FUNCTION THEORY RELATED TO THE GROUP $\mathrm{PSL}_2(\mathbb{R})$

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Introduction

The aim of this article is to discuss some of the analytic aspects of the group $G = \mathrm{PSL}_2(\mathbb{R})$ acting on the hyperbolic plane and its boundary. Everything we do is related in some way with the (spherical) principal series representations of the group G.

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These principal series representations are among the best known and most basic objects of all of representation theory. In this article, we will review the standard models used to realize these representations and then describe a number of new properties and new models. Some of these are surprising and interesting in their own right, while others have already proved useful in connection with the study of cohomological applications of automorphic forms [2] and may potentially have other applications in the future. The construction of new models may at first sight seem superfluous, since by definition any two models of the same representation are equivariantly isomorphic, but nevertheless gives new information because the isomorphisms between the models are not trivial and also because each model consists of the global sections of a certain G-equivariant sheaf, and these sheaves are completely different even if they have isomorphic spaces of global sections.

The principal series representations of G are indexed by a complex number s, called the *spectral parameter*, which we will always assume to have real part between 0 and 1. (The condition $\mathrm{Re}\,(s) = \frac{1}{2}$, corresponding to unitarizability, will play no role in this paper.) There are two basic realizations. One is the space $\mathcal{V}_s$ of functions on $\mathbb{R}$ with the (right) action of G given by

$$(\varphi\,|\,g)(t) \;=\; |ct+d|^{-2s}\,\varphi\Big(\frac{at+b}{ct+d}\Big) \qquad \Big(t\in\mathbb{R},\; g=\begin{bmatrix} a\,b\\ c\,d\end{bmatrix}\in G\Big). \tag{1}$$

The other is the space $\mathcal{E}_s$ of functions u on $\mathfrak{H}$ (complex upper half-plane) satisfying

$$\Delta\,u(z) \;=\; s(1-s)\,u(z) \qquad (z\in\mathfrak{H}), \tag{2}$$

where $\Delta = -y^2\big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\big)$ $(z = x+iy\in\mathfrak{H})$ is the hyperbolic Laplace operator, with the action $u\mapsto u\circ g$. They are related by Helgason's *Poisson transform* (thus named because it is the analogue of the corresponding formula given by Poisson for holomorphic functions)

$$\varphi(t) \quad\mapsto\quad (\mathsf{P}_s\varphi)(z) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(t)\,R(t;z)^{1-s}\,dt\,, \tag{3}$$

where $R(t;z) = R_t(z) = \frac{y}{(z-t)(\bar z-t)}$ for $z = x+iy\in\mathfrak{H}$ and $t\in\mathbb{C}$. The three main themes of this paper are: the explicit inversion of the Poisson transformation, the study of germs of Laplace eigenfunctions near the boundary $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ of $\mathfrak{H}$, and the construction of a new model of the principal series representation which is a kind of hybrid of $\mathcal{V}_s$ and $\mathcal{E}_s$. We now describe each of these briefly.

- *Inverse Poisson transform.* We would like to describe the inverse map of P_s explicitly. The right-hand side of (3) can be interpreted as it stands if φ is a smooth vector in $\mathcal{V}_s$ (corresponding to a function $\varphi(x)$ which is C^∞ on $\mathbb{R}$ and such that $t\mapsto |t|^{-2s}\varphi(1/t)$ is C^∞ at $t=0$). To get an isomorphism between $\mathcal{V}_s$ and all of $\mathcal{E}_s$, one has to allow *hyperfunctions* $\varphi(t)$. The precise definition, which is somewhat subtle in the model used in (1), will be reviewed in §1.2; for now we recall only that a hyperfunction on $I \subset \mathbb{R}$ is represented by a holomorphic function on $U \smallsetminus I$, where U is a neighborhood U of I in $\mathbb{C}$ with $U \cap \mathbb{R} = I$ and where two holomorphic functions represent the same hyperfunction if their difference is holomorphic on all of U. We will show in §3 that for $u \in \mathcal{E}_s$ the vector $\mathsf{P}_s^{-1} u \in \mathcal{V}_s$ can be represented by the hyperfunction

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$$h_{z_0}(\zeta) \;=\; \begin{cases} u(z_0) \;+\; \int_{z_0}^{\zeta} [u(z),\, (R_\zeta(z)/R_\zeta(z_0))^s] & \text{if } \zeta \in U \cap \mathfrak{H} \\ \int_{\bar\zeta}^{z_0} [(R_\zeta(z)/R_\zeta(z_0))^s,\, u(z)] & \text{if } \zeta \in U \cap \mathfrak{H}^- \end{cases} \tag{4}$$

for any $z_0 \in U \cap \mathfrak{H}$, where $\mathfrak{H}^- = \{z = x + iy \in \mathbb{C} \;:\; y < 0\}$ denotes the lower half-plane and $[u(z), v(z)]$ for any functions u and v in $\mathfrak{H}$ is the *Green's form*

$$[u(z), v(z)] \;=\; \frac{\partial u(z)}{\partial z}\, v(z)\, dz \;+\; u(z)\, \frac{\partial v(z)}{\partial \bar z}\, d\bar z\,, \tag{5}$$

which is a closed 1-form if u and v both satisfy the Laplace equation (2). The asymmetry in (4) is necessary because, although $R(\zeta; z)^s$ tends to zero at $z = \zeta$ and $z = \bar\zeta$, both its z-derivative at ζ and its $\bar z$-derivative at $\bar\zeta$ become infinite, forcing us to change the order of the arguments in the Green's form in the two components of $U \smallsetminus I$. That the two different-looking expressions in (4) are nevertheless formally the same follows from the fact that $[u, v] + [v, u] = d(uv)$ for any functions u and v.

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• *Boundary eigenfunctions.* If one looks at known examples of solutions of the Laplace equation (2), then it is very striking that many of these functions decompose into two pieces of the form $y^s A(z)$ and $y^{1-s} B(z)$ as $z = x + iy$ tends to a point of $\mathbb{R} \subset \mathbb{P}^1_{\mathbb{R}} = \partial\mathfrak{H}$, where $A(z)$ and $B(z)$ are functions which extend analytically across the boundary. For instance, the eigenfunctions that occur as building blocks in the Fourier expansions of Maass wave forms for a Fuchsian group $\mathcal{G} \subset G$ are the functions

$$k_{s,2\pi n}(z) \;=\; y^{1/2}\, K_{s-1/2}(2\pi|n|y)\, e^{2\pi i n x} \qquad (z = x + iy \in \mathbb{R},\; n \in \mathbb{Z},\; n \neq 0), \tag{6}$$

where $K_{s-1/2}(t)$ is the standard K-Bessel function which decays exponentially as $t \to \infty$. The function $K_\nu(t)$ has the form $\dfrac{\pi}{\sin \pi\nu}\, (I_\nu(t) - I_{-\nu}(t))$ with

$$I_\nu(t) = \sum_{n=0}^{\infty} \frac{(-1/4)^n\, t^{2n+\nu}}{n!\, \Gamma(n + \nu)}\,,$$

so $k_{s,2\pi n}(z)$ decomposes into two pieces of the form $y^s\times$(analytic near the boundary) and $y^{1-s}\times$(analytic near the boundary). The same is true for other elements of $\mathcal{E}_s$, involving other special functions like Legendre or hypergeometric functions, that play a role in the spectral analysis of automorphic forms. A second main theme of this paper is to understand this phenomenon. We will show that to every analytic function φ on an interval $I \subset \mathbb{R}$ there is a unique solution u of (2) in $U \cap \mathfrak{H}$ (where U as before is a neighborhood of I in $\mathbb{C}$ with $U \cap \mathbb{R} = I$, supposed simply connected and sufficiently small) such that $u(x+iy) = y^s\, \Phi(x+iy)$ for an analytic function Φ on U with restriction $\Phi|_I = \varphi$. In §4 we will call the (locally defined) map $\varphi \mapsto u$ the *transverse Poisson transform* of φ and will show that it can be described by both a Taylor series in y and an integral formula, the latter bearing a striking resemblance to the original (globally defined) Poisson transform (3):

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$$(\mathsf{P}_s^{\dagger}\varphi)(z) \;=\; \frac{-i\,\Gamma(s + \frac12)}{\Gamma(s)\Gamma(\frac12)} \int_{\bar z}^{z} \varphi(\zeta)\, R(\zeta; z)^{1-s}\, d\zeta\,, \tag{7}$$

where the function $\varphi(\zeta)$ in the integral is the unique holomorphic extension of $\varphi(t)$ to U and the integral is along any path connecting $\bar{z}$ and z within U. The transverse Poisson map produces an eigenfunction u from a real-analytic function φ on an interval I in $\mathbb{P}^1_{\mathbb{R}}$. We also give an explicit integral formula representing the holomorphic function φ in U in terms of the eigenfunction $u = \mathsf{P}_s^\dagger \varphi$.

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As an application, we will show in §6 that the elements of $\mathcal{E}_s$ corresponding under the Poisson transform to analytic vectors in $\mathcal{V}_s$ (which in the model (1) are represented by analytic functions φ on $\mathbb{R}$ for which $t \mapsto |t|^{-2s}\varphi(1/t)$ is analytic at $t = 0$) are precisely those which have a decomposition $u = \mathsf{P}_s^\dagger \varphi_1 + \mathsf{P}_{1-s}^\dagger \varphi_2$ near the boundary of $\mathfrak{H}$, where φ_1 and φ_2, which are uniquely determined by u, are analytic functions on $\mathbb{P}^1_{\mathbb{R}}$.

• *Canonical model.* We spoke above of two realizations of the principal series, as $\mathcal{V}_s$ (functions on $\partial\mathfrak{H} = \mathbb{P}^1_{\mathbb{R}}$) and as $\mathcal{E}_s$ (eigenfunctions of the Laplace operator in $\mathfrak{H}$). In fact $\mathcal{V}_s$ comes in many different variants, discussed in detail in §1, each of which resolves various of the defects of the others at the expense of introducing new ones. For instance, the "line model" (1) which we have been using up to now has a very simple description of the group action, but needs special treatment of the point $\infty \in \mathbb{P}^1_{\mathbb{R}}$, as one could already see several times in the discussion above (e.g. in the description of smooth and analytic vectors or in the definition of hyperfunctions). One can correct this by working on the projective rather than the real line, but then the description of the group action becomes very messy, while yet other models (circle model, plane model, induced representation model, ...) have other drawbacks. In §3 we will introduce a new realization C_s ("canonical model") that has many advantages:

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- all points in hyperbolic space, and all points on its boundary, are treated in an equal way;
- the formula for the group action is very simple;
- its objects are actual functions, not equivalence classes of functions;
- the Poisson transformation is given by an extremely simple formula;
- the canonical model C_s coincides with the image of a canonical inversion formula for the Poisson transformation;
- the elements of C_s satisfy differential equations, discussed below, which lead to a sheaf $\mathcal{D}_s$ that is interesting in itself;
- it uses *two* variables, one in $\mathfrak{H}$ and one in $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}$, and therefore gives a natural bridge between the models of the principal series representations as eigenfunctions in $\mathfrak{H}$ or as hyperfunctions in a deleted neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$.

The elements of the space C_s are precisely the functions $(z, z_0) \mapsto h_{z_0}(z)$ arising as in (4) for some eigenfunction $u \in \mathcal{E}_s$, but also have several intrinsic descriptions, of which perhaps the most surprising is a characterization by a system of two linear differential equations:

$$\frac{\partial h}{\partial z} = -s\,\frac{\zeta - \bar{z}}{z - \bar{z}}\,h^*\,, \qquad \frac{\partial h^*}{\partial \bar{z}} = \frac{s}{(\zeta - \bar{z})(z - \bar{z})}\,h\,, \tag{8}$$

where $h(\zeta, z)$ is a function on $(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H}$ which is holomorphic in the first variable and where $h^*(\zeta, z) := (h(\zeta, z) - h(z, z))/(\zeta - z)$. The "Poisson transform" in this model is very simple: it simply assigns to $h(\zeta, z)$ the function $u(z) = h(z, z)$, which turns out to be an eigenfunction of the Laplace operator. The name "canonical model" refers to the fact that $\mathcal{V}_s$ consists of hyperfunctions and that in C_s we have chosen a family of canonical representatives of these hyperfunctions, indexed in a G-equivariant way by a parameter in the upper half-plane: $h(\,\cdot\,,z)$ for each $z \in \mathfrak{H}$ is the unique representative of the hyperfunction $\varphi(t)R(t;z)^{-s}$ on $\mathbb{P}^1_{\mathbb{R}}$ which is holomorphic in all of $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}$ and vanishes at $\bar{z}$.

• *Further remarks.* The known or potential applications of the ideas in this paper are to automorphic forms in the upper half plane. When dealing with such forms, one needs to work with functions of general weight, not just weight 0 as considered here. We expect that many of our results can be modified to the context of general weights, where the group $G = \mathrm{PSL}_2(\mathbb{R})$ has to be replaced by $\mathrm{SL}_2(\mathbb{R})$ or its universal covering group.

Some parts of what we do in this paper are available in the literature, but often in a different form or with another emphasis. In §4 of the introduction of [6], Helgason gives an overview of analysis on the upper half plane. One finds there the Poisson transformation; the injectivity is proved by a polar decomposition. As far as we know, our approach in Theorem 3.2 with the Green's form is new, and in [2] it is an essential tool to build cocycles. Helgason gives also the asymptotic expansion near the boundary of eigenfunctions of the Laplace operator, from which the results in Section 6 may also be derived. For these asymptotic expansions one may also consult the work of Van den Ban and Schlichtkrull, [1]. A more detailed and deeper discussion can be found in [7], where Section 0 discusses the inverse Poisson transformation in the context of change (referee) the upper half-plane. Our presentation stresses the transverse Poisson transformation, which also seems not to have been treated in the earlier literature and which we use in [2] to recover Maass wave forms from their associated cocycles. Finally, the hybrid models in Section 3 and the related sheaf $\mathcal{D}_s$ are, as far as we know, new.

The paper ends with an appendix giving a number of explicit formulas, including descriptions of various eigenfunctions of the Laplace operator and tables of Poisson transforms and transverse Poisson transforms.

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Conventions and notations. We work with the Lie group

$$G = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm \mathrm{Id}\}\,.$$

We denote the element $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of G by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. A maximal compact subgroup is $K = \mathrm{PSO}(2) = \{k(\theta)\ :\ \theta \in \mathbb{R}/\pi\mathbb{Z}\}$, with

$$k(\theta) \;=\; \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{9a}$$

We also use the Borel subgroup NA, with the unipotent subgroup $N = \{n(x)\ :\ x \in \mathbb{R}\}$ and the torus $A = \{a(y)\ :\ y > 0\}$, with

$$a(y) \;=\; \begin{bmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{bmatrix}, \qquad n(x) \;=\; \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}. \tag{9b}$$

We use $\mathbb{H}$ as a generic letter to denote the hyperbolic plane. We use two concrete models: the unit disk $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$, and the upper half plane $\mathfrak{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. We will denote by x and y the real and imaginary part of $z \in \mathfrak{H}$ respectively. The boundary $\partial\mathbb{H}$ of the hyperbolic plane is in these models $\partial\mathfrak{H} = \mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, the real projective line, and $\partial\mathbb{D} = \mathbb{S}^1$, the unit circle. Both models of $\mathbb{H} \cup \partial\mathbb{H}$ are contained in $\mathbb{P}^1_{\mathbb{C}}$, on which G acts in the upper half plane model by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : z \mapsto \frac{az+b}{cz+d}$, and in the disk model by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : w \mapsto \frac{Aw+B}{\bar{B}w+\bar{A}}$, with $\left[\begin{smallmatrix} A & B \\ \bar{B} & \bar{A} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right]^{-1}$.

All the representations that we discuss in the first five sections depend on $s \in \mathbb{C}$, the *spectral parameter*; it determines the eigenvalue $\lambda_s = s - s^2$ of the Laplace operator Δ, which is given in the upper half plane model by $-y^2\partial_x^2 - y^2\partial_y^2$, and in the disk model by $-(1-|w|^2)^2\,\partial_w\,\partial_{\bar{w}}$. We will always assume $s \notin \mathbb{Z}$, and usually restrict to $0 < \operatorname{Re}(s) < 1$. We work with right representations of G, denoted by $v \mapsto v|_{2s}\,g$ or $v \mapsto v\,|\,g$.

1. The principal series representation $\mathcal{V}_s$

This section serves to discuss general facts concerning the principal series representation. Much of this is standard, but quite a lot of it is not, and the material presented here will be used extensively in the rest of the paper. We will therefore give a self-contained and fairly detailed presentation.

The principal series representations can be realized in various ways. One of the aims of this paper is to gain insight by combining several of these models. Subsection 1.1 gives six standard models for the continuous vectors in the principal series representation. Subsection 1.2 presents the larger space of hyperfunction vectors in some of these models, and in Subsection 1.3 we discuss the isomorphism (for $0 < \operatorname{Re} s < 1$) between the principal series representations with the values s and $1 - s$ of the spectral parameter.

1.1. **Six models of the principal series representation.**

In this subsection we look at six models to realize the principal series representation $\mathcal{V}_s$, each of which is the most convenient in certain contexts. Three of these models are realized on the boundary $\partial\mathbb{H}$ of the hyperbolic plane. Five of the six models have easy algebraic isomorphisms between them. The sixth has a more subtle isomorphism with the others, but gives explicit matrix coefficients. In later sections we will describe more models of $\mathcal{V}_s$ with a more complicated relation to the models here. We also describe the duality between $\mathcal{V}_s$ and $\mathcal{V}_{1-s}$ in the various models. (*Note*. We will use the letter $\mathcal{V}_s$ somewhat loosely to denote "the" principal series representation in a generic way, or when the particular space of functions under consideration plays no role. The spaces $\mathcal{V}_s^\infty$ and $\mathcal{V}_s^\omega$ of smooth and analytic vectors, and the spaces $\mathcal{V}_s^{-\infty}$ and $\mathcal{V}_s^{-\omega}$ of distributions and hyperfunctions introduced in §1.2, will be identified by the appropriate superscript. Other superscripts such as $\mathbb{P}$ and $\mathbb{S}$ will be used to distinguish vectors in the different models when needed.)

- *Line model.* This well-known model of the principal series consists of complex valued functions on $\mathbb{R}$ with the action of G given by

$$\varphi\Big|_{2s}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(x) \;=\; |cx+d|^{-2s}\,\varphi\Big(\frac{ax+b}{cx+d}\Big). \tag{1.1}$$

Since G acts on $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and not on $\mathbb{R}$, the point at infinity plays a special role in this model and a more correct description requires the use of a pair $(\varphi, \varphi_\infty)$ of functions $\mathbb{R} \to \mathbb{C}$ related by $\varphi(x) = |x|^{-2s}\varphi_\infty(-1/x)$ for $x \neq 0$, and with the right hand side in (1.1) replaced by $|ax+b|^{-2s}\varphi_\infty(-\frac{cx+d}{ax+b})$ if $cx+d$ vanishes, together with the obvious corresponding formula for φ_∞. However, we will usually work with φ alone and leave the required verification at ∞ to the reader.

The space $\mathcal{V}_s^\infty$ of *smooth vectors* in this model consists of the functions $\varphi \in C^\infty(\mathbb{R})$ with an asymptotic expansion

$$\varphi(t) \;\sim\; |t|^{-2s} \sum_{n=0}^{\infty} c_n\, t^{-n} \tag{1.2}$$

as $|t| \to \infty$. Similarly, we define the space $\mathcal{V}_s^\omega$ of *analytic vectors* as the space of $\varphi \in C^\omega(\mathbb{R})$ (real-analytic functions on $\mathbb{R}$) for which the series appearing on the right-hand side of (1.2) converges to $\varphi(t)$ for $|t| \geq t_0$ for some t_0. Replacing $C^\infty(\mathbb{R})$ or $C^\omega(\mathbb{R})$ by $C^p(\mathbb{R})$ and the expansion (1.2) with a Taylor expansion of order p, we define the space $\mathcal{V}_s^p$ for $p \in \mathbb{N}$.

- *Plane model.* The line model has the advantage that the action (1.1) of G is very simple and corresponds to the standard formula for its action on the complex upper half-plane $\mathfrak{H}$, but the disadvantage that we have to either cover the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathfrak{H}$ by two charts and work with pairs of functions or else give a special treatment to the point at infinity, thus breaking the inherent G-symmetry. Each of the next five models eliminates this problem, at the expense of introducing complexities elsewhere. The first of these is the plane model, consisting of even functions $\Phi : \mathbb{R}^2 \smallsetminus \{0\} \to \mathbb{C}$ satisfying $\Phi(tx, ty) = |t|^{-2s}\Phi(x,y)$ for $t \neq 0$, with the action

$$\Phi\Big|\begin{bmatrix} a & b \\ c & d \end{bmatrix}(x,y) \;=\; \Phi(ax+by, cx+dy)\,. \tag{1.3}$$

The relation with the line model is

$$\varphi(t) \;=\; \Phi(t,1)\,, \quad \varphi_\infty(t) \;=\; \Phi(-1,t)\,, \quad \Phi(x,y) \;=\; |y|^{-2s}\varphi(x/y)\,, \tag{1.4}$$

and of course the elements in $\mathcal{V}_s^p$, for $p = 0, 1, \ldots, \infty, \omega$, are now just given by $\Phi \in C^p(\mathbb{R}^2 \smallsetminus \{0\})$. This model has the advantage of being completely G-symmetric, but requires functions of two variables rather than just one.

- *Projective model.* If $(\varphi, \varphi_\infty)$ represents an element of the line model, we put

$$\varphi^{\mathbb{P}}(t) \;=\; \begin{cases} (1+t^2)^s\,\varphi(t) & \text{if } t \in \mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\infty\} \;=\; \mathbb{R}\,, \\ (1+t^{-2})^s\,\varphi_\infty(-1/t) & \text{if } t \in \mathbb{P}^1_{\mathbb{R}} \smallsetminus \{0\} \;=\; \mathbb{R}^* \cup \{\infty\}\,. \end{cases} \tag{1.5}$$

The functions $\varphi^{\mathbb{P}}$ form the projective model of $\mathcal{V}_s$, consisting of functions f on the real projective line $\mathbb{P}^1_{\mathbb{R}}$ with the action

$$f\Big|^{\mathbb{P}}_{2s}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(t) \;=\; \Big(\tfrac{t^2+1}{(at+b)^2+(ct+d)^2}\Big)^s\, f\Big(\tfrac{at+b}{ct+d}\Big). \tag{1.6}$$

Note that the factor $\Big(\frac{t^2+1}{(at+b)^2+(ct+d)^2}\Big)^s$ is real-analytic on the whole of $\mathbb{P}^1_{\mathbb{R}}$, since the factor in parentheses is analytic and strictly positive on $\mathbb{P}^1_{\mathbb{R}}$. This model has the advantage that all points of $\mathbb{P}^1_{\mathbb{R}}$ get equal treatment, but the disadvantage that the formula for the action is complicated and unnatural.

• *Circle model.* The transformation $\xi = \frac{t-i}{t+i}$ in $\mathbb{P}^1_{\mathbb{C}}$, with inverse $t = i\frac{1+\xi}{1-\xi}$, maps $\mathbb{P}^1_{\mathbb{R}}$ isomorphically to the unit circle $\mathbb{S}^1 = \{\xi \in \mathbb{C} \;:\; |\xi| = 1\}$ in $\mathbb{C}$, and leads to the circle model of $\mathcal{V}_s$, related to the three previous models by

$$\varphi^{\mathbb{S}}(e^{-2i\theta}) = \varphi^{\mathbb{P}}(\cot\theta) = \Phi(\cos\theta, \sin\theta) = |\sin\theta|^{-2s}\,\varphi(\cot\theta)\,. \tag{1.7}$$

The action of $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$ is described by $\tilde{g} = \left[\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right] g \left[\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} A & B \\ \bar{B} & \bar{A} \end{smallmatrix}\right]$ in $\mathrm{PSU}(1,1) \subset \mathrm{PSL}_2(\mathbb{C})$, with $A = \frac{1}{2}(a + ib - ic + d)$, $B = \frac{1}{2}(a - ib - ic - d)$:

$$f\big|^{\mathbb{S}}_{2s}\, g\,(\xi) = |A\xi + B|^{-2s} f\left(\frac{A\xi+B}{\bar{B}\xi+\bar{A}}\right) \qquad (|\xi| = 1)\,. \tag{1.8}$$

Since $|A|^2 - |B|^2 = 1$, the factor $|A\xi + B|$ is non-zero on the unit circle.

Note that in both the projective and circle models, the elements in $\mathcal{V}_s^p$ are simply the elements of $C^p(\mathbb{P}^1_{\mathbb{R}})$ or $C^p(\mathbb{S}^1)$, so that as vector spaces these models are independent of s.

• *Induced representation model.* The principal series is frequently defined as the induced representation from the Borel group NA to G of the character $n(x)a(y) \mapsto y^{-s}$, in the notation in (9b). (See for instance Chap. VII in [8].) This is the space of functions F on G transforming on the right according to this character of AN, with G acting by left translation. Identifying G/N with $\mathbb{R}^2 \smallsetminus \{(0,0)\}$ leads to the plane model, via $F\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \Phi(a, c)$. On the other hand, the functions in the induced representation model are determined by their values on K, leading to the relation $\varphi^{\mathbb{S}}(e^{2i\theta}) = F(k(\theta))$ with the circle model, with $k(\theta)$ as in (9a).

We should warn the reader that in defining the induced representation one often considers functions whose restrictions to K are square integrable, obtaining a Hilbert space isomorphic to $L^2(K)$. The action of G in this space is a bounded representation, unitary if $\mathrm{Re}\, s = \frac{1}{2}$. Since not all square integrable functions are continuous, this Hilbert space is larger than $\mathcal{V}_s^0$. For $p \in \mathbb{N}$, the space of p times differentiable vectors in this Hilbert space is larger than our $\mathcal{V}_s^p$. (It is between $\mathcal{V}_s^{p-1}$ and $\mathcal{V}_s^p$.) However, $\mathcal{V}_s^\infty$ and $\mathcal{V}_s^\omega$ coincide with the spaces of infinitely-often differentiable, respectively analytic, vectors in this Hilbert space.

• *Sequence model.* We define elements $\mathbf{e}_{s,n} \in \mathcal{V}_s^\omega$, $n \in \mathbb{Z}$, represented in our five models as follows:

$$\mathbf{e}_{s,n}(t) = (t^2+1)^{-s}\left(\frac{t-i}{t+i}\right)^n, \tag{1.9a}$$

$$\mathbf{e}^{\mathbb{R}^2}_{s,n}(x,y) = (x^2+y^2)^{-s}\left(\frac{x-iy}{x+iy}\right)^n, \tag{1.9b}$$

$$\mathbf{e}^{\mathbb{P}}_{s,n}(t) = \left(\frac{t-i}{t+i}\right)^n, \tag{1.9c}$$

$$\mathbf{e}^{\mathbb{S}}_{s,n}(\xi) = \xi^n, \tag{1.9d}$$

$$\mathbf{e}^{\text{ind repr}}_{s,n}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a^2+c^2)^{-s}\left(\frac{a-ic}{a+ic}\right)^n. \tag{1.9e}$$

Fourier expansion gives a convergent representation $\varphi^{\mathbb{S}}(\xi) = \sum_n c_n \mathbf{e}_{s,n}(\xi)$ for each element of $\mathcal{V}_s^0$. This gives the sequence model, consisting of the sequences of coefficients $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$. The action of G is described by $\mathbf{c} \mapsto \mathbf{c}'$ with $\mathbf{c}'_m = \sum_n A_{m,n}(g) c_n$, where the matrix coefficients $A_{m,n}(g)$ are given (by the binomial theorem) in terms of $\tilde{g} = \left[\begin{smallmatrix} A & B \\ \bar{B} & \bar{A}\end{smallmatrix}\right]$ as

$$A_{m,n}(g) \;=\; \frac{(A/B)^m\,(A/\bar{B})^n}{|A|^{2s}} \sum_{l\geq \max(m,n)} \binom{n-s}{l-m}\binom{-n-s}{l-n}\left|\frac{B}{A}\right|^{2l}, \tag{1.10}$$

which can be written in closed form in terms of hypergeometric functions as

$$A_{m,n}(g) \\ = \begin{cases} \frac{A^{n+m}\bar{B}^{m-n}}{|A|^{2s+2m}}\binom{-s-n}{m-n}\; F\big(s-n, s+m; m-n+1; \left|\frac{B}{A}\right|^2\big) & \text{if } m\geq n\,, \\ \frac{A^{n+m}B^{n-m}}{|A|^{2s+2n}}\binom{-s+n}{n-m}\; F\big(s+n, s-m; n-m+1; \left|\frac{B}{A}\right|^2\big) & \text{if } n\geq m\,. \end{cases} \tag{1.11}$$

The description of the smooth and analytic vectors is easy in the sequence model:

$$\begin{aligned} \mathcal{V}_s^{\omega} &= \left\{(c_n) \;:\; c_n = \mathrm{O}\left(e^{-a|n|}\right) \text{ for some } a>0\right\}, \\ \mathcal{V}_s^{\infty} &= \left\{(c_n) \;:\; c_n = \mathrm{O}\left((1+|n|)^{-a}\right) \text{ for all } a\in\mathbb{R}\right\}. \end{aligned} \tag{1.12}$$

The precise description of $\mathcal{V}_s^p$ for finite $p\in\mathbb{N}$ is less obvious in this model, but at least we have $(c_n)\in\mathcal{V}_s^p \Rightarrow c_n = \mathrm{o}(|n|^{-p})$ as $|n|\to\infty$, and, conversely, $c_n = \mathrm{O}(|n|^{-\rho})$ with $\rho > p+1$ implies $(c_n)\in\mathcal{V}_s^p$.

• *Duality.* There is a duality between $\mathcal{V}_s^0$ and $\mathcal{V}_{1-s}^0$, given in the six models by the formulas

$$\langle \varphi,\psi\rangle \;=\; \frac{1}{\pi}\int_{\mathbb{R}} \varphi(t)\,\psi(t)\,dt\,, \tag{1.13a}$$

$$\langle \Phi,\Psi\rangle \;=\; \frac{1}{2\pi}\int_0^{2\pi} \Phi(\cos\theta,\sin\theta)\,\Psi(\cos\theta,\sin\theta)\,d\theta\,, \tag{1.13b}$$

$$\langle \varphi^{\mathbb{P}},\psi^{\mathbb{P}}\rangle \;=\; \frac{1}{\pi}\int_{\mathbb{P}^1_{\mathbb{R}}} \varphi^{\mathbb{P}}(t)\,\psi^{\mathbb{P}}(t)\,\frac{dt}{1+t^2}\,, \tag{1.13c}$$

$$\langle \varphi^{\mathbb{S}},\psi^{\mathbb{S}}\rangle \;=\; \frac{1}{2\pi i}\int_{\mathbb{S}^1} \varphi^{\mathbb{S}}(\xi)\,\psi^{\mathbb{S}}(\xi)\,\frac{d\xi}{\xi}\,, \tag{1.13d}$$

$$\langle F,F_1\rangle \;=\; \int_0^{\pi} F(k(\theta))\,F_1(k(\theta))\,\frac{d\theta}{\pi}\,, \tag{1.13e}$$

$$\langle \mathbf{c},\mathbf{d}\rangle \;=\; \sum_n c_n d_{-n}\,. \tag{1.13f}$$

This bilinear form on $\mathcal{V}_s^0\times\mathcal{V}_{1-s}^0$ is G-invariant:

$$\langle \varphi|_{2s}\,g,\; \psi|_{2-2s}\,g\rangle \;=\; \langle\varphi,\psi\rangle \qquad (g\in G)\,. \tag{1.14}$$

Furthermore we have for $n,m\in\mathbb{Z}$:

$$\langle \mathbf{e}_{1-s,n},\mathbf{e}_{s,m}\rangle \;=\; \delta_{n,-m}\,. \tag{1.15}$$

• *Topology.* The natural topology of $\mathcal{V}_s^p$ with $p\in\mathbb{N}\cup\{\infty\}$ is given by seminorms which we define with use of the action $\varphi\mapsto\varphi|\mathbf{W} = \frac{d}{dt}\varphi|e^{t\mathbf{W}}\big|_{t=0}$ where $\mathbf{W} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$ is in the Lie algebra. The differential operator $\mathbf{W}$ is given by $2i\xi\,\partial_\xi$ in the circle model, by $(1+t^2)\,\partial_t$ in the projective model, and by $(1+x^2)\,\partial_x + 2sx$ in the line model. For

change Reordered and explanation added.

$p \in \mathbb{N}$ the space $\mathcal{V}_s^p$ is a Banach space with norm equal to the sum over $j = 0, \ldots, p$ of the seminorms

$$\|\varphi\|_j = \sup_{x \in \partial\mathbb{H}} \left|\varphi|\mathbf{W}^j(x)\right|. \tag{1.16}$$

The collection of all seminorms $\|\cdot\|_j$, $j \in \mathbb{N}$, gives the natural topology of $\mathcal{V}_s^\infty = \bigcap_{p\in\mathbb{N}} \mathcal{V}_s^p$. In §1.2 we shall discuss the natural topology on $\mathcal{V}_s^\omega$.

Although we have strict inclusions $\mathcal{V}_s^\infty \subset \cdots \subset \mathcal{V}_s^1 \subset \mathcal{V}_s^0$, all these representation spaces of G are irreducible as topological G-representations, due to our standing assumptions $0 < \operatorname{Re} s < 1$, which implies $s \notin \mathbb{Z}$.

- *Sheaf aspects.* In the line model, the projective model and the circle model, we can extend the definition of the G-equivariant spaces $\mathcal{V}_s^p$ for $p = 0, 1, \ldots, \infty, \omega$ of functions on $\partial\mathbb{H}$ to G-equivariant sheaves on $\partial\mathbb{H}$. For instance, in the circle model we can define $\mathcal{V}_s^\omega(I)$ for any open subset $I \subset \mathbb{S}^1$ as the space of real-analytic functions on I. The action of G induces linear maps $f \mapsto f\,|\,g$, from $\mathcal{V}_s^\omega(I)$ to $\mathcal{V}_s^\omega(g^{-1}I)$, so that $I \mapsto \mathcal{V}_s^\omega(I)$ is a G-equivariant sheaf on the G-space $\mathbb{S}^1$ whose space of global sections is the representation $\mathcal{V}_s^\omega$ of G. For the line model and the projective model we proceed similarly.

1.2. **Hyperfunctions.** So far we have considered $\mathcal{V}_s$ as a space of functions. We now want to include generalized functions: distributions and hyperfunctions. We shall be most interested in hyperfunctions on $\partial\mathbb{H}$, in the projective model and the circle model.

- *$\mathcal{V}_s^\omega$ and holomorphic functions.* Before we discuss hyperfunctions, let us first consider $\mathcal{V}_s^\omega$. In the *circle model*, it is the space $C^\omega(\mathbb{S}^1)$ of real-analytic functions on $\mathbb{S}^1$, with the action (1.8). Since the restriction of a holomorphic function on a neighborhood of $\mathbb{S}^1$ in $\mathbb{C}$ to $\mathbb{S}^1$ is real-analytic, and since every real-analytic function on $\mathbb{S}^1$ is such a restriction, $C^\omega(\mathbb{S}^1)$ can be identified with the space $\varinjlim \mathcal{O}(U)$, where U in the inductive limit runs over all open neighborhoods of $\mathbb{S}^1$ and $\mathcal{O}(U)$ denotes the space of holomorphic functions on U.
- *Hyperfunctions.* We can also consider the space $\mathbf{H}(\mathbb{S}^1) = \varinjlim \mathcal{O}(U \smallsetminus \mathbb{S}^1)$ (with U running over the same sets as before) of germs of holomorphic functions in deleted neighborhoods of $\mathbb{S}^1$ in $\mathbb{C}$. The space $C^{-\omega}(\mathbb{S}^1)$ of *hyperfunctions* on $\mathbb{S}^1$ is the quotient in the exact sequence

$$0 \longrightarrow C^\omega(\mathbb{S}^1) \longrightarrow \mathbf{H}(\mathbb{S}^1) \longrightarrow C^{-\omega}(\mathbb{S}^1) \longrightarrow 0\,; \tag{1.17}$$

change Explanation added.

see e.g. §1.1 of [11]. So $C^{-\omega}(\mathbb{S}^1) = \varinjlim_U \mathcal{O}(U \setminus \mathbb{S}^1)/\mathcal{O}(U)$ where U is as above, and where restriction gives an injective map $\mathcal{O}(U) \to \mathcal{O}(U \smallsetminus \mathbb{S}^1)$. Actually, the quotient $\mathcal{O}(U \smallsetminus \mathbb{S}^1)/\mathcal{O}(U)$ does not depend on the choice of U, so it gives a model for $C^{-\omega}(\mathbb{S}^1)$ for any choice of U. Intuitively, a hyperfunction is the jump across $\mathbb{S}^1$ of a holomorphic function on $U \smallsetminus \mathbb{S}^1$.

- *Embedding.* The image of $C^\omega(\mathbb{S}^1)$ in $C^{-\omega}(\mathbb{S}^1)$ in (1.17) is of course zero. There is an embedding $C^\omega(\mathbb{S}^1) \to C^{-\omega}(\mathbb{S}^1)$ induced by

$$(\varphi \in \mathcal{O}(U)) \mapsto (\varphi_1 \in \mathbf{H}_s(\mathbb{S}^1)), \quad \varphi_1(w) = \begin{cases} \varphi(w) & \text{if } w \in U\,,\ |w| < 1\,, \\ 0 & \text{if } w \in U\,,\ |w| > 1\,. \end{cases} \tag{1.18}$$

• *Pairing.* We next define a pairing between hyperfunctions and analytic functions on $\mathbb{S}^1$. We begin with a pairing on $\mathbf{H}(\mathbb{S}^1)\times\mathbf{H}(\mathbb{S}^1)$. Let $\varphi,\,\psi\in\mathbf{H}(\mathbb{S}^1)$ be represented by $f,\,h\in\mathcal{O}(U\smallsetminus\mathbb{S}^1)$ for some U. Let C_+ and C_- be closed curves in $U\smallsetminus\mathbb{S}^1$ which are small deformations of $\mathbb{S}^1$ to the inside and outside, respectively, traversed in the positive direction, e.g. $C_\pm=\{|w|=e^{\mp\varepsilon}\}$ with ε sufficiently small. Then the integral

$$\langle\varphi,\,\psi\rangle\;=\;\frac{1}{2\pi i}\left(\int_{C_+}-\int_{C_-}\right)f(w)\,h(w)\,\frac{dw}{w} \tag{1.19}$$

is independent of the choice of the contours $C_\pm$ and of the neighborhood U. Moreover, if f and h are both in $\mathcal{O}(U)$, then Cauchy's theorem gives $\langle\varphi,\psi\rangle=0$. Hence if $\psi\in C^\omega(\mathbb{S}^1)$, then the right-hand side of (1.19) depends only on the image (also denoted φ) of φ in $C^{-\omega}(\mathbb{S}^1)$ and we get an induced pairing $C^{-\omega}(\mathbb{S}^1)\times C^\omega(\mathbb{S}^1)\to\mathbb{C}$, which we also denote by $\langle\,\cdot\,,\,\cdot\,\rangle$. Similarly, $\langle\cdot\,,\,\cdot\rangle$ gives a pairing $C^\omega(\mathbb{S}^1)\times C^{-\omega}(\mathbb{S}^1)\to\mathbb{C}$. Finally, if φ belongs to the space $C^\omega(\mathbb{S}^1)$, embedded into $C^{-\omega}(\mathbb{S}^1)$ as explained in the preceding paragraph, then it is easily seen that $\langle\varphi,\psi\rangle$ is the same as the value of the pairing $C^\omega(\mathbb{S}^1)\times C^\omega(\mathbb{S}^1)\to\mathbb{C}$ already defined in (1.13d).

