co)homology of $\mathrm{PSL}(2,\mathbb{Z})$. We construct a cocycle class $[\phi_k]\in H^1(\mathrm{PSL}(2,\mathbb{Z}),\mathcal{V}_{2k})$ with coefficients in the space $\mathcal{V}_{2k}$ of continuous functions of weight $2k$ on the projective real line and use the functional equations satisfied by $\mathcal{F}_{2k}$ to show (Theorem 3) that $[\phi_k]$ has rational image

$$\mathfrak{I}([\phi_k])\ \in\ H^1(\mathrm{PSL}(2,\mathbb{Z}),\pi^{2k}V_{2k-2}(\mathbb{Q}))$$

under the integral map $\mathfrak{I}:f\mapsto\int f(x)(X-x)^{2k-2}dx$ from $\mathcal{V}_{2k}$ to the space V_{2k-2} of real polynomials of degree $\leq 2k-2$. These $[\phi_k]$ and $\mathfrak{I}([\phi_k])$ turn out to be the two periods of the nonholomorphic Eisenstein series according to the definition given in [6, 1]. The class $\mathfrak{I}([\phi_k])$ itself is the well known Eisenstein cohomology class studied e.g. in [9] and for the general linear group of degree n in [8, 10]. That is why we call our $[\phi_k]$ the generalized Eisenstein cocycle class. Further, for each narrow ideal class $\mathcal{B}$ we define a cycle class $\eta_{\mathcal{B}}\in H_1(\mathrm{PSL}(2,\mathbb{Z}),\mathcal{V}_{2-2k})$ and show that evaluating $[\phi_k]$ on it gives the value $\zeta(\mathcal{B},k)$ (Theorem 4). Then the rationality of $\mathfrak{I}([\phi_k])$ implies the Siegel-Klingen theorem again.

In the third part of our paper we consider the twisted version of our functions $\mathcal{F}_k^{\chi}$ for a Dirichlet character χ and $k\geq 2$. Such twisted functions can be used to compute zeta values for ray classes. We should mention that a KLF was given for ray classes in real quadratic fields in full generality by Yamamoto [12], but even for this case ($k=1$) our formulas are different. We do not present everything in complete detail but consider a particular example, where we compute Stark's unit and its "higher" version (for $k=2$), which involves dilogarithms instead of logarithms.

We mention that there are similar "higher Kronecker limit formulas" also for imaginary quadratic fields. Let K be an imaginary quadratic field of discriminant $D<-4$ and $\mathcal{B}$ an ideal class of K. Then $\zeta(\mathcal{B},s)=|D/4|^{-s/2}E\big(w,s\big)$, where w is the root in the upper half-plane of any quadratic form of discriminant D in the $\mathrm{PSL}(2,\mathbb{Z})$-orbit corresponding to $\mathcal{B}$ and

$$E(z,s)=\frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}'\,\frac{y^s}{|mz+n|^{2s}}\qquad(z=x+iy\in\mathfrak{H})\tag{9}$$

is the non-holomorphic Eisenstein series. For $k\in\mathbb{Z}_{\geq 2}$ we have

$$E(z,k)=(2i)^{-k}\,\big(\mathcal{D}_{k-1}\,\mathcal{C}_{2k}\big)(z,\bar{z})$$

where $\mathcal{D}_{k-1}$ is the same differential operator as in Theorem 1 and

$$\mathcal{C}_h(z) = \zeta(h) z^{h-1} - \pi \sum_{p=1}^{\infty} \frac{\cot(\pi p z)}{p^{h-1}} \qquad \bigl(z \in \mathbb{C} \smallsetminus \mathbb{R}\bigr)\,.$$

Therefore $|D/4|^{k/2}\,\zeta(\mathcal{B},k) = \mathcal{D}_{k-1}\,\mathcal{C}_{2k}\bigl(w,\overline{w}\bigr)$. The relation between $\mathcal{C}_h$ and $\mathcal{F}_h$ (which is defined in all of $\mathbb{C} \smallsetminus (-\infty;0]$ by the sum (5)) is simply

$$\mathcal{F}_h(z) - \mathcal{F}_h(-z) = \zeta(h)\bigl(z^{h-1} + 1/z\bigr) - \mathcal{C}_h(z)\,.$$

(This is an immediate consequence of $\psi(x) - \psi(-x) = \pi\cot(\pi x) + x^{-1}$.) The reason that the real quadratic case is more complicated than the imaginary quadratic case is that there one has to work with "half-Eisenstein series," i.e. sums over lattice points in a quadrant rather than a whole lattice. This is discussed in §2.

1 The functions $\mathcal{F}_k$ and zeta values

1.1 Properties of the functions $\mathcal{F}_k$

We begin by giving the expansions of the functions $\mathcal{F}_k(x)$ defined in (5) near $x = \infty$, 0 and 1. The result will involve the double zeta values

$$\zeta(m,n) \;=\; \sum_{p>q>0} \frac{1}{p^m q^n}, \qquad (m \ge 2, n \ge 1).$$

They satisfy the well known shuffle relations

$$\begin{aligned} \zeta(m,k-m) \,+\, \zeta(k-m,m) \;&=\; \zeta(m)\,\zeta(k-m) - \zeta(k), \\ \sum_{n=1}^{k-1}\left(\binom{n-1}{m-1} + \binom{n-1}{k-m-1}\right)\zeta(n,k-n) \;&=\; \zeta(m)\,\zeta(k-m) \end{aligned} \tag{10}$$

for each $m = 1,\ldots,k-1$. Here and everywhere below the divergent values $\zeta(1)$ and $\zeta(1,k-1)$ are to be interpreted as γ (Euler's constant) and $\zeta(k-1,1) + \zeta(k) - \gamma\zeta(k-1)$ respectively. We denote by B_n the nth Bernoulli number, defined by the expansion $\dfrac{x}{e^x-1} = \sum\limits_{n=0}^{\infty} B_n \dfrac{x^n}{n!}$, or alternatively by $B_0 = 1$ and $B_n = (-1)^{n-1} n\,\zeta(1-n)$ for $n \ge 1$.

Proposition 1. *The function $\mathcal{F}_k(x)$ has the asymptotic expansions*[1]

$$\mathcal{F}_k(x) \;\sim\; \zeta(k-1)\log x \,-\, \zeta'(k-1) \,+\, \sum_{r=1}^{\infty} \frac{\zeta(1-r)\,\zeta(k+r-1)}{x^r}$$

[1] By an asymptotic expansion $f(x) \sim \sum\limits_{n=n_0}^{\infty} a_n (x-a)^n$ as $x \to a$ we mean that $f(x) - \sum\limits_{n=n_0}^{N-1} a_n(x-a)^n = \mathrm{O}((x-a)^N)$ when $x \to a$ for every N. (We do not require that the series be convergent in any neighborhood of a.) For $a = \infty$ one replaces $x - a$ by $1/x$.

as $x \to \infty$*,*

$$\mathcal{F}_k(x) \;\sim\; -\frac{\zeta(k)}{x} \;+\; \sum_{r=1, r\neq k-1}^{\infty} (-1)^r\, \zeta(r)\, \zeta(k-r) x^{r-1}$$
$$+\,(-1)^{k-1}\big[\zeta(k-1)(\gamma-\log x) \;-\; \zeta'(k-1)\big]x^{k-2}$$

as $x \to 0$*, and*

$$\mathcal{F}_k(x) \;\sim\; -\sum_{r=1}^{k-1}\big[\zeta(r, k-r)+\zeta(k)\big](1-x)^{r-1}$$
$$-\sum_{r=k}^{\infty}\Bigg[\frac{1}{r-k+1}\sum_{l=0}^{r-k}(-1)^l\binom{r-k+1}{l}B_l\,\zeta(k+l-1)\Bigg](1-x)^{r-1}$$

as $x \to 1$*.*

Proof. The standard asymptotic expansion of the digamma function at infinity,

$$\psi(x) \sim \log x + \sum_{r=1}^{\infty} \frac{\zeta(1-r)}{x^r} \qquad (x \to \infty)$$

(the derivative of Stirling's formula), immediately gives the first asymptotic expansion for $\mathcal{F}_k$. For the second, we use the expansion

$$\psi(x) \;=\; -\frac{1}{x} - \gamma + \sum_{r=2}^{\infty}(-1)^r\zeta(r)x^{r-1} \qquad (0 < x < 1)$$

and apply the Euler-Maclaurin summation formula following the method explained in detail in [18] (see Proposition 6.5 of [18] and the second remark after it). When $x \to 1$ we use the recursion $\psi(x+1) = \psi(x) + x^{-1}$ and the expansion of ψ near 0 again. □

Proposition 2. *The function* $\mathcal{F}_k(x)$ *satisfies the two functional equations*

$$\mathcal{F}_k(x) + (-x)^{k-2}\,\mathcal{F}_k\Big(\frac{1}{x}\Big) \;=\; A_k(x) + \zeta(k)\Big((-x)^{k-1} - \frac{1}{x}\Big) \tag{11}$$

and

$$\mathcal{F}_k(x) - \mathcal{F}_k(x+1) + (-x)^{k-2}\,\mathcal{F}_k\Big(\frac{x+1}{x}\Big) = B_k(x) + \zeta(k)\Big(\frac{(-x)^{k-1}}{x+1} - \frac{1}{x}\Big). \tag{12}$$

where $A_k(x)$ *and* $B_k(x)$ *are the polynomials of degree* $k-2$ *given by*

$$A_k(x) \;=\; -\sum_{r=1}^{k-1}\zeta(r)\zeta(k-r)(-x)^{r-1}, \qquad B_k(x) \;=\; -\sum_{r=1}^{k-1}\zeta(k-r, r)(-x)^{r-1}\,.$$

and related by

$$\begin{aligned} B_k(x) + (-x)^{k-2}B_k\Big(\frac{1}{x}\Big) \;&=\; A_k(x) + \zeta(k)\Big((-x)^{k-1} - \frac{1}{x}\Big)\,, \\ B_k(x) + B_k(x+1) \;&=\; (-x)^{k-2}A_k\left(\frac{x+1}{x}\right)\,. \end{aligned} \tag{13}$$

Proof. From $\psi(x) = \sum_{q=0}^{\infty}\Big(\frac{1}{1+q} - \frac{1}{x+q}\Big)$ we get $\frac{(-1)^k}{(k-1)!}\psi^{(k-1)}(x) = \sum_{q=0}^{\infty} \frac{1}{(x+q)^k}$ and hence

$$\frac{(-1)^k}{(k-1)!}\,\mathcal{F}_k^{(k-1)}(x) \;=\; \sum_{p\ge 1,\ q\ge 0} \frac{1}{(px+q)^k}\,.$$

From this we find that

$$\frac{(-1)^k}{(k-1)!}\Big(\mathcal{F}_k^{(k-1)}(x) - \frac{1}{x^k}\,\mathcal{F}_k^{(k-1)}\big(\tfrac{1}{x}\big)\Big)$$
$$= \Big(\sum_{p>0,\ q\ge 0} - \sum_{p\ge 0,\ q>0}\Big)\frac{1}{(px+q)^k} \;=\; \zeta(k)\Big(\frac{1}{x^k}-1\Big)$$

and

$$\frac{(-1)^k}{(k-1)!}\Big(\mathcal{F}_k^{(k-1)}(x) - \mathcal{F}_k^{(k-1)}(x+1) - \frac{1}{x^k}\,\mathcal{F}_k^{(k-1)}\Big(\frac{x+1}{x}\Big)\Big)$$
$$= \Big(\sum_{p>0,\ q\ge 0} - \sum_{q\ge p>0} - \sum_{p\ge q>0}\Big)\,\frac{1}{(px+q)^k} \;=\; \zeta(k)\Big(\frac{1}{x^k} - \frac{1}{(x+1)^k}\Big)\,.$$

Integrating these equations $k-1$ times[2] we obtain formulas (11) and (12) for *some* polynomials A_k and B_k of degree $k-2$, whose coefficients are then determined by the asymptotic expansions given in Proposition 1. The relations (13) between B_k and A_k are equivalent to the shuffle relations (10), and also follow easily from equations (11), (12) and (11) with x replaced by $1/x$. $\square$

Finally, we would like to mention that, although the series in (5) converges only polynomially quickly, we can compute values of the function $\mathcal{F}_k(x)$ (or its derivatives) to high accuracy with little effort. To this end, for integers $M, N \ge 1$ we define

$$\mathcal{F}_{k,N,M}(x) \;=\; \sum_{p=1}^{N}\frac{\psi(px)-\log(px)}{p^{k-1}} \;-\; \zeta'(k-2) \;+\; \zeta(k-1)\,\log x$$
$$+\; \sum_{m=1}^{M}\frac{\zeta(1-m)}{x^m}\bigg(\zeta(k+m-1) \;-\; \sum_{p=1}^{N}\frac{1}{p^{k+m-1}}\bigg)\;,$$

which can be computed in time $\mathrm{O}(MN)$, the calculation of the digamma and zeta functions being standard. Then

$$\Big|\,\mathcal{F}_k(x) - \mathcal{F}_{k,N,M}(x)\Big| \;=\; \bigg|\sum_{p=N+1}^{\infty}\frac{1}{p^{k-1}}\bigg(\psi(px)-\log(px) - \sum_{m=1}^{M}\frac{\zeta(1-m)}{(px)^m}\bigg)\bigg|$$
$$\le\; \sum_{p=N+1}^{\infty}\frac{1}{p^{k-1}}\,\frac{C_M}{(px)^{M+1}} \;<\; \frac{(k+M-1)\,C_M}{x^{M+1}\;N^{k+M-1}}$$

[2] Here we use Bol's identity to write $x^{-k}\,\mathcal{F}_k^{(k-1)}(c+1/x)$ as $(-d/dx)^{k-1}\big(x^{k-2}\,\mathcal{F}_k(c+1/x)\big)$.

with $C_M = |\,\zeta(1-M)| = \mathrm{O}\Big(M^{-\frac{1}{2}}(\frac{M}{2\pi e})^M\Big)$ by Stirling's formula. From this we find that for a given time $T = MN$ the best accuracy is $\mathrm{O}\big(T^{\frac{5}{4}-\frac{k}{2}}\,e^{-\sqrt{8\pi xT/e}}\big)$, achieved by choosing $eM \approx 2\pi xN$.

