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Proof of the gamma conjecture for Fano 3-folds of Picard rank 1

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Abstract. We verify the (first) gamma conjecture, which relates the gamma class of a Fano variety to the asymptotics at infinity of the Frobenius solutions of its associated quantum differential equation, for all 17 of the deformation classes of Fano 3-folds of rank 1. This involves computing the corresponding limits ('Frobenius limits') for the Picard–Fuchs differential equations of Apéry type associated by mirror symmetry with the Fano families, and is achieved using two methods, one combinatorial and the other using the modular properties of the differential equations. The gamma conjecture for Fano 3-folds always contains a rational multiple of the number $\zeta(3)$. We present numerical evidence suggesting that higher Frobenius limits of Apéry-like differential equations may be related to multiple zeta values.

Keywords: gamma class, gamma conjecture, Picard–Fuchs equation, Fano 3-fold.

Dedicated to the memory of Andrei Andreevich Bolibrukh

Introduction

The goal of this paper is twofold. On the one hand, we will calculate certain entries of the transition matrix (or central connection matrix) for the Laplace transform of a number of Apéry-like differential equations and find the expansion coefficients of the most rapidly growing solution at infinity in terms of the Frobenius basis at $z = 0$. On the other hand, these calculations, whose results always involve the number $\zeta(3)$, enable us to verify the prediction of mirror symmetry (the so-called first gamma conjecture) for each of the 17 deformation classes in Iskovskikh's classification of smooth Fano 3-folds of Picard rank 1.

We will present two approaches to computing the limits in question. One of them, which we carry out for the differential equation satisfied by the generating function of the numbers used by Apéry in his famous proof of the irrationality of $\zeta(3)$, is based on the explicit hypergeometric formula that he gave for these numbers. This method is less satisfactory since it depends on complicated formulae that were found experimentally and whose proofs are somewhat artificial. Nevertheless we include

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it because it could also be applied in non-modular situations (like the differential equations associated with most Fano 4-folds). It also works almost automatically whenever the differential equation is of hypergeometric type, which is the case in 10 of the 17 cases in Iskovskikh's list. The second method is based on the modular parametrization of the differential equations in question, like the one found many years ago by Beukers in the Apéry case. This method is much nicer and explains the crucial constant $\zeta(3)$ as a period of an Eisenstein series. It works in a uniform way for each of the differential equations admitting a modular parametrization, which holds for 15 of the 17 families, including all non-hypergeometric ones. This gives the following theorem, which is our main result.

Theorem 1. *The gamma conjecture holds for all Fano* 3*-folds of Picard rank* 1.

Since the themes of the paper cover a wide spectrum and may not all be known to the same readers, we will include in § 1 a review of the main ingredients in the story (Fano varieties, Iskovskikh classification, quantum differential equations, gamma classes, gamma conjecture) for completeness. However, the actual calculations of the limits, which are given in § 2, involve only classical tools from number theory and can be read independently of this material. In the rest of this introduction we give a little more indication of the background of the problem and state the explicit prediction made by the gamma conjecture in the Apéry case.

Mirror symmetry predicts, among other things, that with each of the 17 families of Fano varieties in question there should be associated a family $\mathcal{E}$ of K3 surfaces over $\mathbb{P}^1$ (related by an isogeny to the so-called Landau–Ginzburg model) in such a way that the 'quantum differential equation' on the Fano side is the Laplace transform of the Picard–Fuchs differential equation satisfied by the periods of $\mathcal{E}$. These quantum differential equations, whose definition will be recalled briefly in § 1.4, arise from counting embedded holomorphic curves (Gromov–Witten invariants), so that this is the 'A-side' in the terminology of string theory (where one is usually interested in families of Calabi–Yau 3-folds) while both the Landau–Ginzburg model and our second family $\mathcal{E}$ with its Picard–Fuchs equation would be the 'B-side'. One further expects that the Picard rank of the Fano variety and that of the generic fibre of $\mathcal{E}$ add up to 20. In the 17 cases we are considering, these predictions of mirror symmetry were made precise in [1] and proved in [1] and [2]. In each case $\mathcal{E}$ is a family of Kuga–Sato type whose base space is equal to the modular curve[1] $X_0^*(N) = X_0(N)/W_N$ classifying the unordered pairs (E, E') of N-isogeneous elliptic curves for some N, and the fibre is equal to the smooth resolution of the quotient of $E \times E'$ by (-1), with Picard number $19 = 20 - 1$ as it should be (with 16 algebraic cycles coming from the resolutions of the 16 singularities, and 3 more from the classes of E, E' and the graph of the isogeny). The fact that these families are of Kuga–Sato type means precisely that the solutions of their associated Picard–Fuchs differential equations have modular parametrizations. More specifically, each of these differential equations has a unique holomorphic solution of the form $\Phi(t) = \sum_{n=0}^{\infty} A_n t^n$ with $A_0 = 1$ at the point $t = 0$, where t is a suitably chosen coordinate on the base $\mathbb{P}^1$, and the modular parametrization says that $\Phi(t(\tau))^2$

[1] Or a d-sheeted covering of it, where d is the 'index'. In the introduction we assume that $d = 1$.

is a modular form of weight 4 on $\Gamma_0^*(N)$ for some Hauptmodul $t(\tau)$ on $X_0^*(N)$. The gamma conjecture relates the asymptotics at infinity of the four Frobenius solutions of the differential equation satisfied by $\Psi(z) = \sum_{n=0}^{\infty} A_n z^n/n!$ to the so-called gamma class (a characteristic class in cohomology with real coefficients, whose definition will be recalled in § 1.5) of the corresponding Fano 3-fold.

We now describe the picture in more detail in the Apéry case, corresponding to $N = 6$. Here we have

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad \Phi(t) = 1 + 5t + 73t^2 + 1445t^3 + \cdots . \tag{0.1}$$

The corresponding recursion and differential equation are

$$(n+1)^3 A_{n+1} - P(n)A_n + n^3 A_{n-1} = 0, \qquad \bigl(D^3 - tP(D) + t^2(D+1)^3\bigr)\Phi(t) = 0, \tag{0.2}$$

where $D = t\,d/dt$ and $P(n)$ is the polynomial

$$34n^3 + 51n^2 + 27n + 5 = (2n+1)(17n^2 + 17n + 5).$$

The modular parametrization, as found by Beukers [3], is given by

$$t = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12} = q - 12q^2 + 66q^3 + \cdots ,$$

$$\Phi(t) = \frac{(\eta(2\tau)\eta(3\tau))^7}{(\eta(\tau)\eta(6\tau))^5} = 1 + 5q + 13q^2 + \cdots ,$$

where $\eta(\tau)$ is the Dedekind eta-function and $q = e^{2\pi i\tau}$, as usual. To state the gamma conjecture, we consider instead of $\Phi(t)$ the related power series [2]

$$\Psi(z) = \sum_{n=0}^{\infty} A_n \frac{z^n}{n!} = 1 + 5z + \frac{73}{2}z^2 + \frac{1445}{6}z^3 + \cdots .$$

It also satisfies a linear differential equation, this time of order 4, having no singularities on $\mathbb{C}^*$, a regular singularity at 0 and an irregular one at infinity. (This is what we called the 'Laplace transform' above, although 'Borel transform' might be a better name.) The space of solutions of the transformed equation near $z = 0$ has the standard Frobenius basis $\Psi_j(z)$, $0 \leqslant j \leqslant 3$, where each of the $\Psi_j(z)$ has a singularity of the form $(\log z)^j/j!$ near the origin. We define the *Frobenius limits* κ_j by the formula

$$\kappa_j := \lim_{z\to\infty} \frac{\Psi_j(z)}{\Psi(z)}. \tag{0.3}$$

The gamma conjecture gives their values in terms of $\zeta(3)$ and the Chern numbers of one of the Fano varieties (the one called V_{12}) in the Iskovskikh list. Specifically, it predicts that

$$\kappa_1 = -\gamma, \qquad \kappa_2 = \frac{\gamma^2}{2} - \frac{3}{2}\zeta(2), \qquad \kappa_3 = -\frac{\gamma^3}{6} + \frac{3}{2}\gamma\zeta(2) + \frac{5}{2}\zeta(3), \tag{0.4}$$

[2] We should warn the reader that our z is the reciprocal of the one occurring frequently in the literature.

where γ is Euler's constant. (An easy explicit computation of the gamma class in this case and in the other 16 cases is given in § 1.5, Proposition 2.) This will be proved as a special case of our main theorem (§ 1.3, Theorem 3), which gives the Frobenius limits in all 17 cases.

We end by remarking that the Frobenius basis is a part of an infinite family of functions $\Psi_j(z)$, $j \geqslant 0$, with singularities of type $(\log z)^j$ at the origin, but with Ψ_j, $j \geqslant 4$, satisfying some inhomogeneous versions of the differential equation for $\Psi(z)$. The limits κ_j defined in (0.3) still exist and can be computed numerically to high precision by a method that will be explained in § 2.4. We find that for $j \leqslant 10$ each of these ratios is again a polynomial in γ and the Riemann zeta values, but this is not the case for κ_{11}, which involves a *multiple* zeta value as well. This point, although purely experimental at the moment, seems worth mentioning. It will be described in more detail in § 2.4.

Since the main ingredient in our story is the monodromy of Fuchsian differential equations, we hope that it is a suitable homage to Andrei Andreevich Bolibrukh, who contributed so deeply to this subject.

§ 1. Fano varieties, Apéry-like differential equations, and mirror symmetry

In this section we briefly describe the gamma conjecture (see [4] and [5] for more details) and give a complete statement of the prediction it makes for Fano 3-folds whose Picard groups have rank 1. We also describe the associated Picard–Fuchs equations and their modular parametrizations, following [1].

1.1. Fano varieties and the Iskovskikh classification. In this paper a *Fano variety* X means a smooth projective algebraic variety with ample anticanonical class. The projective line $\mathbb{P}^1$ is the only Fano variety of dimension 1. Fano varieties of dimension 2 are called del Pezzo surfaces; these are either $\mathbb{P}^1 \times \mathbb{P}^1$, or blow-ups of $\mathbb{P}^2$ at d points, $0 \leqslant d \leqslant 8$. By the results of Mori and Mukai [6] there are 105 deformation families of Fano 3-folds.

