SOME PROBLEMS RELATED TO POLYLOGARITHMS

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1) Erdős-Stewart-Tijdeman for function fields (cf [2], pp. 390-391). Let F be a field, $S = \{p_1, \ldots, p_s\}$ a set of primes. We set

$$X(S) = \{x \mid x \text{ and } 1 - x \text{ are } S\text{-units}\}.$$

Question For a given value of $s = |S|$, how big can $|X(S)|$ be?

In the case $F = \mathbb{Q}$ one knows that the number $N(s) := \max_{|S|=s} |X(S)|$ is bounded above by $1000 \cdot 50^s$ (Evertse) and below by $e^{(2-o(1))\sqrt{s/\log s}}$ for s large (Erdős-Stewart-Tijdeman), and one expects that $N(s) = \exp(s^{\frac{2}{3}+o(1)})$. For number fields, Evertse's result still holds (with an appropriately modified bound), but the situation for the lower bound is less clear. (The proof of Erdős-Stewart-Tijdeman applies the pigeonhole principle to the solutions of the equation $A + B = C$ in integers having only prime factors from S, and seems to be specific to $\mathbb{Q}$.) The really interesting question, however, is the function field case:

Problem Is there a lower bound for $|X(S)|$ in the case $F = \mathbb{Q}(t)$ which grows more than polynomially in s?

A consequence of this would be the existence of non-trivial functional equations for polylogarithms at any level, since the number of conditions required to make an element $\sum n_i\,[x_i] \in \mathbb{Q}[X(S)]$ belong to the mth Bloch group grows like s^m, and an element of the mth Bloch group for $\mathbb{Q}(t)$ is a functional equation for $Li_m(z)$.

2) Ladders (cf. [2], pp. 387-389). For $\alpha \in \overline{\mathbb{Q}}$, α not a root of unity, define

$$\delta(\alpha) = \dim \operatorname{Ker}\Bigl(\bigoplus_{i=0}^{\infty} \mathbb{Z} \;\to\; \overline{\mathbb{Q}}^{\times}\Bigr),$$

$$(c_i)_{i \geq 0} \;\mapsto\; \alpha^{c_0} \prod_{i=1}^{\infty} (1 - \alpha^i)^{c_i}\,.$$

It is known that this is always finite. If $\delta(\alpha) \geq k\,[\mathbb{Q}(\alpha)\colon \mathbb{Q}]$ then one gets examples of polylogarithmic relations ("ladders") up to order $m = 2k + 2$, the best example known being provided by Lehmer's algebraic number of degree 10, for which $\delta(\alpha)$ is at least ≥ 73 (Cohen-Lewin-Zagier, with 2 extra ones added by Niklasch), yielding polylogarithmic relations for all Li_m up to $m = 16$.

Problem How big can $\delta(\alpha)$ be? Is there a global upper bound? Find examples with big ratio $\frac{\delta(\alpha)}{\deg(\alpha)}$, where $\deg(\alpha)$ denotes the degree of the minimal polynomial of α over $\mathbb{Q}$.

3) Rational conformal field theories. Let A be a symmetric $n \times n$−matrix with rational coefficients. Then look for $x \in \big(\mathbb{C} - \{0,1\}\big)^n$ such that

$$1 - x \equiv x^A \mod (\mathbb{C}^*)_{\rm tors}\,,$$

i.e. $1 - x_i = \zeta_i \prod_j x_j^{a_{ij}}$ for all i, where ζ_i is a root of unity. Since these are n equations in n unknowns, the solution set is generically 0-dimensional, so that a typical solution $x = (x_1, \ldots, x_n)$ belongs to $\overline{\mathbb{Q}}^n$. One immediately checks that $\sum_{i=1}^n x_i \wedge (1 - x_i) = 0$, so that $\xi := \sum_{i=1}^n [x_i]$ defines an element of the second Bloch group of $\overline{\mathbb{Q}}$.

Problem Find (some, or all) matrices A such that

i) ξ is zero in the Bloch group (i.e. ξ is a linear combination of 5-term relations).

¿From Borel's theorem and the known identification of the Bloch group and the Bloch-Wigner function D with K_3 and the regulator, we see that i) is equivalent to

$$\sum_{i=1}^n D(x_i^\sigma) = 0\,, \qquad \forall \sigma \in {\rm Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\,.$$

Since $D|_{\mathbb{R}} \equiv 0$, one has the related problem

i′) find A with solutions $(x_1, \ldots, x_n)$ which are totally real algebraic numbers.