1.2 The functions $\mathcal{F}_k$ and polylogarithms

For $m,n \geq 1$ consider the function of $x > 0$

$$F_{m,n}(x) = (-x)^n \int_0^\infty \mathrm{Li}_m(e^{-xt})\ \mathrm{Li}_n(e^{-t})\,dt\ , \tag{14}$$

where $\mathrm{Li}_m(x) = \sum_{n=1}^\infty x^n/n^m$ is the mth polylogarithm function. Integrating by parts and using $x\,\mathrm{Li}'_m(x) = \mathrm{Li}_{m-1}(x)$, we obtain

$$\begin{aligned} F_{m+1,n}(x) &= -(-x)^n \int_0^\infty \mathrm{Li}_{m+1}(e^{-xt})\ d\,\mathrm{Li}_{n+1}(e^{-t}) \\ &= \ (-x)^n\,\zeta(m+1)\ \zeta(n+1)\ +\ F_{m,n+1}(x)\ . \end{aligned}$$

Hence if one fixes the sum $m+n$ then there is essentially one such function up to a polynomial. Let us set for $k \geq 3$

$$F_k(x)\ =\ F_{k-2,1}(x)\ .$$

Then $F_{m,n}(x) = F_k(x) - \sum_{r=2}^n \zeta(r)\,\zeta(k-r)(-x)^{r-1}$ with $k = m+n+1$. Substituting $t \to \frac{t}{x}$ in the integral (14) we get the equality

$$F_{m,n}(x) + (-x)^{n+m-1} F_{n,m}\Big(\frac{1}{x}\Big)\ =\ 0\ .$$

Expressing $F_{m,n}$ and $F_{n,m}$ in terms of the function F_k yields

$$F_k(x)\ +\ (-x)^{k-2} F_k\Big(\frac{1}{x}\Big)\ =\ \sum_{r=2}^{k-2} \zeta(r)\ \zeta(k-r)\,(-x)^{r-1}. \tag{15}$$

This functional equation is very similar to (11). And indeed, F_k is the same as our old $\mathcal{F}_k$ up to sign and the addition of simple functions:

Proposition 3. *For $k > 2$ one has*

$$F_k(x)\ =\ -\mathcal{F}_k(x)\ -\ \gamma\,\zeta(k-1)\ -\ \frac{\zeta(k)}{x}\ .$$

Proof. From $\psi(x) = \int_0^\infty \big(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\big)\,dt$ and $\gamma = \int_0^\infty \big(\frac{1}{1-e^{-t}}\ -\ \frac{1}{t}\big)e^{-t}\,dt$ ([11]) we get $\psi(x) + \gamma + \frac{1}{x} = \psi(x+1) + \gamma = \int_0^\infty \frac{1-e^{-xt}}{e^t-1}\,dt = -\int_0^\infty \big(1 - e^{-xt}\big)\,d\,\mathrm{Li}_1(e^{-t})$. Therefore

$$x \int_0^\infty e^{-xt}\ \mathrm{Li}_1(e^{-t})dt = \psi(x) + \gamma + \frac{1}{x}\ ,$$

so $F_k(x) = -x \int\limits_0^\infty \sum\limits_{p=1}^\infty \frac{e^{-xtp}}{p^{k-2}}\ \mathrm{Li}_1(e^{-t})dt = -\sum\limits_{p=1}^\infty \frac{\psi(px)}{p^{k-1}} - \gamma\,\zeta(k-1) - \frac{\zeta(k)}{x}$. $\square$

This proposition together with (15) gives an alternative proof of (11). Let us also sketch a similar proof of the second functional equation (12). The double polylogarithm functions ([4, 3]) are defined for $|z_i| < 1$ by the series

$$\mathrm{Li}_{m,n}(z_1, z_2) \;=\; \sum_{0<p<q} \frac{z_1^p}{p^m} \frac{z_2^q}{q^n}.$$

Let us consider again for $m, n \geq 1$ and $x > -1$

$$G_{m,n}(x) \;=\; (-x)^n \int_0^\infty \mathrm{Li}_{m,n}(e^{-xt}, e^{-t})\, dt\ .$$

This integral is also convergent for $m = 0$ and $n \geq 2$. Integrating for $m \geq 0$, $n \geq 1$ the equality

$$\frac{d}{dt}\, \mathrm{Li}_{m+1,n+1}(e^{-xt}, e^{-t}) \;=\; -x\, \mathrm{Li}_{m,n+1}(e^{-xt}, e^{-t}) - \mathrm{Li}_{m+1,n}(e^{-xt}, e^{-t})$$

we get the relation

$$G_{m+1,n}(x) \;=\; (-x)^n\, \zeta(n+1, m+1) \;+ G_{m,n+1}(x).$$

So, there is one such function $G_{m,n}$ up to a polynomial for a fixed sum $m + n$. We claim this function is again F_{m+n+1}. Let $k = m + n + 1$. Careful integration (paying attention to the singularity at $t = 0$) of the equality

$$\mathrm{Li}_{k-1,0}(e^{-xt}, e^{-t}) = \mathrm{Li}_{k-1}(e^{-(x+1)t})\, \mathrm{Li}_0(e^{-t})$$

yields $G_{k-2,1}(x) = F_k(x{+}1){+}\zeta(k{-}1, 1){+}\zeta(k)$. Hence $G_{m,n}(x) = F_k(x{+}1)$ $+\gamma\, \zeta(k-1) - \sum_{r=1}^{n} \zeta(r, k-r)(-x)^{r-1}$. Now expressing everything in the equality

$$\begin{aligned} F_{m,n}(x) \;&=\; \sum_{p,q\geq 1} (-x)^n \int_0^\infty \frac{e^{-t(px+q)}}{p^m\, q^n} dt = \sum_{p<q} \;+\; \sum_{q<p} \;+\; \sum_{p=q} \\ &=\; G_{m,n}(x) \;-\; (-x)^{m+n-1} G_{n,m}\left(\frac{1}{x}\right) \;+\; \zeta(m+n+1)\frac{(-x)^n}{x+1} \end{aligned}$$

in terms of F_k gives the functional equation

$$\begin{aligned} &F_k(x) \;-\; F_k(x+1) \;+\; (-x)^{k-2} F_k\Big(\frac{x+1}{x}\Big) \\ &\quad=\; \sum_{r=1}^{k-1} \zeta(k-r, r)(-x)^{r-1} \;+\; \frac{\zeta(k)}{x+1} \;-\; \gamma\, \zeta(k-1)(-x)^{k-2}\ , \end{aligned}$$

which, together with Proposition 3, yields (12).

1.3 The differential operator $\mathcal{D}_n$

Definition 2. *Let $\mathcal{D}_n$ for every integer $n \geq 0$ be the differential operator from functions of one variable to functions of two variables defined by*

$$(\mathcal{D}_n\, F)(x, y) \;=\; \sum_{i=0}^{n} \binom{2n-i}{n} \frac{F^{(i)}(x) \;-\; (-1)^i F^{(i)}(y)}{i!\; (y-x)^{n-i}}\ .$$

This can be written as $(\mathcal{D}_n\, F)\,(x,y) = (\mathcal{D}_n^+\, F)(x,y) + (-1)^{n+1} (\mathcal{D}_n^+\, F)(y,x)$, where

$$(\mathcal{D}_n^+\, F)(x,y) \;=\; \sum_{i=0}^{n} \binom{2n-i}{n} \frac{F^{(i)}(x)}{i!\,(y-x)^{n-i}}\;. \tag{16}$$

(The operator $\mathcal{D}_n^+$ will be also used later in the paper.) However, note that, unlike $\mathcal{D}_n\, F$, the function $\mathcal{D}_n^+\, F$ is not differentiable, because it has an nth order pole along the diagonal $x = y$, whereas, as we shall see, $(\mathcal{D}_n\, F)(x,y)$ and even $(\mathcal{D}_n\, F)(x,y)/(x-y)^{n+1}$ remain finite near $x = y$.

Proposition 4. *(i) For any matrix* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})$ *and any smooth function* F *we have*

$$\Big(\mathcal{D}_n^+ \Big[(cx+d)^{2n} F\Big(\frac{ax+b}{cx+d}\Big)\Big]\Big)(x,y) \;=\; (\det g)^n (\mathcal{D}_n^+\, F)\Big(\frac{ax+b}{cx+d}, \frac{ay+b}{cy+d}\Big)\,,$$

and similarly with $\mathcal{D}_n^+$ *replaced by* $\mathcal{D}_n$.

(ii) For any $T \in \mathbb{C}$ *we have*

$$(-1)^n \binom{2n}{n} \left(\frac{(T-x)(T-y)}{x-y}\right)^n = \sum_{m=0}^{2n} (-1)^m \binom{2n}{m} \mathcal{D}_n^+\,(x^m)\; T^{2n-m}\,.$$

(iii) For all integers $m \geq 0$ *we have*

$$\mathcal{D}_n\big(x^m\big) \;=\; (x-y)^{n+1} \sum_{r+s=m-1} \binom{r}{n}\binom{s}{n} x^{r-n} y^{s-n}\,. \tag{17}$$

In particular, $\mathcal{D}_n\big(P\big) = 0$ *for all polynomials* P *of degree* $\leq 2n$ *and*

$$\mathcal{D}_n\big(x^{2n+1}\big) \;=\; (x-y)^{n+1}/;. \tag{18}$$

(iv) For any T *we have*

$$\mathcal{D}_n\left(\frac{1}{T-x}\right) = \left(\frac{x-y}{(T-x)(T-y)}\right)^{n+1}\,.$$

(v) If F *is continuously differentiable* $2n+1$ *times between* x *and* y *then*

$$(\mathcal{D}_n F)(x,y) = \frac{1}{n!^2} \int_y^x \left(\frac{(x-t)(t-y)}{x-y}\right)^n F^{(2n+1)}(t)dt\,. \tag{19}$$

Proof. All of the parts of the proposition follow from the machinery of differential operators given in the appendix. (See the remark after Proposition 7.) Here we give more direct proofs.

(i) The formula

$$\mathcal{D}_n^+ = \frac{1}{n!}\left(\frac{\partial}{\partial x} + \frac{2}{y-x}\right)\left(\frac{\partial}{\partial x} + \frac{4}{y-x}\right)\cdots\left(\frac{\partial}{\partial x} + \frac{2n}{y-x}\right) \tag{20}$$

can be verified by induction. The intertwining property (i) for $\mathcal{D}_n^+$, and hence also for $\mathcal{D}_n$, follows immediately since the operator $\frac{\partial}{\partial x} + \frac{k}{y-x}$ intertwines between weights $-k$ and weight $2-k$ in the variable x.

(ii) From (16) one has obviously $\mathcal{D}_n^+(1) = \binom{2n}{n}(y-x)^{-n}$. Applying (i) with $F(x) = 1$ and $g = \begin{pmatrix} 0 & 1 \\ 1 & -T \end{pmatrix}$ we get

$$\mathcal{D}_n^+ \, (T-x)^{2n} = (-1)^n \binom{2n}{n} \left(\frac{(T-x)(T-y)}{x-y} \right)^n ,$$

which is equivalent to (ii).

(iii), (iv) We first prove the special case (18). Directly from the definition of $\mathcal{D}_n$ we find

$$\frac{n!^2}{(2n+1)!} (x-y)^{-n-1} \, \mathcal{D}_n(x^{2n+1}) = g_n\left(\frac{x}{x-y}\right) + g_n\left(\frac{-y}{x-y}\right)$$

with

$$g_n(u) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{u^{2n+1-i}}{2n+1-i} .$$

Since $g_n'(u) = u^n(1-u)^n$ is symmetric under $u \leftrightarrow 1-u$ the expression $g_n(u) + g_n(1-u)$ is constant. Its value is given by $g_n(0) + g_n(1) = 0 + \int_0^1 u^n(1-u)^n du = \dfrac{n!^2}{(2n+1)!}$ (beta integral). This gives (18). Now (iv) follows using (i) with $F(x) = x^{2n+1}$ and $g = \begin{pmatrix} 0 & 1 \\ 1 & -T \end{pmatrix}$. To get (iii) one lets $T \to \infty$ and expands both sides of (iv) as series in $1/T$.

Finally, for (v) we note that equation (19) can be rewritten as

$$(\mathcal{D}_n \, F)(x,y) \;=\; \frac{(x-y)^{n+1}}{n!^2} \int_0^1 t^n \, (1-t)^n \, F(tx + (1-t)y) \, dt$$

(proving the divisibility by $(x-y)^{n+1}$ mentioned above). This formula is true if F is a polynomial by (17) and the beta integral, and then in the general case by polynomial approximation. $\square$

1.4 The higher Kronecker limit formula

Proof of Theorem 1. The proof is almost immediate from Proposition 4 and the results of [13], where it was shown that the partial zeta function $\zeta(\mathcal{B}, s)$ has the decomposition

$$\zeta(\mathcal{B}, s) = D^{-s/2} \sum_{w \in \mathrm{Red}(\mathcal{B})} Z_s(w, w') , \tag{21}$$

where $Z_s(x,y) = \sum_{p>0,\, q\geq 0} \left(\dfrac{x-y}{(px+q)(py+q)} \right)^s$. In [13], the function $P(w,w')$ from the Kronecker limit formula (1) was obtained as the limiting value of $Z_s(w,w')$ after its pole has been removed. Here there is no pole and we can simply set $s = k$. By part (iv) of Proposition 4,

$$\left(\frac{x-y}{(px+q)(py+q)} \right)^k = -\, \mathcal{D}_{k-1}\Big(\frac{1}{p^{2k-1}(px+q)} \Big) ,$$

and since

$$\mathcal{F}_{2k}(x) = \sum_{p>0, q\geq 0} \frac{1}{p^{2k-1}} \left(\frac{1}{1+q} - \frac{1}{px+q} \right)$$

we find that $Z_k(x,y) = (\mathcal{D}_{k-1}\,\mathcal{F}_{2k})\,(x,y)$. $\square$

We illustrate the theorem with a numerical example. The field $K = \mathbb{Q}(\sqrt{3})$ has two narrow classes, $\mathcal{B}_0$ and $\mathcal{B}_1$, with the corresponding sets of reduced quadratic irrationalities being

$$\mathrm{Red}(\mathcal{B}_0) = \left\{ 2+\sqrt{3} \right\}, \qquad \mathrm{Red}(\mathcal{B}_1) = \left\{ 1+\frac{1}{\sqrt{3}}\,,\ \frac{3+\sqrt{3}}{2} \right\}.$$

Recall that we can compute the values of $\mathcal{F}_k$ and its derivatives with high accuracy. In the table below we give the values involved in the higher KLF for $\zeta(\mathcal{B}_i, 2)$.

x	$\mathcal{F}_4(x)$	$\mathcal{F}_4'(x)$
$2+\sqrt{3}$	$1.6300186927097819021\ldots$	$0.3642242838863980846\ldots$
$2-\sqrt{3}$	$-4.1957798770335434426\ldots$	$16.6663730167640755318\ldots$
$1+1/\sqrt{3}$	$0.3693130036419609196\ldots$	$1.0208418261452850428\ldots$
$1-1/\sqrt{3}$	$-2.4934462840079526270\ldots$	$7.3791974289140308912\ldots$
$(3+\sqrt{3})/2$	$0.9894426455775365625\ldots$	$0.6173597187715989196\ldots$
$(3-\sqrt{3})/2$	$-1.3887304669486036967\ldots$	$3.7664407174461445334\ldots$

With this table we compute the zeta values at $k=2$ by Theorem 1:

$$\zeta(\mathcal{B}_0, 2) = 1.1389225773470523300\ldots$$

$$\zeta(\mathcal{B}_1, 2) = 0.4232764484862273545\ldots$$

Similarly, to compute $\zeta(\mathcal{B}_i, 3)$ we need the following values:

x	$\mathcal{F}_6(x)$	$\mathcal{F}_6'(x)$	$\mathcal{F}_6''(x)$
$2+\sqrt{3}$	$1.2518745037778042\ldots$	$0.3175540356781837\ldots$	$-0.0965606430551642\ldots$
$2-\sqrt{3}$	$-3.9986990011103493\ldots$	$15.4134269703386953\ldots$	$-107.1707338224315422\ldots$
$1+1/\sqrt{3}$	$0.1460504335512779\ldots$	$0.9019022568593455\ldots$	$-0.7493154998661828\ldots$
$1-1/\sqrt{3}$	$-2.4317204973004163\ldots$	$6.7529054090681804\ldots$	$-27.9768290236134814\ldots$
$(3+\sqrt{3})/2$	$0.6918347553446228\ldots$	$0.5414072943018742\ldots$	$-0.2773058112199031\ldots$
$(3-\sqrt{3})/2$	$-1.4261182127652868\ldots$	$3.4079868563585469\ldots$	$-8.6985660821795807\ldots$

Therefore

$$\zeta(\mathcal{B}_0, 3) = 1.0233279526833285\ldots$$

$$\zeta(\mathcal{B}_1, 3) = 0.1667564870865704\ldots$$

With the zeta values computed above one can check numerically that

$$\zeta(\mathcal{B}_0, 2) + \zeta(\mathcal{B}_1, 2) = \frac{\pi^4}{18\sqrt{12}} \tag{22}$$

and

$$\zeta(\mathcal{B}_0, 3) - \zeta(\mathcal{B}_1, 3) = \frac{\pi^6}{324\sqrt{12}}\,. \tag{23}$$

One sees that the combinations $\zeta(\mathcal{B}, k) + (-1)^k\,\zeta(\mathcal{B}^*, k)$ in both cases belong to $\pi^{2k}\sqrt{D}\,\mathbb{Q}$. This statement is true in general and is equivalent to the Siegel-Klingen theorem by the functional equations for L-functions. We now show how to deduce explicit formulas for such rational combinations from our higher KLF.