We will be interested in Fano 3-folds whose Picard lattices have rank 1. (Since all cohomology classes of degree 2 are algebraic here, this simply means that $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$.) According to Iskovskikh [7], [8] (see also [9]) there are exactly 17 such varieties up to deformation. The relevant numerical invariants for this classification are the *index* $d = [H^2(X,\mathbb{Z})\colon \mathbb{Z}c_1]$, where c_1 is the anticanonical divisor, and the *level* $N = \frac{1}{2d^2}\langle c_1^3, [X]\rangle$, which is always a positive integer. The 17 possible pairs of invariants (N,d) are then

$$\begin{gathered}(1,1),\ (2,1),\ (3,1),\ (4,1),\ (5,1),\ (6,1),\ (7,1),\ (8,1),\ (9,1),\ (11,1),\\ (1,2),\ (2,2),\ (3,2),\ (4,2),\ (5,2),\ (3,3),\ (2,4).\end{gathered} \tag{1.1}$$

For example, the pair $(N,d) = (2,4)$ corresponds to the Fano variety $X = \mathbb{P}^3$ (so here the deformation family has a 0-dimensional base), whose associated Picard–Fuchs differential equation is hypergeometric. The pair $(N,d) = (6,1)$ determines a family which is called V_{12} (its geometric definition plays no role for us and will be omitted) and corresponds to the Apéry differential equation as described in the introduction. This case will hereafter be referred to as the 'Apéry case'.

1.2. Apéry-like differential equations. Here we describe the Picard–Fuchs differential equations associated with the 17 cases. The modular properties of these equations are presented in the next subsection, and the relation of their Laplace transforms to the quantum cohomology of Fano varieties is described in § 1.4.

The differential equations occurring are of type D3. Here 'type Dn' (the full name is 'determinantal differential equations of order n') is a specific class of linear differential equations, introduced in [10], which includes the Picard–Fuchs differential equations of certain families of Calabi–Yau varieties of dimension $n-1$ and, in its Laplace-transformed version, the quantum differential equations of certain n-dimensional Fano varieties. Operators of type D2, which are of the form

$$D^2 + t(a_1 D(D+1) + b_1) + a_2 t^2 (D+1)^2 + a_3 t^3 (D+1)(D+2), \qquad D = t\frac{d}{dt},$$

are precisely the ones appearing as the 'Apéry-like differential equations' in [11] and [12]. Their prototype was the differential equation of order 2 coming from the coefficients used by Apéry in his new proof of the irrationality of $\zeta(2)$. Since the class D3 also contains the differential equation of order 3 corresponding to Apéry's proof of irrationality of $\zeta(3)$, we will use the terminology 'Apéry-like equations of order n' as an alternative name for the class Dn. The generic D3 operator, which is the case that we will be studying, is of the form

$$\begin{aligned}\mathcal{L} = D^3 &+ t\left(D+\frac{1}{2}\right)(a_1 D(D+1) + b_1) + t^2(D+1)(a_2(D+1)^2 + b_2) \\ &+ a_3 t^3 (D+1)\left(D+\frac{3}{2}\right)(D+2) + a_4 t^4 (D+1)(D+2)(D+3). \qquad (1.2)\end{aligned}$$

It can also be written as tL, where L is a differential operator of the form

$$L = t^2 Q\frac{d^3}{dt^3} + \frac{3}{2}(t^2Q)'\frac{d^2}{dt^2} + \left(\frac{t^2}{2}Q'' + 3tQ' + R\right)\frac{d}{dt} + \left(\frac{t}{2}Q'' + \frac{1}{2}R'\right), \qquad (1.3)$$

in which the polynomials

$$Q = Q(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4, \qquad R = R(t) = 1 + b_1 t + b_2 t^2 \qquad (1.4)$$

of degrees $\leqslant 4$ and $\leqslant 2$ describe the positions of the singularities of the equation and the so-called accessory parameters, respectively.

In general, a Dn equation is obtained from an $(n+1)\times(n+1)$ matrix $\mathcal{A} = (a_{i,j})_{0\leqslant i,j\leqslant n}$ whose entries (which in the quantum cohomology situation arise as correlators in a way recalled briefly in § 1.4) satisfy the equations

$$\begin{gathered} a_{i,j} = 0 \quad \text{for} \quad j < i-1, \qquad a_{i,j} = 1 \quad \text{for} \quad j = i-1, \\ a_{i,j} = a_{n-j,n-i}, \qquad 0 \leqslant i,j \leqslant n, \end{gathered}$$

so that this family of equations has $n^2/4 + O(n)$ parameters. The differential operator $\mathcal{L}$ corresponding to $\mathcal{A}$ and its Laplace transform $\tilde{\mathcal{L}}$ (related to $\mathcal{L}$ by the condition $\tilde{\mathcal{L}}\Psi = 0 \Leftrightarrow \mathcal{L}\Phi = 0$, where $\Phi(t) = \sum A_n t^n$ and $\Psi(z) = \sum A_n z^n/n!$) are then given by

$$\mathcal{L} = D^{-1}\mathrm{det}_R\big((\delta_{i,j}D - a_{i,j}(Dt)^{j-i+1})_{0\leqslant i,j\leqslant n}\big) \qquad (1.5)$$

and (setting $D_z = z\,d/dz$)

$$\tilde{\mathcal{L}} = \det_R\big((\delta_{i,j} D_z - a_{i,j} z^{j-i+1})_{0\leqslant i,j\leqslant n}\big). \tag{1.6}$$

Here the 'right determinant' $\det_R$ of a matrix with non-commuting entries is defined by induction as the alternating sum of the rightmost entries multiplied on the right by the right determinants of the corresponding minors, and formula (1.5) makes sense because every term in the rightmost column of the corresponding matrix is left-divisible by D. An alternative and possibly more natural way to introduce the two operators is to consider the connection given in matrix form by

$$D_z(\xi_0, \dots, \xi_n) = (\xi_0, \dots, \xi_n)(a_{i,j} z^{j-i+1})_{0\leqslant i,j\leqslant n}; \tag{1.7}$$

then $\widetilde{\mathcal{L}}$ is the operator that annihilates ξ_0, and $\mathcal{L}$ arises formally as its 'inverse Laplace transform'. We regard two matrices $\mathcal{A}$ differing by a scalar cI as equivalent, because this corresponds merely to a translation $t^{-1} \mapsto t^{-1} + c$ on the t-side or to multiplication by $\exp(cz)$ on the z-side, with no effect on the Frobenius limits in (0.3). We remark that operators of type Dn are self-dual in the sense that the coefficient of t^i is $(-1)^{n-i}$-symmetric under $D \mapsto -D - i$. The equations of type D3 are also symmetric squares, which is important for us because being a symmetric power of a second-order differential operator is a necessary condition for modularity. The corresponding assertion for higher Dn is completely false, and indeed the D4-equations occurring as the Picard–Fuchs equations of families of Calabi–Yau 3-folds are almost never modular.

The 4×4 matrices corresponding to the 17 Iskovskikh families were listed in [1]. For example, the matrices

$$\begin{pmatrix} 5 & 96 & 1692 & 12816 \\ 1 & 12 & 216 & 1692 \\ 0 & 1 & 12 & 96 \\ 0 & 0 & 1 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 12/5 & 24 & 198 & 880 \\ 1 & 22/5 & 44 & 198 \\ 0 & 1 & 22/5 & 24 \\ 0 & 0 & 1 & 12/5 \end{pmatrix}$$

correspond to the Apéry case $(N, d) = (6, 1)$ and to the most complicated case $(N, d) = (11, 1)$, in which the differential equation corresponds to a 5-term recursion for the coefficients of its holomorphic solution, and the scalar shifts are chosen to make the solutions become Eisenstein series. Instead of giving the matrices in the remaining cases, we give in Table 1 the polynomials Q and R defined by (1.4). These polynomials contain the same information and are much more compact to write down. We give the data only for $d = 1$ since there is a simple algebraic procedure for deducing the differential equation satisfied by a power series $\Phi(t^d)$ from the one satisfied by $\Phi(t)$. (In our cases the passage from $(N, 1)$ to (N, d) simply replaces $Q(t)$ by $Q(t^d)$ while the new R-polynomial equals $1 + (4b_1 - a_1)t^2$ when $d = 2$ and just 1 when $d = 3$ or 4, where a_1 and b_1 are the linear coefficients of the original polynomials Q and R.) The final three columns of the table contain certain invariants $(f_M)_{M|N}$, $(h_M)_{M|N}$ and $\mu_N = \frac{1}{2}\sum_M M h_M$, which will be defined in the next subsection (see (1.14)) and used in the proof of the gamma conjecture.

Table 1

N	$Q(t)$	$R(t)$	$\{f_M\}$	$\{h_M\}$	μ_N
1	$1-1728t$	$1-240t$	—	—	62
2	$1-256t$	$1-48t$	$(24,-24)$	$(-80,80)$	40
3	$1-108t$	$1-24t$	$(12,-12)$	$(-30,30)$	30
4	$1-64t$	$1-16t$	$(8,0,-8)$	$(-16,0,16)$	24
5	$1-44t-16t^2$	$1-12t+4t^2$	$(6,-6)$	$(-10,10)$	20
6	$1-34t+t^2$	$1-10t$	$(5,-1,1,-5)$	$(-7,1,-1,7)$	17
7	$1-26t-27t^2$	$1-8t+3t^2$	$(4,-4)$	$(-5,5)$	15
8	$1-24t+16t^2$	$1-8t$	$(4,-2,2,-4)$	$(-4,1,-1,4)$	13
9	$1-18t-27t^2$	$1-6t$	$(3,0,-3)$	$(-3,0,3)$	12
11	$1-\frac{68}{5}t-\frac{616}{25}t^2-\frac{252}{125}t^3-\frac{1504}{625}t^4$	$1-\frac{24}{5}t-\frac{56}{25}t^2$	$(\frac{12}{5},-\frac{12}{5})$	$(-2,2)$	10

1.3. Modular properties. The operators we are studying are of type D3 by construction. They also have very special modularity properties, discovered in [1], which we now describe.