• *Group action.* We now define the action of G. We had identified $\mathcal{V}_s^\omega$ in the circle model with $C^\omega(\mathbb{S}^1)$ together with the action (1.8) of $G=\mathrm{PSL}_2(\mathbb{R})\cong\mathrm{PSU}(1,1)$. For $\tilde g=\left[\begin{smallmatrix}A&B\\\bar B&\bar A\end{smallmatrix}\right]$ and $\xi\in\mathbb{S}^1$ we have $|A\xi+B|^2=(A+B\xi^{-1})(\bar A+\bar B\xi)$, which is holomorphic and takes values near the positive real axis for ξ close to $\mathbb{S}^1$ (because $|A|>|B|$). So if we rewrite the automorphy factor in (1.8) as $\left[(\bar A+\bar B\xi)(A+B/\xi)\right]^{-s}$, then we see that it extends to a single-valued and holomorphic function on a neighborhood of $\mathbb{S}^1$ (in fact, outside a path from 0 to $-B/A$ and a path from ∞ to $-\bar A/\bar B$). In other words, in the description of $\mathcal{V}_s^\omega$ as $\varinjlim\mathcal{O}(U)$, the G-action becomes

change one-valued changed to single-valued everywhere.

$$\varphi|_{2s}g\,(w)\;=\;\left[(\bar A+\bar Bw)(A+B/w)\right]^{-s}\varphi(\tilde g w)\,. \tag{1.20}$$

This description makes sense on $\mathcal{O}(U\smallsetminus\mathbb{S}^1)$, and hence also on $\mathbf{H}(\mathbb{S}^1)$ and $C^{-\omega}(\mathbb{S}^1)$. We define $\mathcal{V}_s^{-\omega}$ as $C^{-\omega}(\mathbb{S}^1)$ together with this G-action. It is then easy to check that the embedding $\mathcal{V}_s^\omega\subset\mathcal{V}_s^{-\omega}$ induced by the embedding $C^\infty(\mathbb{S}^1)\subset C^{-\omega}(\mathbb{S}^1)$ described above is G-equivariant, and also that the pairing (1.19) satisfies (1.14) and hence defines an equivariant pairing $\mathcal{V}_s^{-\omega}\times\mathcal{V}_{1-s}^\omega\to\mathbb{C}$ extending the pairing (1.13d) on $\mathcal{V}_s^\omega\times\mathcal{V}_{1-s}^\omega$.

Note also that, if we denote by $\mathbf{H}_s$ the space $\mathbf{H}(\mathbb{S}^1)$ equipped with the action (1.20), then (1.17) becomes a short exact sequence

$$0\longrightarrow\mathcal{V}_s^\omega\longrightarrow\mathbf{H}_s\xrightarrow{\pi}\mathcal{V}_s^{-\omega}\longrightarrow 0 \tag{1.21}$$

of G-modules and (1.19) defines an equivariant pairing $\mathbf{H}_s\times\mathbf{H}_{1-s}\to\mathbb{C}$.

The equivariant duality identifies $\mathcal{V}_s^{-\omega}$ with a space of linear forms on $\mathcal{V}_{1-s}^\omega$, namely (in the circle model) the space of all linear forms that are continuous for the inductive limit topology on $C^{-\omega}(\mathbb{S}^1)$ induced by the topologies on the spaces $\mathcal{O}(U)$ given by supremum norms on annuli $1-\varepsilon<|w|<1+\varepsilon$. Similarly, the space $\mathcal{V}_s^{-\infty}$ of distributional vectors in $\mathcal{V}_s$ can be defined in the circle model as the space of linear forms on $\mathcal{V}_s^p$ that are continuous for the topology with supremum norms of all derivatives as its set of seminorms. We thus have an increasing sequence of spaces:

$$\begin{aligned}&\mathcal{V}_s^\omega\text{ (analytic functions) }\subset\mathcal{V}_s^\infty\text{ (smooth functions) }\subset\cdots\\&\qquad\subset\mathcal{V}_s^{-\infty}\text{ (distributions) }\subset\mathcal{V}_s^{-\omega}\text{ (hyperfunctions)}\,,\end{aligned} \tag{1.22}$$

where all of the inclusions commute with the action of G.

• *Hyperfunctions in other models.* The descriptions of the spaces $\mathcal{V}_s^{-\omega}$ and $\mathcal{V}_s^{-\infty}$ in the *projective model* are similar. The space of hyperfunctions $C^{-\omega}(\mathbb{P}^1_{\mathbb{R}})$ is defined similarly to (1.17), where we now let U run through neighborhoods of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. The formula (1.6) describing the action of G on functions on $\mathbb{P}^1_{\mathbb{R}}$ makes sense on a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$ and can be rewritten

$$f|^{\mathbb{P}}_{2s}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau) \;=\; (a^2+c^2)^{-s}\left(\frac{\tau-i}{\tau-g^{-1}(i)}\right)^s\left(\frac{\tau+i}{\tau-g^{-1}(-i)}\right)^s f\left(\tfrac{a\tau+b}{c\tau+d}\right). \tag{1.23}$$

where the automorphy factor now makes sense and is holomorphic and single-valued outside a path from i to $g^{-1}(i)$ and a path from $-i$ to $g^{-1}(-i)$. The duality in this model is given by

$$\langle\varphi,\psi\rangle \;=\; \frac{1}{\pi}\left(\int_{C_+}-\int_{C_-}\right)\varphi(\tau)\,\psi(\tau)\,\frac{d\tau}{1+\tau^2} \tag{1.24}$$

where the contour C_+ runs in the upper half-plane $\mathfrak{H}$, slightly above the real axis in the positive direction, and returning along a wide half circle in the positive direction and the contour C_- is defined similarly, but in the lower half-plane $\mathfrak{H}^-$, going clockwise. Everything else goes through exactly as before.

The kernel function

$$k(\zeta,\tau) \;=\; \frac{(\zeta+i)(\tau-i)}{2i(\tau-\zeta)} \tag{1.25}$$

can be used to obtain a representative in $\mathbf{H}_s$ (in the projective model) for any $\alpha\in\mathcal{V}_s^{-\omega}$: if we think of α as a linear form on $\mathcal{V}^{\omega}_{1-s}$, then

$$g(\zeta) \;=\; \langle k(\zeta,\,\cdot\,),\alpha\rangle \tag{1.26}$$

is a holomorphic function on $\mathbb{P}^1_{\mathbb{C}}\smallsetminus\mathbb{P}^1_{\mathbb{R}}$ such that $\pi(g)=\alpha$. Cauchy's theorem implies that g and any representative $\psi\in\mathbf{H}_s$ of α differ by a holomorphic function on a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$. The particular representative g has the nice properties of being holomorphic on $\mathfrak{H}\cup\mathfrak{H}^-$, and being normalized by $g(-i)=0$.

If one wants to handle hyperfunctions in the line model, one has to use both hyperfunctions φ and φ_∞ on $\mathbb{R}$, glued by $\varphi(\tau)=(\tau^2)^{-s}\varphi_\infty(-1/\tau)$ on neighborhoods of $(0,\infty)$ and $(-\infty,0)$. For instance, for $\operatorname{Re}s<\frac12$ the linear form $\varphi\mapsto\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(t)\,dt$ on $\mathcal{V}^0_{1-s}$ defines a distribution $\mathbf{1}_s\in\mathcal{V}_s^{-\infty}$. In §A.2 we use (1.26) to describe $\mathbf{1}_s\in\mathcal{V}_s^{-\omega}$ in the line model. The plane model seems not to be convenient for working with hyperfunctions.

Finally, in the *sequence model* there is the advantage that one can describe all four of the spaces in (1.22) very easily, since the descriptions in (1.12) applied to $\mathcal{V}^{\omega}_{1-s}$ and $\mathcal{V}^{\infty}_{1-s}$ lead immediately to the descriptions

$$\begin{aligned}\mathcal{V}_s^{-\omega} \;&=\; \left\{(c_n)\;:\;c_n=\mathrm{O}(e^{a|n|})\text{ for all }a>0\right\},\\ \mathcal{V}_s^{-\infty} \;&=\; \left\{(c_n)\;:\;c_n=\mathrm{O}\left((1+|n|)^a\right)\text{ for some }a\in\mathbb{R}\right\}\end{aligned} \tag{1.27}$$

of their dual spaces, where a sequence $\mathbf{c}$ corresponds to the hyperfunction represented by the function which is $\sum_{n\geq0}c_nw^n$ for $1-\varepsilon<|w|<1$ and $-\sum_{n<0}c_nw^n$ for $1<|w|<1+\varepsilon$; the action of G still makes sense here because the matrix coefficients as given in (1.11) decay exponentially (like $(|B|/|A|)^{|n|}$) as $|n|\to\infty$ for any $g\in G$. Thus in the sequence model, the four spaces in (1.22) correspond to sequences $\{c_n\}$

of complex numbers having exponential decay, superpolynomial decay, polynomial growth, or subexponential growth, respectively. (See (1.12) and (1.27).)

1.3. **The intertwining map $\mathcal{V}_s^{-\omega} \to \mathcal{V}_{1-s}^{-\omega}$.** The representations $\mathcal{V}_s^{-\omega}$ and $\mathcal{V}_{1-s}^{-\omega}$, with the same eigenvalue $s(1-s)$ for the Casimir operator, are not only dual to one another, but are also isomorphic (for $s \notin \mathbb{Z}$). Suppose first that $F \in C^p(G)$ is in the induced representation model of $\mathcal{V}_s^p$ with $\mathrm{Re}\, s > \frac{1}{2}$ and $p = 0, 1, \ldots, \infty$. With $n(x) = \left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right]$ as in (9b) and $w = \left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$ we define

$$I_s F(g) \;=\; \frac{1}{b(s-\frac{1}{2})} \int_{-\infty}^{\infty} F(gn(x)w)\, dx\,, \qquad b(s) \;=\; \mathsf{B}(s, \frac{1}{2}) \;=\; \frac{\Gamma(s)\Gamma(\frac{1}{2})}{\Gamma(s+\frac{1}{2})}\,, \tag{1.28}$$

where the gamma factor $b(s-\frac{1}{2})$ is a normalization the reason of which will become clear later. The shift over $\frac{1}{2}$ is chosen since we will meet the same gamma factor unshifted in §4. From $n(x)w \in k(-\operatorname{arccot} x)a(\sqrt{1+x^2})\,N$ we find

$$I_s F(g) \;=\; b(s-\frac{1}{2})^{-1} \int_{-\infty}^{\infty} F(gk(-\operatorname{arccot} x))(1+x^2)^{-s}\, dx\,,$$

which shows that the integral converges absolutely for $\mathrm{Re}\, s > \frac{1}{2}$. By differentiating under the integral in (1.28) we see that $I_s F \in C^p(G)$. From $a(y)n(x)w = n(yx)wa(y)^{-1}$ it follows that $I_s F(ga(y)n(x')) = y^{s-1}F(g)$. The action of G in the induced representation model is by left translation, hence I_s is an intertwining operator $\mathcal{V}_s^p \to \mathcal{V}_{1-s}^p$:

$$(I_s F)\,|_{1-s}\, g_1 \;=\; I_s(F\,|_s\, g_1) \qquad \text{for } g_1 \in G\,. \tag{1.29}$$

To describe I_s in the plane model, we choose for a given $(\xi,\eta) \in \mathbb{R}^2 \smallsetminus \{0\}$ the element $g_{\xi,\eta} = \left[\begin{smallmatrix}\xi & -\eta/(\xi^2+\eta^2)\\ \eta & \xi/(\xi^2+\eta^2)\end{smallmatrix}\right] \in G$ to obtain

$$\begin{aligned} I_s\Phi(\xi,\eta) \;&=\; b(s-\frac{1}{2})^{-1} \int_{-\infty}^{\infty} \Phi\Big(g_{\xi,\eta}\begin{bmatrix} x & -1\\ 1 & 0\end{bmatrix}\Big)\, dx\\ &=\; b(s-\frac{1}{2})^{-1} \int_{-\infty}^{\infty} \Phi\Big(x\,(\xi,\eta) + \frac{1}{\xi^2+\eta^2}(-\eta,\xi)\Big)\, dx\,. \end{aligned} \tag{1.30a}$$

By relatively straightforward computations, we find that the formulas for I_s in the other models (still for $\mathrm{Re}\, s > \frac{1}{2}$) are given by:

$$I_s\varphi(t) \;=\; b(s-\frac{1}{2})^{-1} \int_{-\infty}^{\infty} |t-x|^{2s-2}\, \varphi(x)\, dx\,, \tag{1.30b}$$

$$I_s\varphi^{\mathbb{P}}(t) \;=\; b(s-\frac{1}{2})^{-1} \int_{\mathbb{P}^1_{\mathbb{R}}} \Big(\frac{(t-x)^2}{(1+t^2)(1+x^2)}\Big)^{s-1} \varphi^{\mathbb{P}}(x)\, \frac{dx}{x^2+1}\,, \tag{1.30c}$$

$$I_s\varphi^{\mathbb{S}}(\xi) \;=\; \frac{2^{1-2s}}{i}\, b(s-\frac{1}{2})^{-1} \int_{\mathbb{S}^1} (1-\xi/\eta)^{s-1}(1-\eta/\xi)^{s-1}\varphi^{\mathbb{S}}(\eta)\, \frac{d\eta}{\eta}\,, \tag{1.30d}$$

$$(I_s\mathbf{c})_n \;=\; \frac{\Gamma(s)}{\Gamma(1-s)}\, \frac{\Gamma(1-s+n)}{\Gamma(s+n)}\, c_n \;=\; \frac{(1-s)_{|n|}}{(s)_{|n|}}\, c_n\,, \tag{1.30e}$$

with in the last line the Pochhammer symbol given by $(a)_k = \prod_{j=0}^{k-1}(a+j)$ for $k \geq 1$ and $(a)_0 = 1$. The factor $(1-s)_{|n|}/(s)_{|n|}$ is holomorphic on $0 < \mathrm{Re}\, s < 1$. Hence $I_s\mathbf{e}_{s,n}$ is well defined for these values of the spectral parameter. The polynomial growth of the factor

change Description Pochhammer symbol inserted.

shows that I_s extends to a map $I_s : \mathcal{V}_s^p \to \mathcal{V}_{1-s}^p$ for $0 < \mathrm{Re}\, s < 1$ for $p = \omega, \infty, -\infty, -\omega$, but for finite p we have only $I_s \mathcal{V}_s^p \subset \mathcal{V}_{1-s}^{p-1}$ if $0 < \mathrm{Re}\, s < 1$. See the characterizations (1.12) and (1.27). The intertwining property (1.29) extends holomorphically. The choice of the normalization factor in (1.28) implies that $I_{1-s} \circ I_s = \mathrm{Id}$, as is more easily seen from formula (1.30e). From this formula we also see that $\langle I_{1-s}\varphi, I_s\alpha\rangle = \langle \varphi, \alpha\rangle$ for $\varphi \in \mathcal{V}_{1-s}^{\omega}$, $\alpha \in \mathcal{V}_s^{-\omega}$, and that $I_{1/2} = \mathrm{Id}$.

For $\varphi \in \mathcal{V}_s^p$, $p \geq 1$, we have in the line model $\varphi'(x) = \mathrm{O}(|x|^{-2s-1})$ as $|x| \to \infty$. For $\mathrm{Re}\, s > \frac{1}{2}$, integration by parts gives

$$I_s\varphi(t) \;=\; \frac{-\Gamma(s)}{2\sqrt{\pi}\,\Gamma(s+\frac{1}{2})} \int_{-\infty}^{\infty} \mathrm{sign}\,(t-x)\,|t-x|^{2s-1}\,\varphi'(x)\,dx\,, \tag{1.31}$$

and this now defines $I_s\varphi$ for $\mathrm{Re}\, s > 0$, and shows that $I_s\mathcal{V}_s^1 \subset \mathcal{V}_{1-s}^0$.

We can describe the operator $I_s : \mathcal{V}_s^{-\omega} \to \mathcal{V}_{1-s}^{-\omega}$ on the representatives of hyperfunctions in $\mathbf{H}_s$ by sending a Laurent series $\sum_{n\in\mathbb{Z}} b_n w^n$ on an annulus $\alpha < |w| < \beta$ in $\mathbb{C}^*$ to $\sum_{n\in\mathbb{Z}} \frac{(1-s)_{|n|}}{(s)_{|n|}} b_n w^n$ converging on the same annulus. One can check that this gives an intertwining operator $I_s : \mathbf{H}_s \to \mathbf{H}_{1-s}$. (Since G is connected, it suffices to check this for generators of the Lie algebra, for which the action on the $\mathbf{e}_{s,n}$ is relatively simple. See §A.5.)

2. Laplace eigenfunctions and the Poisson transformation

The principal series representations can also be realized as the space of eigenfunctions of the Laplace operator Δ in the hyperbolic plane $\mathbb{H}$. This model has several advantages: the action of G involves no automorphy factor at all, the model does not give a preferential treatment to any point, all vectors correspond to actual functions, with no need to work with distributions or hyperfunctions, and the values s and $1-s$ of the spectral parameter give the same space. The isomorphism from the models on the boundary used so far to the hyperbolic plane model is given by an simple integral transform (Poisson map). Before discussing this transformation in Section 2.3, we consider in Section 2.1 eigenfunctions of the Laplace operator on hyperbolic space, and discuss in Section 2.2 the Green's form already used in [10].

Finally, in Section 2.4 we consider second order eigenfunctions, i.e., functions on $\mathbb{H}$ that are annihilated by $(\Delta - s(1-s))^2$.

change paragraph inserted

2.1. The space $\mathcal{E}_s$ and some of its elements. We use $\mathbb{H}$ as general notation for the hyperbolic 2-space. For computations it is convenient to work in a realization of $\mathbb{H}$. In this paper we use the realization as the complex upper half-plane and a realization as the complex unit disk.

The upper half-plane model of $\mathbb{H}$ is $\mathfrak{H} = \{z = x+iy \,:\, y > 0\}$, with boundary $\mathbb{P}^1_{\mathbb{R}}$. Lengths of curves in $\mathfrak{H}$ are determined by integration of $y^{-1}\sqrt{(dx)^2+(dy)^2}$. To this metric are associated the Laplace operator $\Delta = -y^2(\partial_x^2 + \partial_y^2) = (z-\bar z)^2\partial_z\partial_{\bar z}$ and the volume element $d\mu = \frac{dx\,dy}{y^2}$. The hyperbolic distance $\mathrm{d}(z,z')$ between two points $z, z' \in \mathbb{H}$ is given in the upper half plane model by

$$\cosh \mathrm{d}^{\mathfrak{H}}(z,z') \;=\; \rho^{\mathfrak{H}}(z,z') \;=\; 1 + \frac{|z-z'|^2}{2yy'} \qquad (z,z' \in \mathfrak{H})\,. \tag{2.1}$$

The isometry group of $\mathfrak{H}$ is the group $G = \mathrm{PSL}_2(\mathbb{R})$, acting as usual by fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$. The subgroup leaving fixed i is $K = \mathrm{PSO}(2)$. So $G/K \cong \mathfrak{H}$. The action of G leaves invariant the metric and the volume element and commutes with Δ.

We use also the disk model $\mathbb{D} = \{w \in \mathbb{C} \, : \, |w| < 1\}$ of $\mathbb{H}$, with boundary $\mathbb{S}^1$. It is related to the upper half-plane model by $w = \frac{z-i}{z+i}$, $z = i\frac{1+w}{1-w}$. The corresponding metric is $\frac{2\sqrt{(d\,\mathrm{Re}\,w)^2+(d\,\mathrm{Im}\,w)^2}}{1-|w|^2}$, and the Laplace operator $\Delta = -(1-|w|^2)^2\partial_w\partial_{\bar{w}}$. The formula for hyperbolic distance becomes

$$\cosh \mathrm{d}^{\mathbb{D}}(w,w') \;=\; \rho^{\mathbb{D}}(w,w') \;=\; 1 + \frac{2|w-w'|^2}{(1-|w|^2)(1-|w'|^2)}\,. \tag{2.2}$$

Here the group of isometries, still denoted G, is the group $\mathrm{PSU}(1,1)$ of matrices $\left[\begin{smallmatrix} A & B \\ \bar{B} & \bar{A}\end{smallmatrix}\right]$ $(A,\, B \in \mathbb{C},\, |A|^2 - |B|^2 = 1)$, again acting via fractional linear transformations.

By $\mathcal{E}_s$ we denote the space of solutions of $\Delta u = \lambda_s\, u$ in $\mathbb{H}$, where $\lambda_s = s(1-s)$. Since Δ is an elliptic differential operator with real analytic coefficients, all elements of $\mathcal{E}_s$ are real-analytic functions. The group G acts by $(u|g)(z) = u(gz)$. (We will use z to denote the coordinate in both $\mathfrak{H}$ and $\mathbb{D}$ when we make statements applying to both models of $\mathbb{H}$.) Obviously, $\mathcal{E}_s = \mathcal{E}_{1-s}$. If U is an open subset of $\mathbb{H}$, we denote by $\mathcal{E}_s(U)$ the space of solutions of $\Delta u = \lambda_s u$ on U.

There are a number of special elements of $\mathcal{E}_s$ which we will use in the sequel. Each of these elements is invariant or transforms with some character under the action of a one-parameter subgroup $H \subset G$. The simplest are $z = x + iy \mapsto y^s$ and $z \mapsto y^{1-s}$, which are invariant under $N = \{\left[\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right] \;:\; x \in \mathbb{R}\}$, and transform according to a character of $A = \{\left[\begin{smallmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y}\end{smallmatrix}\right] \;:\; y > 0\}$. More generally, the functions in $\mathcal{E}_s$ transforming according to non-trivial characters of N are written in terms of Bessel functions. These are important in describing Maass forms with respect to a discrete subgroup of G that contains $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right]$. The functions transforming according to a character of A are described in terms of hypergeometric functions. (The details, and properties of all special functions used, are given in §A.1.)

If we choose the subgroup H to be $K = \mathrm{PSO}(2)$, we are led to the functions $P_{s,n}$ described in the disk model with polar coordinates $w = re^{i\theta}$ by

$$P_{s,n}(re^{i\theta}) \;:=\; P^n_{s-1}\Big(\frac{1+r^2}{1-r^2}\Big)\, e^{in\theta} \qquad (n \in \mathbb{Z})\,, \tag{2.3}$$

where P^n_{s-1} denotes the Legendre function of the first kind. Note the shift of the spectral parameter in P^n_{s-1} and $P_{s,n}$. If $n = 0$ one usually writes P_{s-1} instead of P^0_{s-1}; but to avoid confusion we will not omit the 0 in $P_{s,0}$.

Every function in $\mathcal{E}_s$ can be described in terms of the $P_{s,n}$: if we write the Fourier expansion of $u \in \mathcal{E}_s$ as $u(re^{i\theta}) = \sum_{n\in\mathbb{Z}} A_n(r)e^{in\theta}$, then $A_n(r)$ has the form $a_n\, P^n_{s-1}(\frac{1+r^2}{1-r^2})$ for some $a_n \in \mathbb{C}$, so we have an expansion

$$u(w) \;=\; \sum_{n\in\mathbb{Z}} a_n\, P_{s,n}(w)\,, \qquad a_n \in \mathbb{C}\,. \tag{2.4}$$

Sometimes it will be convenient to consider also subgroups of G conjugate to K. For a given $z' = x' + iy' \in \mathfrak{H}$ we choose $g_{z'} = \left[\begin{smallmatrix} \sqrt{y'} & x'/\sqrt{y'} \\ 0 & 1/\sqrt{y'}\end{smallmatrix}\right] \in NA \subset G$ to obtain an automorphism of $\mathfrak{H}$ sending i to z'. If we combine this with our standard identification of $\mathfrak{H}$ and $\mathbb{D}$, we get a new identification sending the chosen point z' to $0 \in \mathbb{D}$, and the function $P_{s,n}$ on $\mathbb{D}$ becomes the following function on $\mathfrak{H} \times \mathfrak{H}$:

$$p_{s,n}(z,z') \;:=\; P_{s,n}\Bigl(\frac{z-z'}{z-\overline{z'}}\Bigr). \tag{2.5}$$

This definition of $P_{s,n}$ depends in general on the choice of $g_{z'}$ in the coset $g_{z'}K$. In the case $n = 0$ the choice has no influence, and we obtain the the very important *point pair invariant* $p_s(z,z')$, defined, in either the disk or the upper half plane, by the formula

change $\mathbb{H}$ explicitly indicated.

$$p_s(z,z') \;:=\; p_{s,0}(z,z') \;=\; P_{s-1}(\rho^{\mathbb{H}}(z,z')) \qquad (z,z' \text{ in } \mathbb{H}), \tag{2.6}$$

with the argument $\rho(z,z') = \cosh d(z,z')$ of the Legendre function $P_{s-1} = P^0_{s-1}$ being given algebraically in terms of the coordinates of z and z' by formulas (2.1) or (2.2), respectively. This function is defined on the product $\mathbb{H} \times \mathbb{H}$, is invariant with respect to the diagonal action of G on this product, and satisfies the Laplace equation with respect to each variable separately.

The Legendre function Q^n_{s-1} in equation (A.8) in the appendix provides elements of $\mathcal{E}_s(\mathbb{D} \smallsetminus \{0\})$:

$$Q_{s,n}(re^{i\theta}) \;=\; Q^n_{s-1}\Bigl(\frac{1+r^2}{1-r^2}\Bigr)e^{in\theta} \qquad (n \in \mathbb{Z}). \tag{2.7}$$

change $\mathbb{H}$ explicitly mentioned.

The corresponding point-pair invariant with $Q^0_{s-1} = Q_{s-1}$

$$q_s(z,z') \;=\; Q_{s-1}(\rho^{\mathbb{H}}(z,z')) \qquad (z,z' \text{ in } \mathbb{H}) \tag{2.8}$$

is the well-known Green's function for Δ (integral kernel function of $(\Delta - \lambda_s)^{-1}$), has a logarithmic singularity as $z \to z'$ and grows like the s-th power of the Euclidean distance (in the disk model) from z to the boundary as $z \to \partial\mathbb{H}$ with z' fixed. This latter property will be crucial in §4, where we will study a space $\mathcal{W}^\omega_s$ of germs of eigenfunctions near $\partial\mathbb{H}$ having precisely this boundary behavior.

The eigenfunction $R(t;\,\cdot\,)^s$, given in the $\mathfrak{H}$-model by

$$R(t;z)^s \;=\; \frac{y^s}{|t-z|^{2s}} \qquad (t \in \mathbb{R},\ z = x+iy \in \mathfrak{H}), \tag{2.9}$$

is the image under the action of $\left[\begin{smallmatrix} 0 & 1 \\ -1 & t \end{smallmatrix}\right] \in G$ on the eigenfunction $z \mapsto y^s$. This function was already used extensively in [10] (§2 and §5 of Chapter II). For fixed $t \in \mathbb{R}$, the functions $R(t;\,\cdot\,)^s$ and $R(t;\,\cdot\,)^{1-s}$ are both in $\mathcal{E}_s$. For fixed $z \in \mathfrak{H}$, we have $R(\,\cdot\,;z)^s$ in the line model of $\mathcal{V}^\omega_s$. The basic invariance property

$$|ct+d|^{-2s}R(gt;gz)^s \;=\; R(t,z)^s \qquad \Bigl(g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G\Bigr) \tag{2.10}$$

may be viewed as the statement that $(t,z) \mapsto R(t;z)^s$ belongs to $(\mathcal{V}^\omega_s \otimes \mathcal{E}_s)^G$. The function $R(\,\cdot\,;\,\cdot\,)^{1-s}$ is the kernel function of the Poisson transform in §2.3.

We may allow t to move off $\mathbb{R}$, in such a way that $R(t,z)^s$ becomes holomorphic in this variable:

$$R(\zeta;z)^s \;=\; \Bigl(\frac{y}{(\zeta-z)(\zeta-\bar z)}\Bigr)^s \qquad (\zeta \in \mathbb{C},\ z = x+iy \in \mathfrak{H}). \tag{2.11}$$

However, this not only has singularities at $z=\zeta$ or $z=\bar\zeta$, but is also many-valued. To make a well-defined function we have to choose a path C from ζ to $\bar\zeta$, in which case $R(\zeta;\,\cdot\,)^s$ becomes single-valued on $U=\mathfrak{H}\smallsetminus C$ and lies in $\mathcal{E}_s(U)$. (Cf. [10], Chap. II, §1.) Sometimes it is convenient to write R_ζ^s instead of $R(\zeta;\,\cdot\,)^s$.

Occasionally, we will choose other branches of the multivalued function $R(\,\cdot\,;\,\cdot\,)^s$. We have

$$\partial_z R(\zeta;z)^s \;=\; \frac{s}{z-\bar z}\,\frac{\zeta-\bar z}{\zeta-z}\,R(\zeta;z)^s\,,\qquad \partial_{\bar z} R(\zeta;z)^s \;=\; -\frac{s}{z-\bar z}\,\frac{\zeta-z}{\zeta-\bar z}\,R(\zeta;z)^s\,,\tag{2.12}$$

provided we use the same branch on the left and the right.

2.2. **The Green's form and a Cauchy formula for $\mathcal{E}_s$.** Next we recall the bracket operation from [10], which associates to a pair of eigenfunctions of Δ with the same eigenvalue a closed 1-form (Green's form). It comes in two versions, differing by an exact form:

$$[u,v] \;=\; u_z\,v\,dz \,+\, u\,v_{\bar z}\,d\bar z\,,\qquad \{u,v\} \;=\; 2i\,[u,v] \,-\, i\,d(uv)\,.\tag{2.13}$$

Because $[u|g,v|g]=[u,v]\circ g$ for any locally defined holomorphic map g (cf. [10], lemma in §2 of Chapter II), these formulas make sense and define the same 1-form whether we use the $\mathfrak{H}$- or $\mathbb{D}$-model of $\mathbb{H}$, and define G-equivariant maps $\mathcal{E}_s\times\mathcal{E}_s\to\Omega^1(\mathbb{H})$ (or $\mathcal{E}_s(U)\times\mathcal{E}_s(U)\to\Omega^1(U)$ for any open subset U of $\mathbb{H}$). The $\{u,v\}$-version of the bracket, which is antisymmetric, is given in (x,y)-coordinates $z=x+iy\in\mathfrak{H}$ by

$$\{u,\,v\} \;=\; \begin{vmatrix} u & u_x & u_y\\ v & v_x & v_y\\ 0 & dx & dy\end{vmatrix}\tag{2.14}$$

and in (r,θ)-coordinates $w=re^{i\theta}\in\mathbb{D}$ by

$$\{u,\,v\} \;=\; \begin{vmatrix} u & r\,u_r & u_\theta\\ v & r\,v_r & v_\theta\\ 0 & dr/r & d\theta\end{vmatrix}\,.\tag{2.15}$$

We can apply the Green's form in particular to any two of the special functions discussed above, and in some cases the resulting closed form can be written as the total differential of an explicit function. A trivial example is $2i\,[y^s,y^{1-s}]=s\,dz-(1-s)\,d\bar z$, $\{y^s,y^{1-s}\}=(2s-1)\,dx$. A less obvious example is

$$[R_a^s,\,R_b^{1-s}](z) \;=\; \frac{1}{b-a}\,d\Bigl(\frac{(\bar z-a)(z-b)}{z-\bar z}\,R_a^s(z)\,R_b^{1-s}(z)\Bigr),\tag{2.16}$$

where a and b are either distinct real numbers or distinct complex numbers and $z\notin\{a,b,\bar a,\bar b\}$. On both sides we take the same branches of R_a^s and R_b^{1-s}. This formula, which can be verified by direct computation, can be used to prove the Poisson inversion formula discussed below (cf. Remark 1, §3.2). Some other examples are given in §A.4.

We can also consider the brackets of any function $u\in\mathcal{E}_s$ with the point-pair invariants $p_s(z,z')$ or $q_s(z,z')$. The latter is especially useful since it gives us the following $\mathcal{E}_s$-analogue of Cauchy's formula:

Theorem 2.1. *Let C be a piecewise smooth simple closed curve in $\mathbb{H}$ and u an element of $\mathcal{E}_s(U)$, where $U \subset \mathbb{H}$ is some open set containing C and its interior. Then for $w \in \mathbb{H} \smallsetminus C$ we have*

$$\frac{1}{\pi i}\int_C [u, q_s(\,\cdot\,, w)] \;=\; \begin{cases} u(w) & \text{if } w \text{ is inside } C, \\ 0 & \text{if } w \text{ is outside } C, \end{cases} \tag{2.17}$$

where the curve C is traversed is the positive direction.

Proof. Since $[u, q_s(\,\cdot\,, w)]$ is a closed form, the value of the integral in (2.17) does not change if we deform the path C, so long as we avoid the point w where the form becomes singular. The vanishing of the integral when w is outside of C is therefore clear, since we can simply contract C to a point. If w is inside C, then we can deform C to a small hyperbolic circle around w. We can use the G-equivariance to put $w = 0$, so that this hyperbolic circle is also a Euclidean one, say $z = \varepsilon e^{i\theta}$. We can also replace $[u, q_s(\,\cdot\,, 0)]$ by $\{u, q_s(\,\cdot\,, 0)\}/2i$, since their difference is exact. From (2.15) and the asymptotic result (A.11) we find that the closed form $-\frac{i}{2}\{u, q_s(\,\cdot\,, 0)\}$ equals $\left(\frac{i}{2}u(0) + \mathrm{O}(\varepsilon \log \varepsilon)\right) d\theta$ on the circle. The result follows. ∎

The method of the proof just given can also be used to check that for a contour C in $\mathbb{D}$ encircling 0 once in positive direction we have for all $n \in \mathbb{Z}$

$$\int_C [P_{s,n},\, Q_{s,m}] \;=\; \pi i\,(-1)^n\,\delta_{n,-m}\,. \tag{2.18}$$

Combining this formula with the expansion (2.4), we arrive at the following generalization of the standard formula for the Taylor expansion of holomorphic functions:

Proposition 2.2. *For each $u \in \mathcal{E}_s$:*

$$u(w) \;=\; \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\pi i}\, P_{s,n}(w) \int_C [u, Q_{s,-n}]\,. \tag{2.19}$$

If $u \in \mathcal{E}_s(A)$, where A is some annulus of the form $r_1 < |w| < r_2$ in $\mathbb{D}$, there is a more complicated expansion of the form

$$u(w) \;=\; \sum_{n\in\mathbb{Z}} \left(a_n P_{s,n}(w) + b_n Q_{s,n}(w)\right). \tag{2.20}$$

For fixed $w' \in \mathbb{D}$ the function $w \mapsto q_s(w, w')$ has only one singularity, at $w = w'$. So both on the disk $|w| < |w'|$ as on the annulus $|w| > |w'|$ the function $q_s(\,\cdot\,, w')$ has a polar Fourier expansion, which can be given explicitly:

Proposition 2.3. *For $w, w' \in \mathbb{D}$ with $|w| \neq |w'|$:*

$$q_s(w, w') \;=\; \begin{cases} \sum_{n\in\mathbb{Z}} (-1)^n\, P_{s,-n}(w')\, Q_{s,n}(w) & \text{if } |w| > |w'|\,, \\ \sum_{n\in\mathbb{Z}} (-1)^n\, P_{s,-n}(w)\, Q_{s,n}(w') & \text{if } |w| < |w'|\,. \end{cases} \tag{2.21}$$

Proof. Apply (2.20) to $q_s(\,\cdot\,, w')$ on the annulus $A = \{w \in \mathbb{D} \;:\; |w'| < |w|\}$. Since $q_s(\,\cdot\,, w')$ represents an element of $\mathcal{W}_s^\omega$ the expansion becomes

$$q_s(w, w') \;=\; \sum_{n\in\mathbb{Z}} b_n(w')\, Q_{s,n}(w) \qquad (|w| > |w'|)\,.$$

From $q_s(e^{i\theta}w, e^{i\theta}w') = q_s(w,w')$ it follows that $b_n(e^{i\theta}w') = e^{-in\theta}b_n(w)$. For $w \in \mathbb{D} \smallsetminus \{0\}$ we have $q_s(w,\cdot\,) \in \mathcal{E}_s(B)$ with $B = \{w' \in \mathbb{D} \;:\; |w'| < |w|\}$. Then the coefficients b_n are also in $\mathcal{E}_s(B)$. Since $Q_{s,-n}$ has a singularity at $0 \in \mathbb{D}$, the coefficients have the form $b_n(w') = c_n\, P_{s,-n}(w')$. Now we apply (2.17) and (2.18) to obtain with a path C inside the region A:

$$\begin{aligned} P_{s,m}(w') \;&=\; \frac{1}{\pi i}\int_C [P_{s,m}, q_s(\,\cdot\,, w')] \;=\; \frac{1}{\pi i}\sum_{n\in\mathbb{Z}} c_n P_{s,-n}(w') \int_C [P_{s,m}, Q_{s,n}] \\ &=\; c_{-m}\, P_{s,m}(w')\,(-1)^m\,. \end{aligned}$$

Hence $c_m = (-1)^m$, and the proposition follows, with the symmetry of q_s. ∎

2.3. **The Poisson transformation.** There is a well-known isomorphism P_s from $\mathcal{V}_s^{-\omega}$ to $\mathcal{E}_s$. This enables us to view $\mathcal{E}_s$ as a model of the principal series. We first describe P_s abstractly and then more explicitly in various models of $\mathcal{V}_s^{-\omega}$. In Section 3.2 we will describe the inverse isomorphism from $\mathcal{E}_s$ to $\mathcal{V}_s^{-\omega}$.

For $\alpha \in \mathcal{V}_s^{-\omega}$ and $g \in G$

$$(\mathsf{P}_s\alpha)(g) \;=\; \langle \alpha|_{2s}\, g,\, \mathbf{e}_{1-s,0}\rangle \;=\; \langle \alpha,\, \mathbf{e}_{1-s,0}|_{2-2s}\, g^{-1}\rangle \tag{2.22}$$

describes a function on G that is K-invariant on the right. Hence it is a function on $G/K \cong \mathbb{H}$. The center of the enveloping algebra is generated by the Casimir operator. It gives rise to a differential operator on G that gives, suitably normalized, the Laplace operator Δ on the right-K-invariant functions. Since the Casimir operator acts on $\mathcal{V}_{1-s}^{-\omega}$ as multiplication by $\lambda_s = (1-s)s$, the function $\mathsf{P}_s\alpha$ defines an element of $\mathcal{E}_s$. We write in the upper half plane model

$$\mathsf{P}_s\alpha(z) = \mathsf{P}_s\alpha\big(n(x)a(y)\big)\,, \tag{2.23}$$

with the notation in (9b). The definition in (2.22) implies that the Poisson transformation is G-equivariant:

$$\mathsf{P}_s(\alpha|_{2s}g)(z) \;=\; \mathsf{P}_s\alpha(gz)\,. \tag{2.24}$$

The fact that the intertwining operator $I_s : \mathcal{V}_s^{-\omega} \to \mathcal{V}_{1-s}^{-\omega}$ preserves the duality implies that the following diagram commutes:

$$\begin{tikzcd} \mathcal{V}_s^{-\omega} \arrow[dr, "\mathsf{P}_s"] \arrow[dd, "I_s"'] & \\ & \mathcal{E}_s = \mathcal{E}_{1-s} \\ \mathcal{V}_{1-s}^{-\omega} \arrow[ur, "\mathsf{P}_{1-s}"] & \end{tikzcd} \tag{2.25}$$

If $\alpha \in \mathcal{V}_s^{0}$, we can describe $\mathsf{P}_s\alpha$ by a simple integral formula. In the line model this takes the form

$$\begin{aligned} \mathsf{P}_s\alpha(z) \;&=\; \langle \mathbf{e}_{s,0}, \alpha|_{2s}n(x)a(y)\rangle \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \mathbf{e}_{s,0}|_{2-2s}\,\big(n(x)a(y)\big)^{-1}(t)\,\alpha(t)\,dt \\ &=\; \frac{1}{\pi}\int_{-\infty}^{\infty} y^{-1+s}\Big(\big(\frac{t-x}{y}\big)^2+1\Big)^{s-1}\,\alpha(t)\,dt \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} R(t;z)^{1-s}\alpha(t)\,dt\,, \end{aligned} \tag{2.26}$$

so that R^{1-s} is the kernel of the Poisson transformation in the line model. If α is a hyperfunction, the pairing in (2.22) has to be interpreted as discussed in §1.2 as the difference of two integrals over contours close to and on opposite sides of $\partial\mathbb{H}$ (equation (1.19) in the circle model), with $R(\,\cdot\,;z)^{1-s}$ extended analytically to a neighborhood of $\partial\mathbb{H}$.

In the projective model and the circle model we find:

$$\mathsf{P}_s\alpha(z) \;=\; \langle R(\,\cdot\,;z)^{1-s},\alpha\rangle\,, \tag{2.27a}$$

with $R(\,\cdot\,;z)^{1-s}$ in the various models given by

$$R^{\mathbb{P}}(\zeta;z)^{1-s} \;=\; y^{s-1}\Bigl(\frac{\zeta-i}{\zeta-z}\Bigr)^{1-s}\Bigl(\frac{\zeta+i}{\zeta-\bar z}\Bigr)^{1-s} \;=\; \Bigl(\frac{R(\zeta;z)}{R(\zeta;i)}\Bigr)^{1-s}, \tag{2.27b}$$

$$R^{\mathbb{S}}(\xi;w)^{1-s} \;=\; \Bigl(\frac{1-|w|^2}{(1-w/\xi)(1-\bar w\xi)}\Bigr)^{1-s}. \tag{2.27c}$$

By $R(\,\cdot\,,\,\cdot\,)^{1-s}$, without superscript on the R, we denote the Poisson kernel in the line model (as in (2.11)). We take the branch for which $\arg R(\zeta;z)=0$ for ζ on $\mathbb{R}$.