1.5 Rational zeta values

Let us briefly recall the relation between ideal classes and continued fractions given in [13]. Any real quadratic irrationality w has a continued fraction expansion

$$w = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots}}$$

with $b_i \in \mathbb{Z}$ (all i), $b_i \geq 2$ $(i \neq 1)$ and $\{b_i\}$ eventually periodic. The condition of being a root of the quadratic form in a given class $\mathcal{B}$ fixes the "tail" of $\{b_i\}$. The condition of being a larger root of a reduced quadratic form, i.e.

$$w > 1, \quad 1 > w' > 0, \tag{24}$$

is equivalent to $\{b_i\}$ being purely periodic. Let us denote by $(b_1, \ldots, b_l)$ the period of the continued fraction expansion. It is actually defined up to a cyclic shift, so we will say it is a cycle of integers. Then the numbers in $\mathrm{Red}(\mathcal{B})$ are exactly

$$w_i = b_i - \cfrac{1}{b_{i+1} - \cfrac{1}{\ddots}}\,, \quad i \in \mathbb{Z}/l\mathbb{Z},$$

where b_i with $i \in \mathbb{Z}$ stands for $b_{i \pmod{l}}$.

Wide ideal classes $\mathcal{A} \in Cl(K)$ correspond to $\mathrm{GL}(2,\mathbb{Z})$ orbits on the space of integer quadratic forms of discriminant D, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z})$ acts on $\mathcal{Q}^+$ by $Q(X,Y) \mapsto \pm\, Q(aX+bY, cX+dY)$ if $ad - bc = \pm 1$. The quadratic form $Q = AX^2 + BXY + CY^2$ is said to be reduced in the wide sense if $A > 0$, $C < 0$ and $|A + C| < -B$ or, equivalently, when its root $x = \frac{-B+\sqrt{D}}{2A}$ satisfies

$$x > 1, \quad 0 > x' > -1. \tag{25}$$

As in the case of narrow classes, the condition of being a root of the quadratic form in the class $\mathcal{A}$ fixes the "tail" of $\{a_i\}$ in the ordinary continued fraction expansion

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}, \qquad a_i \in \mathbb{Z}, \;\; a_i \geq 1 \; (i \neq 1),$$

while the condition (25) is equivalent to $\{a_i\}$ being purely periodic. Thus to $\mathcal{A}$ there corresponds a cycle of integers $(a_1, \dots, a_m)$ with $a_i \geq 1$ and a cycle of numbers

$$x_i = a_i + \cfrac{1}{a_{i+1} + \cfrac{1}{\ddots}}\,, \quad i \in \mathbb{Z}/m\mathbb{Z},$$

just in the same way as $(b_1, \dots, b_l)$ with $b_i \geq 2$ and $w_1, \dots, w_l$ were assigned to a narrow ideal class. Further, since x_1 satisfies (25) then $w_1 = 1 + x_1$ satisfies (24) and $w_1 > 2$, so w_1 has a purely periodic "minus" continued fraction $w_1 = b_1 - \frac{1}{b_2 - \dots}$ with $b_1 = a_1 + 2 \geq 3$. The narrow ideal class $\mathcal{B}$ corresponding to w_1 then lies in the wide class $\mathcal{A}$ corresponding to x_1 (since both contain the ideal $\mathbb{Z}x_1 + \mathbb{Z} = \mathbb{Z}w_1 + \mathbb{Z}$) and we can reproduce all the numbers $(b_1, \dots, b_l)$ as follows. We consider the minimal *even* period $(a_1, \dots, a_{2r})$, where $r = m$ if m is odd and $r = m/2$ if m is even. The b's and a's are related by

$$\begin{aligned} b_1 &= a_1 + 2, & b_2 &= \dots = b_{a_2} = 2, \\ b_{a_2+1} &= a_3 + 2, & b_{a_2+2} &= \dots = b_{a_2+a_4} = 2, \\ &\vdots \\ b_{l-a_{2r}} &= a_{2r-1} + 2, & b_{l-a_{2r}+1} &= \dots = b_l = 2, \end{aligned} \tag{26}$$

i.e.

$$(b_1, \dots, b_l) \;=\; (\,a_1 + 2, \underbrace{2, \dots, 2}_{a_2 - 1}, a_3 + 2, \underbrace{2, \dots, 2}_{a_4 - 1}, \,\dots,\, a_{2r-1} + 2, \underbrace{2, \dots, 2}_{a_{2r} - 1}\,)\,.$$

In particular, $l(\mathcal{B}) = a_2 + a_4 + \dots + a_{2r}$. If we started with the cyclic shift $(a_2, a_3, \dots, a_{2r}, a_1)$, we would get instead the cycle

$$(\,a_2 + 2, \underbrace{2, \dots, 2}_{a_3 - 1}, \,\dots,\, a_{2r} + 2, \underbrace{2, \dots, 2}_{a_1 - 1}\,)\,.$$

This cycle corresponds to the conjugate narrow class $\mathcal{B}^*$ and $l(\mathcal{B}^*) = a_1 + a_3 + \dots + a_{2r-1}$. In particular, we have $l(\mathcal{B}) + l(\mathcal{B}^*) = \sum_{i=0}^{2r} a_i$. Notice that if the minimal period length m is odd, then $\mathcal{B} = \mathcal{B}^*$; this happens if and only if the fundamental unit of K has negative norm.

Let us denote by $\mathrm{Red}_{\mathrm{w}}(\mathcal{B})$ for a narrow ideal class $\mathcal{B}$ the set of those larger roots of the quadratic forms in $\mathcal{B}$ reduced in the wide sense. (Equivalently, $\mathrm{Red}_{\mathrm{w}}(\mathcal{B})$ consists of the numbers $w - 1$ where $w \in \mathrm{Red}(\mathcal{B})$, $w > 2$.) Then in the notations above we have $\mathrm{Red}_{\mathrm{w}}(\mathcal{B}) = \{x_1, x_3, \dots\}$ and $\mathrm{Red}_{\mathrm{w}}(\mathcal{B}^*) = \{x_2, x_4, \dots\}$. We will also write $Q \in \mathrm{Red}_{\mathrm{w}}(\mathcal{B})$ for such a form.

Lemma 1. *Let $\mathcal{B} \mapsto \mathcal{I}(\mathcal{B})$ be an invariant of narrow ideal classes defined by*

$$\mathcal{I}(\mathcal{B}) = \sum_{w \in \mathrm{Red}(\mathcal{B})} F(w, w') \tag{27}$$

for some function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. Suppose F has the form

$$F(x,y) \;=\; G(x-1,y-1) \;-\; G\Big(\frac{x-1}{x},\frac{y-1}{y}\Big) \tag{28}$$

for some function G. Then

(i) For any narrow ideal class $\mathcal{B}$ we have

$$\mathcal{I}(\mathcal{B}) \;=\; \sum_{x\in \mathrm{Red_w}(\mathcal{B})} G(x,x') \quad-\quad \sum_{x\in \mathrm{Red_w}(\mathcal{B}^*)} G\Big(\frac{1}{x}\,,\,\frac{1}{x'}\Big).$$

(ii) If $G(x,y)$ has the form $H_1(x-y)+H_2\big(\frac{1}{x}-\frac{1}{y}\big)$ for some functions H_1, $H_2 : \mathbb{R} \to \mathbb{C}$, then $\mathcal{I}(\mathcal{B})=0$ for all classes $\mathcal{B}$.

Proof. From (26) we have

$$w_1 = x_1+1 \;=\; a_1+2-\frac{1}{w_2}\,,$$

$$w_2 \;=\; 2-\frac{1}{w_3}\,,\quad \ldots\,,\quad w_{a_2} \;=\; 2-\frac{1}{w_{a_2+1}}\,,$$

$$w_{a_2+1} \;=\; x_3+1 \;=\; a_3+2-\frac{1}{w_{a_2+2}}$$

and therefore

$$\begin{aligned}\sum_{j=2}^{a_2+1} F(w_j,w_j') \;&=\; \sum_{j=2}^{a_2+1}\Big[G(w_j-1,w_j'-1)-G\Big(1-\frac{1}{w_j},1-\frac{1}{w_j'}\Big)\Big]\\ &=\; G(w_{a_2+1}-1,w_{a_2+1}'-1)-G\Big(1-\frac{1}{w_2},1-\frac{1}{w_2'}\Big)\\ &=\; G(x_3,x_3')-G\Big(\frac{1}{x_2}\,,\,\frac{1}{x_2'}\Big)\,.\end{aligned} \tag{29}$$

Summing this cyclically gives statement (i). For statement (ii) we observe that if G has the form given then $F(x,y)=H_1(x-y)-H_1(-1/x+1/y)$. Now from $w_i-w_i' = -1/w_{i+1}+1/w_{i+1}'$ it follows that $\sum_{i \bmod l} F(w_i,w_i') = 0$. ☐

Corollary. *Let the notations be as above. Then*

$$\mathcal{I}(\mathcal{B}) \;\pm\; \mathcal{I}(\mathcal{B}^*) \;=\; \Bigg(\sum_{x\in \mathrm{Red_w}(\mathcal{B})} \;\pm\; \sum_{x\in \mathrm{Red_w}(\mathcal{B}^*)}\Bigg)\, G^{\mp}(x,x')$$

with $G^{\mp}(x,y)=G(x,y)\mp G\big(\frac{1}{x},\frac{1}{y}\big)$.

This corollary is useful because in the application to the following theorem the relevant function G will be transcendental but either G^+ or G^- (depending on the parity of k) will be a rational function.

Theorem 2. *For every narrow ideal class $\mathcal{B}$ and integer $k \geq 2$,*

$$D^{\frac{k}{2}}\left(\zeta(\mathcal{B},k)+(-1)^k\,\zeta(\mathcal{B}^*,k)\right) \;=\; \left(\sum_{x\in\mathrm{Red_w}(\mathcal{B})} \;+\; (-1)^k \sum_{x\in\mathrm{Red_w}(\mathcal{B}^*)}\right) W_k(x,x'),$$

where $W_k(x,y) = \mathcal{D}_{k-1}\Big(-\sum_{r=0}^{k}\zeta(2r)\,\zeta(2k-2r)|x|^{2r-1}\Big)$.

Remark. In the definition of W_k, the absolute value signs are necessary, since $\mathcal{D}_{k-1}$ kills polynomials of degree $\leq 2k-2$, but the restriction of W_k to $\{x > 0 > y\}$, which is all that is used in the theorem, is a rational function (in fact, a polynomial in $x^{\pm 1}$, $y^{\pm 1}$ and $(x-y)^{-1}$).

Proof. We define two functions of one variable by

$$V_0(x) \;=\; -\,x^{2k-2}\,\mathcal{F}_{2k}(1/x) \;-\; \frac{\zeta(2k)}{4}\Big(\frac{1}{x} + x^{2k-1}\Big) \qquad (x>0)$$

and

$$V(x) \;=\; V_0(|x|) \;-\; \frac{\zeta(2k)}{2}\Big(\frac{1}{x} + x^{2k-1}\Big) \qquad (x \in \mathbb{R} \smallsetminus \{0\}).$$

From Proposition 2 we get, after a straightforward but lengthy computation (one has to distinguish the three cases $x > 1$, $0 < x < 1$ and $x < 0$ and to use the functional equations (11)–(13) repeatedly),

$$\mathcal{F}_{2k}(x) \;=\; V(x-1) \;-\; V\Big(\frac{x-1}{x}\Big) \;+\; \big(\text{polynomial of degree} \leq 2k-2\big)$$

and hence, since $\mathcal{D}_{k-1}$ is equivariant and kills polynomials of low degree (parts (i) and (iii) of Proposition 4),

$$\big(\mathcal{D}_{k-1}\,\mathcal{F}_{2k}\big)(x,y) \;=\; \big(\mathcal{D}_{k-1}\,V\big)(x-1,y-1) \;-\; \big(\mathcal{D}_{k-1}\,V\big)\Big(\frac{x-1}{x},\,\frac{y-1}{y}\Big)$$

for $x > 1 > y > 0$. Therefore, by Theorem 1, the conditions of the lemma above are satisfied for $\mathcal{I}(\mathcal{B}) = D^{k/2}\,\zeta(k,\mathcal{B})$ and $G = \mathcal{D}_{k-1}\,V$. Moreover, since $\mathcal{D}_{k-1}\big(x^{2k-1}\big) = (x-y)^k$ and $\mathcal{D}_{k-1}\big(1/x\big) = -\big(-1/x + 1/y\big)^k$ by Proposition 4, part (ii) of Lemma 1 tells us that we can replace G by $G_0(x,y) = \mathcal{D}_{k-1}\big(V_0(|x|)\big)$. Finally, from the functional equation (11) we get

$$\begin{aligned} V_0\big(|x|\big) \;+\; x^{2k-2}\,V_0\big(1/|x|\big) \;&=\; -A_{2k}(|x|) \;+\; \frac{\zeta(2k)}{2}\big(1/|x| + |x|^{2k-1}\big) \\ &=\; -\sum_{r=0}^{k}\zeta(2r)\,\zeta(2k-2r)|x|^{2r-1} \;+\; \big(\text{polynomial of degree} \leq 2k-2\big)\,, \end{aligned}$$

so $G_0(x,y) + (-1)^{k-1}G_0(1/x,1/y) = W_k(x,y)$. The theorem now follows immediately from the corollary to Lemma 1. $\square$

We illustrate the theorem (for $k = 2$ and $k = 3$) using the example from the previous section. We have $\mathrm{Red_w}(\mathcal{B}_0) = \{1+\sqrt{3}\}$, $\mathcal{B}_0^* = \mathcal{B}_1$, $\mathrm{Red_w}(\mathcal{B}_1) = \{(1+\sqrt{3})/2\}$. One can easily compute

$$W_2(x,y) \;=\; \frac{\pi^4}{180}\,\frac{x+y}{x^2y^2(x-y)}\,\big((x^2y^2+1)(x^2-4xy+y^2)+10x^2y^2\big)\,,$$

for $x > y > 0$, so $\zeta(\mathcal{B}_0,2)+\zeta(\mathcal{B}_1,2)$ equals

$$\frac{1}{12}\Big(W_2(1+\sqrt{3},1-\sqrt{3})+W_2\Big(\frac{1+\sqrt{3}}{2},\frac{1-\sqrt{3}}{2}\Big)\Big) \;=\; \frac{\pi^4}{18\sqrt{12}}\,,$$

in accordance with our numerical computation (22). Similarly, using the rather long formula

$$\begin{aligned} W_3(x,y) = \frac{\pi^6}{1890}\,\frac{x+y}{x^3y^3(x-y)^2}\big(&(x^3y^3+1)(x^4-6x^3y+16x^2y^2-6xy^3+y^4)\\ &-\,21\,(xy+1)\,x^3y^3\big) \qquad (x>y>0), \end{aligned}$$

we find that $\zeta(\mathcal{B}_0,3)-\zeta(\mathcal{B}_1,3)$ equals

$$\frac{1}{12\sqrt{12}}\Big(W_3(1+\sqrt{3},1-\sqrt{3})-W_3\Big(\frac{1+\sqrt{3}}{2},\frac{1-\sqrt{3}}{2}\Big)\Big) \;=\; \frac{\pi^6}{324\sqrt{12}}\,,$$

in agreement with (23).