For every $N \geqslant 1$ we have the following modular curves: $X_0(N)$ (which is defined over $\mathbb{C}$ as $\mathbb{H}/\Gamma_0(N)\cup\{\text{cusps}\}$, that is, the completed quotient of the upper half-plane by the congruence subgroup of level N) and $X_0^*(N)$ (the quotient of $X_0(N)$ by the Fricke involution W_N sending $\tau \in \mathbb{H}$ to $-1/N\tau$). As usual, we write $\Gamma_0^*(N)$ for the group generated by $\Gamma_0(N)$ and W_N. As moduli spaces, $X_0(N)$ and $X_0^*(N)$ parametrize the ordered and unordered pairs (respectively) of elliptic curves related by a cyclic isogeny of degree N. The involution W_N acts on the space $M_k(\Gamma_0(N))$ of holomorphic modular forms of weight k on $\Gamma_0(N)$ by the formula $(f|_k W_N)(\tau) = N^{k/2}\tau^k f(-1/N\tau)$ and splits this space into two eigenspaces $M_k^{\pm}(\Gamma_0(N))$ with

$$M_k^+(\Gamma_0(N)) = M_k(\Gamma_0^*(N)), \qquad M_k^-(\Gamma_0(N)) = M_k(\Gamma_0^*(N),\chi),$$

where $\chi\colon \Gamma_0^*(N) \to \{\pm 1\}$ is the homomorphism sending $\Gamma_0(N)$ to $+1$ and W_N to -1. If $N=1$, then $W_N = S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ belongs to $\Gamma_0(N) = \mathrm{SL}(2,\mathbb{Z})$, so that W_N has only the eigenvalue $+1$. In this case, by an abuse of notation, we will write $F \in M_k^-(\Gamma_0(1))$ if $F\sqrt{E_4}$ is a modular form of weight $k+2$ on $\Gamma_0(1)$, where $E_4(\tau) = 1+240q+\cdots$ is the normalized Eisenstein series of weight 4. Then F is not a single-valued function on $\mathbb{H}$ (since E_4 has simple zeros at $\tau = (\pm 1 + i\sqrt{3})/2$), but it is well defined and holomorphic on the union of the closed standard fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ and its image under S (this union is closed under S and contains the fixed point i) and satisfies the functional equation $(F|_k S) = -F$ in this domain.

We now state the main result on the modular properties of the differential equations associated with the 17 families in Iskovskikh's list. The first assertion of this theorem was proved in [1] along with a statement of the second assertion and a sketch of its proof in some cases. The remaining harder cases were checked by Przyjalkowski [2]. The ingredients used were the quantum Lefschetz theorem of Givental [13], the computation of quantum multiplication by the first Chern class in the Grassmannians by Przyjalkowski [14] and Fulton and Woodward [15], Kuznetsov's calculation of GW-invariants for the varieties V_{22} (private communication), and Beauville's result [16] on V_5.

Theorem 2 (see [1], [2]). *Let N and d be positive integers such that the curve $\Gamma_0^*(N)$ has genus* 0 *and there is a modular form in $M_2(\Gamma_0(N))^-$ whose differential equation with respect to the dth root of a Hauptmodul*[3] *of the group $\Gamma_0^*(N)$ is of type* D3. *Then (N,d) belongs to the list* (1.1) *and, up to equivalence, the Laplace transform of this differential equation is the quantum differential equation* (*see* § 1.4 *below*) *of the corresponding Fano variety in Iskovskikh's classification.*

The modular form F and the Hauptmodul t occurring in this theorem can be given by a (nearly) uniform formula. If $N > 1$, we define a modular form $F_N \in M_2(\Gamma_0(N))^-$ as the unique Eisenstein series of weight 2 on $\Gamma_0(N)$ which is equal to $+1$ at $\tau = \infty$, -1 at $\tau = 0$, and 0 at the other cusps. If $N = 1$, this definition makes no sense, but we can set $F_1 = \sqrt{E_4}$, and this again belongs to $M_2^-(\Gamma_0(1))$ in the sense just introduced. Then, in every case, the modular form $F = F_{N,d}$ and the Hauptmodul $t = t_{N,d}$, whose existence is asserted by the theorem, are given (after the normalizations $F(\tau) = 1 + O(q)$ and $t(\tau) = q + O(q^2)$ and up to the equivalence $t \mapsto t/(1+ct)$, $F \mapsto (1+ct)F$) by the uniform formulae

$$F_{N,d}(\tau) = F_N(d\tau), \qquad t_{N,d}(\tau) = t_N(d\tau)^{1/d} \tag{1.8}$$

for all N and d, where the power series $t_N(\tau)$ in q is defined by the formula

$$F_N(\tau) t_N(\tau)^{(N+1)/12} = \eta(\tau)^2 \eta(N\tau)^2. \tag{1.9}$$

Conversely, it is an elementary exercise to show that the only pairs $(N,d) \in \mathbb{N}^2$ for which the function $t = t_{N,d}$ defined by (1.8) and (1.9) is a modular function on $\Gamma_0(dN)$ are those listed in (1.1). We sketch the argument. If N is sufficiently large, then the well-known 'valency formula' (a formula giving the number of zeros of a holomorphic modular form in a fundamental domain for the group) implies that the order of at least one zero of F_N in $\mathbb{H}$ is not divisible by $(N+1)/\mathrm{GCD}(N+1, 12)$, and then the function $t_N(\tau)$ defined for large $\mathrm{Im}(\tau)$ by (1.9) cannot be extended to the upper half-plane even as a single-valued meromorphic function, let alone a modular function. Checking the remaining cases by computer, we find that the only N for which the required root can be extracted are the ten values $N = 1, \dots, 9, 11$ occurring in (1.1) and four further values $N = 12$, 16, 18 and 36. But in each of the latter cases, the function $t_N(\tau)$ defined by (1.9) is not modular on $\Gamma_0(N)$. This fixes all possible values of N, and then for every N a similar argument shows that for all values of d except those occurring in (1.1), the function $t_{N,d}$ defined by (1.8) is only a root of a modular function on $\Gamma_0(dN)$.

We note that in each of the four cases $N = 12$, 16, 18 and 36 mentioned above, F_N is an eta-product and t^{N+1} is also an eta-product and is a modular function

[3] That is, a modular function of weight 0 parametrizing the quotient of the upper half-plane with respect to this group.

on $\Gamma_0^*(N)$, even though t itself is not. For example, if $N = 12$, then

$$F_{12}(\tau) = \frac{\eta(2\tau)^4\eta(3\tau)\eta(4\tau)\eta(6\tau)^4}{\eta(\tau)^3\eta(12\tau)^3},$$
$$t_{12}(\tau) = \left(\frac{\eta(\tau)^{60}\eta(12\tau)^{60}}{\eta(2\tau)^{48}\eta(3\tau)^{12}\eta(4\tau)^{12}\eta(6\tau)^{48}}\right)^{1/13}, \tag{1.10}$$

so t^{13} is a modular function on $\Gamma_0(12)$, but t is invariant only under a non-congruence subgroup of $\Gamma_0(12)$ of index 13.

As a side remark, we mention that there are exactly eight values of N for which the modular form $F_N \in M_2^-(\Gamma_0(N))$ can be written as a quotient of products of eta-functions:

N	4	6	8	9	12	16	18	36
F_N	$\frac{2^{20}}{1^8 4^8}$	$\frac{2^7 3^7}{1^5 6^5}$	$\frac{2^6 4^6}{1^4 8^4}$	$\frac{3^{10}}{1^3 3^3}$	$\frac{2^4 3^1 4^1 6^4}{1^3 12^3}$	$\frac{2^1 4^6 8^1}{1^2 16^2}$	$\frac{2^1 3^3 6^3 9^1}{2^2 18^2}$	$\frac{4^1 6^4 9^1}{1^1 36^1}$

The notation in this table becomes clear from a comparison of the $N = 12$ entry with the formula (1.10) for F_{12}, or the $N = 6$ entry with the formula for $\Phi(t)$ in the Apéry case that was given in the introduction.

The modular form $F = F_{N,d}$ can be written in all cases as a power series $\Phi(t)$ in the Hauptmodul $t = t_{N,d}$, where $\Phi = \Phi_{N,d}$ is a power series with integer coefficients and leading coefficient 1. This integrality, which is clear from the modular description, is not at all obvious from the recursion for the coefficients that comes from the differential equation for Φ. This was one part of the 'Apéry miracle' that was demystified by Beuker's modular interpretation. The operator $\mathcal{L}$, which annihilates Φ, then has the form tL with L given by (1.3), as explained in the previous subsection. The following proposition expresses it in purely modular terms.

Proposition 1. *The differential operator $\mathcal{L}$ is given in terms of the modular variable τ by the formula*

$$\mathcal{L} = \frac{1}{H(\tau)}\left(\frac{1}{2\pi i}\,\frac{d}{d\tau}\right)^3 \frac{1}{F(\tau)}, \tag{1.11}$$

where $F = F_{N,d} \in M_2^-(\Gamma_0(N))$ is the modular form defined above and $H = H_{N,d}$ is a modular form belonging to $M_4(\Gamma_0(N))^-$. This form is given explicitly by the formula $H_{N,d}(\tau) = H_N(d\tau)$, where H_N for $N > 1$ is the unique Eisenstein series of weight 4 on $\Gamma_0(N)$ which equals $+1$ at $\tau = \infty$, -1 at $\tau = 0$, and 0 at the other cusps, and $H_1 = E_6/\sqrt{E_4}$.

Proof. We first note that the space of solutions of the differential equation is spanned by $F(\tau)$, $\tau F(\tau)$ and $\tau^2 F(\tau)$. Hence it consists of precisely those functions whose quotient by $F(\tau)$ is annihilated by $(d/d\tau)^3$. Therefore $\mathcal{L}$ must be of the form (1.11) for some function $H(\tau)$. Equating the symbols (that is, the coefficients of $(d/dt)^3$) on both sides of (1.11), using (1.3), we find that

$$t(\tau)^2 Q(t(\tau)) = \frac{1}{H(\tau)}\,\frac{t'(\tau)^3}{t(\tau)F(\tau)}, \tag{1.12}$$

where $t' = \frac{1}{2\pi i}\frac{dt}{d\tau}$, and the expression (1.12) is at least a meromorphic modular form of weight 4 on $\Gamma_0(N)$ with W_N-eigenvalue -1. On the other hand, since the differential operator is of type D3, a purely algebraic calculation shows that

$$H(\tau) = F(\tau)\frac{t'(\tau)}{t(\tau)}. \tag{1.13}$$

Comparing these two formulae, we see that the modular form H (for $N > 1$) is holomorphic and takes the prescribed values at the cusps. This yields the desired conclusion since $S_4(\Gamma_0(N))^-$ is always equal to 0 here. Alternatively, one can simply check case-by-case that the right-hand side of any of the equations above coincides with $E^-_{4,N}$ when $N > 1$ and with $E_6(\tau)/\sqrt{E_4(\tau)}$ when $N = 1$. The cases with $d > 1$ follow easily from the cases with $d = 1$, with $H(\tau) = H_{N,d}(\tau)$ defined as $H_N(d\tau)$. □

Note that the equality of the equations (1.12) and (1.13) for H implies that the curve $u^2 = Q(t)$, which is either rational or elliptic depending on the degree of Q, has a modular parametrization by $t = t(\tau)$, $u = t'/tF$, and in particular gives the Taniyama–Weil (Taylor–Wiles) parametrization when $\deg Q > 2$ (which here happens in only four cases, $(N,d) \in \{(11,1), (5,2), (3,3), (2,4)\}$). In this case the function $f = tF$ is the cusp form of weight 2 for which the expression $f(\tau)\,d\tau = t'(\tau)/u(\tau)$ is the Weierstrass differential dt/u on the corresponding elliptic curve. Hence this function is a Hecke eigenform with multiplicative coefficients. In this connection we mention the joint paper [17] of one of us and Vlasenko, which contains among other things a classification of all D3-equations with five distinct singularities for which the function $t\Phi(t)$ has an expansion with respect to $\exp(\Phi_1/\Phi_0)$ with multiplicative coefficients (which in the modular case again means that this function is a Hecke eigenform).