For instance, for $n = 1$, $A = (a_{11}) = (\frac{p}{q})$, the problem is to find p, q such that all roots of $x^p + x^q = 1$ are real, and it is easily seen that the only solutions are $a_{11} = 1, \frac{1}{2}, 2$.

The problem arises in conformal field theory [1] where it is conjectured that the classification of so-called rational conformal field theories (which includes the famous ADE classification with Dynkin diagrams as a very special case) is *equivalent* to the classification of matrices A satisfying i) or i′). A third property which is believed to be equivalent to these two is that for some vector $b \in \mathbb{Q}^n$ and number $c \in \mathbb{Q}$ the series

$$f_{A,b,c}(q) := \sum_{r_1,\ldots,r_n=0}^{\infty} \frac{q^{\frac{1}{2}rAr^t + br^t + c}}{(q)_{r_1} \cdots (q)_{r_n}}\,, \qquad (q)_r := (1-q)(1-q^2)\cdots(1-q^r),$$

is a modular function. In the example $A = (1)$ above, the corresponding values of (b, c) are $(0, -\frac{1}{48})$, $(\frac{1}{2}, \frac{1}{24})$, and $(-\frac{1}{2}, \frac{1}{24})$, the modularity of (for instance) the first of these following from the classical identity

$$f_{1,0,-\frac{1}{48}}(q) \;=\; \frac{\eta(\tau)^2}{\eta(\tau/2)\eta(2\tau)}\,,$$

where $\eta(\tau)$ denotes the Dedekind eta function and $q = e^{2\pi i\tau}$. For $A = (2)$ the function $f_{A,b,c}$ is modular for $(b, c) = (0, -\frac{1}{60})$ or $(1, \frac{11}{60})$ by virtue of the well-known Rogers-Ramanujan identities.

4) K−groups by hand. Show "by hand" that the second Bloch group of $\mathbb{Q}$ is torsion i.e. any linear combination $\sum n_i[x_i]$, ($n_i \in \mathbb{Q}$, $x_i \in \mathbb{Q} - \{0,1\}$) which satisfies

$$\sum n_i\big(x_i \wedge (1 - x_i)\big) = 0$$

is a rational combination of 5-term relations.

(This corresponds to the fact that $K_3(\mathbb{Q})\otimes\mathbb{Q} = 0$. The analogous problem for the vanishing of $K_2(\mathbb{Q})\otimes\mathbb{Q}$ would be to show that any element in $\bigwedge^2(\mathbb{Q}^\times)$ is a rational linear combination of elements $x \wedge (1-x)$, and this *does* have an elementary solution. See Milnor's book on algebraic $K-$theory, chap.11.)

5) Enlarged Mordell-Weil group. There is a well-known analogy

$$\begin{array}{rcccl} \text{(elliptic curve)} & E/\mathbb{Q} & \longleftrightarrow & F & \text{(number field)} \\ \text{(rational points)} & E(\mathbb{Q}) & \longleftrightarrow & \mathcal{O}_F^\times & \text{(unit group)} \\ \text{(Shafarevich group)} & \text{Ш} & \longleftrightarrow & Cl(F) & \text{(class group)} \\ \text{(integral points)} & E(\mathbb{Z}) & \longleftrightarrow & X(\emptyset) & \text{(exceptional units).} \end{array} \qquad (*)$$

(Here "integral points" means integral points of an affine model of E and "exceptional units" are elements u of F such that both u and $1-u$ are units.) This analogy is most visible and best known in the context of the Birch-Swinnerton-Dyer conjecture, in which the rank, covolume, and torsion subgroup of $E(\mathbb{Q})$ and the (conjecturally finite) order of Ш enter in the same way as the corresponding invariants of $\mathcal{O}_F^\times$ and the order of Cl_F enter the Dirichlet class number formula.