We end by discussing the relation of Theorem 2 with the Klingen-Siegel theorem and with the functional equations of $\zeta(\mathcal{B},s)\pm\zeta(\mathcal{B}^*,s)$. Consider, for $k \geq 2$, the rational function

$$\mathcal{P}_k(x) \;=\; \sum_{r=0}^{2k}\frac{B_r\,B_{2k-r}}{r!\,(2k-r)!}\,x^{r-1} \;=\; \frac{4}{(2\pi i)^{2k}}\sum_{r=0}^{k}\zeta(2r)\;\zeta(2k-2r)\,x^{2r-1}\,,$$

where B_r are the Bernoulli numbers. This function appeared in the literature (e.g. in [15]) as a generalized period polynomial associated to the Eisenstein series of weight $2k$. It satisfies the relations

$$\mathcal{P}_k(x) = x^{2k-2}\,\mathcal{P}_k\left(\frac{1}{x}\right),\quad \mathcal{P}_k(x)-\mathcal{P}_k(x-1)+x^{2k-2}\,\mathcal{P}_k\left(\frac{x-1}{x}\right) = 0. \tag{30}$$

For $k \geq 2$ define $\widetilde{W}_k(x,y) \in \mathbb{Q}\big[x^{\pm1},y^{\pm1},(x-y)^{-1}\big]$ by

$$\widetilde{W}_k(x,y) \;=\; (-1)^{k-1}\frac{(k-1)!^2}{2}\,\Big((\mathcal{D}^+_{k-1}\,\mathcal{P}_k)(x,y)+(-1)^{k-1}\,(\mathcal{D}^+_{k-1}\,\mathcal{P}_k)(y,x)\Big)\,.$$

The function $\widetilde{W}_k$ is related to the function W_k of Theorem 2 by

$$W_k(x,y) = \frac{(2\pi)^{2k}}{2\,(k-1)!^2}\widetilde{W}_k(x,y) \quad \text{if } x>0>y\,. \tag{31}$$

Moreover, since $\widetilde{W}_k$ is $(-1)^{k-1}$-symmetric in x and y, the number defined by

$$\widetilde{W}_k(Q) := D^{\frac{k-1}{2}}\,\widetilde{W}_k\left(\frac{-B+\sqrt{D}}{2A},\frac{-B-\sqrt{D}}{2A}\right)$$

for an integer quadratic form $Q = AX^2 + BXY + CY^2$ is rational. Note that the normalization here is different from the one in (8), reflecting the different symmetry. Now we can formulate the following

Corollary. *For any narrow ideal class $\mathcal{B}$ and integer $k \geq 2$,*

$$\zeta(\mathcal{B}, 1-k) = \left(\sum_{Q \in \mathrm{Red_w}(\mathcal{B})} + (-1)^k \sum_{Q \in \mathrm{Red_w}(\mathcal{B}^*)} \right) \widetilde{W}_k(Q) \;=\; \sum_{Q \in \mathrm{Red}(\mathcal{B})} \widetilde{W}_k(Q)\,.$$

Proof. From the functional equations

$$\pi^{-s}\,\Gamma\big(\frac{s+\varepsilon}{2}\big)^2\, D^{+s/2} \Big(\zeta(\mathcal{B}, s) \;+\; (-1)^{\varepsilon}\, \zeta(\mathcal{B}^*, s) \Big) \;=\; \text{same with } s \mapsto 1-s \tag{32}$$

($\varepsilon = 0,\, 1$) we have $\zeta(\mathcal{B}, 1-k) = (-1)^k\, \zeta(\mathcal{B}^*, 1-k)$ and

$$D^{k/2} \Big(\zeta(\mathcal{B}, k) \;+\; (-1)^k\, \zeta(\mathcal{B}^*, k) \Big) \;=\; \frac{(2\pi)^{2k}}{4(k-1)!^2}\, D^{(1-k)/2} \;\times$$

$$\times \Big(\zeta(\mathcal{B}, 1-k) \;+\; (-1)^k\, \zeta(\mathcal{B}^*, 1-k) \Big) \;=\; \frac{(2\pi)^{2k}}{2(k-1)!^2}\, D^{(1-k)/2}\, \zeta(\mathcal{B}, 1-k).$$

Combining the latter with Theorem 2 and using (31) give

$$\zeta(\mathcal{B}, 1-k) \;=\; D^{(k-1)/2} \left(\sum_{x \in \mathrm{Red_w}(\mathcal{B})} + (-1)^k \sum_{x \in \mathrm{Red_w}(\mathcal{B}^*)} \right) \widetilde{W}_k(x, x'),$$

what is exactly the second statement of the corollary. Further, using the equivariance property of $\mathcal{D}_{k-1}$ (Proposition 4, (i)) we get from (30)

$$\widetilde{W}_k(x, y) \;=\; (-1)^{k-1} \widetilde{W}_k\Big(\frac{1}{x}\,,\,\frac{1}{y}\Big)\,,$$

$$\widetilde{W}_k(x, y) \;-\; \widetilde{W}_k(x-1, y-1) \;+\; \widetilde{W}_k\Big(\frac{x-1}{x}\;,\;\frac{y-1}{y}\Big) \;=\; 0\,.$$

From Lemma 1 and its corollary with $G(x, y) = \widetilde{W}_k(x, y)$ it follows that

$$\mathcal{I}(\mathcal{B}) \;:=\; \sum_{w \in Red(\mathcal{B})} \widetilde{W}_k(w, w') \;=\; (-1)^k\, \mathcal{I}(\mathcal{B}^*)$$

$$=\; \left(\sum_{x \in \mathrm{Red_w}(\mathcal{B})} \;+\; (-1)^k \sum_{x \in \mathrm{Red_w}(\mathcal{B}^*)} \right) \widetilde{W}_k(x, x'),$$

since $\widetilde{W}_k^{\pm} = 2\,\widetilde{W}_k$ when $\pm 1 = (-1)^{k-1}$. Therefore $\zeta(\mathcal{B}, 1-k) = D^{(k-1)/2}\, \mathcal{I}(\mathcal{B}) = \sum_{Q \in Red(\mathcal{B})} \widetilde{W}_k(Q)$. $\square$

Let us compute the rational values $\zeta(\mathcal{B}, 1-k)$ explicitly. If we define homogenous polynomials $d_{r,n}(A, B, C)$ and $f_n(A, B, C)$ by

$$(AX^2 + BXY + CY^2)^n = \sum_{r=0}^{2n} d_{r,n}(A, B, C) X^{2n-r} (-Y)^r,$$

$$f_n(A,B,C) = \sum_{r=0}^{n} (-1)^r \frac{(n-r)!}{r!(2n+1-2r)!} A^r B^{2n+1-2r} C^r,$$

then for $Q = AX^2 + BXY + CY^2$ we have

$$D^{n/2}\,\mathcal{D}_n^+(x^r)\left(\frac{-B+\sqrt{D}}{2A}, \frac{-B-\sqrt{D}}{2A}\right) \;=\; (-1)^n \frac{r!(2n-r)!}{n!^2}\, d_{r,n}(A,B,C)\,,$$

$$D^{n/2}\,\mathcal{D}_n\left(\frac{1}{|x|}\right)\left(\frac{-B+\sqrt{D}}{2A}, \frac{-B-\sqrt{D}}{2A}\right) \;=\; -\frac{(2n+1)!}{n!}\,\frac{f_n(A,B,C)}{C^{n+1}}$$

and hence

$$\begin{aligned} \widetilde{W}_k(Q) \;&=\; \sum_{r=1}^{2k-1} \frac{B_r B_{2k-r}}{r(2k-r)} d_{r-1,k-1}(A,B,C) \\ &\quad + \frac{(-1)^k (k-1)! B_{2k}}{4k} \left(\frac{f_{k-1}(A,B,C)}{C^k} + \frac{f_{k-1}(C,B,A)}{A^k}\right). \end{aligned} \tag{33}$$

Example. Let us once more go back to the example from the previous section and compute $\zeta(\mathcal{B}_0, 1-k)$ and $\zeta(\mathcal{B}_1, 1-k)$ for $k = 2, 3$. From (33)

$$\widetilde{W}_2(Q) = -\frac{B}{144}\left(1 + \frac{1}{10}\Big(\frac{B^2}{C^2} + \frac{B^2}{A^2}\Big) - \frac{3}{5}\Big(\frac{A}{C} + \frac{C}{A}\Big)\right).$$

In $\mathcal{B}_0$ there is only one reduced quadratic form $Q_0 = X^2 - 4XY + Y^2$ and in $\mathcal{B}_1$ there are two reduced forms $Q_1 = 3X^2 - 6XY + 2Y^2$ and $Q_2 = 2X^2 - 6XY + 3Y^2$. Therefore

$$\begin{aligned} \zeta(\mathcal{B}_0, -1) \;&=\; \widetilde{W}_2(Q_0) \;=\; \frac{1}{12}\,, \\ \zeta(\mathcal{B}_1, -1) \;&=\; \widetilde{W}_2(Q_1) \,+\, \widetilde{W}_2(Q_2) \;=\; \frac{1}{24} \,+\, \frac{1}{24} \;=\; \frac{1}{12}\,. \end{aligned}$$

For the Dedekind zeta function we then have $\zeta_{\mathbb{Q}(\sqrt{3})}(-1) = 1/6$. When $k = 3$

$$\widetilde{W}_3(Q) \;=\; \frac{B(A+C)}{720} - \frac{1}{252}\Big(\frac{B^5}{60} - \frac{AB^3C}{6} + \frac{A^2BC^2}{2}\Big)\Big(\frac{1}{A^3} + \frac{1}{C^3}\Big)$$

and

$$\begin{aligned} \zeta(\mathcal{B}_0, -2) \;&=\; \widetilde{W}_3(Q_0) \;=\; \frac{1}{18}\,, \\ \zeta(\mathcal{B}_1, -2) \;&=\; \widetilde{W}_3(Q_1) \,+\, \widetilde{W}_3(Q_2) \;=\; -\frac{1}{36} \,-\, \frac{1}{36} \;=\; -\frac{1}{18}\,, \end{aligned}$$

in accordance with the fact that the Dedekind zeta function $\zeta_{\mathbb{Q}(\sqrt{3})}(s)$ has zero (of the second order) at $s = -2$.

Remark. The right-hand side of (33) already appeared in [14]. To formulate the statement given there, let us first rewrite the decomposition (21) as

$$\zeta(\mathcal{B}, s) \;=\; \sum_{Q \in Red(\mathcal{B})} Z_Q(s) \qquad \text{for} \qquad \mathrm{Re}(s) > 1, \tag{34}$$

where

$$Z_Q(s) \;=\; D^{-s/2} Z_s\left(\frac{-B+\sqrt{D}}{2A}, \frac{-B-\sqrt{D}}{2A}\right).$$

Theorem 2 from [14] states that $Z_Q(s)$ can be continued to a meromorphic function on the whole of $\mathbb{C}$ with its only pole at $s = 1$ and, in our notations [3], one has

$$Z_Q(1-k) = \widetilde{W}_k(Q)\,.$$

One of the statement of our corollary follows from this and we now see that the individual terms $\widetilde{W}_k(Q)$ come from analytic continuation of zeta functions of reduced quadratic forms $Z_Q(s)$ in the decomposition (34). Another way to say this is that the results of this paper and the results of [14] together give a proof of the functional equations (32) as a consequence of functional equations for the "cone zeta functions" $Z_Q(s)$ at integer arguments. It would be interesting to see whether one can give a proof of the functional equations for arbitrary s in the same way (i.e., by writing the difference of the right and left sides of (32) as an invariant of the form $I(\mathcal{B}) = \sum F(w, w')$ and showing that F has the form required in Lemma 1 to force $I \equiv 0$), but we did not succeed in doing this.

2 Homological aspects

One may notice that the values $\mathcal{D}_{k-1}(\mathcal{F}_{2k})(w, w')$ in the higher KLF depend only on the $(2k-1)$st derivative of $\mathcal{F}_{2k}$ (see (19)). In this section we construct a cocycle class using this derivative and represent our formula as a homological pairing.

2.1 Functional equations and cohomology of modular group

Recall that the modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ is freely generated by the two elements

$$S = \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\; U = \pm\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

of orders 2 and 3 correspondingly. A 1-cocycle on Γ with coefficients in some Γ-module M is a function $\phi : \Gamma \to M$ satisfying the relation

$$\phi(gh) = g\phi(h) + \phi(g) \quad \text{for any } g, h \in \Gamma. \tag{35}$$

Hence, if $m_1 = \phi(S)$ and $m_2 = \phi(U)$ then

$$\begin{aligned} (1+S)m_1 &= 0, \\ (1+U+U^2)m_2 &= 0. \end{aligned} \tag{36}$$

And conversely, as soon as the two elements $m_1, m_2 \in M$ are given such that the conditions (36) are satisfied, the map $\phi(S) = m_1$ and $\phi(U) = m_2$ can be uniquely extended to a 1-cocycle on Γ by (35).

[3] Our notations slightly differ from [14], namely our polynomial $d_{r,n}$ is $d_{2n-r,n}$ there and our Z_Q is $Z_{\widetilde{Q}}$ where $\widetilde{Q} = CX^2 - BXY + AY^2$.