In the circle model we have for each $\alpha\in\mathcal{V}_s^{-\omega}$:

$$\mathsf{P}_s\alpha(w) \;=\; \frac{(1-|w|^2)^{1-s}}{2\pi i}\Bigl(\int_{C_+}-\int_{C_-}\Bigr)g(\xi)\,\bigl((1-w/\xi)(1-\bar w\xi)\bigr)^{s-1}\,\frac{d\xi}{\xi}\,, \tag{2.28}$$

with C_+ and C_- as in (1.19), adapted to the domain of the representative $g\in\mathbf{H}_s$ of the hyperfunction α.

For the values of s we are interested in, Helgason has shown that the Poisson transformation is an isomorphism:

Theorem 2.4. (Theorem 4.3 in [5]). *The Poisson map $\mathsf{P}_s:\mathcal{V}_s^{-\omega}\to\mathcal{E}_s$ is a bijection for* $0<\operatorname{Re}s<1$.

The usual proof of this uses the K-Fourier expansion, where K ($\cong \mathrm{PSO}(2)$) is the standard maximal compact subgroup of G. One first checks by explicit integration the formula

$$\mathsf{P}_s\mathbf{e}_{s,m} \;=\; (-1)^m\,\frac{\Gamma(s)}{\Gamma(s+m)}\,P_{s,m} \qquad (n\in\mathbb{Z}), \tag{2.29}$$

with $\mathbf{e}_{s,m}$ and $P_{s,m}$ as defined in (1.9d) and (2.3) respectively. (Indeed, with (2.27b) and (2.28) we obtain the Poisson integral

$$\mathsf{P}_s\mathbf{e}_{s,m}(w) \;=\; \frac{(1-|w|^2)^{1-s}}{2\pi i}\int_{|\xi|=1}\xi^m\,\bigl((1-w/\xi)(1-\bar w\xi)\bigr)^{s-1}\,\frac{d\xi}{\xi}\,.$$

Since $|w/\xi|<1$ and $|\bar w\xi|<1$ this leads to the expansion

$$\begin{aligned}&(1-|w|^2)^{1-s}\sum_{n_1,n_2\ge0,\,-n_1+n_2=m}\frac{(1-s)_{n_1}\,(1-s)_{n_2}}{n_1!\,n_2!}\,w^{n_1}\,\bar w^{n_2}\\ &\quad=\;(1-|w|^2)^{1-s}\,\frac{(1-s)_{|m|}}{|m|!}\sum_{n\ge0}\frac{(1-s)_n\,(1-s+|m|)_n}{(1+|m|)_n\,n!}\,|w|^{2n}\cdot\begin{cases} w^m & \text{if } m\ge0\,,\\ \bar w^{-m} & \text{if } m\le 0\,.\end{cases}\end{aligned}$$

This is $(-1)^m\Gamma(s)/\Gamma(s+m)$ times $P_{s,m}$ as defined in (A.8) and (A.9).) Then one uses the fact that the elements of $\mathcal{V}_s^{-\omega}$ are given by sums $\sum c_n\,\mathbf{e}_{s,n}$ with coefficients c_n of subexponential growth (eq. (1.27)), and shows that the coefficients in the expansion (2.19) also have subexponential growth for each $u\in\mathcal{E}_s$. This is the analogue of the fact that a holomorphic function in the unit disk has Taylor coefficients at 0 of subexponential growth, and can be proved the same way. An alternative proof of Theorem 2.4 will follow from the results of §3.2, where we shall give an explicit inverse map for P_s.

Thus, $\mathcal{E}_s$ is a model of the principal series representation $\mathcal{V}_s^{-\omega}$, and also of $\mathcal{V}_{1-s}^{-\omega}$, that does not change under the transformation $s \mapsto 1-s$ of the spectral parameter. It is completely G-equivariant. The action of G is simply given by $u\,|\,g = u \circ g$.

As discussed in §1, the space $\mathcal{V}_s^{-\omega}$ (hyperfunctions on $\partial\mathbb{H}$) contains three canonical subspaces $\mathcal{V}_s^{-\infty}$ (distributions), $\mathcal{V}_s^{\infty}$ (smooth functions) and $\mathcal{V}_s^{\omega}$ (analytic functions on $\partial\mathbb{H}$), and we can ask whether there is an intrinsic characterization of the corresponding subspaces $\mathcal{E}_s^{-\infty}$, $\mathcal{E}_s^{\infty}$, and $\mathcal{E}_s^{\omega}$ of $\mathcal{E}_s$. For $\mathcal{E}_s^{-\infty}$ the answer is simple and depends only on the asymptotic properties of the eigenfunctions near the boundary, namely:

Theorem 2.5. ([9], Theorem 4.1 and Theorem 5.3.) *Let* $0 < \mathrm{Re}\, s < 1$. *The space* $\mathcal{E}_s^{-\infty} = \mathsf{P}_s(\mathcal{V}_s^{-\infty})$ *consists of the functions in* $\mathcal{E}_s$ *having at most polynomial growth near the boundary.*
("At most polynomial growth near the boundary" means $\ll (1-|w|^2)^{-C}$ for some C in the disk model and $\ll ((|z+i|^2)/y)^C$ in the upper half-plane model.)

The corresponding theorems for the spaces $\mathcal{E}_s^{\infty}$ and $\mathcal{E}_s^{\omega}$, which do not only involve estimates of the speed of growth of functions near $\partial\mathbb{H}$, are considerably more complicated. We will return to the description of these spaces in Section 6.

- *Explicit examples.* One example is given in (2.29). Another example is

$$\mathsf{P}_s\delta_{s,\infty}(z) \;=\; R^{\mathbb{P}}(\infty; z)^{1-s} \;=\; y^{1-s}\,, \tag{2.30}$$

where $\delta_{s,\infty} \in \mathcal{V}_s^{-\infty}$ is the distribution associating to $\varphi \in \mathcal{V}_{1-s}^{\infty}$, in the projective model, its value at ∞. As a third example we consider the element $R(\,\cdot\,; z_0)^s \in \mathcal{V}_s^{\omega}$. For convenience we use the circle model. Then $a(w, w') = (\mathsf{P}_s R^{\mathbb{S}}(\,\cdot\,; w')^s)(w)$ satisfies the relation $a(gw', gw) = a(w', w)$ for all $g \in G$, by equivariance of the Poisson transform and of the function R^s. So a is a point-pair invariant. Since $a(w, \cdot\,) \in \mathcal{E}_s$, it has to be a multiple of p_s. We compute the factor by taking $w' = w = 0 \in \mathbb{D}$:

$$\begin{aligned}\mathsf{P}_s R^{\mathbb{S}}(\,\cdot\,; 0)^s(0) \;&=\; \frac{1}{2\pi i}\int_{\mathbb{S}^1} R^{\mathbb{S}}(\xi; 0)^s R^{\mathbb{S}}(\xi; 0)^{1-s}\,\frac{d\xi}{\xi}\\ &=\; \frac{1}{2\pi i}\int_{\mathbb{S}^1} 1\cdot 1\,\frac{d\xi}{\xi} \;=\; 1\,.\end{aligned}$$

Thus we have

$$\mathsf{P}_s R(\,\cdot\,; w')^s(w) \;=\; p_s(w', w)\,. \tag{2.31}$$

With (2.25) and the fact that $P_{1-s,0} = P_{s,0}$ this implies

$$I_s\, R(\,\cdot\,; w')^s \;=\; R(\,\cdot\,; w')^{1-s}\,. \tag{2.32}$$

2.4. **Second order eigenfunctions.** The Poisson transformation allows us to prove results concerning the space

$$\mathcal{E}_s' \;:=\; \mathrm{Ker}\left((\Delta - \lambda_s)^2 : C^\infty(\mathbb{H}) \longrightarrow C^\infty(\mathbb{H})\right). \tag{2.33}$$

Proposition 2.6. *The following sequence is exact:*

$$0 \longrightarrow \mathcal{E}_s \longrightarrow \mathcal{E}'_s \xrightarrow{\Delta-\lambda_s} \mathcal{E}_s \longrightarrow 0\,. \tag{2.34}$$

Proof. Only the surjectivity of $\mathcal{E}'_s \to \mathcal{E}_s$ is not immediately clear.

Let $0 < \mathrm{Re}\, s_0 < 1$. Suppose we have a family $s \mapsto f_s$ on a neighborhood of s_0 such that $f_s \in \mathcal{E}_s$ for all s near s_0, and suppose that this family is C^∞ in (s, z) and holomorphic in s. Then

$$(\Delta - \lambda_{s_0})(\partial_s f_s|_{s=s_0}) - (1-2s_0) f_{s_0} = 0\,.$$

For $s_0 \neq \frac{1}{2}$, this gives an element of $\mathcal{E}'_{s_0}$ that is mapped to f_{s_0} by $\Delta - \lambda_{s_0}$. If $s_0 = \frac{1}{2}$, we replace f_s by $\frac{1}{2}(f_s + f_{1-s})$, and differentiate twice.

To produce such a family, we use the Poisson transformation. By Theorem 2.4 there is a unique $\alpha \in \mathcal{V}^{-\omega}_{s_0}$ such that $f = \mathsf{P}_{s_0}\alpha$. We fix a representative $g \in \mathcal{O}(U \smallsetminus \mathbb{S}^1)$ of α in the circle model, which represents a hyperfunction α_s for all $s \in \mathbb{C}$. (The projective model works as well.) We put

$$f_s(w) \;=\; \mathsf{P}_s\alpha_s(w) \;=\; \frac{1}{2\pi i}\left(\int_{C_+} - \int_{C_-}\right) R^{\mathbb{S}}(\zeta; w)^{1-s} g(\zeta)\,\frac{d\zeta}{\zeta}\,.$$

The contours C_+ and C_- have to be adapted to w, but can stay the same when w varies through a compact subset of $\mathfrak{H}$. Differentiating this family provides us with a lift of f in $\mathcal{E}'_{s_0}$. ∎

This proof gives an explicit element

$$\tilde f(w) \;=\; \frac{-1}{2\pi i}\left(\int_{C_+} - \int_{C_-}\right) R^{\mathbb{S}}(\zeta; w)^{1-s_0}\,(\log R^{\mathbb{S}}(\zeta; w))\; g(\zeta)\,\frac{d\zeta}{\zeta} \tag{2.35}$$

of $\mathcal{E}'_s$ with $(\Delta - \lambda_s)\tilde f = (1-2s) f$. Note that for $s = \frac{1}{2}$ the function $\tilde f$ belongs to $\mathcal{E}_{1/2}$, giving an interesting map $\mathcal{E}_{1/2} \to \mathcal{E}_{1/2}$. As an example, if $f(z) = y^{1/2}$, then we can take $g(\zeta) = \frac{\zeta}{2i}$ as the representative of the hyperfunction $\alpha = \delta_{1/2,\infty}$ with $\mathsf{P}_{1/2}\alpha = h$, and by deforming the contours C_+ and C_- into one circle $|\zeta| = R$ with R large we obtain (in the projective model):

$$\begin{aligned} \tilde f(z) \;&=\; \frac{-1}{\pi}\int_{|\zeta|=R} R^{\mathbb{P}}(\zeta; z)^{1/2}\left(\log y + \log\frac{\zeta^2+1}{(\zeta - z)(\zeta - \bar z)}\right)\frac{\zeta}{2i}\,\frac{d\zeta}{\zeta^2+1} \\ &=\; -y^{1/2}\log y\,. \end{aligned} \tag{2.36}$$

In part C of Table 1 in §A.2 we describe the distribution in $\mathcal{V}^{-\infty}_{1/2}$ corresponding to this element of $\mathcal{E}_{1/2}$.

Theorem 2.5 shows that the subspace $\mathcal{E}^{-\infty}_s$ corresponding to $\mathcal{V}^{-\infty}_s$ under the Poisson transformation is the space of elements of $\mathcal{E}_s$ with polynomial growth. We define $(\mathcal{E}'_s)^{-\infty}$ as the subspace of $\mathcal{E}'_s$ of elements with polynomial growth. The following proposition, including the somewhat technical second statement, is needed in Chapter V of [2].

Proposition 2.7. *The sequence*

$$0 \longrightarrow \mathcal{E}^{-\infty}_s \longrightarrow (\mathcal{E}'_s)^{-\infty} \xrightarrow{\Delta-\lambda_s} \mathcal{E}^{-\infty}_s \longrightarrow 0 \tag{2.37}$$

is exact. All derivatives $\partial_w^l \partial_{\bar w}^m f(w)$, $l, m \geq 0$, of $f \in (\mathcal{E}'_s)^{-\infty}$, in the disk model, have polynomial growth.

Proof. We use the construction in the proof of Proposition 2.6. We use

$$f_s(w) \;=\; \langle R^{\mathbb{S}}(\,\cdot\,; w)^{1-s}, \alpha\rangle\,,$$

with $\alpha \in \mathcal{V}_{s_0}^{-\infty}$. For $\zeta \in \mathbb{S}^1$ we obtain by differentiating the expression for $R^{\mathbb{S}}$ in (2.27c)

$$(\zeta\partial\zeta)^n \partial_w^l \partial_{\bar w}^m \, R^{\mathbb{S}}(\zeta; w)^{1-s} \;\ll_{n,l,m}\; (1-|w|^2)^{s-l-m-n}\,.$$

With the seminorm $\|\cdot\|_n$ in (1.16) we can reformulate this as

$$\|\partial_w^l \partial_{\bar w}^m R^{\mathbb{S}}(\,\cdot\,; w)^{1-s}\|_n \;\ll_{n,l,m}\; (1-|w|^2)^{s-l-m-n}\,. \tag{2.38}$$

Since α determines a continuous linear form on $\mathcal{V}_s^p$ for some $p \in \mathbb{N}$, this gives an estimate

$$\partial_w^l \partial_{\bar w}^m f(w) \;\ll_{\alpha,l,m}\; (1-|w|^2)^{\mathrm{Re}\, s-1-l-m-p}$$

for $f \in \mathcal{E}_{s_0}^{-\infty}$.

Differentiating $R^{\mathbb{S}}(\,\cdot\,; w)^{1-s}$ once or twice with respect to s multiplies the estimate in (2.38) with at most a factor $\left|\log(1-|w|^2)\right|^2$. The lift $\tilde f \in \mathcal{E}'_{s_0}$ of f_{s_0} in the proof of Proposition 2.6 satisfies

$$\partial_w^l \partial_{\bar w}^m \tilde f(w) \;\ll_{\alpha,l,m,\varepsilon}\; (1-|w|^2)^{\mathrm{Re}\, s-1-l-m-p-\varepsilon}$$

for each $\varepsilon > 0$. ∎

3. Hybrid models for the principal series representation

In this section we introduce the canonical model of the principal series, discussed in the introduction. In Subsection 3.1 we define first two other models of $\mathcal{V}_s$ in functions or hyperfunctions on $\partial\mathbb{H} \times \mathbb{H}$, which we call *hybrid models*, since they mix the properties of the model of $\mathcal{V}_s$ in eigenfunctions, as discussed in Section 2, with the models discussed in Section 1. The second of these, called the flabby hybrid model, contains the *canonical model* as a special subspace. The advantage of the canonical model becomes very clear in Subsection 3.2, where we give an explicit inverse for the Poisson transformation whose image coincides exactly with the canonical model.

In Subsection 3.3 we will characterize the canonical model as a space of functions on $(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H}$ satisfying a certain system of differential equations. We use these differential equations to define a sheaf $\mathcal{D}_s$ on $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$, the *sheaf of mixed eigenfunctions*. The properties of this sheaf and of its sections over other natural subsets of $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$ are studied in the remainder of the subsection and in more detail in Section 5.

3.1. **The hybrid models and the canonical model.** The line model of principal series representations is based on giving $\infty \in \partial\mathfrak{H}$ a special role. The projective model eliminated the special role of the point at infinity in the line model, at the expense of a more complicated description of the action of $G = \mathrm{PSL}_2(\mathbb{R})$, but it also broke the G-symmetry in a different way by singling out the point $i \in \mathfrak{H}$. The corresponding point $0 \in \mathbb{D}$ plays a special role in the circle model. The sequence model is based on the characters of the specific maximal compact subgroup $K = \mathrm{PSO}(2) \subset G$ and not of its conjugates, again breaking the G-symmetry. The induced representation model depends on the choice of the Borel group NA. Thus none of the models for $\mathcal{V}_s$ discussed in Section 1 reflects fully the intrinsic symmetry under the action of G.

To remedy these defects, we will replace our previous functions φ on $\partial\mathbb{H}$ by functions $\widetilde{\varphi}$ on $\partial\mathbb{H} \times \mathbb{H}$, where the second variable plays the role of a base point, with $\widetilde{\varphi}(\,\cdot\,, i)$ being equal to the function $\varphi^{\mathbb{P}}$ of the projective model. This has the disadvantage of replacing functions of one variable by functions of two, but gives a very simple formula for the G-action, is completely symmetric, and will also turn out to be very convenient for the Poisson transform. Explicitly, given $(\varphi, \varphi_\infty)$ in the line model, we define $\widetilde{\varphi} : \mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H} \to \mathbb{C}$ by

$$\widetilde{\varphi}(t,z) \;=\; \begin{cases} \left(\dfrac{|z-t|^2}{y}\right)^s \varphi(t) & \text{if } t \in \mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\infty\}\,, \\ \left(\dfrac{|1+z/t|^2}{y}\right)^s \varphi_\infty\Bigl(-\dfrac{1}{t}\Bigr) & \text{if } t \in \mathbb{P}^1_{\mathbb{R}} \smallsetminus \{0\} \end{cases} \tag{3.1}$$

(here $y = \operatorname{Im}(z)$ as usual), generalizing equation (1.5) for $z = i$. The function $\widetilde{\varphi}$ then satisfies

$$\widetilde{\varphi}(t,z_1) \;=\; \left(\frac{|z_1-t|^2/y_1}{|z_2-t|^2/y_2}\right)^s \widetilde{\varphi}(t,z_2) \;=\; \left(\frac{R(t;z_2)}{R(t;z_1)}\right)^s \widetilde{\varphi}(t,z_2) \tag{3.2}$$

for $t \in \mathbb{P}^1_{\mathbb{R}}$ and $z_1, z_2 \in \mathfrak{H}$. A short calculation, with use of (2.10), shows that the action of G becomes simply

$$\widetilde{\varphi|g}(t,z) \;=\; \widetilde{\varphi}(gt,\, gz) \qquad (t \in \mathbb{P}^1_{\mathbb{R}},\; z \in \mathfrak{H},\; g \in G) \tag{3.3}$$

in this model. From equations (3.1), (1.5) and (3.2), we find

$$\varphi^{\mathbb{P}}(t) \;=\; \varphi(t,i)\,, \qquad \widetilde{\varphi}(t,z) \;=\; \left(\frac{(t-z)(t-\bar{z})}{(t^2+1)\,y}\right)^s \varphi^{\mathbb{P}}(t)\,, \tag{3.4}$$

giving the relation between the new model and the projective model. And we see that only the complicated factor relating $\widetilde{\varphi}$ to $\varphi^{\mathbb{P}}$ is responsible for the complicated action of G in the projective model.

We define the *rigid hybrid model* to be the space of functions $h : \mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H} \to \mathbb{C}$ satisfying (3.2) with $\widetilde{\varphi}$ replaced by h. The G-action is given by $F \mapsto F \circ g$, where G acts diagonally on $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}$. The smooth (resp. analytic) vectors are those for which $F(\,\cdot\,, z)$ is smooth (resp. analytic) on $\mathbb{P}^1_{\mathbb{R}}$ for any $z \in \mathfrak{H}$; this is independent of the choice of z because the expression in parentheses in (3.2) is analytic and strictly positive on $\mathbb{P}^1_{\mathbb{R}}$. These spaces are models for $\mathcal{V}_s^\infty$ and $\mathcal{V}_s^\omega$, respectively, but when needed will be denoted $\mathcal{V}_s^{\infty,\mathrm{rig}}$ and $\mathcal{V}_s^{\omega,\mathrm{rig}}$ to avoid confusion. We may view the elements of the rigid hybrid model as a family of functions $t \mapsto \widetilde{\varphi}(t,z)$ in projective models with a varying special point $z \in \mathfrak{H}$. The isomorphism relating the rigid hybrid model and the line (respectively projective) model is then given by (3.1) (respectively (3.4)).

In the case of $\mathcal{V}_s^{\omega,\mathrm{rig}}$ we can replace t in (3.2) or (3.4) by a variable ζ on a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$. We observe that, although $R(\zeta;z)^s$ is multivalued in ζ, the quotient $\left(\frac{R(\zeta;z_1)}{R(\zeta;z)}\right)^s$ in (3.2) is holomorphic in ζ on a neighborhood (depending on z and z_1) of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. In the rigid hybrid model, the space $\mathbf{H}_s^{\mathrm{rig}}$ consists of germs of functions h on a deleted neighborhood $U \smallsetminus (\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H})$ which are holomorphic in the first variable and satisfy

$$h(\zeta, z_1) \;=\; \Bigl(\frac{R(\zeta; z_2)}{R(\zeta; z_1)}\Bigr)^s h(\zeta, z_2) \qquad (z_1, z_2 \in \mathfrak{H},\ \zeta \text{ near } \mathbb{P}^1_{\mathbb{R}}), \tag{3.5}$$

where "near $\mathbb{P}^1_{\mathbb{R}}$" means that ζ is sufficiently far in the hyperbolic metric from the geodesic joining z_1 and z_2. This condition ensures that (ζ, z_1) and (ζ, z_2) belong to U and the multiplicative factor in (3.5) is a power of a complex number not in $(-\infty, 0]$ and is therefore well defined. The action of G on $\mathbf{H}^{\mathrm{rig}}_s$ is given by $h(\zeta, z) \mapsto h(g\zeta, gz)$. In this model $\mathcal{V}^{-\omega}_s$ is represented as $\mathbf{H}^{\mathrm{rig}}_s / \mathcal{V}^{\omega,\mathrm{rig}}_s$. The pairing between hyperfunctions and test functions in this model is given by

$$\bigl\langle h, \widetilde{\psi} \bigr\rangle \;=\; \frac{1}{\pi}\Bigl(\int_{C_+} - \int_{C_-}\Bigr) h(\zeta, z)\, \widetilde{\psi}(\zeta, z)\, R(\zeta; z)\, d\zeta \tag{3.6}$$

with the contours C_+ and C_- as in (1.24). Provided we adapt the contours to z, we can use any $z \in \mathfrak{H}$ in this formula for the pairing.

The rigid hybrid model, as described above, solves all of the problems of the various models of $\mathcal{V}_s$ as function spaces on $\partial\mathbb{H}$, but it is in some sense artificial, since the elements h depend in a fixed way on the second variable, and the use of this variable is therefore in principle superfluous. We address the remaining artificiality by replacing the rigid hybrid model by another model. The intuition is to replace functions satisfying (3.5) by hyperfunctions satisfying this relation. change This and the two following paragraphs reorganized.

Specifically, we define the *flabby hybrid model* as

$$\mathcal{M}^{-\omega}_s \;:=\; \mathcal{H}_s / \mathcal{M}^{\omega}_s\,,$$

where $\mathcal{H}_s$ is the space of functions[1] $h(\zeta, z)$ that are defined on $U \smallsetminus (\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H})$ for some neighborhood U of $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}$ in $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$, are holomorphic in ζ, and satisfy

$$\zeta \mapsto h(\zeta, z_1) - \Bigl(\frac{R(\zeta; z_2)}{R(\zeta; z_1)}\Bigr)^s h(\zeta, z_2) \;\in\; \mathcal{O}(U_{z_1, z_2}) \quad \text{ for all } z_1, z_2 \in \mathfrak{H}\,, \tag{3.7}$$

where $U_{z_1,z_2} = \{\zeta \in \mathbb{P}^1_{\mathbb{C}} \;:\; (\zeta, z_1), (\zeta, z_2) \in U\}$, while $\mathcal{M}^{\omega}_s$ consists of functions defined on a neighborhood U of $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}$ in $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$ and holomorphic in the first variable. The action of G in $\mathcal{H}_s$ is by $h\,|\,g\,(\zeta, z) = h(g\zeta, gz)$. The pairing between hyperfunctions and analytic functions is given by the same formula (3.6) as in the rigid hybrid model.

An element $h \in \mathcal{H}_s$ can thus be viewed as a family $\{h(\,\cdot\,, z)\}_{z \in \mathfrak{H}}$ of representatives of hyperfunctions parametrized by $\mathfrak{H}$. Adding an element of $\mathcal{M}^{\omega}_s$ does not change this family of hyperfunctions. The requirement (3.7) on h means that the family of hyperfunctions satisfies (3.5) in hyperfunction sense.

Finally, we describe a subspace $C_s \subset \mathcal{H}_s$ which maps isomorphically to $\mathcal{V}^{-\omega}_s$ under the projection $\mathcal{H}_s \twoheadrightarrow \mathcal{V}^{-\omega}_s$ and hence gives a canonical choice of representatives of the hyperfunctions in $\mathcal{M}^{-\omega}_s$. We will call C_s the *canonical hybrid model*, or simply the *canonical model* for the principal series representation $\mathcal{V}^{-\omega}_s$. To define C_s we

[1] Here one has the choice to impose any desired regularity conditions (C^0, C^∞, C^ω, …) in the second variable or in both variables jointly. We do not fix any such choice since none of our considerations depend on which choice is made and since in any case the most interesting elements of this space, like the canonical representative introduced below, are analytic in both variables.

recall that any hyperfunction on $\mathbb{P}^1_{\mathbb{R}}$ can be represented by a holomorphic function on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}$ with the freedom only of an additive constant. One usually fixes the constant by requiring $h(i) = 0$ or $h(i) + h(-i) = 0$, which is of course not G-equivariant. Here we can exploit the fact that we have two variables to make the normalization in a G-equivariant way, by requiring that

$$h(\bar{z}, z) \;=\; 0\,. \tag{3.8}$$

We thus define C_s as the space of functions on $\left(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}\right) \times \mathfrak{H}$ that are holomorphic in the first variable and satisfy (3.7) and (3.8). We will see below (Theorem 3.2) that the Poisson transform $\mathsf{P}_s : \mathcal{V}_s^{-\omega} \to \mathcal{E}_s$ becomes extremely simple when restricted to C_s, and also that C_s coincides with the image of a canonical lifting of the inverse Poisson map $\mathsf{P}_s{}^{-1} : \mathcal{E}_s \to \mathcal{V}_s^{-\omega}$ from the space of hyperfunctions to the space of hyperfunction representatives.

Remark. We will also occasionally use the slightly larger space C_s^+ (no longer mapped injectively to $\mathcal{E}_s$ by P_s) consisting of functions in $\mathcal{H}_s$ that that are defined on all of $(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H}$, without the requirement (3.8). Functions in this space will be called *semicanonical* representatives of the hyperfunctions they represent. The decomposition $h(\zeta, z) = \big(h(\zeta, z) - h(\bar{z}, z)\big) + h(\bar{z}, z)$ gives a canonical and G-equivariant splitting of C_s^+ as the direct sum of C_s and the space of functions on $\mathfrak{H}$, so that there is no new content here, but specific hyperfunctions sometimes have a particularly simple semicanonical representative (an example is given below), and it is not always natural to require (3.8).

• *Summary.* We have introduced a “rigid”, a “flabby”, and a “canonical” hybrid model, related by

$$\mathcal{V}_s^{-\omega} \;\cong\; \mathbf{H}_s^{\mathrm{rig}}/\mathcal{V}_s^{\omega,\mathrm{rig}} \;\cong\; \mathcal{M}_s^{-\omega} \;=\; \mathcal{H}_s/\mathcal{M}_s^{\omega} \;\cong\; C_s \;\subset\; \mathcal{H}_s\,. \tag{3.9}$$

In the flabby hybrid model, the space $\mathcal{H}_s$ consists of functions on a deleted neighborhood $U \smallsetminus (\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H})$ that may depend on the function, holomorphic in the first variable, and satisfying (3.7). The subspace $\mathcal{M}_s^{\omega}$ are functions on the whole of some neighborhood U of $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}$, holomorphic in the first variable.

For the elements of $\mathcal{H}_s$ and $\mathcal{M}_s^{\omega} \subset \mathcal{H}_s$ we do not require any regularity in the second variable. In the rigid hybrid model the spaces $\mathbf{H}_s^{\mathrm{rig}} \subset \mathcal{H}_s$ and $\mathcal{V}_s^{\omega,\mathrm{rig}} \subset \mathcal{M}_s^{\omega}$ are characterized by the condition in (3.5), which forces a strong regularity in the second variable. The canonical hybrid model C_s consists of a specific element from each class of $\mathcal{H}_s/\mathcal{M}_s^{\omega}$ that is defined on $(\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}) \smallsetminus (\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H})$ and is normalized by (3.8). In §3.3 we will see that this implies analyticity in both variables jointly.

• *Examples.* As an example we represent the distribution $\delta_{s,\infty}$ in all three hybrid models. This distribution, which was defined by $\varphi^{\mathbb{P}} \mapsto \varphi^{\mathbb{P}}(\infty)$ in the projective model (cf. (2.30)), is represented in the projective model by $h^{\mathbb{P}}(\zeta) = \frac{1}{2i}\zeta$, and hence, by (3.4), by

$$\widetilde{h}(\zeta, z) \;=\; \frac{\zeta}{2i}\, y^{-s} \left(\frac{(\zeta - z)(\zeta - \bar{z})}{(\zeta - i)(\zeta + i)}\right)^{s} \tag{3.10}$$

in the rigid hybrid model. Since the difference $\frac{\zeta}{2iy^s}\Bigl(\bigl(\frac{1-z/\zeta}{1-i/z}\bigr)^s\bigl(\frac{1-\bar z/\zeta}{1+i\zeta}\bigr)^s-1\Bigr)$ is holomorphic in ζ on a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$ for each z, we obtain the much simpler semicanonical representative

$$h_{\mathrm{s}}(\zeta,z) \;=\; \frac{\zeta}{2i\,y^s}\,, \tag{3.11}$$

of $\delta_{s,\infty}$ in the flabby hybrid model. Finally, subtracting $h_{\mathrm{s}}(\bar z,z)$ we obtain the (unique) representative of $\delta_{s,\infty}$ in the canonical hybrid model:

$$h_{\mathrm{c}}(\zeta,z) \;=\; \frac{\zeta-\bar z}{2i}\,y^{-s}\,. \tag{3.12}$$

We obtain other elements of C_s by the action of G. For $g\in G$ with $g\,\infty = a\in\mathbb{R}$ we get

$$h_{\mathrm{c}}|g^{-1}(\zeta,z) \;=\; \frac{\zeta-\bar z}{z-\bar z}\,\frac{z-a}{\zeta-a}\;R(a;z)^{1-s}\,. \tag{3.13}$$

Here property (3.8) is obvious, and (3.7) holds because the only singularity of (3.13) on $\mathbb{P}^1_{\mathbb{R}}$ is a simple pole of residue $(i/2)\,R(a;z)^{-s}$ at $\zeta=a$.

• *Duality and Poisson transform.* From (1.24) we find that if $h\in\mathcal{H}_s$ and $f\in\mathcal{M}^{\omega}_{1-s}$ are defined on $U\smallsetminus(\mathbb{P}^1_{\mathbb{R}}\times\mathfrak{H})$, respectively U, for the same neighborhood U of $\mathbb{P}^1_{\mathbb{R}}\times\mathfrak{H}$, then

$$\langle f,h\rangle \;=\; \frac{1}{\pi}\Bigl(\int_{C_+}-\int_{C_-}\Bigr)f(\zeta,z)\,h(\zeta,z)\,R(\zeta;z)\,d\zeta\,, \tag{3.14}$$

where C_+ and C_- are contours encircling z in $\mathfrak{H}$ and $\bar z$ in $\mathfrak{H}^-$, respectively, such that $C_+\times\{z\}$ and $C_-\times\{z\}$ are contained in U. The result $\langle f,h\rangle$ does not change if we replace h by another element of $h+\mathcal{M}^{\omega}_s\subset\mathcal{H}_s$.

We apply this to the Poisson kernel $f^{\mathbb{P}}_z(\zeta)=R^{\mathbb{P}}(\zeta;z)^{1-s}=\bigl(\frac{R(\zeta;z)}{R(\zeta;i)}\bigr)^{1-s}$, for $z\in\mathfrak{H}$. The corresponding element in the rigid pair model is

$$\tilde f_z(\zeta,z_1) \;=\; \Bigl(\frac{R(\zeta;z)}{R(\zeta;i)}\Bigr)^{1-s}\Bigl(\frac{R(\zeta;i)}{R(\zeta;z_1)}\Bigr)^{1-s} \;=\; \Bigl(\frac{R(\zeta;z)}{R(\zeta,z_1)}\Bigr)^{1-s}\,.$$

Applying (3.14) we find for $z,z_1\in\mathfrak{H}$:

$$\begin{aligned}\mathsf{P}_s h(z) \;&=\; \frac{1}{\pi}\Bigl(\int_{C_+}-\int_{C_-}\Bigr)\Bigl(\frac{R(\zeta;z)}{R(\zeta;z_1)}\Bigr)^{1-s}\,h(\zeta,z_1)R(\zeta;z_1)\,d\zeta\\ &=\; \frac{1}{\pi}\Bigl(\int_{C_+}-\int_{C_-}\Bigr)\Bigl(\frac{R(\zeta;z_1)}{R(\zeta;z)}\Bigr)^{s}\,h(\zeta,z_1)\,R(\zeta;z)\,d\zeta\,,\end{aligned} \tag{3.15}$$

where C_+ encircles z and z_1, and C_- encircles $\bar z$ and $\overline{z_1}$. Since this does not depend on z_1, we can choose $z_1=z$ to get

$$\begin{aligned}\mathsf{P}_s h(z) \;&=\; \frac{1}{\pi}\Bigl(\int_{C_+}-\int_{C_-}\Bigr)\,h(\zeta,z)\;R(\zeta;z)\;d\zeta\\ &=\; \frac{1}{2\pi i}\Bigl(\int_{C_+}-\int_{C_-}\Bigr)\,h(\zeta,z)\,\frac{(z-\bar z)}{(\zeta-z)(\zeta-\bar z)}\,d\zeta\,.\end{aligned} \tag{3.16}$$

The representation of the Poisson transformation given by formula (3.16) has a very simple form. The dependence on the spectral parameter s is provided by the model, not by the Poisson kernel. But a really amazing simplification occurs if we assume that the function $h\in\mathcal{H}_s$ belongs to the subspace C_s of canonical hyperfunction representatives.

In that case $h(\zeta, z)$ is holomorphic in ζ in all of $\mathbb{C} \smallsetminus \mathbb{R}$, so we can evaluate the integral by Cauchy's theorem. In the lower half-plane there is no pole, since $h(\bar{z}, z)$ vanishes, so the integral over C_- vanishes. In the upper half-plane there is a simple pole of residue $h(z,z)$ at $\zeta = z$. Hence we obtain:

Proposition 3.1. *The Poisson transform of a function $h \in C_s$ is the function*

$$\mathsf{R}_s h(z) \;=\; h(z,z)\,, \tag{3.17}$$

defined by restriction to the diagonal.

As examples of the proposition we set $\zeta = z$ in equations (3.12) and (3.13) to get

$$\begin{aligned} u(z) = y^{1-s} \qquad &\Rightarrow \quad (\mathsf{P}_s{}^{-1}u)_{\mathrm{can}}(\zeta, z) \;=\; \frac{\zeta - \bar{z}}{2i}\, y^{-s} \\ u(z) = R(a;z)^{1-s} \quad &\Rightarrow \quad (\mathsf{P}_s{}^{-1}u)_{\mathrm{can}}(\zeta, z) \;=\; \frac{\zeta - \bar{z}}{z - \bar{z}}\, \frac{z-a}{\zeta - a}\, u(z)\,. \end{aligned} \tag{3.18}$$

Finally, we remark that on the larger space C_s^+ introduced in the Remark above, we have *two* restriction maps

$$\mathsf{R}_s^+ h(z) \;=\; h(z,z)\,, \qquad \mathsf{R}_s^- h(z) \;=\; h(\bar{z}, z) \tag{3.19}$$

to the space of functions on $\mathfrak{H}$. The analogue of the proposition just given is then that the restriction of P_s to C_s^+ equals the difference $\mathsf{R}_s = \mathsf{R}_s^+ - \mathsf{R}_s^-$.

3.2. **Poisson inversion and the canonical model.** The canonical model is particularly suitable to give an integral formula for the inverse Poisson transformation, as we see in the main result of this subsection, Theorem 3.2. In Proposition 3.4 we give an integral formula for the canonical representative of a hyperfunction in terms of an arbitrary representative in $\mathcal{H}_s$. Proposition 3.6 relates, for $u \in \mathcal{E}_s$, the Taylor expansions in the upper and lower half plane of the canonical representative of $\mathsf{P}_s{}^{-1}u$ to the polar expansion of u with the functions $p_{s,n}$.

To determine the image $\mathsf{P}_s{}^{-1}u$ under the inverse Poisson transform for a given $u \in \mathcal{E}_s$, we have to construct a hyperfunction on $\partial\mathbb{H}$ which maps under P_s to u. A first attempt, based on [10], Chap. II, §2, would be (in the line model) to integrate the Green's form $\{u, R(\zeta;\,\cdot\,)^s\}$ from some base point to ζ. This does not make sense at ∞ since $R(\zeta;\,\cdot\,)$ has a singularity there and one cannot take a well-defined s-th power of it, so we should renormalize by dividing $R(\zeta;\,\cdot\,)^s$ by $R(\zeta;i)^s$, or better, to avoid destroying the G-equivariance of the construction, by $R(\zeta;z)^s$ with a variable point $z \in \mathfrak{H}$. This suggests the formula

$$h(\zeta, z) \;=\; \begin{cases} \int_{z_0}^{\zeta} \{u, (R_\zeta(\,\cdot\,)/R_\zeta(z))^s\} & \text{if } \zeta \in \mathfrak{H}\,, \\ \int_{z_0}^{\bar{\zeta}} \{u, (R_\zeta(\,\cdot\,)/R_\zeta(z))^s\} & \text{if } \zeta \in \mathfrak{H}^-\,, \end{cases} \tag{3.20}$$

in the hybrid model, where $z_0 \in \mathfrak{H}$ is a base point, as a second attempt. This almost works: the fact that the Green's form is closed implies that the integrals are independent of the path of integration, and changing the base-point z_0 changes $h(\,\cdot\,, z)$ by a function holomorphic near $\mathbb{P}^1_{\mathbb{R}}$ and hence does not change the hyperfunction it represents. The problem is that both integrals in (3.20) diverge, because $R(\zeta;z')^s$ has a singularity like $(\zeta - z')^{-s}$ near ζ and like $(\zeta - \overline{z'})^{-s}$ near $\bar\zeta$ and the differentiation implicit in the bracket $\{\,\cdot\,,\,\cdot\,\}$ turns these into singularities like $(\zeta - z')^{-s-1}$ and $(\zeta - \overline{z'})^{-s-1}$ which are no longer integrable at $z' = \zeta$ or $z' = \bar\zeta$, respectively. To remedy this in the upper half-plane, we replace $\{u, (R_\zeta(\,\cdot\,)/R_\zeta(z))^s\}$ by $[u, (R_\zeta(\,\cdot\,)/R_\zeta(z))^s]$, which differs from it by a harmless exact 1-form but is now integrable at ζ. (The same trick was already used in §2, Chap. II of [10], where z_0 was ∞.) In the lower half-plane, $[u, (R_\zeta(\,\cdot\,)/R_\zeta(z)^s]$ is not small near $z' = \bar\zeta$, so here we must replace the differential form $\{u, (R_\zeta(\,\cdot\,)/R_\zeta(z))^s\} = -\{(R_\zeta(\,\cdot\,)/R_\zeta(z))^s, u\}$ by $-[(R_\zeta(\,\cdot\,)/R_\zeta(z))^s, u]$ instead. (We recall that $\{\,\cdot\,,\,\cdot\,\}$ is antisymmetric but $[\,\cdot\,,\,\cdot\,]$ is not.) However, since the differential forms $[u, (R_\zeta(\,\cdot\,)/R_\zeta(z))^s]$ and $-[(R_\zeta(\,\cdot\,)/R_\zeta(z))^s, u]$ differ by the exact form $d(u\,(R_\zeta(\,\cdot\,)/R_\zeta(z))^s)$, this change requires correcting the formula in one of the half-planes. (We choose the upper half plane.) This gives the formula

$$h(\zeta; z) \;=\; \begin{cases} u(z_0)\big(\frac{R(\zeta;z_0)}{R(\zeta;z)}\big)^s + \int_{z_0}^{\zeta}[u, (\frac{R(\zeta;\,\cdot\,)}{R(\zeta;z)})^s] & \text{if } \zeta \in \mathfrak{H}\,, \\ \int_{\bar\zeta}^{z_0}[(\frac{R(\zeta;\,\cdot\,)}{R(\zeta;z)})^s, u] & \text{if } \zeta \in \mathfrak{H}^-\,. \end{cases} \tag{3.21}$$

We note that in this formula, $h(\overline{z_0}, z) = 0$. So we can satisfy (3.8) by choosing $z_0 = z$, at the same time restoring the G-symmetry which was broken by the choice of a base point z_0. We can then choose the continuous branch of $(R_\zeta/R_\zeta(z))^s$ that equals 1 at the end-point z of the path of integration. Thus we have arrived at the following Poisson inversion formula, already given in the Introduction (eq. (4)):

Theorem 3.2. *Let $u \in \mathcal{E}_s$. Then the function $\mathsf{B}_s u \in \mathcal{H}_s$ defined by*

$$(\mathsf{B}_s u)(\zeta, z) \;=\; \begin{cases} u(z) + \int_z^{\zeta}[u, (R_\zeta/R_\zeta(z))^s] & \textit{if } \zeta \in \mathfrak{H}\,, \\ \int_{\bar\zeta}^{z}[(R_\zeta/R_\zeta(z))^s, u] & \textit{if } \zeta \in \mathfrak{H}^- \end{cases} \tag{3.22}$$

along any piecewise C^1-path of integration in $\mathfrak{H} \smallsetminus \{\zeta\}$, respectively $\mathfrak{H} \smallsetminus \{\bar\zeta\}$, with the branch of $(R_\zeta/R_\zeta(z))^s$ chosen to be 1 at the end-point z, belongs to C_s and is a representative of the hyperfunction $\mathsf{P}_s^{-1}u \in \mathcal{M}_s^{-\omega} = \mathcal{H}_s/\mathcal{M}_s^{\omega}$.