If $N > 1$, then since all Eisenstein series on $\Gamma_0(N)$ come from level 1 in the cases considered, the functions F_N and H_N are given by formulae of the form

$$F_N(\tau) = \sum_{M|N} M f_M G_2(M\tau), \qquad H_N(\tau) = \sum_{M|N} M^2 h_M G_4(M\tau), \tag{1.14}$$

where

$$G_2(\tau) = -\frac{1}{24}E_2(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},$$
$$G_4(\tau) = \frac{1}{240}E_4(\tau) = \frac{1}{240} + \sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}$$

are the Hecke-normalized Eisenstein series of level 1 and weights 2 and 4, and the coefficients f_M, h_M $(M|N)$ are rational and possess the antisymmetry property

$$f_{N/M} = -f_M, \qquad h_{N/M} = -h_M \qquad (M|N) \tag{1.15}$$

because F and H are in the (-1)-eigenspace of the involution W_N. (Note that the function $G_2(\tau)$ is only quasimodular, but since (1.15) implies that $\sum f_M$ vanishes, the right-hand side of the formula for F_N in (1.14) is modular.) The coefficients f_M

and h_M were tabulated in § 1.2 together with the coefficients of the corresponding D3-operators. The related numbers

$$\mu_N = \begin{cases} 62 & \text{for } N = 1, \\ \dfrac{1}{2} \displaystyle\sum_{M|N} M h_M & \text{for } N > 1, \end{cases} \tag{1.16}$$

which were also tabulated in § 1.2, appear in the following theorem that gives the values of the Frobenius limits occurring in the gamma conjecture.

Theorem 3. *Let $\Psi_0 = \Psi$, Ψ_1, Ψ_2, Ψ_3 be the Frobenius solutions of the fourth-order linear differential equation satisfied by $\Psi(z) = \sum A_n z^n/n!$, where $F_{N,d}(\tau) = \sum A_n t_{N,d}(\tau)^n$ for one of the 17 pairs (N, d) in (1.1). Define the Frobenius limits κ_j, $j = 1, 2, 3$, by equation (0.3). Then*

$$\kappa_1 = -\gamma, \qquad \kappa_2 = \frac{\gamma^2}{2} - \left(\frac{12}{d^2N} - \frac{1}{2}\right)\zeta(2),$$

$$\kappa_3 = -\frac{\gamma^3}{6} + \left(\frac{12}{d^2N} - \frac{1}{2}\right)\gamma\zeta(2) + \left(\frac{\mu_N}{d^3N} - \frac{1}{3}\right)\zeta(3),$$

where μ_N are the numbers defined in (1.16).

This theorem will be proved in §§ 2.2, 2.3. We remark that the formulae in the theorem can be written in the simpler-looking form

$$\Gamma(1+\varepsilon)^{-1} \sum_{j=0}^{\infty} \kappa_j \varepsilon^j = 1 - \frac{2}{d^2N}\pi^2\varepsilon^2 + \frac{\mu_N}{d^3N}\zeta(3)\varepsilon^3 + O(\varepsilon^4). \tag{1.17}$$

It is in fact the expression on the left-hand side of (1.17) which is naturally computed from the modular side, since this expression describes certain limits associated with Frobenius solutions of the differential equation for Φ itself, rather than for its Laplace transform Ψ. On the topological side, this is related to the 'modified gamma class' as defined in (1.19) below.

1.4. Quantum differential equations of Fano varieties. In this subsection we briefly explain the definition of the differential equations whose asymptotic properties play a role in the gamma conjecture. We first describe the meaning of the entries of the 4×4 matrix $\mathcal{A} = (a_{i,j})$ that specifies the quantum differential equation by the formula (1.6). Then, for the benefit of the interested reader, we briefly explain how this determinantal equation arises.

Very roughly, the entry $a_{i,j}$ with $j \geqslant i$ (the other entries of $\mathcal{A}$ are 0 or 1 by definition) gives the (correctly interpreted) 'number' of rational curves of anticanonical degree $j - i + 1$ that intersect algebraic cycles of codimension specified by i and j. We state this more precisely in the case of interest to us, when X is a Fano 3-fold with $H^2(X; \mathbb{Q}) = \mathbb{Q}c_1$. The *three-point correlator* $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ of three (effective and homogeneous) classes a_1, α_2, α_3 in $H^*(X)$ is defined as the polynomial $\sum c_n z^n$, where c_n is the expected number of holomorphic maps $\mathbb{P}^1 \to X$ of anticanonical degree n sending three fixed generic points P_i of $\mathbb{P}^1$ to cycles representing the Poincaré duals of the classes α_i. Using this correlator, one defines the *quantum cohomology ring* of X as $H^*(X)\otimes\mathbb{C}[z]$ with quantum multiplication $\star$, where $\alpha_1\star\alpha_2$ is defined by regarding $\langle\alpha_1,\alpha_2,\,\cdot\,\rangle$ as a linear functional on $H^*(X)\otimes\mathbb{C}[z]$ with values in $\mathbb{C}[z]$ and dualizing it with respect to the Poincaré pairing: $\int_{[X]}(\alpha_1\star\alpha_2)\cup\alpha_3=\langle\alpha_1,\alpha_2,\alpha_3\rangle$. Then the matrix $G(z)=(g_{i,j}(z))$ of the operator $c_1\star$ with respect to the basis $\mathbf{c}=(1,c_1,c_1^2,c_1^3)$ is of the form $g_{i,j}=a_{i,j}z^{j-i+1}$ for some matrix $\mathcal{A}$, and the desired fourth-order differential operator $\tilde{\mathcal{L}}$ is associated with the system $D_z\vec{\zeta}(z)=\vec{\zeta}(z)G(z)$ of first-order differential equations in the standard way (with 1 as the cyclic vector). The Frobenius basis $\{\Psi_i\}$ of the space of solutions of the differential equation $\tilde{\mathcal{L}}\Psi=0$ then corresponds to the basis of $H_*(X,\mathbb{Q})$ that is Poincaré dual to the basis $\mathbf{c}$. The expressions for the coefficients of $\tilde{\mathcal{L}}$ (or for the expansion coefficients of the Frobenius solutions) in terms of the Gromov–Witten invariants of X can be found in [18].

The identification described enables us to regard the linear functional $\Psi\mapsto\lim_{z\to\infty}\frac{\Psi(z)}{\Psi_0(z)}$ on the space of solutions of the differential equation $\tilde{\mathcal{L}}\Psi=0$ as a cohomology class A_X (called the 'principal asymptotic class' of X). In our case, it is given explicitly by the formula

$$A_X=\sum_{j=0}^{3}\kappa_j c_1^j\in H^*(X;\mathbb{C}),$$

where the numbers κ_j are the Frobenius limits as defined in (0.3). The gamma conjecture, to be described now, says that this class is equal to a certain characteristic class of X called the gamma class.

1.5. The gamma class and the gamma conjecture. The gamma class of a holomorphic vector bundle E over a topological space X is the multiplicative characteristic class (in the sense of Hirzebruch) associated with the power series expansion $\Gamma(1+x)=1-\gamma x+\frac{\gamma^2+\zeta(2)}{2}x^2+\cdots$ of the gamma function at 1. In other words, it sends a holomorphic vector bundle E/X to the cohomology class $\widehat{\Gamma}(E)=\prod_i\Gamma(1+\tau_i)\in H^*(X,\mathbb{R})$ if the total Chern class of E has a formal factorization $c(E)=\prod(1+\tau_i)$ with elements τ_i of degree 2. When E is the tangent bundle of X, we write simply $\widehat{\Gamma}_X$ instead of $\widehat{\Gamma}(E)$. The terms of degree $\leqslant 3$ in this class, which are the only ones needed for our purposes, are given by the formula

$$\begin{aligned}\widehat{\Gamma}_X&=1-\gamma c_1+\left(-\zeta(2)c_2+\frac{\zeta(2)+\gamma^2}{2}c_1^2\right)\\&\quad+\left(-\zeta(3)c_3+(\zeta(3)+\gamma\zeta(2))c_1c_2-\frac{2\zeta(3)+3\gamma\zeta(2)+\gamma^3}{6}c_1^3\right)+\cdots,\end{aligned}\tag{1.18}$$

where $c_i=c_i(TX)\in H^{2i}(X)$ are the Chern classes of X. If we introduce the *modified gamma class* $\widehat{\Gamma}_X^0$ by putting

$$\widehat{\Gamma}_X=\Gamma(1+c_1)\widehat{\Gamma}_X^0,\tag{1.19}$$

then the equality (1.18) takes a simpler form:

$$\widehat{\Gamma}_X^0=1-\zeta(2)c_2+\zeta(3)(c_1c_2-c_3)+\cdots.\tag{1.20}$$

(The passage from $\widehat{\Gamma}_X$ to $\widehat{\Gamma}^0_X$ in our cases reflects a relation between the topology of a Fano variety and that of its K3-surface hyperplane sections. On the other side of the gamma conjecture, as already mentioned in connection with Theorem 3, this passage corresponds to a relation between the Frobenius limits for the quantum differential equation associated with the Fano 3-fold and the Frobenius limits for the Picard–Fuchs differential equations of its mirror dual.) The gamma class of a variety may be regarded as a 'half' of the Todd class occurring in the Hirzebruch–Riemann–Roch theorem, or, more precisely, a 'half' of the $\widehat{A}$-class $\widehat{A}_X = e^{-c_1}\operatorname{Td}_X$ occurring in the Atiyah–Hirzebruch theorem, since the Γ-function identity $\Gamma(1+z)\Gamma(1-z) = \pi z/\sin(\pi z)$ implies that we can factorize the $\widehat{A}$-class in the form $\mu_+(\widehat{\Gamma}_X)\mu_-(\widehat{\Gamma}_X)$, where $\mu_\pm$ are the rescaling operators of multiplication by $(\pm 2\pi i)^{-m}$ in $H^{2m}(X;\mathbb{C})$.