Let us now take for M the space of functions on $\mathbb{P}^1(\mathbb{R})$ with the action of Γ in weight $2k$. Then a 1-cocycle ϕ is the same as a pair of functions $F = \phi(S)$ and $G = \phi(U)$ satisfying

$$F(x) + \frac{1}{x^{2k}} F\left(-\frac{1}{x}\right) = 0, \tag{37}$$

$$G(x) + \frac{1}{(1-x)^{2k}} G\left(\frac{1}{1-x}\right) + \frac{1}{x^{2k}} G\left(\frac{x-1}{x}\right) = 0.$$

Consider $T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $U = TS$ and therefore $G(x) = F(x - 1) + \phi(T)(x)$. Using this and the first of above equations we rewrite the second one as

$$F(x) - F(x+1) + \frac{1}{x^{2k}} F\left(-\frac{x+1}{x}\right) = B(x), \tag{38}$$

where $B = -T^{-1}(1 + U + U^2)\phi(T)$. Equations (37) and (38) remind us correspondingly of equations (11) and (12) for the function $\mathcal{F}_{2k}$. And indeed, one can observe that the proof of Proposition 2 is based on the fact that similar equations are satisfied by the $(k-1)$st derivative of $\mathcal{F}_k$. Consider the function

$$\psi_{2k}(x) = \operatorname{sign}(x) \sum_{p,\, q \geq 0}^{*} \frac{1}{(p\,|x| + q)^k},$$

where here and below "$*$" on the summation sign means that we omit the term with $p = q = 0$ and take the "boundary" terms (where either $p = 0$ or $q = 0$) with the coefficient $\frac{1}{2}$. For positive arguments $\psi_{2k}(x)$ equals $\frac{1}{(2k-1)!}\left(\frac{d}{dx}\right)^{2k-1}\left(\mathcal{F}_{2k}(x) + \frac{\zeta(2k)}{2}\left(\frac{1}{x} + x^{2k-1}\right)\right)$. This function satisfies (37) and (38) with $B \equiv 0$. Therefore we have the following

Lemma 2. *For every $k \geq 2$ the map*

$$\phi_k(S) = \psi_{2k}, \quad \phi_k(T) = 0$$

can be uniquely extended to a 1-cocycle for Γ with coefficients in the space of functions on $\mathbb{P}^1(\mathbb{R})$ with the action in weight $2k$.

Let us give $\phi_k(g)$ for $g \in \Gamma$ explicitly. To this end let us consider for $\alpha \neq \beta \in \mathbb{P}^1(\mathbb{R})$ the function on $\mathbb{P}^1(\mathbb{R}) \setminus \{\alpha, \beta\}$ defined by

$$f_{\alpha,\beta}(x) = \begin{cases} \displaystyle\sum_{\frac{p}{q} \in [\alpha,\beta]}^{*} \frac{1}{(qx-p)^{2k}}, & x \in (\beta, \alpha) \\ \displaystyle -\sum_{\frac{p}{q} \in\, [\beta,\alpha]}^{*} \frac{1}{(qx-p)^{2k}}, & x \in (\alpha, \beta)\,, \end{cases} \tag{39}$$

where $[\alpha, \beta]$, $[\beta, \alpha]$, (β, α), (α, β) are to be taken on $\mathbb{P}^1(\mathbb{R})$, e.g., $[\alpha, \beta]$ has its usual meaning if $-\infty \leq \alpha < \beta \leq \infty$ but means $[\alpha, \infty] \cup (-\infty, \beta]$ if $-\infty < \beta < \alpha \leq \infty$.

Then with the convention that $f_{\alpha,\alpha} \equiv 0$, we have the equality

$$f_{\alpha,\beta}(x) + f_{\beta,\gamma}(x) = f_{\alpha,\gamma}(x), \quad x \in \mathbb{P}^1(\mathbb{R}) \setminus \{\alpha, \beta, \gamma\}.$$

Also there is an equivariance $f_{g\alpha,g\beta}(x) = (gf_{\alpha,\beta})\,(x)$ for $g \in \Gamma$. Therefore for any $\alpha \in \mathbb{P}^1(\mathbb{R})$ the map

$$g \mapsto f_{\alpha,g\alpha} \tag{40}$$

is a 1-cocycle. Since $\phi_k(S) = \psi_{2k}(x) = f_{\infty,S\infty}$ and $\phi_k(T) = 0 = f_{\infty,T\infty}$, we have $\phi_k(g) = f_{\infty,g\infty}$ for any $g \in \Gamma$. And for any $\alpha \in \mathbb{P}^1(\mathbb{R})$ the cocycle (40) is homologous to ϕ_k.

Finally, we remark that the properties of $f_{\alpha,\beta}$ given above mean exactly that the map $(\alpha,\beta) \mapsto f_{\alpha,\beta}$ is a modular pseudo-measure on $\mathbb{P}^1(\mathbb{R})$ in the sense of [7] taking values in the space $\mathcal{V}^0_{2k}$ defined in the next section.

2.2 The generalized Eisenstein cocycle class

Definition 3. *For $k \in \mathbb{Z}$, let $\mathcal{V}_{2k}$ be the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that the limit $\lim\limits_{x\to\infty} x^{2k} f(x)$ exists and is finite.*

This is a representation of $\mathrm{SL}(2,\mathbb{R})$ with the action in weight $2k$. [4]

Let V_{2k-2} be the space of real polynomials of degree $\leq 2k-2$. This is a finite-dimensional representation of $\mathrm{SL}(2,\mathbb{R})$ (with the action in weight $2-2k$, so $V_{2k-2} \subset \mathcal{V}_{2-2k}$) and there is an equivariant map $\mathfrak{I} : \mathcal{V}_{2k} \to V_{2k-2}$ given on $f \in \mathcal{V}_{2k}$ by

$$(\mathfrak{I}f)(X) = \int_{-\infty}^{+\infty} f(x)(X-x)^{2k-2}dx. \tag{41}$$

Definition 4. *Let $\mathcal{V}^\circ_{2k}$ be the space of functions $f : \mathbb{R} \to \mathbb{R}$ continuous at all but a finite number of points with the action of $\mathrm{SL}(2,\mathbb{R})$ in weight $2k$.*

The spaces $\mathcal{V}_{2k} \subset \mathcal{V}^\circ_{2k}$ are representations of Γ, and the 1-cocycle ϕ_k from Lemma 2 takes values in $\mathcal{V}^\circ_{2k}$ since $\psi_{2k} \in \mathcal{V}^\circ_{2k}$. The next theorem shows that ϕ_k can be modified by a coboundary to a $\mathcal{V}_{2k}$-valued cocycle with a rational image under the map (41). Before formulating it, let us notice that in the long exact sequence

$$\dots \to H^0(\Gamma, \mathcal{V}^\circ_{2k}\,/\,\mathcal{V}_{2k}) \to H^1(\Gamma, \mathcal{V}_{2k}) \to H^1(\Gamma, \mathcal{V}^\circ_{2k}) \to \dots$$

the term $H^0(\Gamma, \mathcal{V}^\circ_{2k}\,/\,\mathcal{V}_{2k})$ vanishes (since a Γ-invariant function can't have a finite nonzero number of points of discontinuity), and therefore $H^1(\Gamma, \mathcal{V}_{2k})$ is a subspace of $H^1(\Gamma, \mathcal{V}^\circ_{2k})$.

Theorem 3. *Let $[\phi_k] \in H^1(\Gamma, \mathcal{V}^\circ_{2k})$ be the cohomology class of the 1-cocycle ϕ_k from Lemma 2. Then $[\phi_k] \in H^1(\Gamma, \mathcal{V}_{2k})$, and $\mathfrak{I}\,[\phi_k] \in H^1(\Gamma, \pi^{2k} V_{2k-2}(\mathbb{Q}))$, where $V_{2k-2}(\mathbb{Q}) \subset V_{2k-2}$ is the subspace of polynomials with rational coefficients. Namely, $\mathfrak{I}\,[\phi_k]$ is the class of the 1-cocycle*

$$\begin{aligned} T &\mapsto -\frac{\zeta(2k)}{2k-1}\left(X^{2k-1} - (X-1)^{2k-1}\right), \\ S &\mapsto -\frac{2}{2k-1}\sum_{r=1}^{k-1} \zeta(2r)\,\zeta(2k-2r) X^{2r-1}. \end{aligned} \tag{42}$$

[4] There is a more symmetric definition of this representation as the space of those continuous functions $F : \mathbb{R}^2\backslash\{0\} \to \mathbb{R}$ which are homogenous of degree $-2k$, i.e. $F(\lambda x, \lambda y) = \lambda^{-2k}F(x,y)$. Then the right action of $\mathrm{SL}(2,\mathbb{R})$ is given by $(Fg)(x,y) = F(ax+by, cx+dy)$, and $f(x) = F(x,1)$ is an element of $\mathcal{V}_{2k}$ from Definition 3. One can find several other presentations for $\mathcal{V}_{2k}$ in [1].

Proof. From the asymptotic expansion of $\mathcal{F}_{2k}$ at infinity we conclude that

$$\psi_{2k}(x) = \frac{\zeta(2k)}{2} + \frac{\zeta(2k-1)}{2k-1}\frac{1}{x^{2k-1}} + O\left(\frac{1}{x^{2k+1}}\right), \qquad x \to +\infty.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function such that the limit

$$\lim_{x\to\infty} x^{2k}\left(f(x) + \frac{\zeta(2k)}{2}\operatorname{sign}(x) + \frac{\zeta(2k-1)}{2k-1}\frac{1}{x^{2k-1}}\right) \tag{43}$$

exists and is finite. Then from the equivariance property of the functions $f_{\alpha,\beta}$ in the previous section it follows that $\phi_k(g) + (1-g)f \in \mathcal{V}_{2k}$ for any $g \in \Gamma$. Hence, $\phi_k + \partial f$ is a $\mathcal{V}_{2k}$-valued cocycle, so $[\phi_k] \in H^1(\Gamma, \mathcal{V}_{2k})$.

Let F be any smooth function on $\mathbb{R}$ with $F(x) = -\zeta(2k-1)\log|x| - \frac{\zeta(2k)}{2}|x|^{2k-1} + \frac{1}{x^2}C(1/x)$ as $x \to \infty$ with C smooth near 0. Then the limit (43) for $f(x) = \frac{1}{(2k-1)!}\left(\frac{d}{dx}\right)^{2k-1}F(x)$ equals 0, and using the asymptotic expansion of $\mathcal{F}_{2k}$ near 0 we check that the function

$$\mathcal{F}_{2k}(|x|) + \frac{\zeta(2k)}{2}\left(\frac{1}{|x|} + |x|^{2k-1}\right) - A_{2k}(|x|) + (1-S)F(x)$$

is continuously differentiable $2k-1$ times on $\mathbb{R}$ (including $x = 0$). Its $(2k-1)$st derivative is obviously $\psi_{2k}(x) + (1-S)f$ times $(2k-1)!$ With this, by the formula (54) from the appendix we find that

$$\begin{aligned}\mathfrak{I}\Big(\psi_{2k}(x) + (1-S)f\Big) &= \frac{1}{2k-1}\left(-A_{2k}(X) + A_{2k}(-X)\right)\\ &= -\frac{2}{2k-1}\sum_{r=1}^{k-1}\zeta(2r)\,\zeta(2k-2r)X^{2r-1}\end{aligned}$$

and

$$\mathfrak{I}\Big((1-T)f\Big) = -\frac{\zeta(2k)}{2k-1}\left(X^{2k-1} - (X-1)^{2k-1}\right),$$

veryfying that the cocycle $\mathfrak{I}(\phi_k + \partial f) \in Z^1(\Gamma, \mathcal{V}_{2k})$ is given by equation (42). $\square$

2.3 Two periods of the non-holomorphic Eisenstein series

The cocycle (42) is (a rational multiple of) the representative of the cohomological class which corresponds to the holomorphic Eisenstein series

$$E_{2k}(z) = \sum_{m,n}{}' \frac{1}{(mz+n)^{2k}}$$

under the Eichler-Shimura isomorphism between $M_{2k} \oplus \overline{S}_{2k}$ and $H^1(\Gamma, V_{2k-2}(\mathbb{C}))$. Let us briefy explain the relation of our theory to the Eisenstein series. The nonholomorphic Eisenstein series (9) is a Maass form in a sense that it is Γ-invariant function in the upper halfplane satisfying the Laplace equation $\Delta_z E(z,s) = s(1-s)E(z,s)$ where $\Delta_z = (z-\bar{z})^2\frac{\partial^2}{\partial z\partial\bar{z}}$. Following [6, 1] one can define the two "periods" $\psi_1 \in H^1(\Gamma, \mathcal{V}_{2k})$ and $\psi_2 \in H^1(\Gamma, V_{2k-2})$ of $E(z,k)$. Representatives of ψ_1 and ψ_2 are defined by

$$g \mapsto \int_{z_0}^{gz_0} \left[E(z,k), \left(\frac{|X-z|^2}{y} \right)^{-k} \right], \qquad g \mapsto \int_{z_0}^{gz_0} \left[E(z,k), \left(\frac{|X-z|^2}{y} \right)^{k-1} \right]$$

correspondingly, where z_0 is an arbitrary point in the upper halfplane and the bracket $[\cdot,\cdot]$ is Green's form, $[u(z), v(z)] = \frac{\partial u}{\partial z} v dz + u \frac{\partial v}{\partial \bar{z}} d\bar{z}$. The Green form has the following properties:

(i) $[u,v]$ is closed if both u and v are eigenfunctions of Δ with the same eigenvalue;

(ii) If $z \mapsto g(z)$ is any holomorphic change of variables, then $[u \circ g, v \circ g] = [u,v] \circ g$.

For every fixed X both $\left(\frac{|X-z|^2}{y}\right)^{-k}$ and $\left(\frac{|X-z|^2}{y}\right)^{k-1}$ are eigenfunctions of Δ with the eigenvalue $k(1-k)$. Therefore both Green forms above are closed due to (i). Notice that $z \to \left(\frac{|X-z|^2}{y}\right)^{-k}$ and $z \to \left(\frac{|X-z|^2}{y}\right)^{k-1}$ are $\mathrm{SL}(2,\mathbb{R})$-equivariant maps from the upper halfplane to $\mathcal{V}_{2k}$ and V_{2k-2} correspondingly. Thus it follows from (ii) that both forms are Γ-equivariant (with values in the corresponding spaces), so the above integrals indeed define 1-cocycles. Furthemore, one can check that for every fixed z

$$\mathfrak{I} \left(\frac{|X-z|^2}{y} \right)^{-k} = \frac{(2k-3)!!}{(2k-4)!!}\, \pi \, \left(\frac{|X-z|^2}{y} \right)^{k-1},$$

so the periods are related by

$$\mathfrak{I}\psi_1 = \frac{(2k-3)!!}{(2k-4)!!}\, \pi \, \psi_2. \tag{44}$$

We claim that $\psi_1 = i \frac{k}{2} \frac{(2k-1)!!}{(2k)!!} \pi \, [\phi]$, where $[\phi]$ is the cocycle class from Theorem 3. Indeed, let us take the representative of ψ_1 given above and let $z_0 \to \infty$. Then the limiting 1-cocycle sends $T \mapsto 0$ and

$$\begin{aligned} S \mapsto \int_{\infty}^{0} \left[E(z,k), \left(\frac{|X-z|^2}{y} \right)^{-k} \right] &= ikX \int_0^\infty \frac{t^k}{(X^2+t^2)^{k+1}} E(it,k) dt \\ &= i \frac{k}{2} \frac{(2k-1)!!}{(2k)!!}\, \pi \, \operatorname{sign}(X) \sum_{p,q \geq 0}^{*} \frac{1}{(p\,|X| + q)^{2k}}, \end{aligned}$$

and we see it is a multiple of the cocycle ϕ from Lemma 2. Here the first equality is proven in [6] and the second calculation is due to Chang and Mayer [2, 6].