Corollary 3.3. *The maps $\mathsf{B}_s : \mathcal{E}_s \to C_s$ and $\mathsf{R}_s : C_s \to \mathcal{E}_s$ defined by* (3.22) *and* (3.17) *are inverse isomorphisms, and we have a commutative diagram*

$$\begin{tikzcd} \mathcal{H}_s \arrow[r, two heads] & \mathcal{V}_s^{-\omega} \arrow[d, "\mathsf{P}_s"] \\ C_s \arrow[u, hook] \arrow[ur, "\cong"] \arrow[r, shift right, "\mathsf{R}_s"'] & \mathcal{E}_s \arrow[l, shift right, "\mathsf{B}_s"'] \end{tikzcd} \tag{3.23}$$

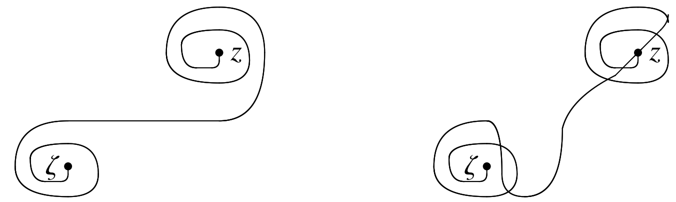
Proof. Let $u \in \mathcal{E}_s$. First we check that $h = \mathsf{B}_s u$ is well defined and determines an element of C_s. The convergence of the integrals in (3.22) requires an estimate of the integrand at the boundaries. For $\zeta \in \mathfrak{H} \smallsetminus \{z\}$ we use

$$[u(z'), (R_\zeta(z')/R_\zeta(z))^s]_{z'} \;=\; \Big(\frac{R(\zeta; z')}{R(\zeta; z)}\Big)^s \Big(u_z(z')\,dz' + \frac{is}{2y'}u(z')\,\frac{\zeta - z'}{\zeta - \overline{z'}}\,d\overline{z'}\Big). \tag{3.24}$$

The factor in front is 1 for $z' = z$ and $\mathrm{O}((\zeta - z')^{-s})$ for z' near ζ. The other contributions stay finite, so the integral for $\zeta \in \mathfrak{H} \smallsetminus \{z\}$ converges. (Recall that $\mathrm{Re}\,(s)$ is always supposed to be < 1.) For $\zeta \in \mathfrak{H}^- \smallsetminus \{\bar z\}$ we use in a similar way

$$[(R_\zeta(z')/R_\zeta(z))^s, u(z')]_{z'} = \Bigl(\frac{R(\zeta;z')}{R(\zeta;z)}\Bigr)^s \Bigl(\frac{is}{2y'} u(z') \frac{\zeta - \overline{z'}}{\zeta - z'}\, dz' + u_{\bar z}(z')\, d\overline{z'}\Bigr).$$

We have normalized the branch of $(R(\zeta;z')/R(\zeta;z))^s$ by prescribing the value 1 at $z' = z$. This choice fixes $(R(\zeta;z')/R(\zeta;z))^s$ as a continuous function on the paths of integration. The result of the integration does not depend on the path, since the differential form is closed and since we have convergence at the other end point ζ or $\bar\zeta$. Any continuous deformation of the path within $\mathfrak{H} \smallsetminus \{\zeta\}$ or $\mathfrak{H} \smallsetminus \{\bar\zeta\}$ is allowed, even if the path intersects itself with different values of $(R(\zeta;z')/R(\zeta;z))^s$ at the intersection point.



If we choose the geodesic path from z to ζ, and if ζ is very near the real line, then the branch of $(R_\zeta(z')/R_\zeta(z))^s$ near $z' = \zeta$ is the principal one (argument between $-\pi$ and π).

The holomorphy in ζ follows from a reasoning already present in [10], Chap. II, §2, and hence given here in a condensed form. Since the form (3.24) is holomorphic in ζ, a contribution to $\partial_{\bar\zeta} h$ could only come from the upper limit of integration, but in fact vanishes since $\mathrm{O}((\zeta - z')^{-s})\,(\zeta - z') = \mathrm{o}(1)$ as $\zeta \to z'$. Hence $h(\,\cdot\,, z)$ is holomorphic on $\mathfrak{H} \smallsetminus \{z\}$. For ζ near z we integrate a quantity $\mathrm{O}((\zeta - z')^{-s})$ from z to ζ, which results in an integral estimated by $\mathrm{O}((\zeta - z)^{1-s})$. So $(\mathsf{B}_s u)(\zeta, z)$ is bounded for ζ near z. Hence $h(\zeta, z) = (\mathsf{B}_s u)(\zeta, z)$ is holomorphic at $\zeta = z$ as well. For the holomorphy on $\mathfrak{H}^-$ we proceed similarly. This also shows that $h(\bar z, z) = 0$, which is condition (3.8) in the definition of C_s.

For condition (3.7) we note that

$$h(\zeta;z) - \Bigl(\frac{R(\zeta;z_1)}{R(\zeta;z)}\Bigr)^s h(\zeta;z_1) = u(z) - u(z_1)\Bigl(\frac{R(\zeta;z_1)}{R(\zeta;z)}\Bigr)^s + \int_z^{z_1} \Bigl[u, \Bigl(\frac{R(\zeta;\,\cdot\,)}{R(\zeta;z)}\Bigr)^s\Bigr]$$

if $\zeta \in \mathfrak{H} \smallsetminus \{z, z_1\}$ and

$$h(\zeta;z) - \Bigl(\frac{R(\zeta;z_1)}{R(\zeta;z)}\Bigr)^s h(\zeta;z_1) = \int_{z_1}^{z} \Bigl[\Bigl(\frac{R(\zeta;\,\cdot\,)}{R(\zeta;z)}\Bigr)^s, u\Bigr]$$

if $\zeta \in \mathfrak{H}^- \smallsetminus \{\bar z, \bar z_1\}$. The right-hand sides both have holomorphic extensions in ζ to a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$, and the difference of these two extensions is seen, using (2.13) and the antisymmetry of $\{\,\cdot\,,\,\cdot\,\}$, to be equal to

$$u(z) - u(z_1)\Bigl(\frac{R(\zeta;z_1)}{R(\zeta;z)}\Bigr)^s + \int_z^{z_1} d\Bigl(u(z')\Bigl(\frac{R(\zeta;z')}{R(\zeta;z)}\Bigr)^s\Bigr) = 0\,.$$

In summary, the function $\mathsf{B}_s u$ belongs to $\mathcal{H}_s$, is defined in all of $(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H}$, and vanishes on the anti-diagonal, so $\mathsf{B}_s u \in C_s$, which is the first statement of the theorem. The second follows immediately from Proposition 3.1, since it is obvious from (3.22) that $\mathsf{R}_s \mathsf{B}_s u = u$ and the proposition says that R_s is the restriction of P_s to C_s. ■

The corollary follows immediately from the theorem if we use Helgason's result (Theorem 2.4) that the Poisson transformation is an isomorphism. However, given that we have now constructed an explicit inverse map for the Poisson transformation, we should be able to give a more direct proof of this result, not based on polar expansions, and indeed this is the case. Since $\mathsf{P}_s\mathsf{B}_s u = \mathsf{R}_s\mathsf{B}_s u = u$, it suffices to show that R_s is injective. To see this, assume that $h \in C_s$ satisfies $h(z,z) = 0$ for all $z \in \mathfrak{H}$. For fixed $z_1, z_2 \in \mathfrak{H}$ let $c(\zeta)$ denote the difference in (3.7). This function is holomorphic near $\mathbb{P}^1_{\mathbb{R}}$, and extends to $\mathbb{P}^1_{\mathbb{C}}$ in a multi-valued way with branch-points of mild growth, $(\zeta - \zeta_0)^{\pm s}$ with $0 < \operatorname{Re} s < 1$, at z_1, $\overline{z_1}$, z_2 and $\overline{z_2}$. Moreover, $c(\zeta)$ tends to 0 as ζ tends to z_1 or $\overline{z_1}$ (because $\operatorname{Re} s > 0$) and also as ζ tends to z_2 or $\overline{z_2}$ (because $h(z_2, z_2) = h(\overline{z_2}, z_2) = 0$ and $\operatorname{Re} s < 1$). Suppose that c is not identically zero. The differential form $d \log c(\zeta)$ is meromorphic on all of $\mathbb{P}^1_{\mathbb{C}}$ and its residues at $\zeta_0 \in \{z_z, z_2, \overline{z_1}, \overline{z_2}\}$ have positive real part. Since c is finite elsewhere on $\mathbb{P}^1_{\mathbb{C}}$, any other residue is non-negative. This contradicts the fact that the sum of all residues of a meromorphic differential on $\mathbb{P}^1_{\mathbb{C}}$ is zero. Hence we conclude that $c = 0$. Then the local behavior of $h(\zeta, z_1) = h(\zeta, z_2)\left(R_\zeta(z_2)/R_\zeta(z_1)\right)^s$ at the branch points shows that both $h(\,\cdot\,, z_1)$ and $h(\,\cdot\,, z_2)$ vanish identically.

Remarks.

1. It is also possible to prove that $\mathsf{B}_s\mathsf{P}_s\varphi = \varphi$ and $\mathsf{P}_s\mathsf{B}_s u = u$ by using complex contour integration and equation (2.16), and our original proof that $\mathsf{B}_s = \mathsf{P}_s{}^{-1}$ went this way, but the above proof using the canonical space C_s is much simpler.

2. Taking $z = i$ in formula (3.22) gives a representative for $\mathsf{P}_s{}^{-1}u$ in the projective model, and using the various isomorphisms discussed in Section 1, we can also adapt it to the other $\partial\mathbb{H}$ models of the principal representation.

We know that each element of $\mathcal{V}_s^{-\omega}$ has a unique canonical representative lying in C_s. The following proposition, in which $k(\tau, \zeta; z)$ denotes the kernel function

$$k(\tau, \zeta; z) \;=\; \frac{1}{2i\,(\tau - \zeta)}\,\frac{\zeta - \bar z}{\tau - \bar z}\,, \tag{3.25}$$

tells us how to determine it starting from an arbitrary representative.

Proposition 3.4. *Suppose that $g \in \mathcal{H}_s$ represents $\alpha \in \mathcal{V}_s^{-\omega}$. The canonical representative $g_c \in C_s$ of α is given, for each $z_0 \in \mathfrak{H}$ by*

$$g_c(\zeta, z) \;=\; \frac{1}{\pi}\Big(\int_{C_+} - \int_{C_-}\Big)\, g(\tau, z_0)\Big(\frac{R(\tau; z_0)}{R(\tau; z)}\Big)^s\, k(\tau, \zeta; z)\, d\tau\,, \tag{3.26}$$

with contours C_+ and C_- homotopic to $\mathbb{P}^1_{\mathbb{R}}$ inside the domain of g, encircling z and z_0, respectively $\bar z$ and $\overline{z_0}$, with C_+ positively oriented in $\mathfrak{H}$ and C_- negatively oriented in $\mathfrak{H}^-$, and ζ inside C_+ or inside C_-.

Note that this can be applied when a representative g_0 of α in the projective model is given: simply apply the proposition to the corresponding representative in the rigid hybrid model as given by (3.4).

Proof. Consider $k(\,\cdot\,, \zeta; z)$ as an element of $\mathcal{V}_s^{\omega}$ in the projective model. Then $g_c(\zeta, z) = \langle \alpha, k(\,\cdot\,, \zeta; z)\rangle$. Adapting the contours, we see that $g_c(\,\cdot\,, z)$ is holomorphic on $\mathfrak{H} \cup \mathfrak{H}^-$.

For a fixed $\zeta \in \operatorname{dom} g$ we deform the contours such that ζ is between the new contours. This gives a term $g(\zeta, z)$ plus the same integral, but now representing a holomorphic function in ζ on the region between C_+ and C_-, which is a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$. So g and g_c represent the same hyperfunction. Condition (3.8) follows from $k(\tau,\bar z;z)=0$. ■

Choosing $z_0=z$ in (3.26) gives the simpler formula

$$g_c(\zeta,z) \;=\; \frac{1}{\pi}\Bigl(\int_{C_+}-\int_{C_-}\Bigr)\,g(\tau,z)\,k(\tau,\zeta;z)\,d\tau\,, \tag{3.27}$$

(which is, of courses, identical to (3.26) if g belongs to $\mathcal{V}_s^{\omega,\mathrm{rig}}$). In terms of $\alpha\in\mathcal{V}_s^{-\omega}$ as a linear form on $\mathcal{V}_{1-s}^{\omega}$ we can write this as

$$g_c(\zeta,z) \;=\; \langle f_\zeta,\alpha\rangle \quad\text{ with }\quad f_\zeta(\tau,z) \;=\; \frac{(\zeta-\bar z)(\tau-z)}{(z-\bar z)(\tau-\zeta)}\,. \tag{3.28}$$

The integral representation (3.26) has the following consequence:

Corollary 3.5. *All elements of C_s are real-analytic on $(\mathbb{P}^1_{\mathbb{C}}\smallsetminus\mathbb{P}^1_{\mathbb{R}})\times\mathfrak{H}$.*

• *Expansions in the canonical model.* For $u\in\mathcal{E}_s$ the polar expansion (2.4) can be generalized, with the shifted functions $p_{s,n}$ in (2.5), to an arbitrary central point:

$$u(z) \;=\; \sum_{n\in\mathbb{Z}} a_n(u,z')\,p_{s,n}(z,z') \qquad (z'\in\mathfrak{H}\text{ arbitrary})..\tag{3.29}$$

Let $h=\mathsf{B}_s u\in C_s$ be the canonical representative of $\mathsf{P}_s^{-1}u\in\mathcal{V}_s^{-\omega}$. For $z'\in\mathfrak{H}$ fixed, $h(\zeta,z')$ is a holomorphic function of $\zeta\in\mathbb{C}\smallsetminus\mathbb{R}$ and has Taylor expansions in $\frac{\zeta-z'}{\zeta-\overline{z'}}$ on $\mathfrak{H}$ and in $\frac{\zeta-\overline{z'}}{\zeta-z'}$ on $\mathfrak{H}^-$. Since $h(\bar z,z)=0$, the constant term in the expansion on $\mathfrak{H}^-$ vanishes. Thus there are $A_n(h,z')\in\mathbb{C}$ such that

$$h(\zeta,z') \;=\; \begin{cases} \sum_{n\ge 0}A_n(h,z')\Bigl(\dfrac{\zeta-z'}{\zeta-\overline{z'}}\Bigr)^n & \text{for }\zeta\in\mathfrak{H}\,,\\ -\sum_{n<0}A_n(h,z')\Bigl(\dfrac{\zeta-z'}{\zeta-\overline{z'}}\Bigr)^n & \text{for }\zeta\in\mathfrak{H}^-\,.\end{cases}\tag{3.30}$$

(We use a minus sign in the expansion on $\mathfrak{H}^-$ because then

$$(\zeta,z')\mapsto \sum_{n\ge n_0}A_n(h,z')\Bigl(\frac{\zeta-z'}{\zeta-\overline{z'}}\Bigr)^n\text{ for }\zeta\in\mathfrak{H}\,,\quad -\sum_{n<n_0}A_n(h,z')\Bigl(\frac{\zeta-z'}{\zeta-\overline{z'}}\Bigr)^n\text{ for }\zeta\in\mathfrak{H}^-\,,$$

represents the same hyperfunction $\mathsf{P}_s^{-1}u$ for any choice of $n_0\in\mathbb{Z}$.) From $\frac{g\zeta-gz'}{g\zeta-g\overline{z'}}=\frac{c\overline{z'}+d}{cz'+d}\,\frac{\zeta-z'}{\zeta-\overline{z'}}$ for $g=\bigl[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\bigr]\in G$ it follows that

$$A_n(h\,|\,g,z') \;=\; \Bigl(\frac{c\overline{z'}+d}{cz'+d}\Bigr)^n A_n(h,gz')\,. \tag{3.31}$$

Similarly, we have from (2.5) and (2.3):

$$a_n(u\,|\,g,z') \;=\; \Bigl(\frac{c\overline{z'}+d}{cz'+d}\Bigr)^n a_n(u,gz')\,. \tag{3.32}$$

In fact, the coefficients $A_n(\,)$ and $a_n(\,)$ are proportional:

Proposition 3.6. *For $u \in \mathcal{E}_s$ and $h = \mathsf{B}_s u \in C_s$, the coefficients in the expansions* (3.29) *and* (3.30) *are related by*

$$a_n(u,z') \;=\; (-1)^n \frac{\Gamma(s)}{\Gamma(s+n)}\, A_n(h,z')\,. \tag{3.33}$$

Proof. The expansion (3.30) for $z' = i$ shows that the hyperfunction $\mathsf{P}_s{}^{-1}u$ has the expansion $\sum_n A_n(h,i)\,\mathbf{e}_{s,n}$ in the basis functions in (1.9). Then (2.28) gives

$$u(z) \;=\; \sum_{n\in\mathbb{Z}} (-1)^n \frac{\Gamma(s)}{\Gamma(s+n)}\, A_n(h,i)\, P_{s,n}(z)\,.$$

This gives the relation in the proposition if $z' = i$, and the general case follows from the transformation rules (3.31) and (3.32). ■

The transition $s \leftrightarrow 1-s$ does not change $\mathcal{E}_s = \mathcal{E}_{1-s}$ or $p_{s,n} = p_{1-s,n}$. So the coefficients $a_n(u,z')$ stay the same under $s \mapsto 1-s$. With the commutative diagram (2.25) we get:

Corollary 3.7. *The operator $C_s \to C_{1-s}$ corresponding to $I_s : \mathcal{V}_s^{-\omega} \to \mathcal{V}_{1-s}^{-\omega}$ acts on the coefficients in* (3.30) *by*

$$A_n(I_s h,\, z') \;=\; \frac{(1-s)_{|n|}}{(s)_{|n|}}\, A_n(h,\, z')\,.$$

We remark that Proposition 3.6 can also be used to give an alternative proof of Corollary 3.5, using (2.19) with u replaced by $u \circ g_{z'}$ to obtain the analyticity of $a_n(u,z')$ in z' and then (3.33) to control the speed of convergence in (3.30).

3.3. **Differential equations for the canonical model and the sheaf of mixed eigenfunctions.** The canonical model provides us with an isomorphic copy C_s of $\mathcal{V}_s^{-\omega} \cong \mathcal{E}_s$ inside the flabby hybrid model $\mathcal{H}_s$. We now show that the elements of the canonical model are real-analytic in both variables jointly, and satisfy first order differential equations in the variable $z \in \mathfrak{H}$ with ζ as a parameter.

The same differential equations can be used to define a sheaf $\mathcal{D}_s$ on $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$. In Proposition 3.10 and Theorem 3.13 we describe the local structure of this sheaf. It turns out that we can identify the space $\mathcal{V}_s^{\omega,\mathrm{rig}}$ of the rigid hybrid model with a space of sections of this sheaf of a special kind. There is a sheaf morphism that relates $\mathcal{D}_s$ to the sheaf $\mathcal{E}_s : U \mapsto \mathcal{E}_s(U)$ of λ_s-eigenfunctions on $\mathfrak{H}$. For elements of the full space $\mathcal{E}_s = \mathcal{E}_s(\mathfrak{H})$ the canonical model gives sections of $\mathcal{D}_s$ over $(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H}$.

Theorem 3.8. *Each $h \in C_s$ and its corresponding eigenfunction $u = \mathsf{P}_s h = \mathsf{R}_s h \in \mathcal{E}_s$ satisfy, for $\zeta \in \mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}$, $z \in \mathfrak{H}$, $\zeta \notin \{z,\bar z\}$, the differential equations*

$$(z-\bar z)\,\partial_z h(\zeta,z) \;+\; s\,\frac{\zeta-\bar z}{\zeta - z}\,\bigl(h(\zeta,z)-u(z)\bigr) \;=\; 0\,, \tag{3.34a}$$

$$(z-\bar z)\,\partial_{\bar z}\bigl(h(\zeta,z)-u(z)\bigr) \;-\; s\,\frac{\zeta - z}{\zeta-\bar z}\, h(\zeta,z) \;=\; 0\,. \tag{3.34b}$$

Conversely, any continuous function h on $(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H}$ that is holomorphic in the first variable, continuously differentiable in the second variable, and satisfies the differential equations (3.34) *for some $u \in C^1(\mathfrak{H})$ belongs to C_s, and u is $\mathsf{P}_s h$.*

The differential equations (3.34) look complicated, but in fact are just the dz- and $d\bar{z}$-components of the identity

$$\big[R(\zeta;\,\cdot\,)^s,\ u(\zeta;\,\cdot\,)\big] \;=\; d\big(\,R(\zeta;\,\cdot\,)^s h(\zeta;\,\cdot\,)\,\big) \tag{3.35}$$

between 1-forms, as one checks easily. (The function $R(\zeta;z)^s$ is multivalued, but if we take the same branch on both sides of the equality, then it makes sense locally.)

Proof of Theorem 3.8. The remark just made shows almost immediately that the function $h = \mathsf{B}_s u \in C_s$ defined by (3.22) satisfies the differential equations (3.34): differentiating (3.22) in z gives

$$d_z(h(\zeta,z)R(\zeta;s)^s) \;=\; \begin{cases} d(u(z)R(\zeta;z)^s) \,-\, [u(z),R(\zeta;z)^s]_z & \text{if } z\in\mathfrak{H},\\ \qquad\qquad\qquad\;\; +\, [R(\zeta;z)^s,u(z)]_z & \text{if } z\in\mathfrak{H}^-,\end{cases}$$

and the right-hand side equals $[R(\zeta;z)^s,u(z)]$ in both cases by virtue of (2.13).

An alternative approach, not using the explicit Poisson inversion formula (3.22), is to differentiate (3.7) with respect to z_1 (resp. $\overline{z_1}$) and then set $z_1 = z_2 = z$ to see that the expression on the left-hand side of (3.34a) (resp. (3.34b)) is holomorphic in ζ near $\mathbb{P}^1_{\mathbb{R}}$. (Here that we use the result proved above that elements of the canonical model are analytical in both variables jointly.) The equations $h(z,z) = u(z)$, $h(\bar{z},z) = 0$ then show that the expressions in (3.34), for z fixed, are holomorphic in ζ on all of $\mathbb{P}^1_{\mathbb{C}}$ and hence constant. To see that both constants vanish, we set $\zeta = \bar{z}$ in (3.34a) (resp. $\zeta = z$ in (3.34b)) and use

$$\partial_z(h(\zeta,z))|_{\zeta=\bar{z}} = \partial_z(h(\bar{z},z)) = \partial_z(0) = 0\,, \qquad \partial_{\bar{z}}(h(\zeta,z))|_{\zeta=z} = \partial_{\bar{z}}(h(z,z)) = \partial_{\bar{z}}u(z)\,.$$

This proves the forward statement of Theorem 3.8. Instead of proving the converse immediately, we first observe that the property of satisfying the differential equations in the theorem is a purely local one and therefore defines a sheaf of functions.

We now give a formal definition of this sheaf and then prove some general statements about its local sections that include the second part of Theorem 3.8.

We note that the differential equations (3.34) make sense, not only on $(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}})\times\mathfrak{H}$, but on all of $\mathbb{P}^1_{\mathbb{C}}\times\mathfrak{H}$, with singularities on the "diagonal" and "antidiagonal" defined by

$$\Delta^+ \;=\; \{\,(z,z)\;:\;z\in\mathfrak{H}\,\}, \qquad \Delta^- \;=\; \{\,(\bar{z},z)\;:\;z\in\mathfrak{H}\,\}. \tag{3.36}$$

We therefore define our sheaf on open subsets of this larger space.

Definition 3.9. For every open subset $U\subset\mathbb{P}^1_{\mathbb{C}}\times\mathfrak{H}$ we define $\mathcal{D}_s(U)$ as the space of pairs (h,u) of functions on U such that

a) h and u are continuous on U,
b) h is holomorphic in its first variable,
c) locally u is independent of the first variable,
d) h and u are continuously differentiable in the second variable and satisfy the differential equations (3.34) on $U\smallsetminus(\Delta^+\cup\Delta^-)$, with $u(z)$ replaced by $u(\zeta,z)$.

This defines $\mathcal{D}_s$ as a sheaf of pairs of functions on $\mathbb{P}^1_{\mathbb{C}}\times\mathfrak{H}$, the sheaf of *mixed eigenfunctions*. In this language, the content of Theorem 3.8 is that C_s can be identified via $h\mapsto(h,\mathsf{R}_s h)$ with the space of global sections of $\mathcal{D}_s((\mathbb{P}^1_{\mathbb{C}}\smallsetminus\mathbb{P}^1_{\mathbb{R}})\times\mathfrak{H})$. The following proposition gives a number of properties of the local sections.

Proposition 3.10. *Let $(h,u) \in \mathcal{D}_s(U)$ for some open set $U \subset \mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$. Then*

i) *The functions h and u are real analytic on U. The function u is determined by h and satisfies $\Delta u = s(1-s)u$.*
ii) *If U intersects $\Delta^+ \cup \Delta^-$, then we have $h = u$ on $U \cap \Delta^+$ and $h = 0$ on $U \cap \Delta^-$.*
iii) *If $u = 0$, then the function h locally has the form $h(\zeta, z) = \varphi(\zeta)\, R(\zeta; z)^{-s}$ for some branch of $R(\zeta; z)^{-s}$, with φ holomorphic.*
iv) *The function h is determined by u on each connected component of U that intersects $\Delta^+ \cup \Delta^-$.*

Proof. The continuity of h and u allows us to consider them and their derivatives as distributions. We obtain from (3.34) the following equalities of distributions on $U \smallsetminus (\Delta^+ \cup \Delta^-)$:

$$\begin{aligned}\partial_z \partial_{\bar z} h \;&=\; \partial_z\Big(\partial_{\bar z} u + \frac{s}{z-\bar z}\,\frac{\zeta - z}{\zeta - \bar z}\, h\Big) \;=\; \partial_z\partial_{\bar z} u - \frac{s}{(z-\bar z)^2}\, h - \frac{s^2}{(z-\bar z)^2}\,(h-u)\,,\\ \partial_{\bar z}\partial_z h \;&=\; \partial_{\bar z}\Big(\frac{-s}{z-\bar z}\,\frac{\zeta-\bar z}{\zeta - z}\,(h-u)\Big) \;=\; \frac{-s}{(z-\bar z)^2}\,(h-u) - \frac{s^2}{(z-\bar z)^2}\, h\,.\end{aligned}$$

The differential operators ∂_z and $\partial_{\bar z}$ on distributions commute. In terms of the hyperbolic Laplace operator $\Delta = (z-\bar z)^2\, \partial_z \partial_{\bar z}$, we have in distribution sense

$$(\Delta - \lambda_{s+1})\, h \;=\; (\Delta + s^2) u \;=\; su\,. \tag{3.37}$$

Since u is an eigenfunction of the elliptic differential operator $\Delta - \lambda_s$ with real analytic coefficients, u and also h are real analytic function in the second variable. To conclude that h is real-analytic in both variables jointly, we note that it is also a solution of the following elliptic differential equation with analytic coefficients

$$(-\partial_\zeta \partial_{\bar\zeta} + \Delta - \lambda_{s+1}) h \;=\; su\,.$$

Near $\infty \in \mathbb{P}^1_{\mathbb{C}}$ we replace ζ by $\upsilon = 1/\zeta$ in the last step.

Since u is locally independent of ζ, we conclude that u is real-analytic on the whole of U, and satisfies $\Delta u = s(1-s)u$ on U. Then (3.37) gives the analyticity of h on U. Now we use (3.34a) to obtain

$$u(\zeta, z) \;=\; h(\zeta, z) + \frac{z-\bar z}{s}\,\frac{\zeta - z}{\zeta - \bar z}\, \partial_z h(\zeta, z)\,.$$

So h determines u on $U \smallsetminus \Delta^-$, and then by continuity on the whole of U. Furthermore $u = h$ on Δ^+. Similarly, (3.34b) implies $h = 0$ on $U \cap \Delta^-$. This proves parts i) and ii) of the proposition.

Under the assumption $u = 0$ in part iii) the differential equations (3.34) become homogeneous in h. For fixed ζ the solutions are multiples of $z \mapsto R(\zeta; z)^{-s}$, as is clear from (3.35). Hence h locally has the form $h(\zeta, z) = \varphi(\zeta)\, R(\zeta; z)^{-s}$, where φ is holomorphic by condition b) in the definition of $\mathcal{D}_s$. It also follows that h vanishes on any connected component of U on which $R(\zeta; z)^{-s}$ is multi-valued, and in particular on any component that intersects $\Delta^+ \cup \Delta^-$. Part iv) now follows by linearity. ■

Proof of Theorem 3.8, converse direction. Functions h and u with the properties assumed in the second part of the theorem determine a section $(h,u) \in \mathcal{D}_s((\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H})$. Proposition 3.10 shows that $u \in \mathcal{E}_s$. By the first part of the theorem, we have $(\mathsf{B}_s u, u) \in$

$\mathcal{D}_s((\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H})$. Since this has the same second component as (h, u), part iv) of the proposition shows that $h = \mathsf{B}_s u \in C_s$, and then part ii) gives $u = \mathsf{R}_s h = \mathsf{P}_s h$. ■

• *Local description of h near the diagonal.* Part iv) of Proposition 3.10 says that the first component of a section (h, u) of $\mathcal{D}_s$ near the diagonal or antidiagonal is completely determined by the second component, but does not tell us explicitly how. We would like to make this explicit. We can do this in two ways, in terms of Taylor expansions or by an integral formula. We will use this in §5

We first consider an arbitrary real-analytic function u in a neighborhood of a point $z_0 \in \mathfrak{H}$ and a real-analytic solution h of (3.34a) near (z_0, z_0) which is holomorphic in the first variable. Then h has a power series expansion $h(\zeta, z) = \sum_{n=0}^{\infty} h_n(z)(\zeta - z)^n$ in a neighborhood of (z_0, z_0), and (3.34a) is equivalent to the recursive formulas

$$h_n(z) = \begin{cases} u(z) & \text{if } n = 0\,, \\ \dfrac{1}{1-s}\,\dfrac{\partial h_0(z)}{\partial z} & \text{if } n = 1\,, \\ \dfrac{1}{n-s}\Bigl(\dfrac{\partial h_{n-1}(z)}{\partial z} + \dfrac{s}{z-\bar z}\,h_{n-1}(z)\Bigr) & \text{if } n \ge 2\,, \end{cases}$$

which we can solve to get the expansion

$$h(\zeta, z) = u(z) + y^{-s} \sum_{n=1}^{\infty} \frac{\partial^{n-1}}{\partial z^{n-1}}\Bigl(y^s \frac{\partial u}{\partial z}\Bigr) \frac{(\zeta - z)^n}{(1-s)_n}\,, \tag{3.38}$$

where $(1-s)_n = (1-s)(2-s)\cdots(n-s)$ as usual is the Pochhammer symbol. Conversely, for any real-analytic function $u(z)$ in a neighborhood of z_0, the series in (3.38) converges and defines a solution of (3.34a) near (z_0, z_0). Thus there is a bijection between germs of real-analytic functions u near z_0 and germs of real-analytic solutions of (3.34a), holomorphic in ζ, near (z_0, z_0). If u further satisfies $\Delta u = \lambda_s u$, then a short calculation shows that the function defined by (3.38) satisfies (3.34b), so we get a bijection between germs of λ_s-eigenfunctions u near z_0 and the stalk of $\mathcal{D}_s$ at (z_0, z_0). An exactly similar argument gives, for any λ_s-eigenfunction u near z_0, a unique solution

$$h(\zeta, z) = -\,y^{-s} \sum_{n=1}^{\infty} \frac{\partial^{n-1}}{\partial \bar z^{n-1}}\Bigl(y^s \frac{\partial u}{\partial \bar z}\Bigr) \frac{(\zeta - \bar z)^n}{(1-s)_n}\,, \tag{3.39}$$

of equations (3.34a) and (3.34b) near the point $(\bar z_0, z_0) \in \Delta^-$. This proves:

Proposition 3.11. *Let $u \in \mathcal{E}_s(U)$ for some open set $U \subset \mathfrak{H}$. Then there is a unique section (h, u) of $\mathcal{D}_s$ in a neighborhood of $\{(z, z) \mid z \in U\} \cup \{(\bar z, z) \mid z \in U\}$, given by equations* (3.38) *and* (3.39)*.*

The second way of writing h in terms of u near the diagonal or anti-diagonal is based on (3.22). This equation was used to lift a global section $u \in \mathcal{E}_s$ to a section $(\mathsf{B}_s u, u)$ of $\mathcal{D}_s$ over all of $(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H}$, but its right-hand side can also be used for functions $u \in \mathcal{E}_s(U)$ for open subsets $U \subset \mathfrak{H}$ to define h near points (z, z) or $(\bar z, z)$ with $z \in U$. This gives a new proof of the first statement in Proposition 3.11, with the advantage that we now also get some information off the diagonal and anti-diagonal:

Proposition 3.12. *If U is connected and simply connected, then the section (h, u) given in Proposition 3.11 extends analytically to $(U \cup \bar U) \times U$.*

• *Formulation with sheaves.* Proposition 3.10 shows that the component h of a local section (h, u) of $\mathcal{D}_s$ determines the component u, which is locally independent of the second variable and satisfies the Laplace equation. So there is a map from sections of $\mathcal{D}_s$ to sections of $\mathcal{E}_s$. To formulate this as a sheaf morphism we need to have sheaves on the same space. We denote the projections from $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$ on $\mathbb{P}^1_{\mathbb{C}}$, respectively $\mathfrak{H}$, by p_1, We use the inverse image sheaf $p_2^{-1}\mathcal{E}_s$ on $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$, associated to the presheaf $U \mapsto \mathcal{E}_s(p_2 U)$. (See, e.g., §1, Chap. II, in [4].) The map p_2 is open, so we do not need a limit over open $V \supset p_2 U$ in the description of the presheaf. Note that the functions in $\mathcal{E}_s(p_2 U)$ depend only on z, but that the sheafification of the presheaf adds sections to $p_2^{-1}\mathcal{E}_s$ that may depend on the first variable. In this way, $(h, u) \mapsto u$ corresponds to a sheaf morphism $C : \mathcal{D}_s \to p_2^{-1}\mathcal{E}_s$. We call the kernel $\mathcal{K}_s$.

We denote the sheaf of holomorphic functions on $\mathbb{P}^1_{\mathbb{C}}$ by $\mathcal{O}$. Then $p_1^{-1}\mathcal{O}$ is also a sheaf on $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$. The following theorem describes $\mathcal{K}_s$ in terms of $p_1^{-1}\mathcal{O}$ and show that the morphism C is surjective.

Theorem 3.13. *The sequence of sheaves on $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$*

$$0 \longrightarrow \mathcal{K}_s \longrightarrow \mathcal{D}_s \xrightarrow{C} p_2^{-1}\mathcal{E}_s \longrightarrow 0 \tag{3.40}$$

is exact. If a connected open set $U \subset \mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$ satisfies $U \cap (\Delta^+ \cup \Delta^-) \neq \emptyset$, then $\mathcal{K}_s(U) = \{0\}$. The restriction of $\mathcal{K}_s$ to $\left(\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}\right) \smallsetminus (\Delta^+ \cup \Delta^-)$ is locally isomorphic to $p_1^{-1}\mathcal{O}$ where holomorphic functions φ correspond to $(\zeta, z) \mapsto (\varphi(\zeta)\, R(\zeta; z)^{-s}, 0)$. The inductive limit of $\mathcal{K}_s(U)$ over all neighborhoods U of $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}$ in $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$ is canonically isomorphic to the space $\mathcal{V}_s^{\omega,\mathrm{rig}}$.

Proof. For the exactness, we only have to check the surjectivity of $C : \mathcal{D}_s \to p_2^{-1}\mathcal{E}_s$. For this we have to verify that for any point $P_0 = (\zeta_0, z_0) \in \mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$, any solution of $\Delta u = \lambda_s u$ lifts to a section $(h, u) \in \mathcal{D}_s(U)$ for some sufficiently small neighborhood U of P_0. If $P_0 \in \Delta^+ \cup \Delta^-$, then this is precisely the content of the first statement of Proposition 3.11. If $P_0 \notin \Delta^+ \cup \Delta^-$, then we define h near P_0 by the formula

$$h(\zeta, z) \;=\; \int_{z_0}^{z} \left[\left(R_\zeta(\,\cdot\,)/R_\zeta(z)\right)^s, u \right] \tag{3.41}$$

instead, again with $\left(R_\zeta(z_1)/R_\zeta(z)\right)^s = 1$ at $z_1 = z$. The next two assertions of the theorem follow from Proposition 3.10. The relation with the rigid hybrid model is based on (3.5). ■

We end this section by making several remarks about the equations (3.34) and their solution spaces C_s and $D_s(U)$.

The first is that there are apparently very few solutions of these equations that can be given in “closed form.” One example is given by the pair $h(\zeta, z) = \frac{\zeta - \bar z}{2i} y^{-s}$, $u(z) = y^{1-s}$ (cf. (3.12)). Of course one also has the translations of this by the action of G, and in Example Example 2 after Theorem 4.5 we will give further generalizations where h is still a polynomial times y^{-s}. One also has the local solutions of the form $(\varphi(\zeta) R(\zeta; z)^{-s}, 0)$ for arbitrary holomorphic functions $\varphi(\zeta)$, as described in Theorem 3.13.

The second observation is that the description of C_s in terms of differential equations can be generalized in a very simple way to the space C_s^+ of semicanonical hyperfunction representatives introduced in the Remark in §3.1: these are simply the functions h on $(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H}$ that satisfy the system of differential equations

$$\begin{aligned} \partial_z\bigl(h(\zeta,z) - u_-(z)\bigr) &= -s\,\frac{\zeta-\bar z}{(z-\bar z)(\zeta-z)}\,\bigl(h(\zeta,z)-u_+(z)\bigr)\,,\\ \partial_{\bar z}\bigl(h(\zeta,z) - u_+(z)\bigr) &= s\,\frac{\zeta-z}{(z-\bar z)(\zeta-\bar z)}\,\bigl(h(\zeta,z)-u_-(z)\bigr) \end{aligned} \tag{3.35}$$

for some function u_+ and u_- of z alone. This defines a sheaf $\mathcal{D}_s^+$ which projects to $\mathcal{D}_s$ by $(h,u_+,u_-) \mapsto (h, u_+ - u_-)$, and we have a map from C_s^+ to the space of global sections of $\mathcal{D}_s^+$ defined by $h \mapsto (h, \mathsf{R}_s^+ h, \mathsf{R}_s^- h)$ with $\mathsf{R}_s^\pm$ defined as in (3.19). In some ways C_s^+ is a more natural space than C_s, but we have chosen to normalize once and for all by $u_-(z) = 0$ in order to have something canonical.