Let $A_X \in H^*(X)$ be the principal asymptotic class of the quantum differential equation associated with a Fano variety X (as explained in § 1.4), and let $\widehat{\Gamma}_X \in H^*(X)$ be the gamma class of X.

We now have all the ingredients necessary to state the gamma conjecture.

Definition. If the equality

$$A_X = \widehat{\Gamma}_X \tag{1.21}$$

holds, then we say that the *gamma conjecture* [4] holds for X.

Theorem 1 says that the gamma conjecture holds in the 17 Iskovskikh cases. To prove it, we compute both sides of (1.21) independently and check that they agree in each case. The result on the Picard–Fuchs side was given in Theorem 3 and will be proved in § 2. The computation on the cohomology side is straightforward and will be given here.

Proposition 2. *The modified gamma class of a Fano* 3*-fold* X *of rank* 1 *is given by*

$$\widehat{\Gamma}^0_X = 1 - \frac{12}{d^2N}\zeta(2)c_1^2 + \frac{h^{1,2}+10}{d^2N}\zeta(3)c_1^3,$$

where c_1 *is the first Chern class of* X *and* $h^{1,2}$ *is the dimension of* $H^{1,2}(X)$.

Proof. The three Chern numbers of X are given by the formulae $\langle c_1^3,[X]\rangle = 2d^2N$ (by the definition of N), $\langle c_1c_2,[X]\rangle = 24$ (since the hyperplane sections of X are K3-surfaces and $\langle c_2,[S]\rangle = e(S) = 24$ for all K3-surfaces S) and $\langle c_3,[X]\rangle = e(X)$, where $e(X)$ is the Euler characteristic of X. Here $e(X) = 4-2h^{1,2}$ since all Hodge numbers of X vanish except for $h^{i,i}=1$ and $h^{1,2}=h^{2,1}$. Since $c_i \in H^{i,i}_{\mathrm{alg}}(X;\mathbb{Q}) = \mathbb{Q}c_1^i$, it follows that

$$c_2 = \frac{12}{d^2N}c_1^2 \quad\text{and}\quad c_1c_2 - c_3 = \frac{h^{1,2}+10}{d^2N}c_1^3.$$

Substituting these values into (1.20), we obtain the desired assertion. □

[4] This is called 'gamma conjecture I' in [5]. Since we will not discuss 'gamma conjecture II' here, we speak simply of the *gamma conjecture*.

Proof of Theorem 1. The numbers $h^{1,2}$ for the 17 Iskovskikh families are given in the table below, which is taken from the comprehensive source [9] on Fano varieties.

N	1	2	3	4	5	6	7	8	9	11	1	2	3	4	5	3	2
d	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	3	4
h^{12}	52	30	20	14	10	7	5	3	2	0	21	10	5	2	0	0	0

We check in all cases that $h^{1,2} + 10 = \mu_N/d$, where μ_N are the integers defined in (1.16) and tabulated in § 1.2.[5] Combining this with Theorem 3 in § 1.3 and Proposition 2, we obtain Theorem 1. □

§ 2. Computation of the Frobenius limits

In this section we give two different approaches to calculating the Frobenius limits. One of them uses the combinatorial description of the coefficients of the holomorphic solution as sums of binomial coefficients (like Apéry's formula in the V_{12}-case or in general Landau–Ginzburg models), and the second uses the modular representation of the solution. In § 2.2 we carry out the first approach in detail in the V_{12}-case, and also verify the prediction of the gamma conjecture in the (easier) hypergeometric cases. The modular approach, which works uniformly in all cases with $N > 1$, will be treated in § 2.3. Finally, in § 2.4 we briefly describe numerical calculations suggesting that the higher Frobenius limits (beyond the dimension of the Fano variety) are also interesting: the first few of them are still polynomials in Riemann zeta values, but further ones apparently involve more complicated kinds of periods like multiple zeta values.

2.1. The two Frobenius bases and their relationship. Here we define the higher Frobenius functions for the Picard–Fuchs type differential equations as well as for their Laplace transforms. We illustrate everything using the Apéry numbers, but the definitions work in all cases.

The definition of the Apéry numbers $A_0 = 1,\ A_1 = 5,\ A_2 = 73,\ \ldots$ and the recursion relations for them were already given in the introduction and will not be repeated here. We consider the Frobenius deformation of Apéry's recursion, that is, the sequence of power series

$$A_n(\varepsilon) = \sum_{j=0}^{\infty} A_n^{(j)} \varepsilon^j \in \mathbb{Q}[[\varepsilon]], \qquad n = 0, 1, \ldots,$$

determined by the initial conditions $A_{-1}(\varepsilon) = 0$, $A_0(\varepsilon) = 1$ and the recursion

$$(n+\varepsilon+1)^3 A_{n+1}(\varepsilon) - P(n+\varepsilon) A_n(\varepsilon) + (n+\varepsilon)^3 A_{n-1}(\varepsilon) = 0, \tag{2.1}$$

[5] Galkin has pointed out to us that the left-hand side of this equality can be interpreted as minus half the Euler characteristic of the open Calabi–Yau 3-fold obtained from X by removing an anticanonical K3-section.

where $P(x)$ is the same polynomial as in (0.2). We assemble the rational numbers $A_n^{(j)}$ into further power series:

$$\Phi_j^{\mathrm{an}}(t) = \sum_{n=0}^{\infty} A_n^{(j)} t^n, \qquad \Phi^{\mathrm{an}}(t,\varepsilon) = \sum_{j=0}^{\infty} \Phi_j^{\mathrm{an}}(t)\varepsilon^j = \sum_{n=0}^{\infty} A_n^{(\varepsilon)} t^n. \tag{2.2}$$

The beginning terms of the first few power series $\Phi_j^{\mathrm{an}}(t)$ are given by

$$\begin{aligned}
\Phi_0^{\mathrm{an}}(t) &= \Phi(t) = 1 + 5t + 73t^2 + 1445t^3 + 33001t^4 + 819005t^5 + \cdots, \\
\Phi_1^{\mathrm{an}}(t) &= 12t + 210t^2 + 4438t^3 + 104825t^4 + \frac{13276637}{5}t^5 + \cdots, \\
\Phi_2^{\mathrm{an}}(t) &= 72t^2 + 2160t^3 + 59250t^4 + 1631910t^5 + \cdots, \\
\Phi_3^{\mathrm{an}}(t) &= -7t - \frac{1011}{8}t^2 - \frac{522389}{216}t^3 - \frac{90124865}{1728}t^4 - \frac{264872026721}{216000}t^5 + \cdots, \\
\Phi_4^{\mathrm{an}}(t) &= 9t + \frac{1437}{16}t^2 + \frac{182489}{144}t^3 + \frac{5753277}{256}t^4 + \frac{663266820361}{1440000}t^5 + \cdots.
\end{aligned} \tag{2.3}$$

Putting $t^\varepsilon = \exp(\varepsilon \log t)$, we define the *Frobenius functions* $\Phi_j(t)$ for all $j \geqslant 0$ by the expansions

$$\Phi(t,\varepsilon) = t^\varepsilon \Phi^{\mathrm{an}}(t,\varepsilon) = \sum_{j=0}^{\infty} \Phi_j(t)\varepsilon^j, \qquad \Phi_j(t) = \sum_{i=0}^{j} \Phi_i^{\mathrm{an}}(t)\frac{(\log t)^{j-i}}{(j-i)!}. \tag{2.4}$$

Then the recursion satisfied by $A_n(\varepsilon)$ translates into the statement that the power series $\Phi_j(t)$ and $\Phi(t,\varepsilon)$ satisfy the differential equations

$$\mathcal{L}(\Phi(t,\varepsilon)) = \varepsilon^3 t^\varepsilon, \qquad \mathcal{L}(\Phi_j(t)) = \frac{(\log t)^{j-3}}{(j-3)!}, \tag{2.5}$$

where

$$\mathcal{L} = D^3 - tP(D) + t(D+1)^3 \qquad \left(D = t\frac{d}{dt}\right)$$

is the differential operator that annihilates $\Phi(t)$, and the right-hand side of the second equality in (2.5) is interpreted as 0 for $j < 3$. In particular, Φ_0, Φ_1 and Φ_2 are solutions of the original differential equation $\mathcal{L}\Phi = 0$ and they constitute the well-known Frobenius basis in the space of solutions. The higher Φ_j are also of interest, as already mentioned in the introduction. In any case, the proof of the gamma conjecture requires the solution Φ_3 of the inhomogeneous differential equation $\mathcal{L}\Phi = 1$ because the Laplace transform establishes a correspondence between the tuples $\{\Phi_0, \ldots, \Phi_J\}$ and $\{\Psi_0, \ldots, \Psi_J\}$ for all J, as we will see now, and the operator $\tilde{\mathcal{L}}$ has order 4 and thus *four* Frobenius solutions $\Psi_0, \ldots, \Psi_3$.

We now do the same thing on the Laplace transform side. The modified numbers $a_n = A_n/n!$ satisfy the modified recursion relations

$$(n+1)^4 a_{n+1} - P(n)a_n + n^2 a_{n-1} = 0 \tag{2.6}$$

with the same polynomial $P(n)$ as before, and their generating function $\Psi(z) = \sum a_n z^n$ therefore satisfies the modified differential equation (the Laplace transform of $\mathcal{L}$)

$$\tilde{\mathcal{L}}(\Psi) = 0, \qquad \tilde{\mathcal{L}} = D_z^4 - zP(D_z) + z^2(D_z+1)^2,$$

where $D = z\,d/dz$. The Frobenius deformation in this case is given by

$$(n+1+\varepsilon)^4 a_{n+1}(\varepsilon) + P(n+\varepsilon)a_n(\varepsilon) + (n+\varepsilon)^2 a_{n-1}(\varepsilon) = 0 \tag{2.7}$$

for $n \geqslant 0$ with initial conditions $a_{-1}(\varepsilon) = 0$ and $a_0(\varepsilon) = 1$. We again set $a_n(\varepsilon) = \sum_j a_n^{(j)} \varepsilon^j$ and define the power series

$$\Psi_j^{\mathrm{an}}(z) = \sum_n a_n^{(j)} z^n, \qquad \Psi^{\mathrm{an}}(z,\varepsilon) = \sum_j \Psi_j^{\mathrm{an}}(z)\varepsilon^j = \sum_n a_n(\varepsilon) z^n.$$