So far from (44) we have that $\mathfrak{I}[\phi] = \frac{i(2k-1)}{4} \, \psi_2$. We now use this to show the following:

Proposition 5. *For $k \in \mathbb{Z}_{\geq 2}$ we have*

$$\left[E(z,k), \left(\frac{|X-z|^2}{y} \right)^{k-1} \right] \equiv \alpha_k \, i \, E_{2k}(z)(X-z)^{2k-2} dz, \tag{45}$$

with $\alpha_k = 2^{2k-1}(1-2k) \in \mathbb{Q}$, where $\omega_1 \equiv \omega_2$ for Γ-equivariant V_{2k-2}-valued **1-forms** *ω_1 and ω_2 means that they differ by the full differential of a Γ-equivariant V_{2k-2}-valued function.*

This proposition implies that $\Im[\phi]$ is a rational multiple of the class of $E_{2k}(z)(X-z)^{2k-2}dz$, so we get an alternative proof of Theorem 3.

Proof. The operators

$$\delta_w = \frac{\partial}{\partial z} + \frac{w}{z-\bar{z}}, \quad \bar{\delta} = (z-\bar{z})^2 \frac{\partial}{\partial \bar{z}}$$

intertwine the action of $\mathrm{PSL}(2,\mathbb{R})$ on the functions in the upper halfplane in weights w, $w+2$ and w, $w-2$ correspondingly. Then $\bar{\delta}\delta_w - \delta_{w-2}\bar{\delta} = w$. Let $\Delta_w = \bar{\delta}\delta_w$, and denote by $\mathcal{H}_{w,\lambda} = \{u \mid \Delta_w u = \lambda u\}$ the space of eigenfunctions in weight w. Then

$$\delta_w : \mathcal{H}_{w,\lambda} \to \mathcal{H}_{w+2,\lambda+w+2}, \qquad \bar{\delta} : \mathcal{H}_{w,\lambda} \to \mathcal{H}_{w-2,\lambda-w}\,.$$

For $u \in \mathcal{H}_{w,\lambda}$ and $v \in \mathcal{H}_{-w,\lambda-w}$ we introduce the generalized Green form as

$$[u,v]_w = \delta_w(u)\ v\ dz + u\ \bar{\delta}(v)\ \frac{d\bar{z}}{(z-\bar{z})^2} = \delta_w(u)\ v\ dz + u\ \frac{\partial}{\partial \bar{z}}(v)\ d\bar{z}.$$

It is a closed form, and for $g \in \mathrm{PSL}(2,\mathbb{R})$ one has $[gu,gv] = g[u,v]$. Also

$$[u,v]_w + [v,u]_{-w} = (\delta_w(u)\ v + \delta_{-w}(v)\ u)dz + \frac{\partial}{\partial \bar{z}}(uv)d\bar{z}$$

$$= (\frac{\partial}{\partial z}(uv) + \frac{w}{z-\bar{z}}uv - \frac{w}{z-\bar{z}}uv)dz + \frac{\partial}{\partial \bar{z}}(uv)d\bar{z} = d(uv)$$

and

$$[\bar{\delta}v, \delta_w u]_{-w-2} = \delta_{-w-2}\bar{\delta}v\ \delta_w u\ dz + \bar{\delta}v\ \bar{\delta}\delta_w u \frac{d\bar{z}}{(z-\bar{z})^2}$$

$$= \lambda v \delta_w u\ dz + \bar{\delta}v\lambda u \frac{d\bar{z}}{(z-\bar{z})^2} = \lambda[u,v]_w.$$

Therefore we have proven that

$$[u,v]_w \equiv -\frac{1}{\lambda}[\delta_w u, \bar{\delta}v]_{w+2} \qquad (u \in \mathcal{H}_{w,\lambda}, \quad v \in \mathcal{H}_{-w,\lambda-w})\,.$$

If we apply this transformation $k-1$ times to the functions

$$u = E(z,k), \quad v = \left(\frac{|X-z|^2}{y}\right)^{k-1} \in \mathcal{H}_{0,k(1-k)}\,.$$

we get (45) since

$$\delta_0^k E(z,k) = (-2i)^k \frac{(2k-1)!}{(k-1)!} E_{2k}(z)$$

and

$$(\bar{\delta})^{k-1}\left(\frac{|X-z|^2}{y}\right)^{k-1} = (2i)^{k-1}(k-1)!(X-z)^{2k-2}.$$

$\square$

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2.4 Homological formulation of the higher "Kronecker limit formula"

Notice that there is an $\mathrm{SL}(2,\mathbb{R})$-equivariant pairing $\mathcal{V}_{2k}\otimes\mathcal{V}_{2-2k}\to\mathbb{R}$ given by

$$\langle f\ ,\ g\rangle=\int_{\mathbb{R}} f(t)g(t)dt.$$

If $g\in V_{2k-2}\subset\mathcal{V}_{2-2k}$ is a polynomial then we have $\langle f,g\rangle=\big(\mathfrak{I}f,g\big)$ where $\mathfrak{I}$ is the map (41) and the $\mathrm{SL}(2,\mathbb{R})$-equivariant pairing on polynomials is given by the formula

$$\Big(\sum_{i=0}^{2k-2}a_iX^i,\ \sum_{i=0}^{2k-2}b_iX^i\Big)=\sum_{i=0}^{2k-2}(-1)^i\ \frac{a_i\,b_{2k-2-i}}{\binom{2k-2}{i}}.$$

We denote by the same brackets the pairings on homology and cohomology

$$\Big\langle\ \cdot\ ,\ \cdot\ \Big\rangle\ :\ H^1\left(\Gamma,\mathcal{V}_{2k}\right)\otimes H_1(\Gamma,\mathcal{V}_{2-2k})\to\mathbb{R},$$

$$\Big(\ \cdot\ ,\ \cdot\ \Big)\ :\ H^1\left(\Gamma,V_{2k-2}\right)\otimes H_1(\Gamma,V_{2k-2})\to\mathbb{R}.$$

Let $Q=AX^2+BXY+CY^2$ be an integer quadratic form of discriminant D. We also denote by Q the function $Q(x,1)$ of $x\in\mathbb{R}$. Then $Q^{k-1}\in V_{2k-2}$. The stabilizer of Q in Γ is isomorphic to $\mathbb{Z}$, and we pick that generator $\gamma_Q\in\mathrm{Stab}_\Gamma\mathrm{Q}$ for which $\gamma_Q(x)<x$ whenever $Q(x)<0$. With this condition we then have $\gamma_{gQ}=g\gamma_Qg^{-1}$, and therefore the class of the 1-cycle $\gamma_Q\otimes Q^{k-1}$ in $H_1(\Gamma,V_{2k-2})$ depends only on the Γ-orbit of Q. Since Γ-orbits correspond to narrow ideal classes, we can give the following definition.

Definition 5. *Let for a narrow class of ideals $\mathcal{B}$ the elements $\xi_{\mathcal{B}}\in H_1(\Gamma,V_{2k-2})$ and $\eta_{\mathcal{B}}\in H_1(\Gamma,\mathcal{V}_{2-2k})$ be the classes of 1-cycles $\gamma_Q\otimes Q^{k-1}$ and $\gamma_Q\otimes Q^{k-1}\mathbf{1}_{\{Q<0\}}$ correspondingly, where Q is any integer quadratic form in $\mathcal{B}$.*

Notice that $-Q$ corresponds to $\mathcal{B}^\dagger=\mathcal{B}'^*$ if Q corresponds to $\mathcal{B}$, and $\gamma_{-Q}=\gamma_Q^{-1}$. Therefore $\xi_{\mathcal{B}^\dagger}=(-1)^k\xi_{\mathcal{B}}$ and

$$\eta_{\mathcal{B}}+(-1)^k\eta_{\mathcal{B}^\dagger}=\xi_{\mathcal{B}}. \tag{46}$$

Theorem 4. *Let $[\phi_k]$ be the cocycle class from Theorem 3. Then for every narrow ideal class $\mathcal{B}$*

$$\Big\langle[\phi_k]\ ,\ \eta_{\mathcal{B}}\Big\rangle=(-1)^{k-1}\frac{(k-1)!^2}{(2k-2)!}D^{k-\frac{1}{2}}\ \zeta(\mathcal{B},k).$$

Proof. Let us pick a form Q in the class $\mathcal{B}$ and a number α such that $Q(\alpha)>0$. Due to Theorem 3 there exists an $f\in\mathcal{V}_{2k}^\circ$ such that the 1-cocycle

$$g\mapsto f_{\alpha,g\alpha}+(1-g)f \tag{47}$$

(with $f_{\alpha,\beta}$ defined by (39)) is $\mathcal{V}_{2k}$-valued. Notice, that we can choose f so that it vanishes outside of any small neighbourhood of α. The 1-cocycle (47) is a representative of $[\phi_k]$, and pairing it with $\gamma_Q\otimes Q^{k-1}\mathbf{1}_{\{Q<0\}}$ gives

$$\big\langle[\phi_k]\ ,\ \eta_{\mathcal{B}}\big\rangle\ =\ \int\limits_{Q<0}Q(x)^{k-1}\sum_{\alpha<\frac{p}{q}<\gamma_Q(\alpha)}^{*}\frac{1}{(qx-p)^{2k}}dx.$$

Notice that (19) can be rewritten as

$$\mathcal{D}_n\, F(Q) \;=\; \frac{(-1)^n\,\mathrm{sign}(A)}{n!} \int\limits_{\frac{Q}{A}<0} Q(x)^n F^{(2n+1)}(x)dx$$

and together with (iv) of Proposition 4 it gives the formula

$$\int\limits_{Q<0} \frac{Q(x)^n}{(x-T)^{2n+2}}dx \;=\; (-1)^n\,\frac{n!^2}{(2n)!}\,\frac{D^{n+\frac{1}{2}}}{Q(T)^{n+1}}.$$

Therefore

$$(-1)^{k-1}\frac{(2k-2)!}{(k-1)!^2}D^{\frac{1}{2}-k}\,\big\langle[\phi_k]\;,\;\eta_{\mathcal{B}}\big\rangle \;=\; \sum_{\alpha<\frac{p}{q}<\gamma_Q(\alpha)}^{*} \frac{1}{Q(p,q)^k}dx$$

$$=\; N(\mathfrak{b})^k \sum_{\lambda\in\mathfrak{b}/\varepsilon,\;\lambda\gg 0} \frac{1}{N(\lambda)^k} \;=\; \zeta(\mathcal{B}',k) \;=\; \zeta(\mathcal{B},k).$$

Here b is any ideal in the class $\mathcal{B}$, and the last equality is true because the zeta functions of $\mathcal{B}$ and $\mathcal{B}'$ coincide. □

Notice that together with (46) this statement yields

$$\zeta(\mathcal{B},k)+(-1)^k\,\zeta(\mathcal{B}^\dagger,k) = (-1)^{k-1}\frac{(2k-2)!}{(k-1)!^2}D^{\frac{1}{2}-k}\,\big(\mathfrak{I}[\phi_k]\;,\;\xi_{\mathcal{B}}\big), \qquad (48)$$

which gives the Siegel-Klingen theorem again, since $\zeta(\mathcal{B}^\dagger,s)=\zeta(\mathcal{B}^*,s)$ and both classes on the right are rational, the cocycle $\mathfrak{I}[\phi_k]$ being given explicitly by equation (42). Equality (48) is very similar to Theorems 5 in [9], where values of L-functions over the real quadratic fields are also given as a homological pairing at those points where the values are rational. The main difference between the two results is that, although both the cycle and the cocycle occurring are the same (we omit the calculation showing this in the case of the cocycle), the explicit descriptions of the Eisenstein cocycle as a rational class (Theorem 3 of [9] and Theorem 3 of our paper) look quite different.

The relation of the above theorem to the higher KLF (Theorem 1) is a consequence of the following proposition.

Proposition 6. *Let* $\mathrm{Red}(\mathcal{B})=\{w_1,\dots,w_{l(\mathcal{B})}\}$ *with* $w_i=b_i-1/w_{i+1}$ *and* Q_i *be the integer quadratic form of discriminant* D *with a root* w_i*. Then*

$$\xi_{\mathcal{B}} \;=\; \Big[\sum_i T^{b_i}S\otimes Q_i^{k-1}\Big] \quad \text{and} \quad \eta_{\mathcal{B}} \;=\; \Big[\sum_i T^{b_i}S\otimes Q_i^{k-1}\cdot\mathbf{1}_{\{Q_i<0\}}\Big].$$

Proof. Since $T^{b_i}S\,Q_i=Q_{i+1}$, one needs to observe that

$$\gamma_{Q_1}=T^{b_{l(\mathcal{B})}}S\,T^{b_{l(\mathcal{B})-1}}S\cdots T^{b_1}S$$

and therefore the 1-cycle $\gamma_{Q_1}\otimes Q_1^{k-1}$ is homologous to $\sum_i T^{b_i}S\otimes Q_i^{k-1}$. In the same way $\gamma_{Q_1}\otimes Q_1^{k-1}\cdot\mathbf{1}_{\{Q_1<0\}}$ is homologous to the other 1-cycle given in the proposition. □

3 Twists of $\mathcal{F}_k$ and applications

In this final section we consider the twisted version of the function $\mathcal{F}_k$ defined by the series

$$\sum_{p=1}^{\infty} \frac{\chi(p)\psi(px)}{p^{k-1}} \qquad (x \in \mathbb{R}_+,\ k \geq 2),$$

where χ is a nontrivial Dirichlet character. Notice that here we do not have to subtract $\log(px)$ from $\psi(px)$ to get convergence when $k = 2$, because $\sum\limits_{p=1}^{\infty} \frac{\chi(p)\log(px)}{p}$ converges (to $L(1,\chi)\log(x) - L'(1,\chi)$). Instead of inserting a character we could have imposed congruence conditions on p. Therefore the new functions are relevant when one wants to study zeta functions for ray classes rather than ordinary ideal classes.

3.1 The twisted function $\mathcal{F}_k^{\chi}$

Definition 6. *For a nontrivial Dirichlet character $\chi : \mathbb{Z} \to \mathbb{C}$ and an integer $k \geq 2$ we define*

$$\mathcal{F}_k^{\chi}(x) = \sum_{p=1}^{\infty} \frac{\chi(p)\psi_0(px)}{p^{k-1}}, \quad x > 0$$

where $\psi_0(x) = \psi(x) + \frac{1}{2x}$.