The third remark concerns the surjectivity of $C : \mathcal{D}_s \to p_2^{-1}\mathcal{E}_s$. We know from Theorem 3.13 that any solution u of the Laplace equation can be completed locally to a solution (h,u) of the differential equations (3.34). We now show that such a lift does not necessarily exist for a u defined on a non-simply connected subset of $\mathfrak{H}$. Specifically, we will show that there is no section of $\mathcal{D}_s$ of the form $(h, q_{1-s}(z,i))$ on any open set $U \subset \mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$ whose image under p_2 contains a hyperbolic annulus with center i.

Now the disk model is more appropriate. We work with coordinates $\xi = i\frac{1+\zeta}{1-\zeta} \in \mathbb{C}^*$ and $w = i\frac{z-i}{z+i} \in \mathbb{D}$. The differential equations (3.34a) and (3.34b) take the form

$$(1-r^2)\,\partial_w h + s\,\frac{1-\bar w\xi}{\xi - w}\,(h-u) \;=\; 0\,, \tag{3.36a}$$

$$(1-r^2)\,\partial_{\bar w}(h-u) + s\,\frac{\xi-w}{1-\bar w\xi}\,h \;=\; 0\,, \tag{3.36b}$$

with $r = |w|$, and (3.35) becomes

$$\bigl[R^{\mathbb{S}}(\xi;\cdot\,)^s, u(\xi,\cdot\,)\bigr] \;=\; d\bigl(R^{\mathbb{S}}(\xi;\cdot\,)\,h(\xi,\cdot\,)\bigr)\,, \tag{3.36c}$$

with the Poisson kernel $R^{\mathbb{S}}$ in the circle model, as in (2.27c).

Proposition 3.14. *Let $A \subset \mathbb{D}$ be an annulus of the form $r_1 < |w| < r_2$ with $0 \le r_1 < r_2 \le 1$, and let $V \subset \mathbb{C}^*$ be a connected open set that intersects the region $r_1 < |\xi| < r_1^{-1}$ in $\mathbb{C}^*$. Then $\mathcal{D}_s(V \times A)$ does not contain sections of the form $(h, Q_{1-s,n})$ for any $n \in \mathbb{Z}$.*

Proof. Suppose that such a section $(h, Q_{1-s,n})$ exists. Take $\rho \in (r_1, r_2)$ such that V intersects the annulus $A_\rho = \{\rho < |\xi| < \rho^{-1}\}$. Let C be the contour $|w| = \rho$. Then the function f given by

$$f(\xi) \;=\; \int_C \bigl[R^{\mathbb{S}}(\xi;\cdot\,)^s, Q_{1-s,n}\bigr]$$

is defined and holomorphic on A_ρ. For $\xi \in V \cap A_\rho$ we know from (3.36c) that the closed differential form $[R^{\mathbb{S}}(\xi;\cdot\,)^s, Q_{1-s,n}]$ on A has a potential. Hence $f(\xi) = 0$ for $\xi \in V \cap A_\rho$, and then $f = 0$ in A_ρ. In particular, $f(\xi) = 0$ for $\xi \in \mathbb{S}^1$. In view of (2.19) this implies that the expansion $R^{\mathbb{S}}(\xi;\cdot\,)^s = \sum_{m\in\mathbb{Z}} a_m(\xi)\,P_{s,m}$ (with $\xi \in \mathbb{S}^1$)

satisfies $a_{-n}(\xi) = 0$. The function $R^{\mathbb{S}}(\xi;\,\cdot)^s$ is the Poisson transform $\mathsf{P}_{1-s}\,\delta_{1-s,\xi}$ of the distribution $\delta_{1-s,\xi} : \varphi^{\mathbb{S}} \mapsto \varphi^{\mathbb{S}}(\xi)$ on $\mathcal{V}^{\omega}_{1-s}$. This delta distribution has the expansion $\delta_{1-s,\xi} = \sum_{m\in\mathbb{Z}} \xi^{-m}\mathbf{e}_{1-s,m}$. Hence $R^{\mathbb{S}}(\xi;\,\cdot)^s = \sum_{m\in\mathbb{Z}} \xi^{-m}\,\frac{(-1)^m\Gamma(1-s)}{\Gamma(1-s+m)} P_{1-s,m}$, in which all coefficients are non-zero. Since $P_{1-s,m} = P_{s,m}$, this contradicts the earlier conclusion. ■

This non-existence result is a monodromy effect. In a small neighborhood of a point $(\xi_0, w_0) \in \mathbb{S}^1 \times A$ we can construct a section $(h, Q_{1-s,n})$ of $\mathcal{D}_s$ as in (3.41):

$$h(\xi,w) \;=\; \int_{w_0}^{w} \big[(R^{\mathbb{S}}(\xi;w')/R^{\mathbb{S}}(\xi;w))^s, Q_{1-s,n}(w')\big]_{w'}\,. \tag{3.37}$$

If we now let the second variable go around the annulus A, then $h(\xi,w)$ is changed to $h(\xi,w) + h_0(\xi,w)$, where h_0 is defined by the same integral as h but with the path of integration being the circle $|w'| = |w_0|$. Using

$$\big[(R^{\mathbb{S}}(\xi;w')/R^{\mathbb{S}}(\xi;w))^s, Q_{1-s,n}(w')\big]_{w'} \;=\; R^{\mathbb{S}}(\xi;w)^{-s}\,\big[R^{\mathbb{S}}(\xi;w')^s, Q_{1-s,n}(w')\big]_{w'}$$

and the absolutely convergent expansion $R^{\mathbb{S}}(\xi;w')^s = \sum_{m\in\mathbb{Z}} \xi^{-m}\,\frac{(-1)^m\,\Gamma(1-s)}{\Gamma(1-s+m)} P_{s,m}(w')$ from the proof above, we find from the explicit potentials in Table 3 in §A.4 that only the term $m = -n$ contributes and that h_0 is given by

$$h_0(\xi,w) \;=\; \pi i\,\frac{(-1)^n\,\Gamma(1-s)}{\Gamma(1-s-n)}\,R^{\mathbb{S}}(\xi;w)^{-s}\,. \tag{3.38}$$

(Here we have also used (2.13) to replace [,] by { , }.)

4. Eigenfunctions near $\partial\mathbb{H}$ and the transverse Poisson transform

The space $\mathcal{E}_s$ of λ_s-eigenfunctions of the Laplace operator embeds canonically into the larger space $\mathcal{F}_s$ of germs of eigenfunctions near the boundary of $\mathbb{H}$. In §4.1 we introduce the subspace $\mathcal{W}^{\omega}_s$ of $\mathcal{F}_s$ consisting of eigenfunction germs that have the behavior $y^s \times$(analytic across $\mathbb{R}$) near $\mathbb{R}$, together with the corresponding property near $\infty \in \mathbb{P}^1_{\mathbb{R}}$, and show that $\mathcal{F}_s$ splits canonically as the direct sum of $\mathcal{E}_s$ and $\mathcal{W}^{\omega}_s$. In §4.2 the space $\mathcal{W}^{\omega}_s$ is shown to be isomorphic to $\mathcal{V}^{\omega}_s$ by integral transformations, one of which is called the *transverse Poisson transformation* because it is given by the same integral as the usual Poisson transformation $\mathcal{V}^{\omega}_s \to \mathcal{E}_s$, but with the integral taken across rather than along $\mathbb{P}^1_{\mathbb{R}}$. This transformation gives another model $\mathcal{W}^{\omega}_s$ of the principal series representation $\mathcal{V}^{\omega}_s$, which has proved to be extremely useful in the cohomological study of Maass forms in [2]. In §4.3 we describe the duality of $\mathcal{V}^{\omega}_s$ and $\mathcal{V}^{-\omega}_{1-s}$ in (1.19) in terms of a pairing of the isomorphic spaces $\mathcal{W}^{\omega}_s$ and $\mathcal{E}_{1-s}$. In §4.4 we construct a smooth version $\mathcal{W}^{\infty}_s$ of $\mathcal{W}^{\omega}_s$ isomorphic to $\mathcal{V}^{\infty}_s$ by using jets of λ_s-eigenfunctions of the Laplace operator. This space is also used in [2].

4.1. **Spaces of eigenfunction germs.** Let $\mathcal{F}_s$ be the space of germs of eigenfunctions of Δ, with eigenvalue $\lambda_s = s(1-s)$, near the boundary of $\mathbb{H}$, i.e.,

$$\mathcal{F}_s \;=\; \varinjlim_{U} \mathcal{E}_s(U \cap \mathfrak{H})\,, \tag{4.1}$$

where the direct limit is taken over open neighborhoods U of $\mathbb{H}$ in $\mathbb{P}^1_{\mathbb{C}}$ (for either of the realizations $\mathfrak{H} \subset \mathbb{P}^1_{\mathbb{C}}$ or $\mathbb{D} \subset \mathbb{P}^1_{\mathbb{C}}$). This space canonically contains $\mathcal{E}_s$ because an eigenfunction in $\mathfrak{H}$ is determined by its values near the boundary (principle of analytic continuation). The action of G in $\mathcal{F}_s$ is by $f\,|\,g(z) = f(gz)$. The functions $Q_{s,n}$ and $Q_{1-s,n}$ in (2.7) represent elements of $\mathcal{F}_s$ not lying in $\mathcal{E}_s$. Clearly we have $\mathcal{F}_{1-s} = \mathcal{F}_s$.

Consider $u, v \in \mathcal{F}_s$, represented by elements of $\mathcal{E}_s(U \cap \mathfrak{H})$ for some neighborhood U of $\partial\mathbb{H}$ in $\mathbb{P}^1_{\mathbb{C}}$. Then the Greens form $[u, v]$ is defined in U, and for a positively oriented closed path C in U which is homotopic to $\partial\mathbb{H}$ in $U \cap (\mathbb{H} \cup \partial\mathbb{H})$, the integral

$$\beta(u, v) \;=\; \frac{1}{\pi i}\int_C [u, v] \;=\; \frac{2}{\pi}\int_C \{u, v\} \tag{4.2}$$

is independent on the choice of C or of the set U on which the representatives of u and v are defined. This defines a G-equivariant antisymmetric bilinear pairing

$$\beta : \mathcal{F}_s \times \mathcal{F}_s \longrightarrow \mathbb{C}\,. \tag{4.3}$$

If both u and v are elements of $\mathcal{E}_s$ we can contract C to a point, thus arriving at $\beta(u, v) = 0$. Hence β also induces a bilinear pairing $\mathcal{E}_s \times (\mathcal{F}_s/\mathcal{E}_s) \to \mathbb{C}$.

For each $z \in \mathbb{H}$, the element $q_s(\,\cdot\,, z)$ of $\mathcal{F}_s$ is not in $\mathcal{E}_s$. By $\Pi_s u(z) = \beta(u, q_s(\,\cdot\,, z))$ we define a G-equivariant linear map $\Pi_s : \mathcal{F}_s \to \mathcal{E}_s$. Explicitly, $u_{\text{in}}(z) := \Pi_s u(z)$ is given by an integral $\frac{1}{\pi i}\int_C [u(z'), q_s(z', z)]_{z'}$, where z is inside the path of integration C. By deforming C we thus obtain $u_{\text{in}}(z)$ for all $z \in \mathfrak{H}$, so $u_{\text{in}} \in \mathcal{E}_s$. We can also define $u_{\text{out}}(z) := \frac{1}{\pi i}\int_C [u(z'), q_s(z', z)]_{z'}$ where now z is between the boundary of $\mathbb{H}$ and the path of integration. For $u \in \mathcal{E}_s$ we see that $u_{\text{out}} = 0$. Theorem 2.1 shows that

$$u_{\text{in}} - u_{\text{out}} \;=\; u\,. \tag{4.4}$$

We now define the subspace $\mathcal{W}_s^\omega$ of $\mathcal{F}_s$. It is somewhat easier in the disk model:

Definition 4.1. The space $\mathcal{W}_s^\omega$ consists of those boundary germs $u \in \mathcal{F}_s$ that are of the form

$$u(w) \;=\; 2^{-2s}(1 - |w|^2)^s A^{\mathbb{S}}(w)$$

where $A^{\mathbb{S}}$ is a real-analytic function on a two-sided neighborhood of $\mathbb{S}^1$ in $\mathbb{P}^1_{\mathbb{C}}$.

In other words, representatives of elements of $\mathcal{W}_s^\omega$, divided by the factor $(1 - |w|)^s$, extend analytically across the boundary $\mathbb{S}^1$. (The factor 2^{-2s} is included for compatibility with other models.)

The next proposition shows that $\mathcal{W}_s^\omega$ is a canonical direct complement of $\mathcal{E}_s$ in $\mathcal{F}_s$.

Proposition 4.2. *The G-equivariant maps $\Pi_s : u \mapsto u_{\text{in}}$ and $1 - \Pi_s : u \mapsto u_{\text{out}}$ split the exact sequence of G-modules*

$$0 \longrightarrow \mathcal{E}_s \underset{\xleftarrow{\;\Pi_s\;}}{\longrightarrow} \mathcal{F}_s \underset{\xleftarrow{\;1-\Pi_s\;}}{\longrightarrow} \mathcal{F}_s/\mathcal{E}_s \longrightarrow 0 \tag{4.5}$$

The image $(1 - \Pi_s)(\mathcal{F}_s/\mathcal{E}_s)$ is equal to the space $\mathcal{W}_s^\omega$, and we have the direct sum decomposition

$$\mathcal{F}_s \;=\; \mathcal{E}_s \oplus \mathcal{W}_s^\omega\,, \tag{4.6}$$

given by $u \leftrightarrow (u_{\text{in}}, u_{\text{out}})$.

Notice that all the spaces in the exact sequence (4.5) are the same for s and $1 - s$, but that Π_s and Π_{1-s} give different splittings and that $\mathcal{W}_{1-s}^\omega \neq \mathcal{W}_s^\omega$ (for $s \neq \frac{1}{2}$).

Proof. The G-equivariance of Π_s, and hence of $1-\Pi_s$, follows from the G-equivariance of $[\cdot,\cdot]$. That these maps split the sequence is clear, since we already saw that Π_s is the identity on $\mathcal{E}_s$. It remains to show that $\mathcal{W}_s^\omega$ is equal to the image of $u \mapsto u_{\mathrm{out}}$.

The asymptotic behavior of Q_{s-1} in (A.13) gives for w' on the path of integration C and w outside C in the definition of $u_{\mathrm{out}}(w)$

$$q_s(w,w') \;=\; \Big(\frac{2}{\rho^{\mathbb{D}}(w',w)+1}\Big)^s f_s\Big(\frac{2}{\rho^{\mathbb{D}}(w',w)+1}\Big)\,,$$

where f_s is analytic at 0. With (2.2):

$$\frac{2}{\rho(w',w)+1} \;=\; \frac{(1-|w'|)^2}{|w-w'|^2+(1-|w'|^2)(1-|w|^2)}\,(1-|w|^2)\,.$$

We conclude that if w' stays in the compact set C, and w tends to $\mathbb{S}^1$, we have

$$u_{\mathrm{out}}(w) \;=\; (1-|w|^2)^s\ (\text{analytic function of } 1-|w|^2)\,.$$

So $u_{\mathrm{out}} \in \mathcal{W}_s^\omega$.

For the converse inclusion it suffices to show that $\mathcal{E}_s \cap \mathcal{W}_s^\omega \;=\; \{0\}$. This follows from the next lemma, which is slightly stronger than needed here. ■

Lemma 4.3. *Suppose that $u \in \mathcal{E}_s$ satisfies on some annulus $1-\delta \le |w|^2 < 1$ with $\delta > 0$*

$$u(w) \;=\; (1-|w|^2)^s A(w) + \mathrm{O}\big((1-|w|^2)^{s+1}\big)\,, \tag{4.7}$$

with a continuous function A on the region $0 \le 1-|w|^2 \le \delta$. Then $u=0$.

Proof. On the annulus the function u is given by its polar Fourier series, with terms

$$u_n(w) \;=\; \int_0^{2\pi} e^{-2in\theta} f(e^{i\theta}w)\,\frac{d\theta}{2\pi}\,.$$

Each u_n satisfies the estimate (4.7), with A replaced by its Fourier term A_n. Moreover, the G-equivariance of Δ implies that u_n is a λ_s-eigenfunction of Δ. It is the term of order n in the expansion (2.19). In particular, u_n is a multiple of $P_{s,n}$. In §A.1.2 we see that $P_{s,n}$ has a term $(1-|w|^2)^{1-s}$ in its asymptotic behavior near the boundary, or a term $(1-|w|^2)^{1/2}\log(1-|w|^2)$ if $s=\frac12$. So u_n can satisfy (4.7) only if it is zero. ■

Remark. The proof of the lemma giver the stronger assertion: If $u \in \mathcal{F}_s$ satisfies (4.7), then $u \in \mathcal{W}_s^\omega$ and $\Pi_s u = 0$.

Returning to the definition of $\mathcal{W}_s^\omega$, we note that the action $g : w \mapsto \frac{Aw+B}{\bar Bw+\bar A}$ in $\mathbb{D}$ gives for the function $A^{\mathbb{S}}$

$$A^{\mathbb{S}}|g(w) \;=\; \big|\bar Bw+\bar A\big|^{-2s} A^{\mathbb{S}}\Big(\frac{Aw+B}{\bar Bw+\bar A}\Big)\,, \tag{4.8}$$

first for $w \in \mathbb{D}$ near the boundary, and by real-analytic continuation on a neighborhood of $\mathbb{S}^1$ in $\mathbb{P}^1_{\mathbb{C}}$. On the boundary, where $|w| = 1$, this coincides with the action of G in the circle model, as given in (1.8). In (1.20) the action in the circle model of $\mathcal{V}_s^\omega$ is extended to holomorphic functions on sets in $\mathbb{P}^1_{\mathbb{C}}$. That action and the action in (4.8) coincide only on $\mathbb{S}^1$, but are different elsewhere. This reflects that $A^{\mathbb{S}}$ is real-analytic, but not holomorphic.

The restriction of $A^{\mathbb{S}}$ to $\mathbb{S}^1$ induces the *restriction map*

$$\rho_s : \mathcal{W}_s^\omega \longrightarrow \mathcal{V}_s^\omega\,, \tag{4.9}$$

which is G-equivariant.

Examples of elements of $\mathcal{W}_s^\omega$ are the functions $Q_{s,n}$, represented by elements of $\mathcal{E}_s(\mathbb{D} \smallsetminus \{0\})$, whereas the functions $Q_{1-s,n}$ belong to $\mathcal{F}_s$ but not to $\mathcal{W}_s^\omega$.

We note that the factor $2^{-2s}(1-|w|^2)^s$ corresponds to $(\frac{y}{|z+i|^2})^s$ on the upper half plane. So in the upper half plane model the elements of $\mathcal{W}_s^\omega$ are represented by functions of the form $u(z) = (\frac{y}{|z+i|^2})^s A^{\mathbb{P}}(z)$ with $A^{\mathbb{P}}$ real-analytic on a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. The transformation behavior for $A^{\mathbb{P}}$ turns out to coincide on $\mathbb{P}^1_{\mathbb{R}}$ with the action of G in the projective model of $\mathcal{V}_s^\omega$ in (1.6). Outside $\mathbb{P}^1_{\mathbb{R}}$ it differs from the action in (1.23) on holomorphic functions. The restriction map $\rho_s : \mathcal{W}_s^\omega \to \mathcal{V}_s^\omega$ is obtained by $u \mapsto A^{\mathbb{P}}|_{\mathbb{P}^1_{\mathbb{R}}}$.

In the line model we have $u(z) = y^s A(z)$ near $\mathbb{R}$ and $u(z) = (y/|z|^2)^s A^\infty(-1/z)$ near ∞, with A and A^∞ real-analytic on a neighborhood of $\mathbb{R}$ in $\mathbb{C}$. The action on A is given by

$$A\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right.(z) \;=\; |cz+d|^{-2s} A\Big(\frac{az+b}{cz+d}\Big), \tag{4.10}$$

coinciding on $\mathbb{R}$ with the action in the line model. Restriction of A to $\mathbb{R}$ induces the description of ρ_s in the line model. The factors $2^{-2s}(1-|w|^2)^s$, $(y/|z+i|^2)^s$, y^s and $(y/|z|)^s$ have been chosen in such a way that $A^{\mathbb{S}}$, $A^{\mathbb{P}}$, A and A^∞ restrict to elements of the circle, projective and line models, respectively, of $\mathcal{V}_s^\omega$, related by (1.8) and (1.5).

The space $\mathcal{W}_s^\omega$ is the space of global sections of a sheaf, also denoted $\mathcal{W}_s^\omega$, on $\partial\mathbb{H}$, where in the disk model $\mathcal{W}_s^\omega(I)$ for an open set I in $\mathbb{S}^1$ corresponds to the real-analytic functions $A^{\mathbb{S}}$ on a neighborhood of I in $\mathbb{P}^1_{\mathbb{C}}$ such that $(1-|w|^2)^s A^{\mathbb{S}}(w)$ is annihilated by $\Delta - \lambda_s$. Restriction gives $\rho_s : \mathcal{W}_s^\omega(I) \to \mathcal{V}_s^\omega(I)$ for each $I \subset \mathbb{S}^1$. In the line model, $\mathcal{W}_s^\omega(I)$ for $I \subset \mathbb{R}$ can be identified with the space of real-analytic functions A on a neighborhood U of I with $y^s A(z) \in \operatorname{Ker}(\Delta - \lambda_s)$ on $U \cap \mathfrak{H}$; for $I \smallsetminus \mathbb{P}^1_{\mathbb{R}} \smallsetminus \{0\}$ we use $(y/|z|^2)^s A^\infty(-1/z)$. The function $z \mapsto y^s$ is an element of $\mathcal{W}_s^\omega(\mathbb{R})$, but not of $\mathcal{W}_s^\omega = \mathcal{W}_s^\omega(\mathbb{P}^1_{\mathbb{R}})$.

Another example is the function $z \mapsto y^{1-s}$, which represents an element of $\mathcal{W}_{1-s}^\omega(\mathbb{R})$, but not of $\mathcal{W}_{1-s}^\omega = \mathcal{W}_{1-s}^\omega(\mathbb{P}^1_{\mathbb{R}})$. It is the Poisson transform of the distribution $\delta_{s,\infty}$, which has support $\{\infty\}$.

The *support* $\operatorname{Supp}(\alpha)$ of a hyperfunction $\alpha \in \mathcal{V}_s^{-\omega}$ is the smallest closed subset X of $\partial\mathbb{H}$ such that each $g \in \mathbf{H}_s$ representing α extends holomorphically to a neighborhood of $\partial\mathbb{H} \smallsetminus X$.

Proposition 4.4. *The Poisson transform of a hyperfunction $\alpha \in \mathcal{V}_s^{-\omega}$ represents an element of $\mathcal{W}_{1-s}^\omega(\partial\mathbb{H} \smallsetminus \operatorname{Supp}(\alpha))$.*

This statement is meaningful only if $\operatorname{Supp}(\alpha)$ is not the whole of $\partial\mathbb{H}$. In Theorem 5.4 we will continue the discussion of the relation between support of a hyperfunction and the boundary behavior of its Poisson transform.

Proof. Let $g \in \mathbf{H}_s$ be a representative of $\alpha \in \mathcal{V}_s^{-\omega}$. In the Poisson integral in (2.28) we can replace the integral over C_+ and C_- by the integral

$$\mathsf{P}_s\alpha(w) \;=\; \frac{(1-|w|^2)^{1-s}}{2\pi i}\int_C g(w)\big((1-w/\xi)(1-\bar{w}\xi)\big)^{s-1}\,\frac{d\xi}{\xi}\,, \tag{4.11}$$

where C is a path inside the domain of g encircling $\operatorname{Supp}(\alpha)$. For w outside C the integral defines a real analytic function on a neighborhood of $\partial\mathbb{D}$, so there the boundary behavior is $(1-|w|^2)^{1-s}\times$ (analytic). Adapting C, we can arrange that any point of $\partial\mathbb{D}\smallsetminus\mathrm{Supp}\,(\alpha)$ is inside this neighborhood. ■

• *Decomposition of eigenfunctions.* We close this subsection by generalizing the decomposition (4.4) from $\mathcal{F}_s$ to $\mathcal{E}_s(R)$, where R is any annulus $0\le r_1<|w|<r_2\le 1$ in $\mathbb{D}$. For $u\in\mathcal{E}_s(R)$ we define

$$u_{\mathrm{in}}\in\mathcal{E}_s(\{|w|<r_2\})\quad\text{ and } u_{\mathrm{out}}\in\mathcal{E}_s(\{|w|>r_1\})\,,$$

each by an integral $\frac{1}{\pi i}\int_C[u,q_s(\,\cdot\,,z)]$, where $C\subset R$ is a circle containing the argument of u_{in} in its interior, respectively the argument of u_{out} in its exterior. Then (4.4) holds in the annulus R. Explicitly, any $u\in\mathcal{E}_s(R)$ has an expansion of the form

$$u\;=\;\sum_{n\in\mathbb{Z}}(a_nQ_{s,n}+b_nP_{s,n})\qquad\text{on } r_1<|w|<r_2\,, \tag{4.12}$$

and u_{in} and u_{out} are then given by

$$u_{\mathrm{in}}\;=\;\sum_{n\in\mathbb{Z}}b_nP_{s,n}\,,\qquad u_{\mathrm{out}}\;=\;-\sum_{n\in\mathbb{Z}}a_nQ_{n,s}\,. \tag{4.13}$$

4.2. **The transverse Poisson map.** In the last subsection we defined restriction maps $\rho_s:\mathcal{W}_s^\omega\to\mathcal{V}_s^\omega$, and more generally $\mathcal{W}_s^\omega(I)\to\mathcal{V}_s^\omega(I)$. We now show that these restriction maps are isomorphisms and construct the explicit inverse maps. We in fact give two descriptions of ρ_s^{-1}, one in terms of power series and one defined by an integral transform (transverse Poisson map); the former is simpler, and also applies in the C^∞ setting (treated in §4.4), while the latter (which is motivated by the power series formula) gives a much stronger statement in the context of analytic functions. The restriction map itself will also be given by an integral transformation. change Reformulated. DON do you agree?

• *Power series version.* Let $u\in\mathcal{W}_s^\omega(I)$, where we work in the line model and can assume that $I\subset\mathbb{R}$ by locality. Write z as $x+iy$ and for $x\in I$ expand the real-analytic function A such that $u(z)=y^sA(z)$ as a power series $\sum_{n=0}^\infty a_n(x)y^n$ in y, convergent in some neighborhood of I in $\mathbb{C}$. By definition, the constant term $a_0(x)$ in this expansion is the image $\varphi=\rho_s(u)$ of u under the restriction map. The differential equation $\Delta u=\lambda_s u$ of u translates into the differential equation

$$y\,(A_{xx}+A_{yy})+2s\,A_y\;=\;0\,. \tag{4.14}$$

Applying this to the power-series expansion of A we find that

$$a_{n-2}''(x)\;+\;n(n+2s-1)\,a_n(x)\;=\;0$$

for $n\ge 2$, and that $a_1\equiv 0$. Together with the initial condition $a_0=\varphi$ this gives

$$a_n(x)\;=\;\begin{cases}\dfrac{(-1/4)^k\,\Gamma(s+\frac12)}{k!\,\Gamma(k+s+\frac12)}\,\varphi^{(2k)}(x) & \text{if } n=2k\,,\\ \qquad\qquad 0 & \text{if } 2\nmid n\,,\end{cases} \tag{4.15}$$

and hence a complete description of A in terms of φ. Conversely, if φ is any analytic function in a neighborhood of $x\in\mathbb{R}$, then its Taylor expansion at x has a positive radius of convergence r_x and we have $\varphi^{(n)}(x)=\mathrm{O}(n!\,c^n)$ for any $c>r_x^{-1}$. From Stirling's formula or the Legendre duplication formula we see that $4^{-k}/k!\,\Gamma(k+s+\frac12)=$

$O(k^{-\mathrm{Re}(s)}/(2k)!)$, so the power series $\sum_{n\ge 0} a_n(x)y^n$ with $a_n(x)$ defined by (4.15) converges for $|y| < r_x$. By a straightforward uniform convergence argument, the function $A(x+iy)$ defined by this power series is real-analytic in a neighborhood of I, and of course it satisfies the differential equation $A_{xx} + A_{yy} + 2sy^{-1}A_y = 0$, so the function $u(z) = y^s A(z)$ is an eigenfunction of Δ with eigenvalue λ_s. This proves:

Theorem 4.5. *Let I be an open subset of $\mathbb{R}$. Define a map from analytic functions on I to the germs of functions on a neighborhood of I in $\mathbb{C}$ by*

$$(\mathsf{P}_s^\dagger\varphi)(x+iy) \;=\; y^s \sum_{k=0}^{\infty} \frac{\varphi^{(2k)}(x)}{k!\,(s+\frac12)_k}\,(-y^2/4)^k\,, \tag{4.16}$$

with the Pochhammer symbol $(\frac12+s)_k = \prod_{j=0}^{k-1}(\frac12+s+j)$. Then $\mathsf{P}_s^\dagger$ is an isomorphism from $\mathcal{V}_s^\omega(I)$ to $\mathcal{W}_s^\omega(I)$ with inverse ρ_s.

Of course, we can now use the G-equivariance to deduce that the local restriction map $\rho_s : \mathcal{W}_s^\omega(I) \to \mathcal{V}_s^\omega(I)$ is an isomorphism for every open subset $I \subset \mathbb{P}^1_{\mathbb{R}}$ and that the global restriction map $\rho_s : \mathcal{W}_s^\omega \to \mathcal{V}_s^\omega$ is an equivariant isomorphism. The inverse maps, which we still denote $\mathsf{P}_s^\dagger$, can be given explicitly in a neighborhood of infinity using the functions φ_∞ and A^∞ as usual for the line model or by the corresponding formulas in the circle model. The details are left to the reader.

Example 1. Take $\varphi(x) = 1$. Then (4.16) gives $\mathsf{P}_s^\dagger\varphi(z) = y^s$ in $\mathcal{W}_s^\omega(\mathbb{R})$. More generally, if $\varphi(x) = e^{i\alpha x}$ with $\alpha\in\mathbb{R}$, then $\mathsf{P}_s^\dagger\varphi$ is the function $i_{s,\alpha}$ defined in (A.3b).

Example 2. We can generalize Example 1 from $\varphi = 1$ to arbitrary polynomials:

$$\varphi(x) \;=\; \binom{-2s}{m}\,x^m \quad\Rightarrow\quad \mathsf{P}_s^\dagger\varphi(z) \;=\; y^s \sum_{k+\ell=m} \binom{-s}{k}\binom{-s}{\ell}\, z^k\,\bar z^\ell\,. \tag{4.17}$$

This can be checked either from formula (4.16) or, using the final statement of Theorem 4.5, by verifying that the expression on the right belongs to $\mathcal{E}_s$ and that its quotient by y^s is analytic near $\mathbb{R}$ and restricts to $\binom{-2s}{m}x^m$ when $y = 0$.

Example 3. Let $a \in \mathbb{C}\smallsetminus I$. Then (4.16) and the binomial theorem give

$$\varphi(x) = (x-a)^{-2s} \quad\Rightarrow\quad \mathsf{P}_s^\dagger\varphi(z) = y^s \sum_{k=0}^{\infty}\binom{-s}{k}\frac{y^{2k}}{(x-a)^{2s+2k}} \;=\; R(a;z)^s\,, \tag{4.18}$$

(Here the branches in $(x-a)^{-2s}$ and $R(a;z)^s$ have to be taken consistently.) Again, we could skip this calculation and simply observe that $R(a;\cdot)^s \in \mathcal{W}_s^\omega(I)$ and that $\varphi(x)$ is the restriction $y^{-s}R(a;x+iy)^s\big|_{y=0}$. If $|a| > |x|$, then expanding the two sides of (4.18) by the binomial theorem gives another proof of (4.17) and makes clear where the binomial coefficients in that formula come from.

Example 4. Our fourth example is

$$\varphi(x) = R(x;z_0)^s \quad\Rightarrow\quad \mathsf{P}_s^\dagger\varphi(z) \;=\; b(s)^{-1}\,q_s(z,z_0) \qquad (z_0\in\mathfrak{H})\,, \tag{4.19}$$

where the constant $b(s)$ is given in terms of beta or gamma functions by

$$b(s) \;=\; \mathsf{B}(s,\tfrac12) \;=\; \frac{\Gamma(s)\,\Gamma(\frac12)}{\Gamma(s+\frac12)}\,. \tag{4.20}$$

Formula (4.19) is proved by remarking that the function on the right belongs to $\mathcal{W}_s^\omega$ and that its image under ρ_s is the function on the left (as one sees easily from the asymptotic behavior of the Legendre function $Q_{s-1}(t)$ as $t \to \infty$). Obtaining it from the power series in (4.16) would probably be difficult, but we will see at the end of the section how to get it from the integral formula for $\mathsf{P}_s^\dagger$ given below.

Remark. Equation (4.15) shows that the function $y^{-s} \cdot \mathsf{P}_s^\dagger\varphi(x+iy)$ is even in y (as is visible in Examples 1 and 2 above). In the projective model, $A^{\mathbb{P}}(z) = \bigl(\frac{y}{|z+i|^2}\bigr)^{-s} u(z) = |z+i|^{2s}A(z)$ is not even in y. In the circle model, related to the projective model by $w = \frac{z-i}{z+i}$, the reflection $z \mapsto \bar z$ corresponds to $w \mapsto 1/\bar w$ (or $r \mapsto r^{-1}$ in polar coordinates $w = re^{i\theta}$), and the function $A^{\mathbb{S}}(w) = 2^{2s}(1-|w|^2)^{-s}u(w)$ satisfies $A^{\mathbb{S}}(1/\bar w) = |w|^{2s}A^{\mathbb{S}}(w)$. For example, equations (A.8) and (A.9) say that the function in $\mathcal{W}_s^\omega(\mathbb{D} \smallsetminus \{0\})$ whose image under ρ_s is w^n ($n \in \mathbb{Z}$) corresponds to $A^{\mathbb{S}}(w) = \bar w^{-n}F(s-n, s; 2s; 1-|w|^2)$, and this equals $w^n|w|^{-2s}F(s-n, s; 2s; 1-|w|^{-2})$ by a Kummer relation. Note that if we had used the factor $\bigl(\frac{1-|w|}{1+|w|}\bigr)^s$ instead of $2^{-2s}(1-|w|^2)^s$ in Definition 4.1, we would have obtained functions $A^{\mathbb{S}}$ that are invariant under $w \mapsto 1/\bar w$.

• *Integral version.* If φ is a real-analytic function on an interval $I \subset \mathbb{R}$, then we can associate to it two extensions, both real-analytic on a sufficiently small complex neighborhood of I: the holomorphic extension, which we will denote by the same letter, and the solution A of the differential equation (4.14) given in Theorem 4.5. The following integral formula describes the isomorphism $\mathsf{P}_s^\dagger : \mathcal{V}_s^\omega \to \mathcal{W}_s^\omega$. change

Theorem 4.6. *Let $U = \bar U \subset \mathbb{C}$ be a connected and simply connected subset of $\mathbb{C}$, with $I := U \cap \mathbb{R}$ non-empty. Then for any holomorphic function φ on U, determining an element of $\mathcal{V}_s^\omega(I)$ in the line model, the eigenfunction $\mathsf{P}_s^\dagger(\varphi) := \mathsf{P}_s^\dagger(\varphi|_I)$ defined near I by* (4.16) *extends analytically to all of $U \cap \mathfrak{H}$ and is given there by the formula*

$$(\mathsf{P}_s^\dagger\varphi)(z) \;=\; \frac{1}{i\,b(s)} \int_{\bar z}^{z} R(\zeta; z)^{1-s}\, \varphi(\zeta)\, d\zeta\,, \tag{4.21}$$

where $b(s)$ is given by (4.20) *and the integral is taken along any piecewise C^1-path in U from $\bar z$ to z intersecting I only once, with the branch of $R(\zeta; z)^{1-s}$ continuous on the path and equal to its standard value at the intersection point with I.*

Note the formal similarity between the formula (4.21) for $\mathsf{P}_s^\dagger\varphi$ and the formula (2.26) for the Poisson map: the integrand is exactly the same, but in the case of P_s the integration is over $\mathbb{P}^1_{\mathbb{R}}$ (or $\mathbb{S}^1$), while in the formula for $\mathsf{P}_s^\dagger$ it is over a path which crosses $\mathbb{P}^1_{\mathbb{R}}$. We therefore call $\mathsf{P}_s^\dagger$ the *transverse Poisson map*.

change

The inverse map $\mathcal{W}_s^\omega \to \mathcal{V}_s^\omega$ is also given by an integral formula:

Theorem 4.7. *Let U and I be as in Theorem 4.6. Then for any real-analytic function A in U for which the function $u = y^sA$ belongs to $\mathcal{E}_s(U \cap \mathfrak{H})$, the function $\varphi = A|_I$*

extends holomorphically to all of U and is given there by the formula

$$\varphi(\zeta) = \begin{cases} \dfrac{2\,b(s)\sin\pi s}{\pi} \displaystyle\int_{\xi_0}^{\zeta} [u(\,\cdot\,), (-R(\zeta;\,\cdot\,))^s] & \text{if } \zeta \in U \cap \mathfrak{H}, \\ A(\zeta) & \text{if } \zeta \in I, \\ \dfrac{2\,b(s)\sin\pi s}{\pi} \displaystyle\int_{\xi_0}^{\bar\zeta} [(-R(\zeta;\,\cdot\,))^s,\, u(\,\cdot\,)] & \text{if } \zeta \in U \cap \mathfrak{H}^-, \end{cases} \tag{4.22}$$

where the integrals are along piecewise C^1-paths in $U \cap \mathfrak{H}$ from any $\xi_0 \in I$ to ζ, respectively $\bar\zeta$, with the branch of $(-R(\zeta;z))^s$ fixed by $|\arg(-R(\zeta;z))| < \pi$ for z near ξ_0.

We have stated these two theorems only for neighborhoods of intervals in $\mathbb{R}$, but because everything is G-equivariant they can easily be transferred to any interval in $\mathbb{P}^1_{\mathbb{R}}$. (Details are left to the reader.) Alternatively, one can work in the projective or the circle model. This will be discussed after we have given the proofs.