Their first values are as follows:

$$\begin{aligned}
\Psi_0^{\mathrm{an}}(z) &= \Psi(z) = 1 + 5z + \frac{73}{2}z^2 + \frac{1445}{6}z^3 + \frac{33001}{24}z^4 + \frac{163801}{24}z^5 + \cdots, \\
\Psi_1^{\mathrm{an}}(z) &= 7z + \frac{201}{4}z^2 + \frac{10733}{36}z^3 + \frac{432875}{288}z^4 + \frac{47115959}{7200}z^5 + \cdots, \\
\Psi_2^{\mathrm{an}}(z) &= -7z - \frac{461}{8}z^2 - \frac{92323}{216}z^3 - \frac{9220085}{3456}z^4 - \frac{6108294133}{432000}z^5 + \cdots, \\
\Psi_3^{\mathrm{an}}(z) &= -\frac{15}{8}z^2 + \frac{169}{4}z^3 + \frac{4285465}{6912}z^4 + \frac{3811075}{768}z^5 + \cdots, \\
\Psi_4^{\mathrm{an}}(z) &= 9z + \frac{2449}{32}z^2 + \frac{441925}{864}z^3 + \frac{52564099}{18432}z^4 + \frac{259795048429}{19200000}z^5 + \cdots.
\end{aligned} \tag{2.8}$$

As above, putting $z^\varepsilon = \exp(\varepsilon \log z)$, we find that the Frobenius functions defined by

$$\Psi(t,\varepsilon) = z^\varepsilon \Psi^{\mathrm{an}}(z,\varepsilon) = \sum_{j=0}^{\infty} \Psi_j(z)\varepsilon^j, \qquad \Psi_j(z) = \sum_{i=0}^{j} \Psi_i^{\mathrm{an}}(z)\frac{(\log z)^{j-i}}{(j-i)!} \tag{2.9}$$

satisfy the inhomogeneous differential equations

$$\tilde{\mathcal{L}}(\Psi(z,\varepsilon)) = \varepsilon^4 z^\varepsilon, \qquad \tilde{\mathcal{L}}(\Psi_i(t)) = \frac{(\log z)^{j-4}}{(j-4)!} \tag{2.10}$$

under the same conventions as before. In particular, $\Psi_0, \ldots, \Psi_3$ form a basis (again called the Frobenius basis) of solutions of the transformed differential equation.

The gamma conjecture concerns the limits κ_j defined in (0.3). However, to calculate them, it is more convenient to work with the numbers $A_n^{(j)}$ and the functions $\Phi_j(t)$, which have better properties (regular singularities, modular parametrization). We must therefore seek how the two sequences of numbers and functions are related. It follows from the recursion that

$$a_n(\varepsilon) = \frac{A_n(\varepsilon)}{(1+\varepsilon)_n} = \frac{A_n(\varepsilon)}{n!}\prod_{k=1}^{n}\left(1+\frac{\varepsilon}{k}\right)^{-1}.$$

Here $(1+\varepsilon)_n$ stands for the ascending Pochhammer symbol $(1+\varepsilon)(2+\varepsilon)\cdots(n+\varepsilon)$. We continue the last formula by inserting the expansion

$$\prod_{k=1}^{n}\left(1+\frac{\varepsilon}{k}\right)^{-1}=\exp\left(-H_n\varepsilon+H_n^{(2)}\frac{\varepsilon^2}{2}-H_n^{(3)}\frac{\varepsilon^3}{3}+\cdots\right),$$

where

$$H_m=1+\frac{1}{2}+\cdots+\frac{1}{m},\qquad H_m^{(2)}=1+\frac{1}{4}+\cdots+\frac{1}{m^2},$$

and so on. This yields

$$\begin{aligned} n!\,a_n^{(0)}&=A_n^{(0)},\\ n!\,a_n^{(1)}&=A_n^{(1)}-H_nA_n^{(0)},\\ n!\,a_n^{(2)}&=A_n^{(2)}-H_nA_n^{(1)}+\frac{H_n^2+H_n^{(2)}}{2}A_n^{(0)},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\tag{2.11}$$

Using the formulae $H_n=\log n+\gamma+O(1/n)$ and $H_n^{(m)}=\zeta(m)+O(1/n)$ for $m>1$ and noticing that the maximum of $a_n^{(j)}z^n$ for large z and fixed j is attained at $n\approx Cz$ with $C=(1+\sqrt{2})^4=17+12\sqrt{2}$ (since $A_n\sim\text{const}\cdot C^n/n^{3/2}$, as discussed in more detail in § 2.4), we see that the asymptotics of the Frobenius solutions are found simply by replacing H_n by $\log(Cz)+\gamma$ and $H_n^{(m)}$ by $\zeta(m)$ for $m>1$. This means that if we define a sequence of Frobenius limits κ_j^0 and their generating function $\kappa^0(\varepsilon)$ in the regular case by the formulae

$$\kappa_j^0=\sum_{i=0}^{j}\left(\frac{(-\log C)^{j-i}}{(j-i)!}\lim_{n\to\infty}\left(\frac{A_n^{(i)}}{A_n}\right)\right),\qquad \kappa^0(\varepsilon)=\sum_{j=0}^{\infty}\kappa_j^0\varepsilon^j=C^{-\varepsilon}\lim_{n\to\infty}\left(\frac{A_n(\varepsilon)}{A_n}\right),\tag{2.12}$$

then the relation between $\kappa^0(\varepsilon)$ and the generating function

$$\kappa(\varepsilon)=\sum_{j=0}^{\infty}\kappa_j\varepsilon^j=\lim_{z\to\infty}\frac{\Psi(z,\varepsilon)}{\Psi(z)}\tag{2.13}$$

is given simply by

$$\kappa^0(\varepsilon)=\frac{1}{\Gamma(1+\varepsilon)}\kappa(\varepsilon).\tag{2.14}$$

This is exactly the same relation (if we replace ε by $c_1(X)$) as that between the gamma class and the modified gamma class of a Fano variety X (see § 1.5). In § 2.4 we will explain how to compute the limits in both (2.12) and (2.13) (which in any case determine each other by (2.14)) quickly and with a very high accuracy. We will also discuss some results of these numerical computations.

2.2. Frobenius limits from the hypergeometric point of view. Here we prove the formula (0.4) for the Frobenius limits in the Apéry case using Apéry's original formula (0.1) for his numbers as finite sums of products of binomial coefficients, or terminating hypergeometric series. This method is fairly computational and uses identities that were found experimentally and whose proofs are not particularly enlightening, but has the advantages of being completely elementary and applicable in principle to all linear differential equations of this type, even if they are not modular. It also gives very easy proofs of the gamma conjecture in the 10 cases of Iskovskikh's list corresponding to hypergeometric differential equations.

The idea is to mimic Apéry's original proof of the irrationality of $\zeta(3)$, where he studied the second solution A_n^* of the recursions (0.2) with initial conditions $A_0^* = 0$, $A_1^* = 1$ (the generating function of this solution also satisfies an inhomogeneous version of the original differential equation, although with the right-hand side t rather than 1) and proved that the limiting ratio $\lim_n A_n^*/A_n$ is equal to $\frac{1}{6}\zeta(3)$. He did this by finding an explicit formula for A_n^* of the form $\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 Q(n,k)$, where $Q(n,k)$ is a suitably chosen elementary function involving partial sums of $\zeta(3)$ (see [19] and [20]). When looking for similar formulae for the numbers $A_n^{(i)}$ with small i, we found experimentally a number of identities of this type that gave the correct values for $1 \leqslant i \leqslant 3$ and for all n up to some large number. Any of these identities could be used to evaluate the Frobenius limits in question. One choice, which has a particularly simple form, is reproduced in Proposition 3 below together with a proof by the standard method of telescoping sums. To find both the identities and their proofs, we made an Ansatz for the functions denoted by $Q_i(n,k)$ and $R_{n,k}(\varepsilon)$ in Proposition 3. It has the form indicated there, but with unknown coefficients (the three numerical coefficients of $H_{n+k}^{(i)}$, $H_k^{(i)}$ and $H_n^{(i)}$ in the case of $Q_i(n,k)$ and the four coefficients in $\mathbb{Q}(n,k)$ of ε^i in the case of $R_{n,k}(\varepsilon)$). Then we determined the values of these coefficients by a computer calculation. This proof is therefore not especially aesthetic, but (as already mentioned) has the advantage of being in principle applicable to the Frobenius deformations of other differential equations that do not necessarily have a modular interpretation.

Proposition 3. *For $n, k \geqslant 0$ set*

$$a_{n,k}(\varepsilon) = \binom{n}{k}^2\binom{n+k}{k}^2 \exp\biggl(\sum_{i=1}^{3} Q_i(n,k)\frac{\varepsilon^i}{i!} + O(\varepsilon^4)\biggr),$$

where

$$Q_1(n,k) = 4H_{n+k} - 4H_k, \qquad Q_2(n,k) = 4H_k^{(2)} - 8H_{n+k}^{(2)},$$
$$Q_3(n,k) = 32H_{n+k}^{(3)} - H_k^{(3)} - 14H_n^{(3)}.$$

Then

$$A_n(\varepsilon) = \sum_{k=0}^{n} a_{n,k}(\varepsilon) + O(\varepsilon^4).$$

Proof. Define $R_{n,k}(\varepsilon) \in \mathbb{Q}(n,k)[\varepsilon]/\varepsilon^4$ by the formula

$$\begin{aligned} R_{n,k}(\varepsilon) = {} & 4(2n+1)\bigl(2k^2+k-(2n+1)^2\bigr) \\ & + \bigl(16k^2+8(4n+3)k-4(2n+1)(12n+5)\bigr)\varepsilon + 16(2k-5n-2)\varepsilon^2 \\ & + \biggl(-16+\frac{14(2n+1)k}{3n^2(n+1)^2}(2n^2+2n-k)\biggr)\varepsilon^3 + O(\varepsilon^4). \end{aligned}$$

Using the easily checked identity

$$(n+1+\varepsilon)^3\frac{a_{n+1,k}(\varepsilon)}{a_{n,k}(\varepsilon)}-P(n+\varepsilon)+(n+\varepsilon)^3\frac{a_{n-1,k}(\varepsilon)}{a_{n,k}(\varepsilon)}=R_{n,k}(\varepsilon)-R_{n,k-1}(\varepsilon)\frac{a_{n,k-1}(\varepsilon)}{a_{n,k}(\varepsilon)}$$

in $\mathbb{Q}(n,k)[\varepsilon]/\varepsilon^4$ and induction on K, we find that

$$\sum_{k=0}^{K}\big((n+1+\varepsilon)^3a_{n+1,k}(\varepsilon)-P(n+\varepsilon)a_{n,k}(\varepsilon)+(n+\varepsilon)^3a_{n-1,k}(\varepsilon)\big)=R_{n,K}(\varepsilon)a_{n,K}(\varepsilon)$$

for all $K\geqslant 0$. Taking $K>n$ shows that the sequence $\sum_{k=0}^{n}a_{n,k}(\varepsilon)$ satisfies the defining recursion for the sequence $A_n(\varepsilon)$. Since these sequences also have the same initial values (0 for $n=-1$, 1 for $n=0$), they are equal. □

Corollary. *We have*

$$\lim_{n\to\infty}\frac{A_n(\varepsilon)}{C^{\varepsilon}A_n}=\exp\Big(-2\zeta(2)\varepsilon^2+\frac{17}{6}\zeta(3)\varepsilon^3+O(\varepsilon^4)\Big).$$

Proof. For large n, the maximum of $a_{n,k}=\binom{n}{k}^2\binom{n+k}{k}^2$ over $0\leqslant k\leqslant n$ is sharply peaked at $k=n\big(1/\sqrt{2}+o(1)\big)$ since the ratio $a_{n,k}/a_{n,k-1}$ is equal to

$$\frac{(n+k)^2(n-k+1)^2}{k^4}\approx\Big(\frac{n^2}{k^2}-1\Big)^2.$$

It follows that

$$\sum_{k=0}^{n}a_{n,k}(\varepsilon)\sim\Big(\sum_{k=0}^{n}a_{n,k}\Big)\exp\Big(4\log\big(1+\sqrt{2}\big)\varepsilon-2\zeta(2)\varepsilon^2+\frac{17}{6}\zeta(3)\varepsilon^3+O(\varepsilon^4)\Big).\quad\square$$

By (2.12) and (2.14), the statement of the corollary is equivalent to the formula (0.4) predicted by the gamma conjecture. This completes the proof of this conjecture in the Apéry case.