Here we replace ψ by ψ_0 in order to have the formula

$$\left(\mathcal{D}_{k-1}\,\mathcal{F}_{2k}^{\chi}\right)(x,y) = \sum_{p,q\geq 0}{}^{*} \chi(p) \left(\frac{x-y}{(px+q)(py+q)}\right)^{k}. \tag{49}$$

(Recall that $*$ on the summation sign means counting the boundary terms with the coefficient $\frac{1}{2}$.) As in Theorem 2, we immediately get

$$\mathcal{F}_k^{\chi}(x) \;\sim\; L(k,\chi)\log(x) - L'(k,\chi) + \sum_{r=2}^{\infty} \frac{\zeta(1-r)L(k+r-1,\chi)}{x^r}$$

as $x \to \infty$, and

$$\mathcal{F}_k^{\chi}(x) \;\sim\; -\frac{L(k,\chi)}{2x} + \sum_{r=1}^{\infty} (-1)^r\,\zeta(r)L(k-r,\chi)x^{r-1}$$

as $x \to 0$. Concerning the functional equations, there are now many of them and the theory in general depends on the conductor of χ. Instead of studing the picture in full generality, let us consider a particular and probably the simplest example $\chi = \left(\frac{\cdot}{3}\right)$. To get the functional equation for

$$\psi_k^{\chi}(x) := \frac{(-1)^k}{(k-1)!}\left(\frac{d}{dx}\right)^{k-1} \mathcal{F}_k^{\chi}(x) = \sum_{\frac{p}{q}\in[0,\infty]}{}^{*} \frac{\chi(p)}{(px+q)^k}$$

(here and below, the summation is taken over *all* fractions $\frac{p}{q}$, not necessarily reduced, but with $q \geq 0$) we break up the quadrant of summation as

$$\frac{p}{q} \in \Big[0,\infty\Big] = \Big[0,1\Big] \cup \Big[1,\frac{3}{2}\Big] \cup \Big[\frac{3}{2},2\Big] \cup \Big[2,3\Big] \cup \Big[3,\infty\Big] . \tag{50}$$

The condition $\dfrac{c}{d} \leq \dfrac{p}{q} \leq \dfrac{a}{b}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix in $\mathrm{SL}(2,\mathbb{Z})$ with nonnegative entries, can be written as $(p \quad q) = (P \quad Q)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $P, Q \geq 0$, and the intervals $[0,1]$, …, $[3,\infty]$ in the decomposition (50) have been chosen so that either $3|a$ or $3|c$ in each case. Now using

$$\sum^{*}_{\frac{c}{d}\leq\frac{p}{q}\leq\frac{a}{b}} \frac{\chi(p)}{(px+q)^k} = \sum^{*}_{P,Q\geq 0} \frac{\chi(p)}{((aP+cQ)x+bP+dQ)^k}$$

$$= \sum^{*}_{P,Q\geq 0} \frac{\chi(aP+cQ)}{(P(ax+b)+Q(cx+d))^k} = \begin{cases} \frac{\chi(a)}{(cx+d)^k}\ \psi_k^{\chi}\left(\frac{ax+b}{cx+d}\right), & \text{if } 3|c, \\ \frac{\chi(c)}{(ax+b)^k}\ \psi_k^{\chi}\left(\frac{cx+d}{ax+b}\right), & \text{if } 3|a \end{cases}$$

and summing over the five intervals in (50), we obtain

$$\psi_k^{\chi}(x) = \psi_k^{\chi}(x+1) + \frac{1}{(3x+2)^k}\ \psi_k^{\chi}\left(\frac{x+1}{3x+2}\right) - \frac{1}{(3x+2)^k}\ \psi_k^{\chi}\left(\frac{2x+1}{3x+2}\right)$$

$$-\frac{1}{(3x+1)^k}\ \psi_k^{\chi}\left(\frac{2x+1}{3x+1}\right) + \frac{1}{(3x+1)^k}\ \psi_k^{\chi}\left(\frac{x}{3x+1}\right). \tag{51}$$

Integrating this equation $k-1$ times gives the 6-term functional equation for $\mathcal{F}_k^{\chi}$ with an unknown polynomial of degree $k-2$, whose coefficients can be found using the asymptotic expansions. For example, when $k=2$ we have

$$\mathcal{F}_2^{\chi}(x) + \mathcal{F}_2^{\chi}\left(\frac{x+1}{3x+2}\right) + \mathcal{F}_2^{\chi}\left(\frac{2x+1}{3x+2}\right)$$

$$= \mathcal{F}_2^{\chi}(x+1) + \mathcal{F}_2^{\chi}\left(\frac{2x+1}{3x+1}\right) + \mathcal{F}_2^{\chi}\left(\frac{x}{3x+1}\right).$$

(In this case there is no additive correction term.)

Remark. Actually, one can define 1-cocycles for $\Gamma_1(3)$ using the sums $\sum^{\star}_{p/q\in[\alpha,\beta]} \dfrac{\chi(p)}{(px+q)^k}$ as above. Let $\mathcal{W}_k^{\circ}$ be the space of functions $f:\mathbb{R}\to\mathbb{R}$ continuous at all but finite number of points with the action of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ given by

$$(f\,|\,g)\,(x) = \frac{\mathrm{sign}(cx+d)}{|cx+d|^k}\ f\Big(\frac{ax+b}{cx+d}\Big).$$

Analogously to (39), the formula

$$f_{\alpha,\beta}(x) = \begin{cases} \displaystyle\sum^{*}_{\frac{p}{q}\in[\alpha,\beta]} \frac{\chi(p)\ \mathrm{sign}(qx-p)}{|qx-p|^k}, & x \in (\beta,\alpha) \\ \displaystyle -\sum^{*}_{\frac{p}{q}\in\,[\beta,\alpha]} \frac{\chi(p)\ \mathrm{sign}(qx-p)}{|qx-p|^k}, & x \in (\alpha,\beta) \end{cases}$$

defines a $\Gamma_1(3)$-equivariant pseudo-measure on $\mathbb{P}^1(\mathbb{R})$ with values in $\mathcal{W}_k^\circ$, hence we get a cocycle class in $H^1(\Gamma_1(3), \mathcal{W}_k^\circ)$.

3.2 Computation of Stark's units and Stark-Gross regulators

We now show how to apply our formulas to verifications of Stark's conjecture and its higher weight analogs. Stark's conjecture describes the special values at $s = 1$ of Artin L-functions $L(s, \rho)$ in terms of logarithms of units in the appropriate abelian extension H of the base field F. A generalization was given by Gross ([5]) in which the values $L(k, \rho)$ for $k \geq 1$ are given in terms of Borel regulators on the K-groups $K_{2k-1}(H)$. Combining this with the conjecture of the second author expressing Borel regulators in terms of higher polylogarithms ([16]), one is led to the conjecture that all values $L(k, \chi)$ of Artin L-functions at positive integer arguments can be written in terms of the kth polylogarithm function $Li_k(x)$ evaluated at algebraic arguments x (again belonging to the same abelian extension H of the base field). For the case of imaginary quadratic fields, where the corresponding Kronecker limit formula is the one described briefly in the introduction, this was discussed in detail in [19]. Here we give the corresponding discussion for real quadratic fields using the twisted Kronecker limit formula. Our examples will be for $k = 1$ (the case of the original Stark conjecture) and $k = 2$, where the polylogarithms become dilogarithms and the K-group $K_{2k-1}(H) = K_3(H)$ is simply the Bloch group $B(H)$, for whose definition we again refer to [16] or [19].

Let $\mathcal{O}$ be the ring of integers in the real quadratic field K. We need an ideal $\mathfrak{f}$ of $\mathcal{O}$ such that the condition $\varepsilon \equiv 1 \pmod{\mathfrak{f}}$, ε a unit, implies $\varepsilon' > 0$. Then there are characters on the ray class group modulo $\mathfrak{f}$ ramified at only one of the two infinite places of K. We will choose $K = \mathbb{Q}(\sqrt{13})$ and $\mathfrak{f} = \left(\frac{1-\sqrt{13}}{2}\right)$. Then $\mathcal{O}^\times = \langle \pm\varepsilon_0 \rangle$ with $\varepsilon_0 = \frac{3+\sqrt{13}}{2}$, and $\varepsilon_0 \equiv -1 \pmod{\mathfrak{f}}$. Therefore the condition $\varepsilon \equiv 1 \pmod{\mathfrak{f}}$ is equivalent to $\varepsilon \in (-\varepsilon_0)^{\mathbb{Z}} = (\varepsilon_0')^{\mathbb{Z}}$, hence $\varepsilon' > 0$.

For each ray class $\mathfrak{C}$ modulo $\mathfrak{f}$, choose a representative $\mathfrak{a}$ of $\mathfrak{C}$ (so $(\mathfrak{a}, \mathfrak{f}) = 1$ and we can change $\mathfrak{a}$ to $\lambda\mathfrak{a}$ with $\lambda \equiv 1 \pmod{\mathfrak{f}}$) and pick an ideal $\mathfrak{b}$ in the (ordinary) inverse ideal class to $\mathfrak{a}$ and a generator α of $\mathfrak{ab}$. Then

$$L(\mathfrak{C}, s) = N(\mathfrak{b})^s \operatorname{sign}(\alpha) \sum_{\substack{\beta \in \mathfrak{b}/\varepsilon_0', \\ \beta \equiv \alpha (\mathrm{mod}\ \mathfrak{bf})}} \frac{\operatorname{sign} \beta'}{|N(\beta)|^s}$$

depends only on $\mathfrak{C}$ and $\exp\Big(\zeta'(\mathfrak{C}, 0)\Big)$ is expected to be a unit. In our example, choose simply $\mathfrak{a} = \mathfrak{b} = \mathcal{O}$ and $\alpha = 1$. Then

$$L(\mathcal{O}, s) = \sum_{\substack{\beta \in \mathcal{O}/\varepsilon_0', \\ \beta \equiv 1 (\mathrm{mod}\ \mathfrak{f})}} \frac{\operatorname{sign} \beta'}{|N(\beta)|^s} = \sum_{p,q \geq 0}{}^{*} \left[\frac{\chi(p) + \chi(p-q)}{(p^2 + 5pq + 3q^2)^s} - \frac{\chi(p)}{(3p^2 + 7pq + 3q^2)^s} \right]$$

with $\chi = \left(\frac{\cdot}{3}\right)$, and

$$\sum_{p,q\geq 0}{}^{*} \frac{\chi(p-q)}{(p^2+5pq+3q^2)^s} = \sum_{p\geq q\geq 0}{}^{*} + \sum_{q\geq p\geq 0}{}^{*}$$

$$= \sum_{p,q\geq 0}{}^{*} \left[\frac{\chi(p)}{(p^2+7pq+9q^2)^s} - \frac{\chi(p)}{(3p^2+11pq+9q^2)^s} \right] ,$$

where we made substitutions $(p,q) \mapsto (p+q,q)$ and $(p,q) \mapsto (q,p+q)$ in the first and the second sums correspondingly. Hence for $k \geq 1$ we get using (49) that

$$13^{\frac{k}{2}} L(\mathcal{O},k) = \mathcal{D}_{k-1}(\mathcal{F}^{\chi}_{2k}) \left(\frac{5+\sqrt{13}}{6}, \frac{5-\sqrt{13}}{6} \right) + \mathcal{D}_{k-1}(\mathcal{F}^{\chi}_{2k}) \left(\frac{7+\sqrt{13}}{18}, \frac{7-\sqrt{13}}{18} \right)$$

$$- \mathcal{D}_{k-1}(\mathcal{F}^{\chi}_{2k}) \left(\frac{11+\sqrt{13}}{18}, \frac{11-\sqrt{13}}{18} \right) - \mathcal{D}_{k-1}(\mathcal{F}^{\chi}_{2k}) \left(\frac{7+\sqrt{13}}{6}, \frac{7-\sqrt{13}}{6} \right).$$

Let us compute $L(\mathcal{O},1)$ and $L(\mathcal{O},2)$. From the tables of values

x	$\mathcal{F}^{\chi}_{2}(x)$	$\mathcal{F}^{\chi}_{2}(x')$
$(5+\sqrt{13})/6$	$-0.221258722400679\dots$	$-2.087528683598576\dots$
$(7+\sqrt{13})/18$	$-0.901613498337444\dots$	$-2.501022141187798\dots$
$(11+\sqrt{13})/18$	$-0.631117746987411\dots$	$-1.269461158816449\dots$
$(7+\sqrt{13})/6$	$-0.083545973555317\dots$	$-0.939154027903259\dots$

x	$\mathcal{D}_1\, \mathcal{F}^{\chi}_{4}(x,x')$
$(5+\sqrt{13})/6$	$6.253557907531478\dots$
$(7+\sqrt{13})/18$	$6.143437556837824\dots$
$(11+\sqrt{13})/18$	$0.700242692864336\dots$
$(7+\sqrt{13})/6$	$0.758740696880221\dots$

we find

$$L(\mathcal{O},1) = 0.546858715139901\dots$$

and

$$L(\mathcal{O},2) = 0.841385544201903\dots$$

(For $k=2$ we do not give all 16 values $\mathcal{F}^{\chi}_{4}(w_i)$, $\mathcal{F}^{\chi}_{4}(w'_i)$, $(\mathcal{F}^{\chi}_{4})'(w_i)$ and $(\mathcal{F}^{\chi}_{4})'(w'_i)$ separately, but only tabulate the combinations $\mathcal{D}_1\, \mathcal{F}^{\chi}_{4}(w_i,w'_i)$.)

Since $39^{\frac{s}{2}} \frac{\Gamma(s)}{(2\pi)^s} L(\mathcal{O},s)$ is symmetric with respect to $s \longleftrightarrow 1-s$, we have

$$L'(\mathcal{O},0) = \frac{\sqrt{39}}{2\pi} L(\mathcal{O},1) = 0.543535072497869\dots$$

This equals numerically to $\log(\alpha)$, where

$$\alpha = \frac{1}{2}\left(\frac{1+\sqrt{13}}{2} + \sqrt{\frac{-1+\sqrt{13}}{2}}\right)$$

is a real solution of $x^4 - x^3 - x^2 - x + 1 = 0$, therefore a unit in accordance with Stark's conjecture (which can be proven in this case).