Proof of Theorem 4.6. Define $\mathsf{P}_s^\dagger\varphi$ locally by (4.16). For $x \in I$ we denote by r_x the radius of the largest open disk with center x contained in U. Using the identity

$$\frac{(2k)!}{4^k\,k!\,\Gamma(k+s+\frac12)} = \frac{\Gamma(k+\frac12)}{\Gamma(\frac12)\,\Gamma(k+s+\frac12)} = \frac{1}{\Gamma(s)\,\Gamma(\frac12)} \int_0^1 (1-t)^{s-1}\,t^{k-\frac12}\,dt$$

(duplication formula and beta function), we find for $x \in I$ and $0 < y < r_x$ the formula

$$\begin{aligned} b(s)\,(\mathsf{P}_s^\dagger\varphi)(x+iy) &= y^s \int_0^1 (1-t)^{s-1}\,t^{-1/2}\Bigl(\sum_{k=0}^\infty \frac{\varphi^{(2k)}(x)}{(2k)!}(-ty^2)^k\Bigr)\,dt \\ &= \frac12\,y^s \int_0^1 (1-t)^{s-1}\,t^{-1/2}\,\bigl(\varphi(x+iy\sqrt{t}) + \varphi(x-iy\sqrt{t})\bigr)\,dt \\ &= y^s \int_{-1}^1 (1-t^2)^{s-1}\varphi(x+iyt)\,dt \qquad (t=t^2) \\ &= y^s \int_{-y}^{y} \Bigl(\frac{y^2}{y^2-\eta^2}\Bigr)^{1-s} \varphi(x+i\eta)\,y^{-1}\,d\eta \qquad (t=\eta/y) \\ &= -i \int_{x-iy}^{x+iy} \Bigl(\frac{y}{(\zeta-z)(\zeta-\bar z)}\Bigr)^{1-s} \varphi(\zeta)\,d\zeta\,, \end{aligned} \tag{4.23}$$

where the path of integration is the vertical line from $x - iy$ to $x + iy$. The integral converges at the end points. The value of the factor $\bigl(y/(\zeta-z)(\zeta-\bar z)\bigr)^{1-s}$ is based on the positive value y^{-1} of $y/(\zeta-z)(\zeta-\bar z)$ at $\zeta = x$. Continuous deformation of the path does not change the integral, as long as we anchor the branch of the factor $\bigl(y/(\zeta-z)(\zeta-\bar z)\bigr)^{1-s}$ at the intersection point with I. (This holds even though that factor is multivalued on $U \smallsetminus \{z, \bar z\}$. We could also allow multiple crossings of I, but then would have to prescribe the crossing point at which the choice of the branch of the Poisson kernel is anchored.) This proves (4.21) for points $z \in U \cap \mathfrak{H}$ sufficiently near to I, and the extension to all of $U \cap \mathfrak{H}$ is then automatic since the integral makes sense in the whole of that domain and is real-analytic in z. ∎

Proof of Theorem 4.7. As in the previous proof, we consider first the case that $\zeta = X + iY \in U \cap \mathfrak{H}$ and that the vertical segment from X to ζ is contained in U. Since we want to integrate up to $z = \zeta$, we will use the Green's form $\omega_\zeta(z) = [u(z), (-R(\zeta; z))^s]$ rather than $[(-R(\zeta; z))^s, u(z)]$ or $\{u(z), (-R(\zeta; z))^s\}$, which would have non-integrable singularities at this end-point. (The minus sign is included because $R(\zeta; z)$ is negative on the segment.) Explicitly, this Green's form is given for $z = x + iy \in U \cap \mathfrak{H}$ by

$$\begin{aligned}
\omega_\zeta(z) &= (-R(\zeta; z))^s \Bigl(\frac{\partial u}{\partial z}\, dz + \frac{is}{2y}\, \frac{z - \zeta}{\bar z - \zeta}\, u\, d\bar z\Bigr) \\
&= (-y\, R(\zeta; z)^s) \Bigl(\frac{\partial A}{\partial z}\, dz - \frac{is}{2y}\, A\, dz + \frac{is}{2y}\, \frac{z - \zeta}{\bar z - \zeta}\, A\, d\bar z\Bigr) \\
&= (-y\, R(\zeta; z))^s \Bigl[\Bigl(\frac{\partial A}{\partial z} - \frac{s}{\bar z - \zeta}\, A\Bigr) dx + \Bigl(i\frac{\partial A}{\partial z} + \frac{s}{y}\, \frac{x - \zeta}{\bar z - \zeta}\, A\Bigr) dy\Bigr]. \qquad (4.24)
\end{aligned}$$

If we restrict this to the vertical line $z = X + itY$ $(0 < t < 1)$ joining X and ζ, it becomes

$$\begin{aligned}
\omega_\zeta(X + itY) &= \Bigl(\frac{iY}{2}\, A_x(X + itY) + \frac{Y}{2}\, A_y(X + itY) + \frac{s}{t(1 + t)}\, A(X + itY)\Bigr) \frac{t^{2s}\, dt}{(1 - t^2)^s} \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(s + \frac12)}{k!\, \Gamma(k + s + \frac12)} \Bigl[\varphi^{(2k+1)}(X)\, (iY/2)^{2k+1} \\
&\qquad + \Bigl(\frac{k}{t} + \frac{s}{t(1 + t)}\Bigr) \varphi^{(2k)}(X)\, (iY/2)^{2k}\Bigr] t^{2k}\, \frac{t^{2s}}{(1 - t^2)^s}\, dt\,,
\end{aligned}$$

where φ is the holomorphic function near I with $\varphi = A$ on I.

Now we use the beta integrals

$$\begin{aligned}
\int_0^1 t^{2k}\, \frac{t^{2s}}{(1 - t^2)^s}\, dt &= \frac12 \int_0^1 t^{k+s-\frac12} (1 - t)^{-s} dt = \frac{\Gamma(k + s + \frac12)\, \Gamma(1 - s)}{2\, \Gamma(k + \frac32)} \\
&= \frac{\Gamma(s + \frac12)\, \Gamma(1 - s)}{2\, \Gamma(\frac12)} \cdot \frac{k!\, \Gamma(k + s + \frac12)}{\Gamma(s + \frac12)} \cdot \frac{2^{2k+1}}{(2k + 1)!}\,, \\
\int_0^1 \Bigl(\frac{k}{t} + \frac{s}{t(1 + t)}\Bigr) t^{2k}\, \frac{t^{2s}}{(1 - t^2)^s}\, dt &= \int_0^1 \Bigl(k\, \frac{t^{2k+2s-1}}{(1 - t^2)^s} + s\, \frac{t^{2k+2s-1} - t^{2k+2s}}{(1 - t^2)^{s+1}}\Bigr) dt \\
&= \frac{k}{2} \int_0^1 t^{k+s-1} (1 - t)^{-s} dt + \frac{s}{2} \int_0^1 (t^{k+s-1} - t^{k+s-\frac12})\, (1 - t)^{-s-1}\, dt \\
&= \frac{k}{2}\, \frac{\Gamma(k + s)\, \Gamma(1 - s)}{\Gamma(k + 1)} + \frac{s}{2} \Bigl(\frac{\Gamma(k + s)\, \Gamma(-s)}{\Gamma(k)} - \frac{\Gamma(k + s + \frac12)\, \Gamma(-s)}{\Gamma(k + \frac12)}\Bigr) \\
&= \frac{\Gamma(s + \frac12)\, \Gamma(1 - s)}{2\, \Gamma(\frac12)} \cdot \frac{k!\, \Gamma(k + s + \frac12)}{\Gamma(s + \frac12)} \cdot \frac{2^{2k}}{(2k)!}
\end{aligned}$$

(the second calculation is valid initially for $\mathrm{Re}\,(s) < 0$, $\mathrm{Re}\,(k + s) > 0$, but then by analytic continuation for $\mathrm{Re}\,(s) < 1$, $\mathrm{Re}\,(k + s) > 0$, where the left-hand side converges) to get

$$\int_X^\zeta \omega_\zeta = \frac{\Gamma(s + \frac12)\, \Gamma(1 - s)}{2\, \Gamma(\frac12)} \sum_{n=0}^{\infty} \varphi^{(n)}(X) \frac{(iY)^n}{n!} = \frac{\pi}{2\, b(s)\, \sin \pi s}\, \varphi(\zeta)\,. \qquad (4.25)$$

Furthermore, we see from (4.24) that the dx-component of the 1-form $\omega_\zeta(x+iy)$ extends continuously to $U\cap\overline{\mathfrak{H}}$ and vanishes on I, so $\int_{x_0}^X \omega_\zeta$ vanishes for any $\xi_0\in I$ and we can replace the right-hand side of (4.25) by $\int_{\xi_0}^\zeta \omega_\zeta$. On the other hand, the fact that the 1-form is closed means that we can integrate along any path from ξ_0 to ζ inside $U\cap\mathfrak{H}$, not just along the piecewise linear path just described, and hence also that we can move ζ anywhere within $U\cap\mathfrak{H}$, thus obtaining the analytic continuation of φ to this domain as stated in (4.22).

If $\zeta = X-iY$ $(Y>0)$ belongs to $\mathfrak{H}^-\cap U$, then the calculation is similar. We suppose that the segment from X to $\bar\zeta$ is in U, and parametrize it by $z = X+itY$. The differential form is

$$\begin{aligned}
&[(-R(\zeta;z))^s, u]\\
&\quad = (-y\,R(\zeta;z))^s\Big(\Big(\frac{\partial A}{\partial\bar z}(z)+\frac{s}{\zeta-z}A(z)\Big)dx+\Big(-i\frac{\partial A}{\partial\bar z}(z)+\frac{s}{y}\,\frac{\zeta-x}{\zeta-z}A(z)\Big)dy\Big),
\end{aligned}$$

which leads to the integral

$$\begin{aligned}
\int_X^{\bar\zeta}\omega_\zeta &= \int_0^1 \frac{t^{2s}}{(1-t^2)^s}\Big(-\frac{iY}{2}A_x(X+itY)+\frac{Y}{2}A_y(X+itY)+\frac{s}{t}\,\frac{1}{1+t}A(X+itY)\Big)\,dt\\
&= \sum_{k\ge0}\frac{\Gamma(s+\frac12)}{k!\,\Gamma(s+\frac12+k)}\Big(\varphi^{(2k+1)}(X)\,(-iY/2)^{2k+1}\\
&\qquad +\Big(\frac{k}{t}+\frac{s}{t(1+t)}\Big)\varphi^{(2k)}(X)\,(-iY/2)^{2k}\Big)\frac{t^{2s+2k}}{(1-t^2)^s}\,dt\,,
\end{aligned}$$

which is the expression that we obtained in the previous case with Y replaced by $-Y$. We replace Y by $-Y$ in (4.25), and obtain the statement in (4.22) on $U\cap\mathfrak{H}^-$ as well. ■

It is not easy to find examples that illustrate the integral transformation (4.22) explicitly, i.e., examples of functions in $\mathcal{W}_s^\omega$ for which the Green's form $[u(\cdot), R(\zeta;\cdot)^s]$ can be written explicitly as dF for some potential function $F(\cdot)$. One case which works, though not without some effort, is $u(z) = y^s = \mathsf{P}_s^\dagger(1)$ (Example 1). Here the needed potential function is given by Entry 6 in Table 3 in §A.4, and a somewhat lengthy calculation, requiring careful consideration of the branches and of the behavior at the end points of the integral, lets us deduce from (4.22) that the inverse transverse Poisson transform of the function $y^s\in\mathcal{W}_s^\omega(\mathbb{R})$ is indeed the constant function 1.

• *Other models.* The two integral formulas above were formulated in the line model. To go to the projective model, we consider first $U\subset\mathbb{C}$ as in the theorems not intersecting the half-line $i\,[1,\infty)$. In that case we find by (1.5) and (2.27b)

$$\mathsf{P}_s^\dagger\varphi^{\mathbb{P}}(z) = \frac{1}{i\,b(s)}\int_{\bar z}^z R^{\mathbb{P}}(\zeta;z)^{1-s}\,\varphi^{\mathbb{P}}(\zeta)\,\frac{d\zeta}{1+\zeta^2}\qquad (z\in U\cap\mathfrak{H}),\tag{4.26a}$$

$$\varphi^{\mathbb{P}}(\zeta) = \begin{cases} \dfrac{2\,b(s)\sin\pi s}{\pi}\displaystyle\int_{\xi_0}^{\zeta}[u(\cdot),(-R^{\mathbb{P}}(\zeta;\ \cdot\ ))^s] & \text{if } \zeta\in U\cap\mathfrak{H},\\ A^{\mathbb{P}}(\zeta) & \text{if } \zeta\in U\cap\mathbb{R}=I,\\ \dfrac{2\,b(s)\sin\pi s}{\pi}\displaystyle\int_{\xi_0}^{\bar\zeta}[(-R^{\mathbb{P}}(\zeta;\ \cdot\ ))^s,\,u(\cdot)] & \text{if } \zeta\in U\cap\mathfrak{H}^-,\end{cases}\tag{4.26b}$$

with $u(z) = \bigl(\frac{y}{|z+i|^2}\bigr)^s A^{\mathbb{P}}(z)$, where the paths of integration and the choices of branches in the Poisson kernels are as in the theorems, suitably adapted. These formulas then extend by G-equivariance to any connected and simply connected open set $U = \bar U \subset \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i,-i\}$ and any $\xi_0 \in U \cap \mathbb{P}^1_{\mathbb{R}}$, giving a local description of the isomorphism $\mathcal{V}^\omega_s \cong \mathcal{W}^\omega_s$ on all of $\mathbb{P}^1_{\mathbb{R}}$. Note that the integrals in (4.26) make sense if we take for U an annulus $1-\varepsilon < \bigl|\frac{z-i}{z+i}\bigr| < 1+\varepsilon$ in $\mathbb{P}^1_{\mathbb{C}}$, which is not simply connected, but the theorem then has to modified. We will explain this in a moment.

In the circle model, we have

$$\mathsf{P}^\dagger_s \varphi^{\mathbb{S}}(w) \;=\; \frac{1}{2\,b(s)} \int_w^{1/\bar w} R^{\mathbb{S}}(\eta;w)^{1-s}\, \varphi^{\mathbb{S}}(\eta)\,\frac{d\eta}{\eta} \qquad (w \in U\cap\mathbb{D})\,, \tag{4.27a}$$

$$\varphi^{\mathbb{S}}(\eta) \;=\; \begin{cases} \dfrac{2\,b(s)\sin\pi s}{\pi} \displaystyle\int_{\xi_0}^{\eta} [u(\,\cdot\,), (-R^{\mathbb{S}}(\eta;\,\cdot\,))^s] & \text{if } \eta\in U\,,\ |\eta|<1\,,\\ \qquad\qquad A^{\mathbb{S}}(\zeta) & \text{if } \eta \in U\cap \mathbb{S}^1\,,\\ \dfrac{2\,b(s)\sin\pi s}{\pi} \displaystyle\int_{\xi_0}^{1/\bar\eta} [(-R^{\mathbb{S}}(\eta;\,\cdot\,))^s,\, u(\,\cdot\,)] & \text{if } \eta\in U\,,\ |\eta|>1\,,\end{cases} \tag{4.27b}$$

with $u(w) = 2^{-2s}(1-|w|)^s A^{\mathbb{S}}(w)$, for $w\in U\cap\mathbb{D}$, with U open in $\mathbb{C}\smallsetminus\{0\}$, connected, simply connected and invariant under $w\mapsto 1/\bar w$, and with $\xi_0 \in U\cap\mathbb{S}^1$, with the paths of integration and the choice of branches of the Poisson kernel again suitably adapted from the versions in the line model.

If U is an annulus of the form $\varepsilon < |w| < \varepsilon^{-1}$ with $\varepsilon \in (0,1)$, we still can apply the relations in (4.27), provided we take in (4.27a) the path from w to $1/\bar w$ homotopic to the shortest path. If we change to a path that goes around a number of times, the result differs from $\mathsf{P}^\dagger_s\varphi(w)$ by an integral multiple of $\frac{\pi i}{b(s)}\,\mathsf{P}_s\varphi^{\mathbb{S}}(w)$. In (4.27b) we can freely move the point ξ_0 in $\partial\mathbb{D}$, without changing the outcome of the integral.

Let us use (4.27a) to verify the formula for $\mathsf{P}^\dagger_s(R(\,\cdot\,;z_0)^s)$ given in Example 3. By G-equivariance, we can suppose that $z_0 = i$. Now changing to circle model coordinates, we find with the help of (2.27c) that the function $\varphi(x) = R(x;i)^s$ corresponds to $\varphi^{\mathbb{S}}(\xi) = 1$ and that the content of formula (4.19) is equivalent to the formula

$$\begin{aligned}\int_r^{1/r} \Bigl(\frac{(1-r/\eta)(1-r\eta)}{1-r^2}\Bigr)^{s-1}\frac{d\eta}{\eta} \;&=\; (1-r^2)^s \int_0^1 \frac{(1-t)^{s-1}\,t^{s-1}\,dt}{(1-t(1-r^2))^s}\\ &=\; (1-r^2)^s\,\frac{\Gamma(s)^2}{\Gamma(2s)}\,F(s,s;2s;1-r^2) \;=\; 2\,Q_{s-1}\Bigl(\frac{1+r^2}{1-r^2}\Bigr),\end{aligned}$$

where in the first line we have made the substitution $\eta = (1-t)r^{-1} + tr$.

4.3. **Duality.** We return to the bilinear form β on $\mathcal{F}_s$ defined in (4.2). We have seen that β is zero on $\mathcal{E}_s\times\mathcal{E}_s$. The next result describes β on other combinations of elements of $\mathcal{F}_s$ in terms of the duality map $\langle\ ,\ \rangle : \mathcal{V}^\omega_s \times \mathcal{V}^{-\omega}_{1-s} \to \mathbb{C}$ defined in (1.19).

Proposition 4.8. *Let u, $v\in\mathcal{F}_s$.*

a) *If $u\in\mathcal{E}_s$ and $v\in\mathcal{W}^\omega_s$, then*

$$\beta(u,v) \;=\; b(s)^{-1}\,\langle\varphi,\alpha\rangle \tag{4.28}$$

with $b(s)$ as in (4.20)*, where $u = \mathsf{P}_{1-s}\alpha$ with $\alpha\in\mathcal{V}^{-\omega}_{1-s}$ and $v = \mathsf{P}^\dagger_s\varphi$ with $\varphi\in\mathcal{V}^\omega_s$.*

b) *If* $u, v \in \mathcal{W}_s^\omega$, *then* $\beta(u,v) = 0$.
c) *If* $u \in \mathcal{W}_{1-s}^\omega$ *and* $v \in \mathcal{W}_s^\omega$, *then*

$$\beta(u,v) = (s - \frac{1}{2})\,\langle \varphi, \psi \rangle\,, \tag{4.29}$$

with $\varphi \in \mathcal{V}_{1-s}^\omega$, $\psi \in \mathcal{V}_s^\omega$ *such that* $u = \mathsf{P}_{1-s}^\dagger \varphi$ *and* $v = \mathsf{P}_s^\dagger \psi$.

Proof. The bijectivity of the maps $\mathsf{P}_{1-s} : \mathcal{V}_s^{-\omega} \to \mathcal{E}_{1-s} = \mathcal{E}_s$ and $\mathsf{P}_s^\dagger : \mathcal{V}_s^\omega \to \mathcal{W}_s^\omega$ implies that we always have φ and α as indicated in a). All transformations involved are continuous for the topologies of $\mathcal{V}_{1-s}^{-\omega}$ and $\mathcal{V}_s^\omega$, so it suffices to check the relation for $\varphi = \mathbf{e}_{s,m}$ and $\alpha = \mathbf{e}_{1-s,m}$. Now we use (2.29), the result for $\mathsf{P}_s^\dagger e_{s,n}$ in §A.3, and equations (2.18) and (1.15) to get the factor in (4.28).

For part b) we write $u = (1-|w|^2)^s A(w)$ and $v = (1-|w|^2)^s B(w)$ with A and B extending in a real-analytic way across $\partial\mathbb{D}$. If we take for C a circle $|w| = r$ with r close to 1, then

$$[u,v] = \frac{1}{2i}\, r(1-r^2)^{2s}\,(AB_r - BA_r)\,d\theta\,.$$

It follows that the integral is $\mathrm{O}((1-r^2)^s)$ as $r \uparrow 1$, and hence vanishes.

In view of b) we can restrict ourselves for c) to the case $s \neq \frac{1}{2}$. As in part a) it suffices to consider the relation for basis vectors. We derive the relation from (A.14):

$$\beta(Q_{1-s,m}, Q_{s,n}) = \beta\big(\pi \cot \pi s\; P_{s,m} + Q_{s,m},\, Q_{s,n}\big) = \pi\, \cot \pi s\,(-1)^n \delta_{n,-m}\,. \qquad \blacksquare$$

4.4. **Transverse Poisson map in the differentiable case.** The G-module $\mathcal{W}_s^\omega$, which is isomorphic to $\mathcal{V}_s^\omega$, turns out to be very useful for the study of cohomology with coefficients in $\mathcal{V}_s^\omega$, discussed in detail in [2]. There we also study cohomology with coefficients in $\mathcal{V}_s^p$, with $p = 2, 3, \ldots, \infty$, and for this we need an analogue $\mathcal{W}_s^p$ of $\mathcal{W}_s^\omega$ related to p times differentiable functions. In this subsection we define such a space and show that there is an equivariant isomorphism $\mathsf{P}_s^\dagger : \mathcal{V}_s^p \to \mathcal{W}_s^p$. To generalize the restriction $\rho_s : \mathcal{W}_s^\omega \to \mathcal{V}_s^\omega$ we will define $\mathcal{W}_s^p$ not as a space of boundary germs, but as a quotient of G-modules. In fact, we give a uniform discussion, covering also the case $p = \omega$ treated in the previous subsections.

Definition 4.9. For $p = 2, 3, \ldots, \infty, \omega$ we define $\mathcal{G}_s^p$ and $\mathcal{N}_s^p$ as spaces of functions $f \in C^2(\mathbb{D})$ for which there is a neighborhood U of $\partial\mathbb{D} = \mathbb{S}$ in $\mathbb{C}$ such that the function $\tilde f(w) = (1-|w|)^{-s} f(w)$ extends as an element of $C^p(U)$ and satisfies on U the conditions

p	for $\mathcal{G}_s^p$	for $\mathcal{N}_s^p$
$\in \mathbb{Z}_{\geq 2}$	$\tilde\Delta_s \tilde f(w) = \mathrm{o}((1-\lvert w\rvert^2)^p)$	$\tilde f(w) = \mathrm{o}((1-\lvert w\rvert^2)^p)$
∞	the above condition for all $p \in \mathbb{N}$	the above condition for all $p \in \mathbb{N}$
ω	$\tilde\Delta_s \tilde f(w) = 0$	$\tilde f(w) = 0$

where $\tilde\Delta_s$ is the differential operator corresponding to $\Delta - \lambda_s$ under the transformation $f \mapsto \tilde f$.

In the analytic case $p = \omega$, the space $\mathcal{G}_s^\omega$ consists of C^2-representatives of germs in $\mathcal{W}_s^\omega$, and $\mathcal{N}_s^\omega$ consists of C^2-representatives of the zero germ in $\mathcal{W}_s^\omega$, i.e., $\mathcal{N}_s^\omega = C_c^2(\mathbb{D})$. Any representative of a germ can be made into a C^2-germ by multiplying it by a suitable cut-off function. Thus $\mathcal{W}_s^\omega$ as as in Definition 4.1 is isomorphic to $\mathcal{G}_s^\omega/\mathcal{N}_s^\omega$. We take C^2-representatives to be able to apply Δ without the need to use a distribution interpretation.

In the upper half plane model, there is an equivalent statement with $f^{\mathbb{S}}$ replaced by $f^{\mathbb{P}}$, and $2^{-2}(1-|w|^2)$ by $\frac{y}{|z+i|^2}$. The group G acts on $\mathcal{G}_s^p$ and $\mathcal{N}_s^p$ for $p=2,\ldots,\infty,\omega$, by $f\,|\,g(z)=f(gz)$, and the operator corresponding to $\Delta-\lambda_s$ is $\tilde\Delta_s=-y^2(\partial_y^2+\partial_x^2(-2sy\,\partial_y$ (cf. (4.14)).

The definition works locally: $\mathcal{G}_s^p(I)$ and $\mathcal{N}_s^p(I)$, with $I\subset\partial\mathbb{H}$ open, are defined in the same way, with Ω now a neighborhood of I in $\mathbb{P}^1_{\mathbb{C}}$. In the case that $I\subset\mathbb{R}$ in the upper half plane model, we have $f(z)=y^s\tilde f(z)$ on $\Omega\cap\mathfrak{H}$ with $\tilde f\in C^p(\Omega)$. On can check that $\mathcal{G}_s^p$ and $\mathcal{N}_s^p$ are sheaves on $\partial\mathbb{H}$.

- *Examples.* The function $z\mapsto y^s$ is in $\mathcal{G}_s^\omega(\mathbb{R})$. The function $Q_{s,n}$ in (2.7) has the right boundary behavior, but is not defined at $w=0\in\mathbb{D}$. We can multiply it by $re^{i\theta}\mapsto\chi(r)$ with a smooth function χ that vanishes near zero and is equal to one on a neighborhood on 1. In this way we obtain an element of $\mathcal{G}_s^\omega$.
- *Restriction to the boundary.* For $f\in\mathcal{G}_s^p$ the corresponding function $f^{\mathbb{S}}$ on Ω has a restriction to $\mathbb{S}^1$ that we denote by $\rho_s f$. It is an element of $\mathcal{V}_s^p$. In this way, restriction to the boundary gives a linear map

$$\rho_s:\mathcal{G}_s^p\longrightarrow\mathcal{V}_s^p \tag{4.30}$$

that turns out to intertwine the actions of G, and that behaves compatibly with respect to the inclusions $\mathcal{G}_s^p\to\mathcal{G}_s^q$ and $\mathcal{V}_s^p\to\mathcal{V}_s^q$ if $q<p$. This restriction map can be localized to give $\rho_s:\mathcal{G}_s^p(I)\to\mathcal{V}_s^p(I)$ for open intervals $I\subset\partial\mathbb{H}$.

Lemma 4.10. *Let $I\subset\partial\mathbb{H}$ be open. For $p=2,\ldots,\infty,\omega$ the space $\mathcal{N}_s^p(I)$ is a subspace of $\mathcal{G}_s^p(I)$. It is equal to the kernel of $\rho_s:\mathcal{G}_s^p(I)\to\mathcal{V}_s^p(I)$.*

Proof. The sheaf properties imply that we can work with $I\neq\partial\mathbb{H}$. The action of G can be used to arrange $I\subset\mathbb{R}$ in the upper half plane model.

Let first $p\in\mathbb{N}$, $p\geq 2$. Suppose that $f(z)=y^s\tilde f(z)$ on $\Omega\cap\mathfrak{H}$ for some $\tilde f\in C^p(\Omega)$, with Ω a neighborhood of I in $\mathbb{C}$. The Taylor expansion at $x\in I$ gives for $i,j\geq 0$, $i+j\leq p$

$$\partial_x^i\partial_y^j\tilde f(x+iy) \;=\; \sum_{n=i+j}^{p}\frac{(n-i)!}{(n-i-j)!}a_{n-i}^{(i)}(x)\,y^{n-i-j}+\mathrm{o}(y^{p-i-j}) \tag{4.31}$$

on Ω, with

$$a_n(x) \;=\; \frac{1}{n!}\partial_y^n\tilde f(x)\,.$$

The differential operator $\Delta-\lambda_s$ applied to f corresponds to the operator $\tilde\Delta_s=-y^2\partial_x^2-y^2\partial_y^2-2sy\partial_y$ applied to $\tilde f$ on the region $\Omega\cap\mathfrak{H}$. Thus we find

$$\tilde\Delta_s\tilde f(x+iy) \;=\; -2sa_1(x)-\sum_{n=2}^{p}\Bigl(a_{n-2}''(x)+n(n+2s-1)a_n(x)\Bigr)y^n+\mathrm{o}(y^p)\,. \tag{4.32}$$

If $f\in\mathcal{N}_s^p(I)$, then $a_n=0$ for $0\leq n\leq p$, and $\tilde\Delta_s\tilde f(z)=\mathrm{o}(y^p)$. So $f\in\mathcal{G}_s^p(I)$, and $\rho_s f(x)=\tilde f(x)=a_0(x)$. Hence $\mathcal{N}_s^p(I)\subset\mathrm{Ker}\,\rho_s$.

Suppose that $f \in \mathcal{G}_s^p(I)$ is in the kernel of ρ_s. Then $a_0 = 0$. From (4.32) we have $a_1 = 0$ and $a''_{n-2} = n(1-2s-n)a_n$ for $2 \leq n \leq p$. Hence $a_n = 0$ for all $n \leq p$, and $f \in \mathcal{N}_s^p(I)$.

The case $p = \infty$ follows directly from the result for $p \in \mathbb{N}$.

In the *analytic case*, $p = \omega$, the inclusions $\mathcal{N}_s^\omega(I) \subset \mathcal{G}_s^\omega(I)$ and $\mathcal{N}_s^p(I) \subset \operatorname{Ker}\rho_s$ are clear. If $f \in \mathcal{G}_s^\omega(I) \cap \operatorname{Ker}\rho_s$, then $\tilde{f}$ is real analytic on Ω, and instead of the Taylor expansion (4.31), we have a power series expansion with the same structure. Since $(\operatorname{Ker}\rho_s) \cap \mathcal{G}_s^\omega(I) \subset \mathcal{N}_s^\infty(I)$, we have $a_n = 0$ for all n, hence the analytic function $\tilde{f}$ vanishes on the connected component of Ω containing I. Thus, $f \in \mathcal{N}_s^\omega$. ∎

Relation (4.32) in this proof also shows that any $f \in \mathcal{G}_s^p(I)$ with $I \subset \mathbb{R}$ has the expansion

$$f(x+iy) = \sum_{0 \leq k \leq p/2} \frac{(-1/4)^k\, \Gamma(s+\frac{1}{2})}{k!\, \Gamma(s+k+\frac{1}{2})} \varphi^{(2k)}(x)\, y^{s+2k} + \mathrm{o}(y^{s+p}) \quad (y \downarrow 0,\ x \in I), \tag{4.33}$$

with $\varphi = \rho_s f \in \mathcal{V}_s^p(I)$.

- *Boundary jets.* For $p = 2, \ldots, \infty$ we define $\mathcal{W}_s^p$ as the quotient in the exact sequence of sheaves on $\partial\mathbb{H}$

$$0 \longrightarrow \mathcal{N}_s^p \longrightarrow \mathcal{G}_s^p \longrightarrow \mathcal{W}_s^p \longrightarrow 0\,. \tag{4.34}$$

In the analytic case, $p = \omega$, we have already seen that $\mathcal{W}_s^\omega$ is the quotient of $\mathcal{G}_s^\omega/\mathcal{N}_s^\omega$.

In the differentiable case $p = 2, \ldots, \infty$, an element of $\mathcal{W}_s^p(I)$ is given on a covering $I = \bigcup_j I_j$ by open intervals I_j by a collection of $f_j \in \mathcal{G}_s^p(I_j)$ such that $f_j \equiv f'_j \bmod \mathcal{N}_s^p(I_j \cap I_{j'})$ in $\mathcal{G}_s^p(I_j \cap I_{j'})$ if $I_j \cap I_{j'} \neq \emptyset$. To each j is associated a neighborhood Ω_j of I_j in $\mathbb{P}^1_\mathbb{C}$ on which $f_j^\mathbb{S}$ is p times differentiable. Add an open set $\hat{\Omega} \subset \mathbb{H}$ such that $\mathbb{H} \subset \hat{\Omega} \cup \bigcup_j \Omega_j$. With a partition of unity subordinate to the collection $\{\hat{\Omega}\} \cup \{\Omega_j : j\}$ we build one function f on $\mathbb{H}$ such that $f^\mathbb{S} = (1-|w|^2)^{-s} f(w)$ differs from $f_j^\mathbb{S}$ on Ω_j by an element of $\mathcal{N}_s^p(I_j)$. In this way we obtain $\mathcal{W}_s^p(I) = \mathcal{G}_s^p(I)/\mathcal{N}_s^p(I)$ in the differentiable case as well.

We have also

$$0 \longrightarrow \mathcal{N}_s^p(\partial\mathbb{H}) \longrightarrow \mathcal{G}_s^p(\partial\mathbb{H}) \longrightarrow \mathcal{W}_s^p(\partial\mathbb{H}) \longrightarrow 0$$

as an exact sequence of G modules. We call elements of $\mathcal{W}_s^p$ *boundary jets* if $p = 2, \ldots, \infty$. The G-morphism ρ_s induces a G-morphism $\rho_s : \mathcal{W}_s^p(I) \to \mathcal{V}_s^p(I)$ for $p = 2, \ldots, \infty, \omega$. The morphism is injective by Lemma 4.10. In fact it is also surjective:

Theorem 4.11. *The restriction map $\rho_s : \mathcal{W}_s^p(I) \to \mathcal{V}_s^p(I)$ is an isomorphism for every open set $I \subset \partial\mathbb{H}$, for $p = 2, \ldots, \infty, \omega$.*

The case $p = \omega$ was the subject of §4.2. Theorem 4.6 described the inverse $\mathsf{P}_s^\dagger$ explicitly with a transverse Poisson integral, and Theorem 4.5 works with a power series expansion. It is the latter approach that suggests how to proceed in the differentiable and smooth cases. We denote the inverse by $\mathsf{P}_s^\dagger$, or by $\mathsf{P}_{s,p}^\dagger$ if it is desirable to specify p.

Proof. In the *differentiable case* $p \in \mathbb{N} \cup \{\infty\}$, it suffices to consider $\varphi \in C_c^p(I)$ where I is an interval in $\mathbb{R}$. The obvious choice would be to define near I

$$f(z) \;=\; \sum_{0\le k\le p/2} \frac{(-1)^k}{4^k\, k!\, (s+\frac12)_k}\, \varphi^{(2k)}(x)\, y^{s+2k}\,. \tag{4.35}$$

However, this is in general not in $C^p(\mathfrak{H})$, because each term $\varphi^{(2k)}(x)\, y^{s+2k}$ is only in C^{p-2k}. Instead we set

$$f(z) \;=\; y^s \int_{-\infty}^{\infty} \omega(t)\, \varphi(x+yt)\, dt \;=\; y^{s-1} \int_{-\infty}^{\infty} \omega(\tfrac{t-x}{y})\, \varphi(t)\, dt\,, \tag{4.36}$$

where φ has been extended by zero outside its support, and where ω is an even real-analytic function on $\mathbb{R}$ with quick decay that has prescribed moments

$$M_{2k} \;:=\; \int_{-\infty}^{\infty} t^{2k}\, \omega(t)\, dt \;=\; \frac{(-1)^k(\frac12)_k}{(s+\frac12)_k} \qquad \text{for even } k \ge 0\,. \tag{4.37}$$

(For instance we could take ω to be the Fourier transform of the product of the function $u \mapsto \Gamma(s+\frac12)\left(\frac{|u|}{2}\right)^{\frac12 - s} I_{s-1/2}(|u|)$ and an even function in $C_c^\infty(\mathbb{R})$ that is equal to 1 on a neighborhood of 0 in $\mathbb{R}$. This choice is even real-analytic.) Replacing φ in (4.36) by its Taylor expansion up to order p, we see that this formally matches the expansion (4.35), but it now makes sense and is C^∞ in all of $\mathfrak{H}$, as we see from the second integral. The first integral shows that

$$\tilde f(z) \;=\; y^{-s} f(z) = \int_{-\infty}^{\infty} \omega(t)\varphi(x+yt)\, dt \tag{4.38}$$

extends as a function in $C^\omega(\mathbb{C})$.

Inserting the power series expansion of order p of φ at $x \in I$ in (4.38) we arrive at $\tilde\Delta_s \tilde f(z) = \mathrm{O}(y^p)$. This finishes the proof in the differentiable and smooth cases. ∎

In the proof of Theorem 4.11 we have chosen an real-analytic Schwartz function ω with prescribed moments. In the case $p = 2, 3, \ldots$ we may use the explicit choice in the following lemma, which will be used in the next chapter:

Lemma 4.12. *For any $s \notin \frac12\mathbb{Z}$ and any integer $N \ge 0$ there is a unique decomposition*

$$(t^2+1)^{s-1} \;=\; \frac{d^N \alpha(t)}{dt^N} \;+\; \beta(t) \tag{4.39}$$

where $\alpha(t) = \alpha_{N,s}(t)$ is $(t^2+1)^{s-1}$ times a polynomial of degree N in t and $\beta(t) = \beta_{N,s}(t)$ is $\mathrm{O}(t^{2s-N-3})$ as $|t| \to \infty$.

We omit the easy proof. The first few examples are

$$\begin{aligned}
(t^2+1)^{s-1} &= \frac{d}{dt}\Big[\frac{t(t^2+1)^{s-1}}{2s-1}\Big] + \frac{2s-2}{2s-1}(t^2+1)^{s-2},\\
(t^2+1)^{s-1} &= \frac{d^2}{dt^2}\Big[\frac{(t^2+1)^{s-1}}{(2s-1)(2s-3)} + \frac{(t^2+1)^s}{2s(2s-1)}\Big] + \frac{4(s-1)(s-2)}{(2s-1)(2s-3)}(t^2+1)^{s-3},\\
(t^2+1)^{s-1} &= \frac{d^3}{dt^3}\Big[\frac{2t(t^2+1)^{s-1}}{(2s+1)(2s-1)(2s-3)} + \frac{t(t^2+1)^s}{2s(2s+1)(2s-1)}\Big]\\
&\qquad + \frac{4(s-1)(s-2)}{(2s+1)(2s-1)}\Big(\frac{2s+3}{2s-3} + 3t^2\Big)(t^2+1)^{s-4}.
\end{aligned}$$

In general we have

$$\alpha_{N,s}(t) = \frac{1}{2^N}\sum_{j=0}^{N/2}\binom{N-j-1}{N/2-1}\frac{(t^2+1)^{s-1+j}}{(s)_j\,(s-\frac{N+1}{2}+j)_{N-j}}$$

if $N \geq 2$ is even, where $(s)_j = s(s+1)\cdots(s+j-1)$ is the ascending Pochhammer symbol, and a similar formula if N is odd, as can be verified using the formula

$$\frac{1}{n!}\frac{d^n}{dt^n}(t^2+1)^{s-1} = \sum_{0\leq j\leq n/2}\binom{n-j}{j}\binom{s-1}{n-j}(2t)^{n-2j}(t^2+1)^{s-n+j}.$$

Let us compute the moments of $\beta = \beta_{N,s}$ as in (4.39). For $0 \leq n < N$ we have

$$\int_{-\infty}^{\infty} t^n\beta(t)\,dt = \int_{-\infty}^{\infty}\Big((t^2+1)^{s-1} - \frac{d^N\alpha(t)}{dt^N}\Big)t^n\,dt.$$

This is a holomorphic function of s on $\mathrm{Re}\, s < 1$. We compute it by considering $\mathrm{Re}\, s < -\frac{n}{2}$:

$$\begin{aligned}
\int_{-\infty}^{\infty} t^n(t^2+1)^{s-1}\,dt &= \begin{cases}\int_0^1 x^{\frac{n-1}{2}}(1-x)^{-s-\frac{n+1}{2}}\,dx & \text{if } n \text{ is even},\\ 0 & \text{if } n \text{ is odd};\end{cases}\\
&= \sqrt{\pi}\tan\pi s\,\frac{\Gamma(s)}{\Gamma(s+\frac12)}\cdot\begin{cases}(-1)^k\,\frac{(\frac12)_k}{(s+\frac12)_k} & \text{if } n=2k \text{ is even},\\ 0 & \text{if } n \text{ is odd}.\end{cases}
\end{aligned}\tag{4.40}$$

So a multiple of $\beta_{N,s}$ has the moments that we need in the proof of Theorem 4.11.

5. Boundary behavior of mixed eigenfunctions

In this section we combine ideas from §3 and §4. Representatives u of elements of $\mathcal{W}^{\omega}_{1-s}$ have the special property that $(1-|w|^2)^{s-1}\,u(w)$ (in the circle model) or $y^{s-1}u(z)$ (in the line model) extends analytically across the boundary $\partial\mathbb{H}$. If such an eigenfunction occurs in a section (h,u) of the sheaf $\mathcal{D}_s$ of mixed eigenfunctions we may ask whether a suitable multiple of h also extends across the boundary. In §5.3 we will show that this is true locally (Theorem 5.2), but not globally (Proposition 5.5).

In §5.1 we use the differential equations satisfied by $y^{-s}u$ for representatives u of elements of $\mathcal{W}^{\omega}_s$ to define an extension $\mathcal{A}_s$ of the sheaf $\mathcal{E}_s$ from $\mathfrak{H}$ to $\mathbb{P}^1_{\mathbb{C}}$. We also extend the sheaf $\mathcal{D}_s$ on $\mathbb{P}^1_{\mathbb{C}}\times\mathfrak{H}$ to a sheaf $\mathcal{D}^*_s$ on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ that has the same relation to $\mathcal{A}_{1-s}$ as the relation of $\mathcal{D}_s$ to $\mathcal{E}_s = \mathcal{E}_{1-s}$. In §5.2 we show that the power series expansion of sections of $\mathcal{A}_s$ leads in a natural way to sections of $\mathcal{D}^*_{1-s}$, a result which is needed for the proofs in §5.3, and in §5.4 we give the generalization of Theorem 3.13 to the sheaf $\mathcal{D}^*_s$. Finally, in §5.5 we consider the sections of $\mathcal{D}_s$ near $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}$.

5.1. **Interpolation between sheaves on $\mathfrak{H}$ and its boundary.** In this subsection we formulate results from §4.2 in terms of a sheaf on $\mathbb{P}^1_{\mathbb{C}}$ that is an extension of the sheaf $\mathcal{E}_s$. This will be used in the rest of this section to study the behavior of mixed eigenfunctions near the boundary $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{R}}$ of $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$ and to extend them across this boundary. We also define an extension $\mathcal{D}^*_s$ of the sheaf $\mathcal{D}_s$ of mixed eigenfunctions.

For an open set $U \subset \mathbb{C}$, let $\mathcal{A}_s(U)$ be the space of real-analytic solutions $A(z)$ of (4.14) in U. For $U \subset \mathbb{P}^1_{\mathbb{C}}$ containing ∞ the definition is the same except that the solutions have the form $A(z) = |z|^{-2s} A^\infty(-1/z)$ for some real-analytic function A^∞ near 0 (which then automatically satisfies the same equation). The action (4.10) makes $\mathcal{A}_s$ into a G-equivariant sheaf: $A \mapsto A|g$ defines an isomorphism $\mathcal{A}_s(U) \cong \mathcal{A}_s(g^{-1}U)$ for any open $U \subset \mathbb{P}^1_{\mathbb{C}}$ and $g \in G$.