We mention that for each of our 17 families as well as for other Dn-equations arising as Picard–Fuchs equations, there are formulae like (0.1) expressing the coefficients A_n as ordinary or multiple finite sums of products of binomial coefficients (terminating hypergeometric series). For example, the 'Landau–Ginzburg models' yield such formulae expressing A_n as the constant term of $P(x,y,z)^n$ for some Laurent polynomial $P(x,y,z)$. One could therefore in principle study each of the other cases using the same idea of inserting appropriate factors into these formulae. We did not try to do this since the modular approach (discussed in the next subsection) is much simpler and works in a uniform way in all cases. We can, however, use the combinatorial approach to give an easy direct proof of Theorem 3 in all ten cases of Iskovskikh's list for which the corresponding differential operators are hypergeometric. This is useful since it includes the two cases with $N=1$ in which the modular approach fails (or at least needs a modification) because the function $F_1=\sqrt{E_4}$ is not a holomorphic modular form. Note, however, that the case

$(N,d)=(2,4)$ corresponds to the Fano variety $\mathbb{P}^3$ and is therefore a particular case of Dubrovin's results in [21] (see also [5]) which prove the gamma conjecture for all $\mathbb{P}^n$, and all hypergeometric cases are essentially known from the work of Iritani (see § 2.4). We nevertheless include our proof here since it is elementary and fits with the other cases considered.

We begin with a remark that applies to all 17 cases, not only the hypergeometric ones. Namely, it suffices to prove Theorem 3 in those 10 cases of Iskovskikh's list that have $d=1$. Indeed, comparing the conclusion (1.17) of Theorem 3 with the formula (2.14) which relates the Frobenius limits for the differential operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$, we see that the theorem now takes the simple form

$$\kappa^0_{N,d}(\varepsilon)=1+0\cdot\varepsilon+\frac{2}{d^2N}\pi^2\varepsilon^2+\frac{\mu_N}{d^3N}\zeta(3)\varepsilon^3+O(\varepsilon^4), \tag{2.15}$$

where $\kappa^0_{N,d}(\varepsilon)$ means the function $\kappa^0(\varepsilon)$ as defined by (2.12) in the case (N,d) of the list (1.1) (so that the function κ^0 in the Apéry case which was used as an illustration in § 2.1 would be $\kappa^0_{6,1}$), but with the constant $C=17+12\sqrt{2}$ appearing in (2.12) replaced in the other 16 cases by $\lim_{n\to\infty}A_n^{1/n}$, which is the reciprocal of the smallest positive root of $Q(t)$. But the power series $\Phi_{N,d}(t)$ is in this case simply $\Phi_{N,1}(t^d)$. Hence the passage from $(N,1)$ to (N,d) replaces A_n and $A_n(\varepsilon)$ by $A_{n/d}$ and $A_{n/d}(\varepsilon)$ (interpreted as 0 when $d\nmid n$) and C by $C^{1/d}$. It follows that $\kappa^0_{N,d}(\varepsilon)=\kappa^0_{N,1}(\varepsilon/d)$ to all orders, not just up to $O(\varepsilon^4)$.

This remark reduces the number of hypergeometric cases to be studied from 10 to 4, namely, to those when $1\leqslant N\leqslant 4$ and $d=1$. In these cases the coefficients of the power series $\Phi(t)=\sum_{m=0}^{\infty}A_nt^n$ are quotients of products of factorials as given by the following table:

N	1	2	3	4
A_n	$\frac{(6n)!}{(3n)!n!^3}$	$\frac{(4n)!}{n!^4}$	$\frac{(3n)!(2n)!}{n!^5}$	$\frac{(2n)!^3}{n!^6}$

In each case we write the expression for A_n as $\prod_r((rn)!)^{\nu_r}$ and notice that it also makes sense for non-integer values of n if we interpret $x!$ as $\Gamma(1+x)$. Then the Frobenius-deformed numbers $A_n(\varepsilon)$, which are defined by the same recursion (here of length 2 rather than 3 as before) with initial value $A_0(\varepsilon)=1$, can be written simply as $A_n(\varepsilon)=A_{n+\varepsilon}/A_\varepsilon$. On the other hand, since $\sum_r r\nu_r=0$ in all cases (otherwise the series $\Phi(t)$ would have radius of convergence zero), Stirling's formula gives the asymptotics $A_x\sim\alpha x^{\nu/2}C^x$ as $x\to\infty$, where $\alpha=\prod_r(2\pi r)^{\nu_r/2}$, $\nu=\sum_r\nu_r$, $C=\prod_r r^{r\nu_r}$. Hence,

$$\begin{aligned}\kappa^0(\varepsilon)&=C^{-\varepsilon}\lim_{n\to\infty}\frac{A_n(\varepsilon)}{A_n}=C^{-\varepsilon}\lim_{n\to\infty}\frac{A_{n+\varepsilon}}{A_nA_\varepsilon}=\frac{1}{A_\varepsilon}=\prod_r\Gamma(1+r\varepsilon)^{-\nu_r}\\&=\exp\biggl(\sum_r\nu_r\Bigl(-\gamma r\varepsilon+\frac{\zeta(2)}{2}r^2\varepsilon^2-\frac{\zeta(3)}{3}r^3\varepsilon^3+\cdots\Bigr)\biggr)\end{aligned}$$

and, since $\sum r\nu_r=0$, $\sum r^2\nu_r=24/N$ and $\sum r^3\nu_r=3\mu_N/N$ in all four cases $1\leqslant N\leqslant 4$, this completes the proof of Theorem 3 in the ten hypergeometric cases.

2.3. Frobenius limits from the modular point of view. The method described in the preceding subsection in Apéry's case depends on a complicated and artificial-looking identity. We now give a more natural proof using the modular parametrizations of our differential equations as discussed in § 1.3. This approach works in a uniform way in all cases with $N > 1$ in Iskovskikh's list, that is, in 15 of the 17 cases. Since the remaining two cases with $N = 1$ are hypergeometric and have already been treated, this completes the proofs of Theorems 3 and 1.

By the remark made in the end of the previous subsection, it suffices to prove the theorem in the nine cases with $d = 1$ and N belonging to the list $\{2, \ldots, 9, 11\}$. After our preparation, this is fairly easy now. In each case the differential operator $\mathcal{L}$ is of the form indicated in Proposition 1 (equation (1.11)), where $H(\tau) = H_N(\tau)$ is the Eisenstein series of weight 4 defined in (1.14). By (2.5) the first four Frobenius functions satisfy the differential equations $\mathcal{L}\Phi_0 = \mathcal{L}\Phi_1 = \mathcal{L}\Phi_2 = 0$ and $\mathcal{L}\Phi_3 = 1$, which on the modular side take the form $(\Phi_j(t(\tau))/F(t))''' = \delta_{j,3} H(\tau)$ for $0 \leqslant j \leqslant 3$. (Here "$'$" means $\frac{1}{2\pi i}\frac{d}{d\tau}$ as in § 1.3.) In view of the asymptotic form of the Frobenius functions $\Phi_j(t)$ at the point $t = 0$, which corresponds to $q = 0$ under the change of variables $2\pi i\tau = \log q = \log t + O(t)$, this means that

$$\Phi_j(t(\tau)) = \frac{(2\pi i\tau)^j}{j!} F(\tau), \quad j = 0, 1, 2, \qquad \Phi_3(t(\tau)) = \widetilde{H}(\tau) F(\tau), \tag{2.16}$$

where $\widetilde{H}(\tau)$ is the *Eichler integral* of $H(\tau)$, defined by $\widetilde{H}''' = H$ and normalized by

$$\widetilde{H}(\tau) = \frac{(2\pi i\tau)^3}{3!} + O(q) \quad \text{as} \quad q \to 0.$$

But from (1.14) we have

$$\widetilde{H}(\tau) = \sum_{M|N} \frac{h_M}{M} \widetilde{G}_4(M\tau), \tag{2.17}$$

where

$$\widetilde{G}_4(\tau) = \frac{(2\pi i\tau)^3}{1440} + \sum_{n \geqslant 1} \frac{n^{-3} q^n}{1 - q^n}$$

is the corresponding normalized Eichler integral of $G_4(\tau)$. But it is well known and elementary to prove that $\widetilde{G}_4$ satisfies the functional equation

$$\widetilde{G}_4(\tau) - \tau^2 \widetilde{G}_4\left(-\frac{1}{\tau}\right) = \frac{\zeta(3)}{2}(\tau^2 - 1) - \frac{\pi^3 i}{6}\tau \tag{2.18}$$

for all $\tau \in \mathbb{H}$. (This follows from the transformation property $G_4|_4 S = G_4$ by 3-fold integration using 'Bol's identity', which asserts that $(F|_{-2}\gamma)''' = F'''|_4\gamma$ for all holomorphic functions F and any Möbius transformation γ. This identity implies that the expression on the left of (2.18) is at most a quadratic polynomial, which we then calculate using the fact that the L-function of G_4 is equal to $\zeta(s)\zeta(s-3)$.)