Let $H = \mathbb{Q}(\alpha)$, an extension of K of order 2. It has two real places and a complex one. Therefore its Bloch group $B_2(H)$ has rank 1. The element

$$\xi_2 \;=\; -2\,[\alpha] \,+\, 2\,[\alpha^3] \,+\, [\alpha(1-\alpha)^2]$$

then generates $B_2(H)$ up to torsion, and we check numerically that

$$L(\mathcal{O},2) \;=\; \frac{16\,\pi^2}{39^{3/2}}\,\mathrm{D}(\sigma(\xi_2)) \tag{52}$$

where $\sigma : H \to \mathbb{C}$ is the complex embedding in which $\operatorname{Im}\sigma(\alpha) > 0$ and $\mathrm{D}(z) = \operatorname{Im}(\operatorname{Li}_2(z) + \log|z|\log(1-z))$ is the Bloch-Wigner dilogarithm function. This particular identity can be proved rigorously (because $L(\mathcal{O},s) = \zeta_H(s)/\zeta_K(s)$ and it is known how to express values of Dedekind zeta functions at $s = 2$ in terms of dilogarithms), but the numerical method illustrated here would also work for fields K with $h(K) > 1$ or for $s = k \geq 3$, where the corresponding identities would in general only be conjectural. For instance, for the L-function treated in this subsection we find after some computation

$$L(\mathcal{O},3) \;=\; 0.948978902888892\cdots \;=\; \frac{16\,\pi^3}{39^{5/2}}\,\mathcal{L}_3(\xi_3)$$

and

$$L(\mathcal{O},4) \;=\; 0.983924877405875\cdots = \frac{128\,\pi^4}{3\cdot 39^{7/2}}\,\mathcal{L}_4\big(\sigma(\xi_4)\big)$$

with

$$\begin{aligned}\xi_3 \;=\; & 7\,[1-\alpha^{-1}] \,-\, 7\,[1-\alpha] \,+\, [\alpha^{-3}(1-\alpha)^2] \,-\, [\alpha\,(1-\alpha)^2] \\ & +\, 3\,[\alpha\,(\alpha-1)] \,-\, 3\,[\alpha^{-2}(1-\alpha)]\,,\end{aligned}$$

$$\xi_4 \;=\; -75[\alpha] - 102[\alpha^2] - 81[-\alpha^3] - 39[-\alpha^4] - 21[-\alpha^7] + 4[-\alpha^9] + 3[\alpha^{11}]$$

and $\mathcal{L}_m(z) = \operatorname{Re}_m\Big(\sum_{k=0}^{m-1}\dfrac{2^k\,B_k}{k!}\log(|z|)^k\operatorname{Li}_{m-k}(z)\Big)$ for $m \geq 2$, where Re_m denotes Im or Re according m is even or odd. We have checked that the elements ξ_3 and ξ_4 indeed belong to the higher Bloch groups $B_3(H)$ and $B_4(H)$ correspondingly. Since the $\operatorname{Gal}(H/K)$-conjugate to α is $1/\alpha$ one easily sees that ξ_3 is antiinvariant with respect to the Galois involution. Similarly, we find

$$L(\mathcal{O},5) \;=\; 0.994927260665932\cdots = \frac{160\,\pi^5}{3\cdot 39^{9/2}}\,\mathcal{L}_5\big(\xi_5\big)$$

and

$$L(\mathcal{O},6) \;=\; 0.998386179720050\cdots = \frac{512\,\pi^6}{15\cdot 39^{11/2}}\,\mathcal{L}_6\big(\xi_6\big)$$

for elements of the higher Bloch groups $B_5(H)$ and $B_6(H)$, namely,

$$\begin{aligned}
\xi_5 \;=\; & -294[\alpha(1-\alpha)^2] \,+\, 602[\alpha(\alpha-1)] \,+\, 6[\alpha^3(\alpha-1)^5] \,+\, 468[\alpha-1] \\
& +\, 375[\alpha(1-\alpha)] \,+\, 4[(\alpha-1)^3] \,+\, 117[\alpha^2(\alpha-1)] \,+\, 60[\alpha^3(1-\alpha)] \\
& +\, 24[\alpha^3(1-\alpha)^2] \,+\, 8[\alpha^3(1-\alpha)^3] \,-\, 81[\alpha^2(\alpha-1)^3] \,-\, 13[\alpha^4(\alpha-1)] \\
& -\, 354[-\alpha(1-\alpha)^2] \,+\, 2[-\alpha^3(1-\alpha)^6] \,-\, 6[-\alpha^5(1-\alpha)^2] \,+\, 240[1-\alpha] \\
& \qquad\qquad - \quad (\text{Galois conjugate})\,,
\end{aligned}$$

$$\begin{aligned}
\xi_6 \;=\; & -3409[\alpha] \,-\, 1068[\alpha^3] \,-\, 835[\alpha^{11}] \,+\, 1405[-\alpha^4] \,+\, 1608[\alpha^5] \,+\, 4015[\alpha^2] \\
& -\, 1455[\alpha^4] \,-\, 1047[\alpha^6] \,-\, 653[\alpha^8] \,-\, 34[\alpha^{10}] \,+\, 48[\alpha^{12}] \,-\, 29[\alpha^{20}] \,+\, 3503[-\alpha] \\
& -\, 4059[-\alpha^2] \,+\, 1017[-\alpha^3] \,-\, 1623[-\alpha^5] \,+\, 1044[-\alpha^6] \,+\, 77[-\alpha^7] \,-\, 2509[-\alpha^9] \\
& +\, 20[-\alpha^{10}] \,+\, 373[-\alpha^{15}] \,+\, 13[-\alpha^{18}] \,-\, 96[\alpha^7] \,+\, 3[\alpha^{14}].
\end{aligned}$$

Appendix: Differentiation and integration operators

In this appendix we study differentiation and integration operators which allow us, in particular, to pass back and forth between cocycles in weight $2k$ and in weight $2-2k$.

The matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$ acts on functions of two real variables in weight (m,n) by the formula

$$\big(F\big|_{(m,n)}\, g\big)(x,y) = (cx+d)^{-m}(cy+d)^{-n} F\left(\frac{ax+b}{cx+d}, \frac{ay+b}{cy+d}\right).$$

Then $gF = F\Big|_{(m,n)} g^{-1}$ is the left group action, and we denote the space of smooth functions of two variables equipped with this action by $\mathcal{W}_{m,n}$. The operator

$$\delta_m = \frac{\partial}{\partial x} + \frac{m}{x-y} \;:\; \mathcal{W}_{m,n} \to \mathcal{W}_{m+2,n},$$

satisfies $\delta_m \circ g = (\det g)\; g \circ \delta_m$ for every $g \in \mathrm{GL}(2,\mathbb{R})$. Let us try to invert δ_m. For $b \in \mathbb{Z}$, we denote by $\mathcal{W}^b_{m,n} \subset \mathcal{W}_{m,n}$ the subspace of functions F such that for every y the expression $(x-y)^b F(x,y)$ is bounded as x tends to y. Consider

$$I_m : \mathcal{W}_{m,n} \to \mathcal{W}_{m-2,n}, \qquad (I_m F)\,(x,y) = \int_y^x \left(\frac{t-y}{x-y}\right)^{m-2} F(t,y)dt.$$

This operator is well defined on $\mathcal{W}^b_{m,n}$ when $b < m-1$ and $I_m(\mathcal{W}^b_{m,n}) \subset \mathcal{W}^{b-1}_{m-2,n}$. Also it satisfies $I_m \circ g = (\det g)^{-1}\; g \circ I_m$ for every $g \in \mathrm{GL}(2,\mathbb{R})$ and

$$\begin{aligned}
I_{m+2}\; \delta_m\; F\;(x,y) &= \int_y^x \frac{\partial}{\partial t}\left(\left(\frac{t-y}{x-y}\right)^m F(t,y)\right) dt \\
&= F(x,y) - \frac{\lim_{t\to y}(t-y)^m F(t,y)}{(x-y)^m} = F(x,y)
\end{aligned}$$

when $F \in W^b_{m,n}$ with $b < m$. Notice that $\delta_{m-2} I_m F = F$ whenever $I_m F$ is defined. We denote by δ^r_m and I^r_m $(r \geq 0)$ the iterated operators $\delta_{m+2r-2} \circ \cdots \circ \delta_{m+2} \circ \delta_m$ and $I_{m-2r+2} \circ \cdots \circ I_{m-2} \circ I_m$, respectively.

From now on we restrict to the case when the weight in y is 0. We denote $\mathcal{W}_{m,0}$ by $\mathcal{W}_m$ and $\mathcal{W}^b_{m,0}$ by $\mathcal{W}^b_m$. Then the first interesting identity is

$$\delta^{2n+1}_{-2n} = \left(\frac{\partial}{\partial x}\right)^{2n+1} : \mathcal{W}_{-2n} \to \mathcal{W}_{2n+2}\,.$$

One can easily check it using induction. Suppose we are given the function of one variable $F(x)$, and we consider it as an element of $\mathcal{W}_{-2n}$. Then its $(2n+1)$st derivative $F^{(2n+1)}(x)$ belongs to $\mathcal{W}^0_{2n+2}$. Thus $I^{2n+1}_{2n+2} F^{(2n+1)} \in \mathcal{W}^{-2n-1}_{-2n}$. Since

$$F(x) - \sum_{i=0}^{2n} \frac{F^{(i)}(y)}{i!}(x-y)^i \in \mathcal{W}^{-2n-1}_{-2n}$$

and the $(2n+1)$st derivative of this expression in x is $F^{(2n+1)}(x)$ again, we have proved the following statement.

Lemma 3. *For every* $0 \leq m \leq 2n+1$

$$\delta^m_{-2n}\left(F(x) - \sum_{i=0}^{2n} \frac{F^{(i)}(y)}{i!}(x-y)^i\right) = I^{2n+1-m}_{2n+2} F^{(2n+1)}.$$

Let us write a formula for the iterate I^m_{2n+2}. To this end, and also for other applications, we consider the simple rational functions of three variables x, y and T

$$(T-x)^a (T-y)^b (x-y)^c.$$

This function is equivariant in (x, y, T) with triple weight $(-a-c, -b-c, -a-b)$ and one can easily check that

$$\begin{aligned} \delta_{-a-c}(T-x)^a(T-y)^b(x-y)^c &= \left(\frac{\partial}{\partial x} - \frac{a+c}{x-y}\right)(T-x)^a(T-y)^b(x-y)^c \\ &= -a(T-x)^{a-1}(T-y)^{b+1}(x-y)^{c-1}. \end{aligned} \tag{53}$$

It has weight $(-a-c+2, -b-c, -a-b)$ as expected. Then

$$\delta^m_{2(n-m)} \left(\frac{T-y}{x-y}\right)^{2n-m} (T-x)^m = (-1)^m\ m!\ \left(\frac{T-y}{x-y}\right)^{2n},$$

and the following lemma easily follows.

Lemma 4. *For* $0 \leq m \leq 2n$

$$I^{m+1}_{2n+2}\ F\ (x,y) \ = \ \frac{1}{m!} \int_y^x \left(\frac{t-y}{x-y}\right)^{2n-m} (x-t)^m F(t,x) dt.$$

We would like also to have an explicit formula for δ_{-2n}^m. Consider the operation

$$\delta^- = (x-y)^2 \frac{\partial}{\partial y} \ : \ \mathcal{W}_m \to \mathcal{W}_{m-2}\,.$$

It intertwines the group action decreasing the weight in the variable x by 2. The following operator

$$\xi_m : \mathcal{W}_m \to \mathcal{W}_{-m}, \qquad (\xi_m F)(x,y) = (x-y)^m F(y,x)$$

is also equivariant and satisfies $\xi_{-m} \circ \xi_m = (-1)^m$. The operators δ and $\bar{\delta}$ are dual with respect to ξ, namely $\xi_{m+2} \circ \delta_m = \bar{\delta} \circ \xi_m \ : \ \mathcal{W}_m \to \mathcal{W}_{-m-2}$.

Lemma 5. *For every* $0 \le m \le 2n$

$$\delta_{-2n}^m \Big(\sum_{i=0}^{2n} \frac{F^{(i)}(y)}{i!} (x-y)^i \Big) \ = \ (-1)^m \frac{m!}{(2n-m)!} \left(\delta^-\right)^{2n-m} \Big(\frac{F(y)}{(x-y)^{2n}} \Big)\,.$$

Proof. The left-hand side can be computed easily since

$$\delta_{-2n}^m (x-y)^i = (i-2n)(i-2n+1) \cdots (i-2n+m-1)(x-y)^{i-m}.$$

Thus the statement is true for $m = 2n$. Other cases follow from this because $f(x,y) = \sum\limits_{i=0}^{2n} \dfrac{F^{(i)}(y)}{i!}(x-y)^i$ satisfies the equation $\bar{\delta}\delta_{-2n} f = -2nf$ and δ_m transforms the eigenfunctions of $\bar{\delta}\delta_m$ to the eigenfunctions of $\bar{\delta}\delta_{m+2}$ (since $\delta^- \delta_m = \delta_{m-2}\delta^- + m$). ☐

Proposition 7. (i) $\quad \delta_{-2n}^m \ = \ \dfrac{m!}{(2n-m)!} \sum\limits_{i=0}^{m} \dfrac{(2n-i)!}{i!(n-i)!} \dfrac{1}{(y-x)^{m-i}} \Big(\dfrac{\partial}{\partial x} \Big)^i.$

(ii) For a function of one variable $F(x)$ we have

$$\begin{aligned}(2n-m)!(\delta_{-2n}^m F)(x,y) + (-1)^{m-1} \, m! \, (x-y)^{2(n-m)} (\delta_{-2n}^{2n-m} F)(y,x) \\ = \int_y^x \left(\frac{t-y}{x-y} \right)^m (x-t)^{2n-m} F^{(2n+1)}(t) dt.\end{aligned}$$

Proof. (i) can be easily checked by induction (one can guess the formula directly from Lemma 5), and (ii) immediately follows from the three lemmas above. ☐

Let us now apply these statements to prove Proposition 4. From the first part of Proposition 7 we see that $\delta_{-2n}^n = n! \, \mathcal{D}_n^+$, which proves (i) there. From (53)

$$\frac{1}{n!} \, \delta_{-2n}^n \big((T-x)^{2n} \big) \ = \ (-1)^n \binom{2n}{n} \Big(\frac{(T-x)(T-y)}{x-y} \Big)^n,$$

whence (ii) follows. In order to prove (iv) we observe that

$$D_n \left(\frac{1}{T-x} \right) \ = \ \frac{1}{n!} \, \delta_{-2n}^n \left(\frac{1}{T-x} - \sum_{i=0}^{2n} \frac{(x-y)^i}{(T-y)^{i+1}} \right)$$

$$= \frac{1}{n!}\,\delta^n_{-2n}\left(\frac{1}{T-x}\left(\frac{x-y}{T-y}\right)^{2n+1}\right) = \left(\frac{x-y}{(T-x)(T-y)}\right)^{n+1},$$

the latter equality being a consequence of (53). And (v) is a special case of the second part of Proposition 7.

As another application, we would like to compute the integral

$$\mathfrak{I}f(X) = \int_{\mathbb{R}} (X-t)^{2k-2} f(t)dt$$

for $f \in \mathcal{V}_{2k}$ (see Definition 3).

Proposition 8. *For $f \in \mathcal{V}_{2k}$ let F be any function on $\mathbb{R}$ such that $f(t) = \frac{1}{(2k-2)!}F^{(2k-1)}(t)$ and define $P_\pm$ as the two polynomials of degree at most $2k-2$ such that $F(t) = P_\pm(t) + \mathrm{O}(1/t)$ as $t \to \pm\infty$. Then $\mathfrak{I}f(X) = P_+(X) - P_-(X)$.*

Notice that, since F is defined up to a polynomial of degree at most $2k-2$, the difference $P_+(X) - P_-(X)$ is independent of the choice of F.

Proof. If we apply $\bar{\delta}$ to the statement of Lemma 5 with $n = k-1$ and $m = 0$ we get

$$(x-y)^2\frac{\partial}{\partial y}\Big(\sum_{i=0}^{2k-2}\frac{F^{(i)}(y)}{i!}(x-y)^i\Big) = \frac{1}{(2k-2)!}\left(\delta^-\right)^{2k-1}\Big(\frac{F(y)}{(x-y)^{2k-2}}\Big)$$

$$= \frac{1}{(2k-2)!}(x-y)^{2k}\left(\delta^{2k-1}F\right)(y,x) = (x-y)^{2k}f(y).$$

Hence

$$\mathfrak{I}f(X) = \sum_{i=0}^{2n}\frac{F^{(i)}(t)}{i!}(X-t)^i\,\Big|_{t=-\infty}^{t=+\infty} = P_+(X) - P_-(X). \tag{54}$$

□

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