For any $U \subset \mathbb{P}^1_{\mathbb{C}}$, the space $\mathcal{A}_s(U)$ can be identified via $A(z) \mapsto u(z) = |y|^s A(z)$ with a subspace of the space $\mathcal{E}_s(U \smallsetminus \mathbb{P}^1_{\mathbb{R}})$ of λ_s-eigenfunctions of the Laplace operator $\Delta = -y^2(\partial_x^2 + \partial_y^2)$ in $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}$ (up to now we have considered the operator Δ and the sheaf $\mathcal{E}_s$ only on $\mathfrak{H}$), namely the subspace consisting of functions which are locally of the form $|y|^s \times (\text{analytic})$ near $\mathbb{R}$ and of the form $|y/z^2|^s \times (\text{analytic})$ near ∞.

If $U \subset \mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}$, then the map $A \mapsto u$ is an isomorphism between $\mathcal{A}_s(U)$ and $\mathcal{E}_s(U)$. (In this case the condition "real-analytic" in the definition of $\mathcal{A}_s(U)$ can be dropped, since C^2 or even distributional solutions of the differential equation are automatically real-analytic.) At the opposite extreme, if U meets $\mathbb{P}^1_{\mathbb{R}}$ in a non-empty set I, then any section of $\mathcal{A}_s$ over U restricts to a section of $\mathcal{V}^\omega_s$ over I, and for any $I \subset \mathbb{P}^1_{\mathbb{R}}$ we obtain from Theorem 4.5 an identification between $\mathcal{V}^\omega_s(I)$ and the inductive limit of $\mathcal{A}_s(U)$ over all neighborhoods $U \supset I$. The sheaf $\mathcal{A}_s$ thus "interpolates" between the sheaf $\mathcal{E}_s$ on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}$ and the sheaf $\mathcal{V}^\omega_s$ on $\mathbb{P}^1_{\mathbb{R}}$. At points outside $\mathbb{P}^1_{\mathbb{R}}$ the stalks of $\mathcal{A}_s$ are the same as those of $\mathcal{E}_s$, while at points in $\mathbb{P}^1_{\mathbb{R}}$ the stalks of the sheaves $\mathcal{A}_s$, $\mathcal{V}^\omega_s$ and $\mathcal{W}^\omega_s$ are all canonically isomorphic. At the level of open sets rather than stalks, Theorems 4.6 and 4.7 say that the space $\mathcal{A}_s(U)$ for suitable U intersecting $\mathbb{P}^1_{\mathbb{R}}$ is isomorphic to $\mathcal{O}(U)$ by a unique isomorphism compatible with restriction to $U \cap \mathbb{R}$, the isomorphisms in both directions being given by explicit integral transforms. Finally, from (4.15) we see that if U is connected and invariant under conjugation, then any $A \in \mathcal{A}_s(U)$ is invariant under $z \mapsto \bar{z}$. In the language of sheaves, this says that, if we denote by $c : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ the complex conjugation, the induced isomorphism $c : c^{-1}\mathcal{A}_s \to \mathcal{A}_s$ is the identity when restricted to $\mathbb{P}^1_{\mathbb{R}}$.

We now do the same construction for the sheaf $\mathcal{D}_s$ of mixed eigenfunctions, defining a sheaf $\mathcal{D}^*_s$ on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ which bears the same relation to $\mathcal{D}_s$ as $\mathcal{A}_s$ has to $\mathcal{E}_s$. (We could therefore have used the notation $\mathcal{E}^*_s$ instead of $\mathcal{A}_s$, but since $\mathcal{A}_s$ interpolates between two very different subsheaves $\mathcal{E}_s$ and $\mathcal{V}^\omega_s$, we preferred a neutral notation which does not favor one of these aspects over the other. Also, $\mathcal{E}_s = \mathcal{E}_{1-s}$, but $\mathcal{A}_s \neq \mathcal{A}_{1-s}$.)

Let (h, u) be a section of the sheaf $\mathcal{D}_s$ in $U \cap (\mathbb{C} \times \mathfrak{H})$, where U is a neighborhood in $\mathbb{C} \times \mathbb{C}$ of a point (x_0, x_0) with $x_0 \in \mathbb{R}$. The function $u(z)$ is a λ_s-eigenfunction of Δ, and we can ask whether it ever has the form $y^s\, A(z)$ or $y^{1-s}\, A(z)$ with $A(z)$ (real-) analytic near x_0. It turns out that the former does not happen, but the latter does, and moreover that in this case the function $h(\zeta, z)$ has the form $y^{-s}\, B(\zeta, z)$ where $B(\zeta, z)$ is also analytic in a neighborhood of $(x_0, x_0) \in \mathbb{C} \times \mathbb{C}$. To see this, we make the substitution

$$u(z) \;=\; y^{1-s}\, A(z)\,, \qquad h(\zeta, z) \;=\; y^{-s}\, B(\zeta, z) \tag{5.1}$$

in the differential equations (3.34) to obtain that these translate into the differential equations

$$(\zeta - z)\, \partial_z B \;=\; -s\, B - \frac{is}{2}\, (\zeta - \bar{z})\, A\,, \tag{5.2a}$$

$$(\zeta - \bar{z})\, \partial_{\bar{z}}(B - yA) \;=\; -s\, B - \frac{is}{2}\, (\zeta - \bar{z})\, A\,, \tag{5.2b}$$

for A and B, in which there is no singularity at $y = 0$. (This would not work if we had used $u = y^s A$, $h = y^* B$ instead.)

As long as z is in the upper half plane, equations (5.1) define a bijection between pairs (h, u) and pairs (B, A), and it makes no difference whether we study the original differential equations (3.34) or the new ones (5.2). The advantage of the new system is that it makes sense for all $z \in \mathbb{C}$ and defines a sheaf $\mathcal{D}_s^*$ on $\mathbb{C} \times \mathbb{C}$ whose sections over $U \subset \mathbb{C} \times \mathbb{C}$ are real-analytic solutions (B, A) of (5.2) in U with B holomorphic in the first variable and A locally constant in the first variable. This sheaf is G-equivariant with respect to the action $(B, A)|g = (B|g, A|g)$ given for $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ by

$$B|g(\zeta, z) \;=\; |cz + d|^{2s} B(g\zeta, gz)\,, \qquad A|g(z) \;=\; |cz + d|^{2s-2} A(gz)\,, \tag{5.3}$$

so it extends to a sheaf on all of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ by setting $\mathcal{D}_s^*(U) = \mathcal{D}_s^*(g^{-1}U)|g$ if U is a small neighborhood of a point (ζ_0, ∞) or (∞, z_0) and g is chosen with $g^{-1}U \subset \mathbb{C} \times \mathbb{C}$.

In (3.38) and (3.39) we give a formula for h in terms of u near the diagonal and the antidiagonal where (h, u) is a section of $\mathcal{D}_s$. In terms of $A = y^{s-1} u$ and $B = y^s h$ this formula becomes

$$B(\zeta, z) \;=\; \begin{cases} -\dfrac{i}{2}(\zeta - \bar{z}) \displaystyle\sum_{n \ge 0} \frac{\partial^n A}{\partial z^n}(z)\, \frac{(\zeta - z)^n}{(1-s)_n} & \text{for } \zeta \text{ near } z\,, \\ -\dfrac{i}{2}(\zeta - z) \displaystyle\sum_{n \ge 1} \frac{\partial^n A}{\partial \bar{z}^n}(z)\, \frac{(\zeta - \bar{z})^n}{(1-s)_n} - \frac{i}{2}(\zeta - \bar{z}) A(z) & \text{for } \zeta \text{ near } \bar{z}\,. \end{cases} \tag{5.4}$$

Now inspection shows that the right hand side of (5.4) satisfies the differential equations (5.2), whether $z \in \mathfrak{H}$ or not, so (B, A) with B as in (5.4) gives a section of $\mathcal{D}_s^*$ on neighborhoods of points (z, z) and $(\bar{z}, z)$ for all $z \in \mathbb{C}$. From (5.4) it is not clear that for $z \in \mathbb{R}$ both expressions define the same function on a neighborhood of z. In the next subsection we will see that they do.

5.2. **Power series expansion.** Sections of $\mathcal{A}_s$ are real-analytic functions of one complex variable, and hence can be seen as power series in two variables. In this subsection we show that the coefficients in these expansions have interesting properties. They will be used in §5.3 to study the structure of sections of $\mathcal{D}_s$ and $\mathcal{D}_s^*$ near the diagonal of $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$.

Let $U \subset \mathbb{C}$ be open, and let $z_0 \in U$. We write the expansion of a section A of $\mathcal{A}_s$ at a point z_0 in the strange form (the reason for which will become apparent in a moment)

$$A(z) \;=\; \sum_{m,n\ge 0} \binom{m+s-1}{m} \binom{n+s-1}{n} c_{m,n}(z_0)\,(z-z_0)^m\,\overline{(z-z_0)}^n\,. \tag{5.5}$$

Then we have the following result.

Theorem 5.1. *Let $U \subset \mathbb{C}$, $A \in \mathcal{A}_s(U)$, and for $z_0 \in U$ define the coefficients $c_{m,n}(z_0)$ for $m,n \ge 0$ by* (5.5)*. Let $r : U \to \mathbb{R}_+$ be continuous. Then the series* (5.5) *converges in $|z-z_0| < r(z_0)$ if and only if the series*

$$\Phi_A(z_0;v,w) \;:=\; \sum_{m,n\ge 0} c_{m,n}(z_0)\, v^m w^n \tag{5.6}$$

converges for $|v|,|w| < r(z_0)$. The function defined by (5.6) *has the form*

$$\Phi_A(z_0;v,w) \;=\; \frac{B(z_0+v,z_0) - B(\bar z_0 + w, z_0)}{y_0 + (v-w)/2i} \tag{5.7}$$

for a unique analytic function B on

$$U' \;=\; \{(\zeta,z) \in \mathbb{C}\times U \;:\; |\zeta - z| < r(z)\} \;\cup\; \{(\zeta,z)\in\mathbb{C}\times U \;:\; |\zeta-\bar z| < r(z)\}$$

satisfying $B(\zeta,z) = y\,A(z)$ and $B(\bar z, z) = 0$ for $z \in U$, and the pair (B,A) is a section of $\mathcal{D}^*_{1-s}$ over U'.

Proof. The fact that $\binom{m+s-1}{m} = m^{\mathrm{O}(1)}$ as $m \to \infty$ implies the relation between the convergence of (5.5) and (5.6). (We use here that a power series $\sum c_{mn,n} v^n w^m$ in two variable converges for $|v|,|w| < r$ if and only if its restriction to $w = \bar v$ converges for $|v| < r$.) The differential equation (4.14) is equivalent to the very simple recursion

$$2iy_0\, c_{m,n}(z_0) \;=\; c_{m,n-1}(z_0) - c_{m-1,n}(z_0) \qquad (m,n \ge 1) \tag{5.8}$$

for the coefficients $c_{m,n}(z_0)$. (This was the reason for the choice of the normalization in (5.5).) This translates into the fact that $(2iy_0 + v - w)\,\Phi_A(z_0;v,w)$ is the sum of a function of v alone and a function of w alone, i.e., we have

$$\Phi_A(z_0;v,w) \;=\; \frac{L_A(z_0;v) - R_A(z_0;w)}{y_0 + (v-w)/2i}\,, \tag{5.9}$$

where, if we use the freedom of an additive constant to normalize $R_A(z_0;0) = 0$, the functions L_A and R_A are given explicitly in terms of the boundary coefficients $\{c_{j,0}(z_0)\}_{j\ge 0}$ and $\{c_{0,j}(z_0)\}_{j\ge 1}$ by

$$\begin{aligned} 2i\,L_A(z_0;v) \;&=\; (v + 2iy_0) \sum_{m\ge 0} c_{m,0}(z_0)\, v^m\,, \\ 2i\,R_A(z_0;w) \;&=\; c_{0,0}(z_0)\, w + (w - 2iy_0) \sum_{n\ge 1} c_{0,n}(z_0)\, w^n\,. \end{aligned} \tag{5.10}$$

(Multiplied out, this says that coefficients $c_{m,n}(z_0)$ satisfying (5.8) are determined by their boundary values by

$$\begin{aligned} c_{m,n}(z_0) \;=\; & \sum_{j=1}^{m} \frac{(-1)^n}{(2iy_0)^{m+n-j}} \binom{m-j+n-1}{m-j} c_{j,0}(z_0) \\ & + \sum_{j=1}^{n} \frac{(-1)^{n-j}}{(2iy_0)^{m+n-j}} \binom{n-j+m-1}{n-j} c_{0,j}(z_0)\,, \end{aligned} \tag{5.11}$$

which of course can be checked directly.)

We define B on U' (now writing z instead of z_0) by

$$B(\zeta, z) \;=\; \begin{cases} L_A(z, \zeta - z) & \text{if } |\zeta - z| < r(z)\,, \\ R_A(z, \zeta - \bar z) & \text{if } |\zeta - \bar z| < r(z)\,. \end{cases} \tag{5.12}$$

These two definitions are compatible if the discs in question overlap (which happens if $r(z) > |y_0|$), because the convergence of (5.6) for $|v|$, $|w| < r(z)$ implies that the fraction in (5.9) is holomorphic in this region, and hence that its numerator vanishes if $z_0 + v = \bar z_0 + w$.

Surprisingly, the function B thus defined constitutes, together with the given section A of $\mathcal{A}_s$, a section (B, A) of $\mathcal{D}^*_{1-s}$ for ζ near z or $\bar z$. To see this, we apply the formulas (5.4), with s replaced by $1 - s$, and express the derivatives of A in the coefficients $c_{m,n}(z)$ with help of (5.5). We find that the first expression in (5.4) is equal to $L_A(z; \zeta - z)$, and the second one to $R_A(z; \zeta - \bar z)$. ■

Example 1. Let $A(z) \in \mathcal{A}_s(\mathbb{C} \smallsetminus \{0\})$ be the function $|z|^{-2s}$. For any $z_0 \neq 0$ the binomial theorem gives $c_{m,n}(z) = (-1)^{m+n}\, |z_0|^{-2s}\, z_0^{-m}\, \bar z_0^{-n}$ and

$$\Phi_A(z_0; v, w) \;=\; \frac{|z_0|^{2-2s}}{(z_0 + v)(\bar z_0 + w)} \;=\; \frac{1}{2iy_0 + v - w}\Bigl(\frac{|z_0|^{2-2s}}{\bar z_0 + w} - \frac{|z_0|^{2-2s}}{z_0 + v}\Bigr), \tag{5.13}$$

in accordance with (5.7) with the solution $B(\zeta, z) = \frac{i}{2}\, |z|^{2-2s} \left(\zeta^{-1} - \bar z^{-1}\right)$, defined on $z \neq 0$, $\zeta \neq 0$. (The regions $|\zeta - z| < |z|$ and $|\zeta - \bar z| < z$ do not overlap.)

Example 2. If $z_0 \in \mathbb{R}$, then (5.8) says that $c_{m,n}(z_0)$ depends only on $n + m$, so the generating function Φ_A has an expansion of the form

$$\Phi_A(z_0; v, w) \;=\; \sum_{N=0}^{\infty} C_N(z_0)\, \frac{v^{N+1} - w^{N+1}}{v - w}\,.$$

Hence in this case we have $A(z) \;=\; \sum_{N \ge 0} C_N(z_0) P_N(z - z_0)$ where P_N is the section of $\mathcal{A}_s$ defined by

$$P_N(z) \;:=\; (-1)^N \sum_{m,n \ge 0,\, m+n=N} \binom{-s}{m} \binom{-s}{n} z^m\, \bar z^n\,, \tag{5.14}$$

a polynomial that already occurred in (4.17).

Example 3. Let $A(z) = y^{-s} p_{s,k}(z, i)$, defined in (2.5), with $z_0 = i$ and $k \ge 0$. We describe $A(z) = \frac{\Gamma(s+k)}{k!\, \Gamma(s-k)}\, \tilde A(w)$ first in the coordinate $w = \frac{z-i}{z+i}$ of the disk model. Taking into account (A.8) and (A.9) we obtain

$$\tilde{A}(w) \;=\; w^k \Big(\frac{1-w\,\bar{w}}{|1-w|^2}\Big)^{-s} (1-w\,\bar{w})^s \; F(s,s+k;1+k,w\,\bar{w})\,.$$

Set $p=(z-i)/2i$, so that $w=p/(p+1)$. Then

$$\begin{aligned}
\tilde{A}(w) \;&=\; (1-w)^s\,(1-\bar{w})^s \sum_{l\geq 0} \frac{(s)_\ell\,(s+k)_\ell}{(1+k)_\ell\,\ell!}\, w^{k+\ell}\bar{w}^\ell \\
&=\; \sum_{\ell\geq 0} \binom{-s}{\ell}\binom{-s-k}{\ell}\binom{\ell+k}{\ell}^{-1} p^{k+\ell}\,\bar{p}^\ell\,(1+p)^{-s-k-\ell}\,(1+\bar{p})^{-s-\ell} \\
&=\; \sum_{\ell,i,j\geq 0} \binom{-s}{\ell}\binom{-s-k}{\ell}\binom{\ell+k}{\ell}^{-1} \binom{-s-k-\ell}{i}\binom{-s-\ell}{j} p^{k+\ell+i}\,\bar{p}^{\ell+j} \\
&=\; \sum_{m\geq k,\,n\geq 0} p^m\,\bar{p}^n \binom{-s-k}{m-k}\binom{-s}{n}\binom{n+k}{k}^{-1} \sum_{l=0}^{n}\binom{m-k}{\ell}\binom{n+k}{n-\ell} \\
&=\; \sum_{m\geq k,\,n\geq 0} \Big(\frac{z-i}{2i}\Big)^m \Big(\frac{\bar{z}+i}{-2i}\Big)^n \binom{-s-k}{m-k}\binom{-s}{n}\binom{n+k}{k}^{-1}\binom{m+n}{n}.
\end{aligned}$$

Hence A has an expansion as in (5.5) with

$$c^{[k]}_{m,n}(i) \;:=\; c_{m,n}(i) \;=\; (-1)^m\,(2i)^{-m-n}\,(1-s)_k \binom{m+n}{n+k} \tag{5.15a}$$

($=0$ if $m<k$), which satisfies the recursion (5.8).

The analogous computation for $k<0$ gives

$$c^{[k]}_{m,n}(i) \;=\; (-1)^m(2i)^{-m-n}\,(1-s)_k \binom{m+n}{m-k} \tag{5.15b}$$

($=0$ if $n<-k$).[2]

In this example we can describe the form of the function B up to a factor without computation by equivariance: since $z\mapsto p_{s,n}(z,i)$ transforms according to the character $\left[\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right] \mapsto e^{2ik\theta}$, the function $h=y^{s-1}B$ should do the same near points of the diagonal or the antidiagonal. Thus, for $k\geq 0$ we know that $B(\zeta,i)$ is a multiple of $(\frac{\zeta-i}{\zeta+i})^k$ near $\zeta=i$, and vanishes near $\zeta=-i$, while for $k<0$, we have $B(\zeta,i)=0$ for ζ near i, and $B(\zeta,i)$ is a multiple of $(\frac{\zeta-i}{\zeta+i})^k$ for ζ near $-i$. The explicit computation using (5.12), (5.10) and (5.15) confirms these predictions, giving $B(\zeta,i)=(-1)^k\,(1-s)_k\,(\frac{\zeta-i}{\zeta+i})^k$ if $k\geq 0$ and ζ is near i, and $B(\zeta,i)=-(-1)^k\,\frac{\Gamma(k+1-s)}{\Gamma(1-s)}\,(\frac{\zeta-i}{\zeta+i})^k$ if $k<0$ and ζ is near $-i$.

Note that, since any holomorphic function of ζ near i (resp. $-i$) can be written as a power series in $\frac{\zeta-i}{\zeta+i}$ (resp. $\frac{\zeta+i}{\zeta-i}$) we see that this example is generic for the expansions of A and B for any section (B,A) of $\mathcal{D}_s$ near $(\zeta,z)=(\pm i,i)$, and hence by G-equivariance for z near any $z_0\in\mathfrak{H}$ and ζ near z_0 or $\bar{z}_0$.

Remark. We wrote formula (5.5) as the expansion of a fixed section $A\in\mathcal{A}_s(U)$ around a variable point $z_0\in U$. If we simply define a function $A(z)$ by (5.5), where z_0 (say

[2] The Pochhammer symbol $(x)_k$ is defined for $k<0$ as $(x-1)^{-1}\cdots(x-|k|)^{-1}$, so that $(x)_k=\Gamma(x+k)/\Gamma(x)$ in all cases.

in $\mathfrak{H}$) is fixed, then we still find that the differential equation $(\Delta - \lambda_s)(y^s A(\,\cdot\,, z_0)) = 0$ is equivalent to the recursion (5.8) and to the splitting (5.9) of the generating function Φ_A defined by (5.6). In this way we have constructed a very large family of (locally defined) λ_s-eigenfunctions of Δ: for any $z_0 \in \mathfrak{H}$ and any holomorphic functions $L(v)$ and $R(w)$ defined on discs of radius $r \le y_0$ around 0, we define coefficients $c_{m,n}$ either by (5.9) and (5.6) or by (5.10) and (5.11); then the function $u(z) = y^s A(z)$ with A given by (5.5) is a λ_s-eigenfunction of Δ in the disk of radius r around z_0.

5.3. **Mixed eigenfunctions near the diagonal of $\mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$.** Parts i) and iv) of Proposition 3.10 show that if (h, u) is a section of $\mathcal{D}_s$ near a point $(z, z) \in \mathfrak{H} \times \mathfrak{H}$ of the diagonal or a point $(\bar z, z) \in \mathfrak{H}^- \times \mathfrak{H}$ of the antidiagonal, then the function h and u determine each other. Diagonal points $(\xi, \xi) \in \mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$ are not contained in the set $\mathbb{P}^1_\mathbb{C} \times \mathfrak{H}$ on which the sheaf $\mathcal{D}_s$ is defined. Nevertheless, there is a relation between the analytic extendability of h and u near such points, which we now study.

Theorem 5.2. *Let $\xi \in \mathbb{P}^1_\mathbb{R}$. Suppose that (h, u) is a section of $\mathcal{D}_s$ over $U \cap (\mathfrak{H} \times \mathfrak{H})$ for some neighborhood U of (ξ, ξ) in $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$. Then the following statements are equivalent:*

a) *The function $y^{s-1}u$ extends real-analytically to a neighborhood of ξ in $\mathbb{P}^1_\mathbb{C}$.*
b) *The function $y^s h$ extends real-analytically to a neighborhood of (ξ, ξ) in $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$.*
c) *The function $y^s h$ extends real-analytically to $U' \cap (\mathbb{P}^1_\mathbb{C} \times \mathfrak{H})$ for some neighborhood U' of (ξ, ξ) in $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$.*

The theorem can be formulated partly in terms of stalks of sheaves. In particular, the functions u in a) represent elements of the stalk $(\mathcal{W}^\omega_{1-s})_\xi$, and the pairs $(y^s h, y^{s-1}u)$ with $y^{s-1}u$ as in a) and $y^s h$ as in b) represent germs in the stalk $(\mathcal{D}^*_s)_{(\xi,\xi)}$. The theorem has the following consequence:

Corollary 5.3. *For each $\xi \in \mathbb{P}^1_\mathbb{R}$ the morphism $C : \mathcal{D}_s \to p_2^{-1}\mathcal{E}_s$ in Theorem 3.13 induces a bijection*

$$\varinjlim_U \mathcal{D}_s(U \cap (\mathbb{P}^1_\mathbb{R} \times \mathfrak{H})) \;\cong\; (\mathcal{W}^\omega_{1-s})_\xi\,,$$

where U runs over the open neighborhoods of (ξ, ξ) in $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$, and

$$(\mathcal{W}^\omega_{1-s})_\xi \;\cong\; (\mathcal{D}^*_s)_{(\xi,\xi)}\,.$$

Proof of Theorem 5.2. We observe that since $U \cap (\mathfrak{H} \times \mathfrak{H})$ intersects the diagonal, the functions h and u in the theorem determine each other near (ξ, ξ) by virtue of parts i) and iv) of Proposition 3.10. Hence the theorem makes sense.

Clearly b) $\Rightarrow$ c). We will prove a) $\Rightarrow$ b) and c) $\Rightarrow$ a). By G-equivariance we can assume that $\xi = 0$.

For a) $\Rightarrow$ b) we write $u = y^{1-s}A$ with A real-analytic on a neighborhood of 0 in $\mathbb{C}$. We apply Theorem 5.1. The power series (5.5) converges for $|z_0| \le R$, $|z - z_0| < r$ for some $r, R > 0$. (Choose r to be the minimum of $r(z_0)$ in $|z_0| \le R$ for R small.) The theorem gives us an analytic function B on the region $W = \{(\zeta, z) \in \mathbb{C} \times \mathbb{C} \;:\; |z| < R, |\zeta - z| < r\}$ such that $(B, A) \in \mathcal{D}^*_s(W)$. By the uniqueness clause of Proposition 3.11, the restriction of B to $W \cap (\mathbb{C} \times \mathfrak{H})$ is $y^s h$. Since $(0, 0) \in W$ this gives b).

For c) $\Rightarrow$ a) we start with a section $(y^s B, y^{1-s} A)$ of $\mathcal{D}_s(U_R \times U_R^+)$ for some $R > 0$, where $U_R = \{z \in \mathbb{C} \ : \ |z| < R\}$ and $U_R^+ = U_R \cap \mathfrak{H}$. Then $A \in \mathcal{A}_{1-s}(U_R^+)$. We apply Theorem 5.1 again, with $z_0 \in U_R^+$. By the uniqueness clause in Proposition (3.11) the function B appearing in (5.7) is the same as the given B in a neighborhood of $\{(z,z) \ : \ z \in U_R^+\} \cup \{(\bar{z},z) \ : \ z \in U_R^+\}$. Since $B(\,\cdot\,, z_0)$ is holomorphic in U_R for each $z_0 \in U_R^+$, the right hand side of (5.7) is holomorphic for all v, w with $|z_0+v|, |\bar{z}_0+w| < R$. (The denominator does not produce any poles since the numerator vanishes whenever the denominator does.) Hence the first statement of Theorem 5.1 shows that the series (5.5) represents $A(z)$ on the open disk $|z - z_0| < R - |z_0|$. For $|z_0| < \frac{1}{2}R$ this disc contains 0, so A is real-analytic at 0. ■

In Proposition 4.4 we showed that the Poisson transform of a hyperfunction represents an element of $\mathcal{W}^{\omega}_{1-s}$ outside the support of the hyperfunction. With Theorem 5.2 we arrive at the following more complete result.

Theorem 5.4. *Let $I \subset \partial\mathbb{H}$ be open, and let $\alpha \in \mathcal{V}_s^{-\omega}$. Then $\mathsf{P}_s\alpha$ represents an element of $\mathcal{W}^{\omega}_{1-s}(I)$ if and only if $I \cap \operatorname{Supp}(\alpha) = \emptyset$.*

Proof. Proposition 4.4 gives the implication $\Leftarrow$. For the other implication suppose that $\mathsf{P}_s\alpha$ represents an element of $\mathcal{W}^{\omega}_{1-s}(I)$. Let g_{can} be the canonical representative of α, defined in §3.1. Then $(g_{\mathrm{can}}, \mathsf{P}_s\alpha) \in \mathcal{D}_s((\mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}}) \times \mathfrak{H})$ by Theorem 3.8 and Definition 3.9. The implication a) $\Rightarrow$ b) in Theorem 5.2 gives the analyticity of $y^s g_{\mathrm{can}}$ on a neighborhood of (ξ,ξ) in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ for each $\xi \in I$. It follows that for $z_0 \in \mathfrak{H}$ sufficiently close to ξ the function $g_{\mathrm{can}}(\,\cdot\,, z_0)$ is holomorphic at ξ. It then follows from the definition of the mixed hybrid model in §3.1 that $g_{\mathrm{can}}(\,\cdot\,, z)$ is holomorphic at ξ for all $z \in \mathfrak{H}$. Thus ξ cannot be in $\operatorname{Supp}(\alpha)$. ■

Theorem 5.2 is a local statement. We end this subsection with a generalization of Proposition 3.14, which shows that the results of Theorem 5.2 have no global counterpart. For convenience we use the disk model.

Proposition 5.5. *Let $A \subset \mathbb{D}$ be an annulus of the form $r_1 < |w| < 1$ with $0 \leq r_1 < 1$, and let $V \subset \mathbb{P}^1_{\mathbb{C}}$ be a connected open set that intersects the region $r_1 < |w| < r_1^{-1}$. Then $\mathcal{D}_s(V \times A)$ does not contain non-zero sections of the form (h,u) where $u \in \mathcal{E}_s(A)$ represents an element of $\mathcal{W}^{\omega}_{1-s}$.*

Proof. The proof is similar to that of Proposition 3.14. Suppose that $(h,u) \in \mathcal{D}_s(V \times A)$ where u represents an element of $\mathcal{W}^{\omega}_{1-s}$. By (3.36c) the holomorphic function $\xi \mapsto \int_C [R^{\mathbb{S}}(\xi;\,\cdot\,)^s, u]$ is identically zero on some neighborhood $\rho < |\xi| < \rho^{-1}$ of the unit circle. We have the absolutely convergent representation $u = \sum_n b_n Q_{1-s,n}$ on A for a sequence (b_n) of complex numbers. Combining this with the expansion $R^{\mathbb{S}}(\xi;\,\cdot\,)^s = \sum_m \frac{(-\xi)^{-m}}{(1-s)_m} P_{1-s,m}$ and (2.18), we obtain

$$\sum_n b_n \frac{(-\xi)^n}{(1-s)_m} \ = \ 0$$

for all $\xi \in \mathbb{S}^1$. Hence all b_n vanish, so u and hence also h are zero. ■

Corollary 5.6. *If V is a neighborhood of $\partial\mathbb{D}$ in $\mathbb{P}^1_{\mathbb{C}}$, then $\mathcal{D}_s(V \times (V \cap \mathfrak{H})) = \{0\}$.*

Proof. Let $(h,u) \in \mathcal{D}_s(V\times(V\cap\mathfrak{H}))$. Corollary 5.3 implies that $u \in \mathcal{E}_s(B\cap\mathbb{D})$ represents an element of $\mathcal{W}_\omega^{1-s}$. The neighborhood V contains an annulus of the form $r_1 < |w| < r_1^{-1}$, and Proposition 5.5 shows that $(h,u) = (0,0)$. ■

5.4. **The extended sheaf of mixed eigenfunctions.** In §5.1 we defined an extension $\mathcal{D}_s^*$ of the sheaf of mixed eigenfunctions from $\mathbb{P}^1_\mathbb{C} \times \mathfrak{H}$ to $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$. We now prove an analogue of Theorem 3.13, the main result on the sheaf $\mathcal{D}_s$, for $\mathcal{D}_s^*$.

We denote by $\mathcal{O}$ the sheaf of holomorphic functions on $\mathbb{P}^1_\mathbb{C}$, by $p_j : \mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C} \to \mathbb{P}^1_\mathbb{C}$ the projection of the j-th factor ($j = 1,2$), and put $\Delta^\pm = \{(z,z)\}_{z\in\mathbb{P}^1_\mathbb{C}} \cup \{(z,\bar z)\}_{z\in\mathbb{C}}$. We define $\mathcal{K}_s^*$ to be the subsheaf of $\mathcal{D}_s^*$ whose sections have the form $(B,0)$.

Theorem 5.7. *The sheaf $\mathcal{K}_s^*$ is the kernel of the surjective sheaf morphism $C : \mathcal{D}_s^* \to p_2^{-1}\mathcal{A}_s$ that sends $(B,A) \in \mathcal{D}_s^*(U)$ ($U \subset \mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ open) to A. The restriction of $\mathcal{K}_s^*$ to $\Delta^\pm$ vanishes, and its restriction to $\left(\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}\right) \smallsetminus \Delta^\pm$ is locally isomorphic to $p_1^{-1}\mathcal{O}$.*

This theorem gives us the exact sequence

$$0 \longrightarrow \mathcal{K}_s^* \longrightarrow \mathcal{D}_s^* \xrightarrow{C} p_2^{-1}\mathcal{A}_{1-s} \longrightarrow 0$$

generalizing the exact sequence in Theorem 3.13.

Proof. By G-equivariance we can work on open $U \subset \mathbb{C}\times\mathbb{C}$. The differential equations (5.2) imply that sections $(B,0)$ of $\mathcal{D}_s^*$ on U have the form $B(\zeta,z) = \varphi(\zeta)\,(\zeta - z)^s$ $(\zeta-\bar z)^s$ for some function φ. The analyticity of B implies that $\varphi = 0$ near points of $\Delta^\pm$, and the holomorphy of B in its first variable implies that φ is holomorphic. Thus $\mathcal{K}_s^*$ is locally isomorphic to $\partial_1^{-1}\mathcal{O}$ outside Δ^* and its stalks at points of Δ^* vanish.

Let (h,u) be a section of $\mathcal{D}_s^*$ over some open $U \subset \mathbb{C}\times\mathbb{C}$. Denote by $D_{(a)}$ and $D_{(b)}$ the expressions in the left hand sides of (5.2). A computation shows that

$$\big((\zeta-\bar z)\partial_{\bar z} + s\big)\,D_{(a)} - \big((\zeta - z)\partial_z + s\big)\,D_{(b)}$$

is $\frac{1}{2i}\,(\zeta-z)\,(\zeta-\bar z)$ times $(z-\bar z)A_{z\bar z} - (1-s)\,A_z + (1-s)\,A_{\bar z}$. The vanishing of the latter is the differential equation defining $\mathcal{A}_{1-s}$. So A is a section of $\mathcal{A}_{1-s}$ on $p_2U \smallsetminus \Delta^*$. By analyticity it is in $\mathcal{A}_{1-s}(p_2U)$. Hence $C : (B,A) \mapsto A$ determines a sheaf morphism between the restrictions of $\mathcal{D}_s^*$ and $p_2^{-1}\mathcal{A}_{1-s}$ on $\mathbb{C}\times\mathbb{C}$, and by G-equivariance on $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$.

To prove the surjectivity of C we have construct for each $(\zeta_0, z_0) \in \mathbb{C}\times\mathbb{C}$ and each $A \in \mathcal{A}_s(U)$ for some neighborhood U of z_0 a section (B,A) of $\mathcal{D}_s^*$ on a possibly smaller neighborhood of (ζ_0,z_0). This suffices by G-equivariance.

For $(\zeta_0,z_0) \in \Delta^*$ this construction is carried out in (5.4). Let $(\zeta_0,z_0) \notin \Delta^*$. The integral in (3.41) suggests that we should consider the differential form

$$\begin{aligned}\omega &= y^s\,\big[(R(\zeta;z_1)/R(\zeta;z))^s, y_1^{1-s}\,A(z_1)\big]_{z_1}\\ &= \Big(\frac{(\zeta-z)(\zeta-\bar z)}{(\zeta-z_1)(\zeta-\bar z_1)}\Big)^s \Big(\frac{s(\zeta-\bar z_1)}{2i(\zeta-z_1)}\,A(z_1)\,dz_1 + \big(\frac{i}{2}(1-s)A(z_1) + y_1\,A_{\bar z}(z_1)\big)\,d\bar z_1\Big).\end{aligned}$$

Choosing continuous branches of $\big(\frac{(\zeta-z)(\zeta-\bar z)}{(\zeta-z_1)(\zeta-\bar z_1)}\big)^s$ near (ζ_0,z_0) we obtain $B(\zeta,z) = \int_{z_0}^z \omega$ such that (B,A) satisfies (5.2) near (ζ_0,z_0), which can be checked by a direct computation, and follows from the proof of Theorem 3.13 if $z_0 \in \mathfrak{H}$. ■

Remark. We have define $\mathcal{D}_s^*$ in such a way that the restriction of $\mathcal{D}_s^*$ to $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$ is isomorphic to $\mathcal{D}_s$. Let $c : (\zeta, z) \mapsto (\zeta, \bar{z})$. An isomorphism $\mathcal{D}_s^* \to c^{-1}\mathcal{D}_s^*$ is obtained by $\tilde{B}(\zeta, z) = B(\zeta, \bar{z}) + yA(\bar{z})$, $\tilde{A}(z) = A(\bar{z})$. So the restriction of $\mathcal{D}_s^*$ to $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}^-$ is isomorphic to $c^{-1}\mathcal{D}_s$. New in the theorem is the description of $\mathcal{D}_s^*$ along $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{R}}$. In points (ξ, ξ) with $\xi \in \mathbb{P}^1_{\mathbb{R}}$ the surjectivity of C is the step a) $\Rightarrow$ b) in Theorem 5.2.

5.5. **Boundary germs for the sheaf $\mathcal{D}_s$.** In Subsection 5.3 we considered sections of $\mathcal{D}_s$ that extend across $\partial\mathbb{H}$ and established a local relation between these sections and the sheaf $\mathcal{W}^\omega_{1-s}$. In this subsection we look instead at the sections of $\mathcal{D}_s$ along the inverse image $p_1^{-1}\mathbb{P}^1_{\mathbb{R}}$, where $p_1 : \mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H} \to \mathbb{P}^1_{\mathbb{C}}$ is the projection on the first component. The proofs will be omitted or sketched briefly.

A first natural thought would be to consider the inductive limit $\varinjlim \mathcal{D}_s(U \cap (\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}))$, where U runs through the collection of all neighborhoods of $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, but Corollary 5.6 shows that this space is zero. Instead, we define

$$\mathsf{d}_s \;=\; \varinjlim \mathcal{D}_s(U \smallsetminus (\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}))\,, \qquad \mathsf{h}_s \;=\; \varinjlim \mathcal{D}_s(U)\,, \tag{5.16}$$

where the open sets U run over either

(a) the collection of open neighborhoods of $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}$ in $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$, or
(b) the larger collection of open neighborhoods of $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}'$ in $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}'$ with $\mathfrak{H}'$ the complement of some compact subset of $\mathfrak{H}$.

It turns out that the direct limits in (5.16) are the same for both choices. Clearly d_s contains h_s and the group G acts on both spaces. The canonical model C_s is a subspace of the space d_s.

In Theorem 3.13 we considered the sheaf morphism $C : \mathcal{D}_s \to p_2^{-1}\mathcal{E}_s$ that sends a pair (h, u) to its second coordinate $u = u(\zeta, z)$, which is locally constant in ζ and a λ_s-eigenfunction in z. This morphism induces a surjective map $C : \mathsf{h}_s \to \mathcal{E}_s$ whose kernel is the space $\mathcal{V}_s^{\omega,\mathrm{rig}}$ introduced in §3.1. It also induces a map (still called C) from the larger space d_s to $\mathcal{E}_s \oplus \mathcal{E}_s$ by sending (h, u) to the pair (u_+, u_-), where $u_\pm(\,\cdot\,) = u(\zeta_\pm, \,\cdot\,)$ for any $\zeta_\pm \in \mathfrak{H}^\pm$. This map is again surjective and its kernel is the space $\mathbf{H}_s^{\mathrm{rig}}$ studied in §3.1. Moreover, the results of that subsection show that the kernels of these two maps C are related by the exact sequence

$$0 \longrightarrow \mathcal{V}_s^{\omega,\mathrm{rig}} \longrightarrow \mathbf{H}_s^{\mathrm{rig}} \xrightarrow{\mathsf{P}_s} \mathcal{E}_s \longrightarrow 0\,,$$

where the Poisson map P_s is given explicitly by

$$\mathsf{P}_s h(z, z_1) \;=\; \frac{1}{\pi}\Bigl(\int_{C_+} - \int_{C_-}\Bigr)\Bigl(\frac{R(\zeta; z_1)}{R(\zeta; z)}\Bigr)^s h(\zeta, z_1)\, R(\zeta; z)\, d\zeta \qquad (z,\, z_1 \in \mathfrak{H}),$$

(eq. (3.15)). Here C_+ (resp. C_-) is a closed path in $\mathfrak{H}$ (resp. $\mathfrak{H}^-$) encircling z and z_1 (resp. $\bar{z}$ and $\bar{z}_1$) and the right-hand side is independent of z_1. Now consider an element of h_s represented by the pair $(h, u) \in \mathcal{D}_s(U \smallsetminus (\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}))$ for some open neighborhood U of $\mathbb{P}^1_{\mathbb{R}} \times \mathfrak{H}$ in $\mathbb{P}^1_{\mathbb{C}} \times \mathfrak{H}$, and define $\mathsf{P}_s h(z, z_1)$ by the same formula, where $C_\pm$ are now required to lie in the neighborhood $\{\zeta \in \mathbb{P}^1_{\mathbb{C}} \mid (\zeta, z_1) \in U\}$ of $\mathbb{P}^1_{\mathbb{R}}$ and to be homotopic to $\mathbb{P}^1_{\mathbb{R}}$ in this neighborhood. The right-hand side is still independent of the choice of contours $C_\pm$, and is also independent of the choice of representative (h, u) of $[(h, u)] \in \mathsf{d}_s$, but it is no longer independent of z_1. Instead, we have that the function $\mathsf{P}_s h(\,\cdot\,, z_1)$ belongs to $\mathcal{E}_s$ for each fixed $z_1 \in \mathfrak{H}$ and that its dependence on z_1 is governed by

$$d_{z_1}\bigl(\mathsf{P}_s(h,u)(z,z_1)\bigr) \;=\; \bigl[p_s(\,\cdot\,,z),\; u_+ - u_-\bigr] \tag{5.17}$$

with the Green's form as in (2.13) and the point pair invariant $p_s(\,\cdot\,,\,\cdot\,)$ as in (2.6).