Problem Can one find natural extensions of $E(\mathbb{Q})$, each of finite rank but with the ranks not bounded, which correspond under this analogy to the groups $\mathcal{O}_{F,S}^\times=\{S\text{-units}\}$, S a finite set of prime ideals in F? One aspect of the analogy $(*)$ is that the 4-term exact sequence

$$1 \to \mathcal{O}_F^\times \to F^\times \to \bigoplus_{\mathfrak{p}} \mathbb{Z} \to \mathrm{Cl}_F \to 0 \qquad (1)$$

(sum over prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$), can be used as a model for the sequence

$$0 \longrightarrow E(\mathbb{Q}) \longrightarrow E(\mathbb{A}) \longrightarrow \mathrm{Form}(E/\mathbb{A},\mathbb{Q}) \longrightarrow \text{Ш} \longrightarrow 0 \qquad (2)$$

where $\mathbb{A}$ denotes the ring of adeles of $\mathbb{Q}$ and $\mathrm{Form}(E/\mathbb{A},\mathbb{Q})$ is the group of $\mathbb{Q}$-forms of $E/\mathbb{A}$, i.e. $\mathbb{Q}$-isomorphism classes of pairs (C,ϕ) where C is a curve defined over $\mathbb{Q}$ and ϕ an isomorphism $E \to C$ over $\mathbb{A}$ (cf. [3]). The groups $\mathcal{O}_{F,S}^\times$ lie in the second group of (1) and eventually exhaust it, so it might be reasonable to try to solve the problem by taking some subgroups of $E(\mathbb{A})$. What we *don't* want is to enlarge $E(\mathbb{Q})$ to $E(K)$, $[K:\mathbb{Q}]<\infty$, eventually ending up with $E(\overline{\mathbb{Q}})$ rather than $E(\mathbb{A})$. That would be analogous to replacing $\mathcal{O}_{F,S}^\times$ by the unit group of larger fields, rather than to non-units of the original field F. The problem can therefore be seen as the elliptic analogue to that of finding the whole of a number field F if we have been given only the units (say, numerically).

The connection (and potential application) to polylogarithms is as follows. Generalizing the Bloch-Wigner dilogarithm function $D(z) = \Im\big(Li_2(z) + \log|z|\log(1-z)\big)$ one has the elliptic dilogarithm function $D_E(z) = \sum_{n\in\mathbb{Z}} D(q^n z)$ for a point $z \in \mathbb{C}^\times/q^{\mathbb{Z}} = E(\mathbb{C})$ on an elliptic curve $E/\mathbb{C}$. If E is defined over $\mathbb{Q}$ and $\sum_i n_i P_i$ is a formal integral linear combinations of points of $E(\mathbb{Q})$ satisfying certain auxiliary conditions (found experimentally by H. Cohen and myself and then corrected and refined by Rolshausen, Wildeshaus, Goncharov and Levin), then $\sum_i n_i D_E(P_i)$ is conjecturally a rational multiple of the value at $s=2$ of the Hasse-Weil zeta function at $s = 2$, and there are similar conjectural formulas, involving higher elliptic polylogarithms, for other special values of L-series associated to E. The problem with these conjectures is that, unlike the situation for number fields where there are (conjecturally, and also experimentally) always enough linear combination of elements of the field to satisfy the conditions defining the higher Bloch groups, and hence to give special values of the zeta function of the field in terms of polylogarithms, here it very often happens that E has too few rational points to yield linear combinations satisfying all the requisite conditions. For example, $E(\mathbb{Q})$ may be the trivial group, in which case one certainly cannot find examples, or $E(\mathbb{Q})$ may have rank 1 but with a generator whose multiples all contain different prime factors, in which case one will again not be able to satisfy the needed conditions. This is analogous to the situation we would have in the case of a number field F if we tried to find elements $\sum n_i[x_i] \in \mathbb{Z}[F]$ with all x_i and $1 - x_i$ integral outside a given finite set of primes S: the number of such x_i is always finite, and if it happened to be too small then we could not satisfy all the conditions defining the mth Bloch group. (The number of such conditions grows roughly like $|S|^m$.) The solution there would be obvious: simply replace the set S by a bigger one, thus increasing both the rank of the finitely generated group $\mathcal{O}_{F,S}^\times$ and the cardinality of the finite set $X(S) = \{x \in F \mid x, 1 - x \in \mathcal{O}_{F,S}^\times\}$; doing this also increases the number of conditions that have to be satisfied, but the size of $X(S)$ grows faster and one eventually wins the race. Here we would like to do the same thing, but are stymied by the lack of a natural extension of $E(\mathbb{Q})$ analogous to the extension $\mathcal{O}_{F,S}^\times$ of $\mathcal{O}_F^\times$. If such extensions exist, then it is reasonable to expect that the elliptic dilogarithm function D_E will be defined on these groups also and that for sufficiently large extensions one will be able to find as many combinations as one wants which belong to the image of the Beilinson-Bloch regulator map and hence conjecturally give special values of the L-series associated to E and its symmetric powers.

References

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