Substituting this into (2.17) and using the antisymmetry property (1.15), we get

$$N\tau^2\widetilde{H}\left(-\frac{1}{N\tau}\right)+\widetilde{H}(\tau)=\sum_{M|N}\frac{h_M}{2M}\zeta(3)(M^2\tau^2-1)=\frac{\mu_N}{N}\zeta(3)(N\tau^2+1), \quad (2.19)$$

where μ_N is the same as in (1.16). (A similar calculation is carried out in [3], § 1.2.) The calculation of the Frobenius limit now follows. Indeed, (2.16), (2.19) and the anti-invariance of F under $|_2W_N$ imply that if we set

$$(k_0,k_1,k_2,k_3)=\left(1,0,\frac{2\pi^2}{N},\frac{\mu_N}{N}\zeta(3)\right),$$

then $\Phi_j(t(\tau))-k_jF(\tau)$ is a W_N-invariant function on the upper half-plane for every $j\in\{0,1,2,3\}$. But this means that these four expressions, regarded as functions of $t=t(\tau)$, are regular at the value $t=t(\tau_N)=1/C$ corresponding to the fixed point $\tau_N=i/\sqrt{N}$ of the involution W_N and hence have radius of convergence that is greater than the radius of convergence $1/C$ of the series $\Phi(t)$. It follows that the Frobenius limit $\kappa_j^0=\lim_{t\to1/C}\Phi_j(t)/\Phi(t)$ is equal to k_j. This completes the proof of (1.17) in all cases with $N>1$.

Finally, we make some remarks about the missing case $N=1$. Here the function $H_1(\tau)=E_6(\tau)/\sqrt{E_4(\tau)}=1-624q+64368q^2-\cdots$ is no longer a modular form, but from the transformation property $H_1|_4S=-H_1$ and Bol's identity we still have

$$\widetilde{H}_1(\tau)+\tau^2\widetilde{H}_1\left(-\frac{1}{\tau}\right)=\mu\zeta(3)(\tau^2+1)$$

for some complex number μ, so that the only thing missing is the evaluation $\mu=\mu_1=62$. Of course, this evaluation follows from our alternative proof of the formulae for the Frobenius limits in terms of the hypergeometric expansion of $\sqrt{E_4}$, and we have also checked it numerically to high precision, but we have not found a purely modular proof. Such a proof could possibly be given by imitating the calculations in [22], where the Eichler integral of the very similar almost-modular form $\Delta(\tau)/\sqrt{E_6(\tau)}$ of weight 3 is related to the zeros of the Weierstrass $\wp$-function.

2.4. Higher Frobenius limits: beyond the gamma conjecture. Here we briefly discuss the values of the Frobenius limits κ_j (or rather of the equivalent limits κ_j^0) for $j>3$.

Since all our results here are numerical, we first say briefly how to calculate κ_j^0 and κ_j to very high accuracy and very quickly. (Of course, one does not really need both since they are related by (2.14). But being able to do the calculations in two ways provides a nice verification of the numerical correctness of the procedure.) For κ_j one can use (0.3) directly with a moderately large value of z like $z=100$ (much smaller values actually suffice) and, since the convergence is exponential, this works well. For κ_j^0 one cannot use (2.12) directly because the ratio of $C^{-\varepsilon}A_n(\varepsilon)$ to A_n converges to its limit $\kappa^0(\varepsilon)$ only like $1/n$. Instead we use the fact that A_n has an asymptotic expansion (in Apéry's case; the others are of course similar) of the form

$$A_n\sim\mathbf{A}(n):=2^{-9/4}\pi^{-3/2}\frac{C^{n+1/2}}{(n+1/2)^{3/2}}P\left(\frac{1}{64\left(n+\frac{1}{2}\right)\sqrt{2}}\right)$$

for a certain power series $P(X) = 1 + 30X + 274X^2 - 17132X^3 + \cdots$ with easily computable rational (and conjecturally integral) coefficients determined by the property that $\mathbf{A}(n)$ satisfies the same recursion as A_n. Then

$$\kappa^0(\varepsilon) = C^{-\varepsilon} \lim_{n\to\infty} \frac{A_n(\varepsilon)}{\mathbf{A}(n)} = \lim_{n\to\infty} \frac{A_n(\varepsilon)}{\mathbf{A}(n+\varepsilon)}.$$

The latter expression converges faster than any power of n. Hence taking a moderately large number of coefficients of P and a moderately large value of n, we get very precise values for the power series $\kappa^0(\varepsilon)$. (For example, using 100 terms of P and taking $n = 100$ gives the first 15 coefficients κ_j^0 to 300 decimal digits in less than 10 seconds on a normal PC.)

We did these calculations (to 300 digits) for both κ_j and κ_j^0 in the V_{12} case and several others, each time finding agreement of the two series in (2.14) to the precision of the calculation. We then tried to recognize the coefficients κ_j^0 beyond the range $j \leqslant 3$, where their values were predicted by the gamma conjecture and proved by our calculations in the last two subsections. It turned out that up to $j = 10$ (in the V_{12} case) these values were always polynomials in Riemann zeta values (and also in the Euler constant γ if we work with κ_j instead). The results are cleaner if we use the coefficients λ_j of the generating function

$$\sum_{j=1}^{\infty} \lambda_j \varepsilon^j = \log(\kappa^0(\varepsilon)).$$

In this case the first ten values are given (within the accuracy of the calculation) by the formulae

$$\lambda_1 = 0, \qquad \lambda_2 = -2\zeta(2), \qquad \lambda_3 = \frac{17}{6}\zeta(3), \qquad \lambda_4 = -3\zeta(4),$$
$$\lambda_5 = \frac{7}{3}\zeta(5), \qquad \lambda_6 = -\frac{2}{3}\zeta(6) - \frac{1}{72}\zeta(3)^2, \qquad \lambda_7 = -\frac{5}{3}\zeta(7) + \frac{1}{6}\zeta(3)\zeta(4),$$
$$\lambda_8 = \frac{29}{12}\zeta(8) - \frac{11}{18}\zeta(3)\zeta(5), \qquad \lambda_9 = \frac{8}{9}\zeta(9) + \frac{5}{3}\zeta(3)\zeta(6) + \frac{11}{3}\zeta(4)\zeta(5) + \frac{17}{648}\zeta(3)^3,$$
$$\lambda_{10} = -\frac{147}{5}\zeta(10) - \frac{59}{18}\zeta(3)\zeta(7) - \frac{121}{18}\zeta(5)^2 - \frac{17}{36}\zeta(4)\zeta(3)^2,$$

which involve only Riemann zeta values, as already stated. But for the 11th coefficient we find

$$\lambda_{11} = 66\zeta(11) + \frac{59}{3}\zeta(4)\zeta(7) + \frac{110}{3}\zeta(5)\zeta(6) + \frac{215}{36}\zeta(8)\zeta(3) + \frac{187}{108}\zeta(3)^2\zeta(5) + \frac{2}{3}\zeta(3,5,3),$$

where the last term involves the multiple zeta value

$$\zeta(3,5,3) = \sum_{0<\ell<m<n} \frac{1}{\ell^3 m^5 n^3} = 0.002630072587647\ldots$$

instead of the ordinary ones. This suggests that the higher Frobenius limits may be interesting periods in general and that at least in some cases they are connected with multiple zeta values. We make a few final remarks in this direction. First, the first weight in which the ring of multiple zeta values is not generated over $\mathbb{Q}$ by Riemann zeta values is 8, and the dimension of the space of multiple zeta values in both this weight and weight 10 is larger by 1 (more properly, by at most 1, since the required statements on linear independence have not yet been proved). However, neither of these values appear in our computations, and the first non-trivial multiple zeta value that we see is $\zeta(3,5,3)$ of weight 11. This suggests a connection with Brown's 'single-valued multiple zeta values' [23], which also diverge from the ring of ordinary zeta values for the first time in weight 11. However, the connection is not quite clear since the new single-valued multiple zeta value in weight 11 (modulo polynomials in Riemann zeta values) is not a rational multiple of $\zeta(3,5,3)$, but rather a rational linear combination of $\zeta(3,5,3)$ and the product of $\zeta(3)$ and the non-trivial double zeta value $\zeta(3,5)$. However, a private communication from Brown suggests that there may be an explanation connected with the duality property of D3-equations (mentioned in § 1.2) and with his calculations in [24]. Finally, we mention that in the other non-hypergeometric cases we looked at we again found polynomials in Riemann zeta values for the Frobenius limits up to a certain weight but not beyond, and that we could not always recognize the higher values (like κ_7 in the case $(N,d) = (9,1)$). This phenomenon is presumably related to the fact that Fano varieties like V_{12} can be obtained as successive hyperplane sections of Fano varieties of higher dimension, but only up to a certain point, and this is precisely the range in which the gamma conjecture for these higher-dimensional Fano varieties can predict the values expressible as polynomials in Riemann zeta values. For example, the variety V_{12} can be obtained as a 7-fold iterated hyperplane section of a certain 10-dimensional Fano variety with Picard rank 1 (namely, the orthogonal Grassmannian of isotropic 5-planes in $\mathbb{C}^{10}$). We have checked in this case that the prediction of the gamma conjecture does indeed agree with the values of λ_j found numerically for all $j \leqslant 10$.

2.5. Related work. Further references. The gamma conjecture in our hypergeometric cases (which correspond to complete intersections in toric varieties) follows essentially from the celebrated quantum Lefschetz theorem of Givental [25] and Iritani's work [26]; we have given a proof for the sake of completeness. Dubrovin [21] has computed all other asymptotic expansions in the case of projective spaces. Another exposition of this case is given in [4], and our statement of the gamma conjecture may be regarded as an explicit and Fano-specific version of the condition of 'compatibility with the Stokes structure' in that paper. Przyjalkowski [27] defined weak Landau–Ginzburg models for Fano varieties and proposed candidates for weak Landau–Ginzburg models in our 17 cases. He discovered that the number of irreducible components in the resolution of the central fibre (which corresponds to the point $t = \infty$ in our notation) is one more than the number $h^{1,2}$ of the corresponding Fano variety. The relation between the Hodge numbers of Fano varieties and reducible fibres of their Landau–Ginzburg models is explained in [28]. Galkin established the modularity of 'G-Fano varieties' [29]. He computed the 'Apéry constants' for many homogeneous spaces and introduced what he called the 'Apéry class' in [30]. The work of van Enckevort and van Straten [31] pertains to the case of Calabi–Yau (rather than Fano) 3-folds, but there is an implicit relation to the topology of Fano 4-folds, again by the quantum Lefschetz principle. Finally, the modularity of Fano 3-folds of all Picard ranks has recently been announced by Doran, Harder, Katzarkov, Lewis, and Przyjalkowski.

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