We therefore define a space $\mathcal{E}_s^+$ consisting of pairs (f,v) where v belongs to $\mathcal{E}_s$ and $f : \mathfrak{H} \times \mathfrak{H} \to \mathbb{C}$ satisfies

$$f(\,\cdot\,,z_1) \;\in \mathcal{E}_s \text{ for each } z_1 \in \mathfrak{H}\,, \tag{5.18a}$$

$$d\bigl(f(z,\,\cdot\,)\bigr) \;=\; [p_s(\,\cdot\,,z), v] \text{ on } \mathfrak{H} \text{ for each } z \in \mathfrak{H}\,. \tag{5.18b}$$

The group G acts on this space by composition (diagonally in the case of f). By the discussion above we can define an equivariant and surjective map $\mathsf{P}_s^+ : \mathsf{h}_s \to \mathcal{E}_s^+$ with kernel h_s by $[(h,u)] \mapsto (\mathsf{P}_s h, u_+ - u_-)$. Finally, the space $\mathcal{E}_s^+$ is mapped to $\mathcal{E}_s$ by $(f,v) \mapsto v$ with kernel $\mathcal{E}_s$ (because $f(\,\cdot\,,z_1)$ is constant if $v = 0$). (In fact, the space $\mathcal{E}_s^+$ is isomorphic to $\mathcal{E}_s \times \mathcal{E}_s$ as a vector space, though not as a G-module, by the map sending (f,v) to $(f(\,\cdot\,,i),v)$.) Putting all these maps together, we can summarize the interaction of the morphisms C and P_s by the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{V}_s^{\omega,\mathrm{rig}} \arrow[r] \arrow[d] & \mathsf{h}_s \arrow[r, "C"] \arrow[d] & \mathcal{E}_s \arrow[r] \arrow[d, "\mathrm{diag}"] & 0 \\
0 \arrow[r] & \mathbf{H}_s^{\mathrm{rig}} \arrow[r] \arrow[d, "\mathsf{P}_s"] & \mathsf{d}_s \arrow[r, "C"] \arrow[d, "\mathsf{P}_s^+"] & \mathcal{E}_s \oplus \mathcal{E}_s \arrow[r] \arrow[d, "(1,-1)"] & 0 \\
0 \arrow[r] & \mathcal{E}_s \arrow[r] \arrow[d] & \mathcal{E}_s^+ \arrow[r, "\mathrm{pr}_2"] \arrow[d] & \mathcal{E}_s \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

6. Boundary splitting of eigenfunctions

In the introduction we mentioned that eigenfunctions often have the local form $y^s \times (\text{analytic}) + y^{1-s} \times (\text{analytic})$ near points of $\mathbb{R}$. Here we consider this phenomenon more systematically in both the analytic context (§6.1) and the differentiable context (§6.2). This will lead in particular to a description of both $\mathcal{E}_s^\omega = \mathsf{P}_s(\mathcal{V}_s^\omega)$ and $\mathcal{E}_s^\infty = \mathsf{P}_s(\mathcal{V}_s^\infty)$ in terms of boundary behavior.

As stated in the introduction, results concerning the boundary behavior of elements of $\mathcal{E}_s$ are known (also for more general groups; see, eg., [7] and [1]). However, our approach is more elementary and also includes several formulas that do not seem to be in the literature and that are useful for certain applications (such as those in [2].

6.1. **Analytic case.** In Proposition 4.2 we showed that the space $\mathcal{F}_s$ of boundary germs is the direct sum of $\mathcal{E}_s$ (the functions that extend to the interior) and $\mathcal{W}_s^\omega$ (the functions that extend across the boundary). We now look at the relation of these spaces with $\mathcal{E}_s^\omega$, the image in $\mathcal{E}_s$ of $\mathcal{V}_s^\omega$ under the Poisson transformation.

If $s \neq \frac{1}{2}$ we denote by $\mathcal{F}_s^\omega$ the direct sum of $\mathcal{W}_s^\omega$ and $\mathcal{W}_{1-s}^\omega$. (That this sum is direct is obvious since for $s \neq \frac{1}{2}$ an eigenfunction u cannot have the behavior $y^s \times$ (analytic) and at the same time $y^{1-s} \times$ (analytic) near points of $\mathbb{R}$.) For $s = \frac{1}{2}$ we will define $\mathcal{F}_{1/2}^\omega$ as a suitable limit of these spaces in the following sense. If $s \neq \frac{1}{2}$, an element of $\mathcal{F}_s^\omega$ is locally (near $x_0 \in \mathbb{R}$) represented by a linear combination of y^s and y^{1-s} with coefficients that are analytic in a neighborhood of x_0. Replacing y^s and y^{1-s} by $\frac{1}{2}\left(y^s + y^{1-s}\right)$ and $\frac{1}{2s-1}\left(y^s - y^{1-s}\right)$, we see that an element of $\mathcal{F}_{1/2}^\omega$ should (locally) have the form $A(z)\,y^{1/2}\,\log y + B(z)\,y^{1/2}$ with A and B analytic at x_0. We therefore define $\mathcal{F}_{1/2}^\omega$ (now using disk model coordinates to avoid special explanations at ∞) as the space of germs in $\mathcal{F}_{1/2}$ represented by

$$f(w) \;=\; (1-|w|^2)^{1/2}\left(A(w)\log(1-|w|^2) + B(w)\right), \tag{6.1}$$

with A and B real analytic on a neighborhood of $\mathbb{S}^1$ in $\mathbb{C}$. We have a G-equivariant exact sequence

$$0 \longrightarrow \mathcal{W}_{1/2}^\omega \longrightarrow \mathcal{F}_{1/2}^\omega \xrightarrow{\tau} \mathcal{W}_{1/2}^\omega \longrightarrow 0 \tag{6.2}$$

where τ sends f in (6.1) to A. The surjectivity of τ is a consequence of the following proposition, which we will prove below. This proposition shows that for all s with $0 < \operatorname{Re} s < 1$ the space $\mathcal{F}_s^\omega$ is isomorphic as a G-module to the sum of two copies of $\mathcal{V}_s^\omega$.

Proposition 6.1. *The exact sequence* (6.2) *splits G-equivariantly.*

The splittings $\mathcal{F}_s = \mathcal{E}_s \oplus \mathcal{W}_s^\omega = \mathcal{E}_s \oplus \mathcal{W}_{1-s}^\omega$ show that non-zero elements of $\mathcal{E}_s$ cannot belong to $\mathcal{W}_s^\omega$ or $\mathcal{W}_{1-s}^\omega$. The following theorem shows that they *can* be in $\mathcal{F}_s^\omega$, and that this happens if and only if they belong to $\mathcal{E}_s^\omega$.

Theorem 6.2. *Let* $0 < \operatorname{Re} s < 1$. *Then*

$$\mathcal{E}_s^\omega = \mathcal{E}_s \cap \mathcal{F}_s^\omega\,,$$

and $\mathcal{F}_s^\omega \;=\; \mathcal{E}_s^\omega \oplus \mathcal{W}_s^\omega = \mathcal{E}_s^\omega \oplus \mathcal{W}_{1-s}^\omega$.

So for $s \neq \frac{1}{2}$, the space $\mathcal{F}_s^\omega$ is the direct some of each two of the three isomorphic subspaces $\mathcal{E}_s^\omega$, $\mathcal{W}_s^\omega$ and $\mathcal{W}_{1-s}^\omega$. For $s = \frac{1}{2}$ two of these subspaces coincide.

We discuss the cases $s \neq \frac{1}{2}$ and $s = \frac{1}{2}$ separately.

Proposition 6.3. *Let* $s \neq \frac{1}{2}$. *For each* $\varphi \in \mathcal{V}_s^\omega$ *we have*

$$\mathsf{P}_s\varphi \;=\; c(s)\,\mathsf{P}_s^\dagger\varphi + c(1-s)\,\mathsf{P}_{1-s}^\dagger I_s\varphi\,, \tag{6.3}$$

where, with $b(s)$ *as in* (4.20)*, the factor* $c(s)$ *is given by*

$$c(s) \;=\; \frac{\tan\pi s}{\pi}\,b(s) \;=\; \frac{1}{\sqrt{\pi}}\,\frac{\Gamma(\frac{1}{2}-s)}{\Gamma(1-s)}\,. \tag{6.4}$$

Proof. Since φ is given by a Fourier expansion which converges absolutely uniformly on the paths of integration in the transformation occurring in (6.3), it is sufficient to prove this relation in the spacial case $\mathbf{e}_{s,n}$ ($n \in \mathbb{Z}$). We have $\mathsf{P}_s \mathbf{e}_{s,n} = (-1)^n \frac{\Gamma(s)}{\Gamma(s+n)} P_{s,n}$ and $\mathsf{P}_s^\dagger \mathbf{e}_{s,n} = (-1)^n \frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(s+n)} Q_{s,n}$. See §A.2. The relations (A.14) and (1.30e) give the lemma for $\varphi = \mathbf{e}_{s,n}$ for all $n \in \mathbb{Z}$. ■

Remark. One can also give a direct (but more complicated) proof of (6.3) for arbitrary $\varphi \in \mathcal{V}_s^\omega$, without using the basis $\{\mathbf{e}_{s,n}\}$, by writing all integral transforms explicitly and moving the contours suitably.

The proof of Theorem 6.2 (for $s \neq \frac{1}{2}$) follows from Proposition 6.3. The inclusion $\mathcal{E}_s^\omega \subset \mathcal{F}_s^\omega$ is a consequence of the more precise formula (6.3). For the reverse inclusion we write an arbitrary $u \in \mathcal{F}_s^\omega$ in the form $c(s)\,\mathsf{P}_s^\dagger \varphi + v$ with $v \in \mathcal{W}_{1-s}^\omega$ and $\varphi \in \mathcal{V}_s^\omega$. If $u \in \mathcal{E}_s$, then $u - \mathsf{P}_s \varphi = v - c(1-s)\mathsf{P}_{s-1}^\dagger I_s \varphi \in \mathcal{E}_s \cap \mathcal{W}_{1-s}^\omega = \{0\}$, so $u = \mathsf{P}_s \varphi \in \mathcal{E}_s^\omega$. This completes the proof.

We can summarize this discussion and its relation with the Poisson transformation in the following commutative diagram of G-modules and canonical G-equivariant morphisms

$$\begin{tikzcd}
 & \mathcal{V}_s^\omega \arrow[rr, shift left, "I_s"] \arrow[dl, bend right, "\mathsf{P}_s^\dagger"'] \arrow[dr, "\mathsf{P}_s"] & & \mathcal{V}_{1-s}^\omega \arrow[ll, shift left, "I_{1-s}"] \arrow[dl, "\mathsf{P}_{1-s}"'] \arrow[dr, bend left, "\mathsf{P}_{1-s}^\dagger"] & \\
\mathcal{W}_s^\omega \arrow[ur, "\rho_s"'] \arrow[rr, dash, "\cong"] & & \mathcal{E}_s^\omega = \mathcal{E}_{1-s}^\omega \arrow[rr, dash, "\cong"] & & \mathcal{W}_{1-s}^\omega \arrow[ul, "\rho_{1-s}"]
\end{tikzcd}$$

together with the fundamental examples (and essential ingredient in the proof):

$$\begin{tikzcd}
 & R(\,\cdot\,; z_0)^s \arrow[rr, leftrightarrow] \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & & R(\,\cdot\,; z_0)^{1-s} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
b(s) q_s(\,\cdot\,, z_0) \arrow[r, leftrightarrow] & {} & \begin{array}{c} p_s(\,\cdot\,, z_0) = p_{1-s}(\,\cdot\,, z_0) = \\ \pi^{-1} \tan \pi s\,(q_s(\,\cdot\,, z_0) + q_{1-s}(\,\cdot\,, z_0)) \end{array} \arrow[r, leftrightarrow] & {} & b(1-s) q_{1-s}(\,\cdot\,, z_0)
\end{tikzcd}$$

We now turn to the case $s = \frac{1}{2}$. We have to prove Proposition 6.1 and Theorem 6.2 in this case.

To construct a splitting $\sigma : \mathcal{W}_{1/2}^\omega \to \mathcal{F}_{1/2}^\omega$ of the exact sequence (6.2) we put $\sigma Q_{1/2,n} = -\frac{\pi^2}{2} P_{1/2,n} \in \mathcal{E}_s$ for $n \in \mathbb{Z}$. Since $P_{1/2,n} \in \mathsf{P}_{1/2} \mathcal{V}_{1/2}^\omega$, we have $\sigma Q_{n,1/2} \in \mathcal{E}_s^\omega$. Further, $\tau\sigma Q_{1/2,n} = Q_{1/2,n}$ by (A.13) and (A.15). The $Q_{1/2,n} \in \mathcal{W}_{1/2}^\omega$ with $n \in \mathbb{Z}$ generate a dense linear subspace of $\mathcal{W}_{1/2}^\omega$ for the topology of $\mathcal{V}_{1/2}^\omega$ transported to $\mathcal{W}_{1/2}^\omega$ by $\mathsf{P}_{1/2}^\dagger : \mathcal{V}_{1/2}^\omega \to \mathcal{W}_{1/2}^\omega$. Hence there is a continuous linear extension $\sigma : \mathcal{W}_{1/2}^\omega \to \mathcal{E}_{1/2}^\omega$. The generators $\mathbf{E}^+$ and $\mathbf{E}^-$ of the Lie algebra of G act in the same way on the system $(Q_{s,n})_n$ as on on the system $(P_{s,n})_n$. (See §A.5, and use case **G** in Table 1 of §A.2 and case **a** in Table 2 of §A.3.) So σ is an infinitesimal G-morphism, and, since G is connected, a G-morphism. The splitting $\sigma : \mathcal{W}^\omega_{1/2} \to \mathcal{F}^\omega_{1/2}$ also gives the surjectivity of τ, and hence the exactness of the sequence (6.2).

Since $\sigma Q_{1/2,n}$ belongs to $\mathcal{E}^\omega_{1/2}$, we have $\sigma(\mathcal{W}^\omega_{1/2}) \subset \mathcal{E}^\omega_{1/2}$. Since $\mathcal{E}^\omega_{1/2}$ is an irreducible G-module this inclusion is an equality. This gives $\mathcal{F}^\omega_{1/2} = \mathcal{E}^\omega_{1/2} \oplus \mathcal{W}^\omega_{1/2}$ and $\mathcal{E}^\omega_{1/2} \subset \mathcal{E}_{1/2} \cap \mathcal{F}^\omega_{1/2}$. The reverse inclusion the follows by the same argument as for $s \neq \frac{1}{2}$. ■

Remark. The case $s = \frac{1}{2}$ could also have been done with explicit elements. For each s with $0 < \mathrm{Re}\, s < 1$ and each $n \in \mathbb{Z}$, the subspace $\mathcal{F}^\omega_{s,n}$ of $\mathcal{F}^\omega_s$ in which the elements $\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right]$ act as multiplication by $e^{2in\theta}$ has dimension 2. In the family $s \mapsto \mathcal{F}^\omega_{s,n}$ there are three families of non-zero eigenfunctions $s \mapsto P_{s,n} \in \mathcal{E}^\omega_s$, $s \mapsto Q_{s,n} \in \mathcal{W}^\omega_s$ and $s \mapsto Q_{1-s,n} \in \mathcal{W}^\omega_{1-s}$. For $s \neq \frac{1}{2}$ each of these functions can be expressed as a linear combination of the other two, as given by (A.14), which is at the basis of our proof of Proposition 6.3. At $s = \frac{1}{2}$, the elements $Q_{s,n}$ and $Q_{1-s,n}$ coincide. This is reflected in the singularities at $s = \frac{1}{2}$ in the relation (A.14). The families $s \mapsto P_{s,n}$ and $s \mapsto Q_{s,n}$ provide a basis of $\mathcal{F}^\omega_s$ for all s, corresponding to the decomposition $\mathcal{F}^\omega_s = \mathcal{E}^\omega_s \oplus \mathcal{W}^\omega_s$. Relation (A.14) implies

$$P_{1/2,n} = \frac{-2}{\pi^2} \frac{d}{ds} Q_{s,n}\Big|_{s=1/2},$$

which explains the logarithmic behavior at the boundary.

6.2. **Differentiable case.** In the previous subsection we described the boundary behavior of elements of $\mathcal{E}^\omega_s = \mathsf{P}_s \mathcal{V}^\omega_s$ in terms of convergent expansions. In the differentiable case the spaces $\mathcal{W}^p_s$ consist of boundary jets, not of boundary germs. So a statement like that in Theorem 6.2 seems impossible. Nevertheless, we have the following generalization of Proposition 6.3:

Proposition 6.4. i) *Let $p \in \mathbb{N}$, $p \geq 2$, and $s \neq \frac{1}{2}$. For each $\varphi \in \mathcal{V}^p_s$ there are $b \in \mathcal{G}^p_s$ representing $c(s)\, \mathsf{P}^\dagger_s \varphi \in \mathcal{W}^p_s$ and $a \in \mathcal{G}^{p-1}_{1-s}$ representing $c(1-s)\, \mathsf{P}^\dagger_{1-s} I_s \varphi \in \mathcal{W}^{p-1}_{1-s}$ such that*

$$\mathsf{P}_s\varphi(w) = b(w) + a(w) + \mathrm{O}((1-|w|^2)^{p-s}) \qquad (|w| \uparrow 1)\,. \tag{6.5}$$

ii) *Let $s \neq \frac{1}{2}$. For each $\varphi \in \mathcal{V}^\infty_s$ there are $b \in \mathcal{G}^\infty_s$ and $a \in \mathcal{G}^\infty_{1-s}$ representing $c(s)\, \mathsf{P}^\dagger_s \varphi$ and $c(1-s)\, \mathsf{P}^\dagger_{1-s} \varphi$, respectively, such that for each $N \in \mathbb{N}$*

$$\mathsf{P}_s\varphi(w) \sim b(w) + a(w) + \mathrm{o}((1-|w|^2)^N) \qquad (|w| \uparrow 1)\,. \tag{6.6}$$

Proof. The proof of Proposition 6.3 used the fact that the $\mathbf{e}_{s,n}$ generate a dense subspace of $\mathcal{V}^\omega_s$ and that the values of Poisson transforms and transverse Poisson transforms are continuous with respect to this topology. That reasoning seems hard to generalize when we work with boundary jets. Instead, we use the explicit Lemma 4.12.

A given $\varphi \in \mathcal{V}^p_s$ can be written as a sum of elements in $\mathcal{V}^p_s$ each with support in a small interval in $\partial\mathbb{H}$. With the G-action this reduces the situation to be considered to $\varphi \in C^p_c(I)$ where I is a finite interval in $\mathbb{R}$. Proposition 4.4 shows that $\mathsf{P}_s\varphi$ represents an element of $\mathcal{W}^\omega_{1-s}(\mathbb{P}^1_\mathbb{R} \smallsetminus I)$. So we can restrict our attention to $\mathsf{P}_s\varphi(z)$ with z near I, and work in the line model.

We take α and β as in Lemma 4.12 with $N = p$. Then

$$\begin{aligned} \mathsf{P}_s\varphi(z) &= \pi^{-1}\, y^{1-s} \int_{-\infty}^{\infty} (t^2+y^2)^{s-1}\, \varphi(t+x)\, dt \\ &= \frac{1}{\pi}\, y^s \int_{-\infty}^{\infty} (t^2+1)^{s-1}\, \varphi(x+yt)\, dt \;=\; y^s A(z) + y^{1-s} B(z)\,, \end{aligned}$$

with

$$B(z) \;=\; \pi^{-1} \int_{-\infty}^{\infty} \beta(t)\, \varphi(x+yt)\, dt\,, \qquad A(z) \;=\; \pi^{-1} y^{2s-1} \int_{-\infty}^{\infty} \alpha^{(p)}(t)\, \varphi(x+yt)\, dt\,.$$

We consider $B(z)$ and $A(z)$ for $x \in I$ and $0 < y \le 1$. The decay of β implies that

$$B(z) \;=\; \frac{1}{\pi} \sum_{n=0}^{p} \frac{\varphi^{(n)}(x)}{n!} y^n \int_{-\infty}^{\infty} t^n\, \beta(t)\, dt \;+\; \mathrm{o}(y^p)\,.$$

In (4.40) we have computed the integrals. We arrive at

$$B(z) \;=\; c(s) \sum_{k=0}^{[p/2]} \frac{(-1/4)^k \Gamma(s+\frac12)}{k!\, \Gamma(k+s+\frac12)}\, \varphi^{(2k)}(x)\, y^{2k} + \mathrm{o}(y^p)\,. \tag{6.7}$$

A comparison with (4.33) shows that $y^s\, B(z)$ has the asymptotic behavior near I of representatives of $c(s)\, \mathsf{P}_s^\dagger \varphi$.

In the second term we apply p-fold integration by parts.

$$A(z) \;=\; (-1)^p \pi^{-1} y^{2s-1+p} \int_{-\infty}^{\infty} \alpha(t)\, \varphi^{(p)}(x+yt)\, dt\,.$$

For fixed φ, this expression is a holomorphic function of s on the region $\mathrm{Re}\, s > 0$. In the computation we shall work with $\mathrm{Re}\, s$ large.

The function $h \mapsto (1+h)^{s-1}$ has a Taylor expansion at $h = 0$ of any order R, with a remainder term $\mathrm{O}(h^{R+1})$ that is uniform for $h \ge 0$. This implies that $\alpha(t)$ has an expansion of the form $\alpha(t) = \sum_{n=0}^{R} b_n |t|^{p+2s-2-2n} + \mathrm{O}(|t|^{p+2s-2R-4})$, uniformly for $t \in \mathbb{R} \smallsetminus \{0\}$. We take $R = [p/2]$, and use the relation $\partial_t^p \alpha(t) = (1+t^2)^{s-1} - \beta(t)$ and the decay of $\beta(t)$ to conclude that

$$\alpha(t) \;=\; \sum_{0 \le n \le p/2} \binom{s-1}{n} \frac{(\mathrm{sign}\, t)^p |t|^{2s-2n+p-2}}{(2s-2n-1)_p} + \mathrm{O}(|t|^{2s-3})\,. \tag{6.8}$$

We compute this with $\mathrm{Re}\, s > 1$. The error term contributes to $A(z)$:

$$y^{2s-1+p} \int_{t=-\infty}^{\infty} \mathrm{O}(|t|^{2s-3})\, \varphi^{(p)}(x+yt)\, dt \;=\; \mathrm{O}(y^{p+1})\,. \tag{6.9}$$

(We have replaced t by t/y in the integral.) The term of order n contributes:

$$\begin{aligned} &\frac{(-1)^p y^{2n}}{\pi} \binom{s-1}{n} \frac{y^{-2s-p+2n+1}}{(2s-2n-1)_p} \int_{-\infty}^{\infty} (\mathrm{sign}\, t)^p |t|^{2s+p-2n-2}\, \varphi^{(p)}(x+t)\, dt \\ &= \frac{(-1)^p y^{2n}\, \Gamma(s)\, \Gamma(2s-2n-1)}{\pi\, n!\, \Gamma(s-n)\, \Gamma(2s-2n-1+p)}\, (-1)^{p-2n} (2s-1)_{p-2n} \\ &\quad \cdot \int_{-\infty}^{\infty} |t|^{2s-2} \varphi^{(2n)}(x+t)\, dt \qquad \text{(partial integration } p-2n \text{ times)}\,. \end{aligned}$$

In (1.30b) we see that the holomorphic function $\int_{-\infty}^{\infty}|t|^{2s-2}\varphi^{(2n)}(x+t)\,dt$ continued to the original value of s gives us $b(s-\frac{1}{2})\,(I_s\varphi)^{(2n)}(x)$, provided $2n<p$. We have $I_s\mathcal{V}_s^p\subset\mathcal{V}_{1-s}^{p-1}$, but not necessarily $I_s\varphi\in\mathcal{V}_s^p$. For even p we move the contribution $\mathrm{O}(y^p)$ to the error term. The terms of order $n<p/2$ give

$$\begin{aligned}&\frac{y^{2n}\,\Gamma(s)\,\Gamma(2s-2n-1)}{\pi\,n!\Gamma(s-n)\,\Gamma(2s-1)}\;\frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,(I_s\varphi)^{(2n)}(x)\\ &\quad=\;\frac{\tan\pi(1-s)}{\sqrt{\pi}}\;\frac{\Gamma(1-s)}{\Gamma(\frac{3}{2}-s)}\;\frac{(-1/4)^k\,\Gamma(\frac{3}{2}-s)}{n!\;\Gamma(\frac{3}{2}-s+n)}\,(I_s\varphi)^{(2n)}(x)\,y^{2k}\,.\end{aligned}$$

Thus we arrive at

$$A(z)\;=\;c(1-s)\sum_{0\le n<p/2}\frac{(-1/4)^n\,\Gamma(\frac{3}{2}-s)}{n!\;\Gamma(\frac{3}{2}-s+n)}\,(I_s\varphi)^{(2n)}(x)y^{2k}+\mathrm{O}(y^{2\left[\frac{p+1}{2}\right]})\,. \tag{6.10}$$

Again we have arrived at the expansion a representative of $\mathcal{W}_{1-s}^{p-1}$ should have according to (4.33). This completes the proof of part i).

In view of Definition 4.9 the estimate (6.5) holds for all representatives $b\in\mathcal{G}_s^p$ and $a\in\mathcal{G}_{1-s}^p$ of $c(s)\,\mathsf{P}_s^\dagger\varphi$, respectively $c(1-s)\,\mathsf{P}_{1-s}^\dagger\varphi$. In particular, for $\varphi\in\mathcal{V}_s^\infty$, this estimate holds for each $p\in\mathbb{N}$, $p\ge 2$, for representatives $b_\infty\in\mathcal{G}_s^\infty$ of $c(s)\,\mathsf{P}_s^\dagger\varphi$ and $a_\infty\in\mathcal{G}_{1-s}^\infty$ of $c(1-s)\,\mathsf{P}_{1-s}^\dagger\varphi$. This implies part ii) of the proposition. ∎

Appendix: Examples and explicit formulas

We end by giving a collection of definitions and formulas that were needed in the main body of the paper or that illustrate its results. In particular, we describe a number of examples of eigenfunctions of the Laplace operator (in A.1), of Poisson transforms (in A.2), of transverse Poisson transforms (in A.3), and of explicit potentials of the Green's form $\{u,v\}$ for various special choices of u and v (in A.4), as well as some formulas for the action of the Lie algebra of G (in A.5).

A.1. **Special functions and equivariant elements of $\mathcal{E}_s$.** Let $H\subset G$ be one of the subgroups $N=\left\{n(x)\;:\;x\in\mathbb{R}\right\}$, $A=\left\{a(y)\;:\;y>0\right\}$ or $K=\{k(\theta)\;:\;\theta\in\mathbb{R}/\mathbb{Z}\}$ with

$$n(x)\;=\;\begin{bmatrix}1&x\\0&1\end{bmatrix},\qquad a(y)\;=\;\begin{bmatrix}\sqrt{y}&0\\0&1/\sqrt{y}\end{bmatrix},\qquad k(\theta)\;=\;\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}. \tag{A.1}$$

For each character χ of H we determine the at most two-dimensional subspace $\mathcal{E}_{s,\chi}^H$ of $\mathcal{E}_s$ transforming according to this character.

A.1.1. *Equivariant eigenfunctions for the unipotent group N.* The characters of N are $\chi_\alpha\;:\;n(x)\mapsto e^{i\alpha x}$ with $\alpha\in\mathbb{R}$. If $u\in\mathcal{E}_{s,\alpha}^N$ (we write $\mathcal{E}_{s,\alpha}^N$ instead of $\mathcal{E}_{s,\chi_\alpha}^N$), then $u(z)=e^{i\alpha x}f(y)$, where f satisfies the differential equation

$$y^2f''(y)\;=\;(s^2-s+\alpha^2y^2)f(y)\,. \tag{A.2}$$

This can also be applied to $\mathcal{E}^N_{s,\alpha}(U)$ for any connected N-invariant subset U of $\mathfrak{H}$. For the trivial character, i.e., $\alpha = 0$, this leads to the basis $z \mapsto y^s$, $z \mapsto y^{1-s}$ of $\mathcal{E}^N_{s,0}$ if $s \neq \frac{1}{2}$, and $z \mapsto y^{1/2}$, $z \mapsto y^{1/2}\log y$ if $s = \frac{1}{2}$. For non-zero α we have:

$$k_{s,\alpha}(z) = \sqrt{y}\, K_{s-1/2}(|\alpha|y)\, e^{i\alpha x} \tag{A.3a}$$
$$= \frac{2^{s-\frac{3}{2}}\,\Gamma(s)}{\sqrt{\pi}\,|\alpha|^{s-\frac{1}{2}}}\, e^{i\alpha x} \int_{-\infty}^{\infty} e^{i\alpha t}\, \frac{y^s\, dt}{(y^2+t^2)^s}\,,$$
$$i_{s,\alpha}(z) = \frac{\Gamma(s+\frac{1}{2})}{|\alpha/2|^{s-\frac{1}{2}}}\, \sqrt{y}\, I_{s-1/2}(|\alpha|y)\, e^{i\alpha x}\,, \tag{A.3b}$$

with the modified Bessel functions

$$I_u(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{u+2n}}{n!\,\Gamma(u+1+n)}\,, \qquad K_u(t) = \frac{\pi}{2}\,\frac{I_{-u}(t)-I_u(t)}{\sin \pi u}\,. \tag{A.4}$$

The element $i_{\alpha,s}$ represents a boundary germ in $\mathcal{W}^\omega_s(\mathbb{R})$. The normalization of $i_{s,\alpha}$ is such that the restriction $\rho_s i_{\alpha,s}(x) = e^{i\alpha x}$ in the line model.

The elements $k_{s,\alpha}$ and $i_{s,\alpha}$ form a basis of $\mathcal{E}^N_{s,\alpha}$ for all s with $0 < \operatorname{Re} s < 1$. For $s \neq \frac{1}{2}$ another basis is $i_{s,\alpha}$ and $i_{1-s,\alpha}$. The element $k_{s,\alpha}$ is invariant under $s \mapsto 1-s$, and

$$k_{s,\alpha} = \frac{\Gamma(\frac{1}{2}-s)}{|\alpha|^{1/2-s}2^{s+1/2}}\, i_{s,\alpha} + \frac{\Gamma(s-\frac{1}{2})}{|\alpha|^{s-1/2}2^{3/2-s}}\, i_{1-s,\alpha} \tag{A.5}$$

gives (for $s \neq \frac{1}{2}$) a local boundary splitting as an element of $\mathcal{W}^\omega_s(\mathbb{R}) \oplus \mathcal{W}^\omega_{1-s}(\mathbb{R})$.

For the trivial character, $k_{s,\alpha}$ may be replaced by

$$\ell_s(z) = \frac{y^s - y^{1-s}}{2s-1} \quad \text{for } s \neq \frac{1}{2}\,, \qquad \ell_{1/2}(z) = y^{1/2}\,\log y\,. \tag{A.6}$$

A.1.2. *Equivariant eigenfunctions for the compact group K.* The characters of K are $k(\theta) \mapsto e^{in\theta}$ with $n \in \mathbb{Z}$ and $k(\theta)$ as in (A.1). If $u(re^{i\theta}) = f(r)e^{in\theta}$ is in $\mathcal{E}^K_{s,n}(U)$, with a K-invariant subset $U \subset \mathfrak{H}$, then f satisfies the differential equation

$$-\frac{1}{4}(1-r^2)^2\big(f''(r) + r^{-1}f'(r) - n^2r^{-2}f(r)\big) = s(1-s)f(r)\,. \tag{A.7}$$

For general annuli in $\mathbb{H}$, the solution space has dimension 2, with basis

$$\begin{aligned} P_{s,n}(re^{i\theta}) &= P_{1-s,n}(re^{i\theta}) = P^n_{s-1}\Big(\frac{1+r^2}{1-r^2}\Big)e^{in\theta}\,,\\ Q_{s,n}(re^{i\theta}) &= Q^n_{s-1}\Big(\frac{1+r^2}{1-r^2}\Big)e^{in\theta}\,, \end{aligned} \tag{A.8}$$

with the Legendre functions

$$
\begin{aligned}
P^m_{s-1}\Bigl(\frac{1+r^2}{1-r^2}\Bigr) &= \frac{\Gamma(s+m)}{|m|!\,\Gamma(s-|m|)}\, r^{|m|} F(1-s,s;1+|m|;\frac{r^2}{r^2-1})\\
&= \frac{\Gamma(s+m)}{|m|!\,\Gamma(s-|m|)}\, r^{|m|}\,(1-r^2)^s\, F(s,s+|m|;1+|m|;r^2)\\
&= \frac{\Gamma(s+m)}{|m|!\,\Gamma(s-|m|)}\, r^{|m|}\,(1-r^2)^{1-s}\, F(1-s,1-s+|m|;1+|m|;r^2)\,, \qquad \text{(A.9)}\\
Q^m_{s-1}\Bigl(\frac{1+r^2}{1-r^2}\Bigr) &= \frac{(-1)^m}{2}\,\frac{\Gamma(s)\Gamma(s+m)}{\Gamma(2s)}\,\frac{(1-r^2)^s}{r^m} F(s-m,s;2s;1-r^2)\\
&= \frac{(-1)^m}{2}\,\frac{\Gamma(s)\Gamma(s+m)}{\Gamma(2s)}\,\frac{(1-r^2)^s}{r^{2s-m}} F(s-m,s;2s;1-r^{-2})\,,
\end{aligned}
$$

and the hypergeometric function $F = {}_2F_1$ given for $|z|<1$ by

$$
F(a,b;c;z) = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n}\,\frac{z^n}{n!}\,, \qquad \text{with} \quad (a)_n = \prod_{j=0}^{n-1}(a+j)\,. \tag{A.10}
$$

(See [3], 2.1, 3.2 (3), 3.3.1 (7), 3.3.1 (1), 3.2 (36) and 2.9 (2) and (3).) The space $\mathcal{E}^K_{s,n}(\mathbb{H})$ is spanned by $P_{s,n}$ alone, since $Q_{s,n}(r)$ has a singularity as $r \downarrow 0$:

$$
Q_{s,n}(r) = \begin{cases} -\log r\,(1 + r\cdot(\text{analytic in } r)) + (\text{analytic in } r) & \text{if } n=0\,,\\ \frac12\,(|n|-1)!\,\frac{\Gamma(s+n)}{\Gamma(s+|n|)} r^{-|n|}(1+r\cdot(\text{analytic in } r)) & \\ \qquad\qquad + \log r\cdot(\text{analytic in } r) & \text{otherwise}\,. \end{cases} \tag{A.11}
$$

See [3], 3.9.2 (5)–(7) for the leading terms, and 2.3.1 for more information. Directly from (A.9) we find for $r \downarrow 0$

$$
P_{s,n}(r) = \frac{\Gamma(s+n)}{|n|!\,\Gamma(s-|n|)}\, r^{|n|}\,(1 + r\cdot(\text{analytic in } r))\,. \tag{A.12}
$$

The solution $Q_{s,n}$ is special near the boundary $\mathbb{S}^1$ of $\mathbb{D}$. As $r \uparrow 1$:

$$
Q_{s,n}(r) = (-1)^n\,\frac{\sqrt{\pi}\,\Gamma(s+n)}{\Gamma(s+\frac12)}\, 2^{-2s}(1-r^2)^s(1+(1-r)\cdot(\text{analytic in } 1-r))\,. \tag{A.13}
$$

Thus, $Q_{s,n} \in \mathcal{W}^\omega_s$, and $\rho_s Q_{s,n}(\xi) = (-1)^n\,\frac{\sqrt{\pi}\,\Gamma(s+n)}{\Gamma(s+\frac12)}\,\xi^n$ on $\mathbb{S}^1$.

For $s \neq \frac12$ we have:

$$
P_{s,n} = \frac{1}{\pi}\tan \pi s\,(Q_{s,n} - Q_{1-s,n})\,. \tag{A.14}
$$

(The formula in [3], 3.3.1, (3) gives this relation with a minus sign in front of $\frac{1}{\pi}$.) This relation confirms that $P_{1-s,n} = P_{s,n}$, and forms the basis of the boundary splitting in (6.3). It shows that in the asymptotic expansion of $P_{s,n}(r)$ as $r \uparrow 0$ there are non-zero terms with $(1-r)^s$ and with $(1-r)^{1-s}$. At $s=\frac12$, we have as $r \uparrow 1$

$$
P_{1/2,n}(re^{i\theta}) = -\frac{(-1)^n\,\Gamma(\frac12+n)}{\pi^{3/2}}\,(1-r^2)^{1/2}\,\log(1-r^2) + \mathrm{O}(1)\,. \tag{A.15}
$$

So $P_{s,n}$ is not in $\mathcal{W}^\omega_s(I)$ for any interval $I \subset \mathbb{S}^1$.

A.1.3. *Equivariant eigenfunctions for the torus A.* The characters of A are of the form $a(t) \mapsto t^{i\alpha}$ with $\alpha \in \mathbb{R}$. We use the coordinates $z = \rho e^{i\phi}$ on $\mathfrak{H}$, for which $a(t)$ acts as $(\rho, \phi) \mapsto (t\rho, \phi)$. If $u(\rho e^{i\phi}) f(\cos\phi)$ is in $\mathcal{E}^A_{s,\alpha}$, then f satisfies on $(-1, 1)$ the differential equation

$$-(1-t^2)^2 f''(t) + t(1-t^2) f'(t) + \big(\alpha^2(1-t^2) - s(1-s)\big) f(t) = 0\,. \tag{A.16}$$

This leads to the following basis of the space $\mathcal{E}^A_{s,\alpha}$:

$$\begin{aligned} f^+_{s,\alpha}(\rho e^{i\phi}) &= \rho^{i\alpha}(\sin\phi)^s F\Big(\frac{s+i\alpha}{2}, \frac{s-i\alpha}{2}; \frac{1}{2}; \cos^2\phi\Big)\,, \\ f^-_{s,\alpha}(\rho e^{i\phi}) &= \rho^{i\alpha}\cos\phi\,(\sin\phi)^s F\Big(\frac{s+i\alpha+1}{2}, \frac{s-i\alpha+1}{2}; \frac{3}{2}; \cos^2\phi\Big)\,. \end{aligned} \tag{A.17}$$

The + or − indicates the parity under $z \mapsto -\bar z$. In particular:

$$f^+_{s,\alpha}(i) = 1\,, \quad \frac{\partial f^+_{s,\alpha}}{\partial\phi}(i) = 0\,, \quad f^-_{s,\alpha}(i) = 0\,, \quad \frac{\partial f^-_{s,\alpha}}{\partial\phi}(i) = -1\,. \tag{A.18}$$

Relation (2), §2.9 in [3] shows that $f^+_{1-s,\alpha} = f^+_{s,\alpha}$ and $f^-_{1-s,\alpha} = f^-_{s,\alpha}$.

For the boundary behavior it is better to employ the Kummer relation (33) in §2.9 of [3], and express these functions in terms of

$$\rho^{i\alpha}(\sin\phi)^s F\Big(\frac{s+i\alpha}{2}, \frac{s-i\alpha}{2}; s+\frac{1}{2}; \sin^2\phi\Big)\,. \tag{A.19}$$

In applying this Kummer relation one has to choose $\sqrt{\sin^2\phi}$. This produces a singularity, and the expression in (A.19) describes an element of $\mathcal{E}_s\,(\mathfrak{H} \smallsetminus i\mathbb{R}_+)$. Denote by $f^R_{s,\alpha}$ the restriction to $0 < \phi < \frac{\pi}{2}$ and by $f^L_{s,\alpha}$ the restriction to $\frac{\pi}{2} < \phi < \pi$. The Kummer relation implies the following equalities:

$$\begin{aligned} f^R_{s,\alpha} &= \frac{\sqrt{\pi}\,\Gamma(s+\frac12)}{\Gamma(\frac{s+i\alpha+1}{2})\Gamma(\frac{s-i\alpha+1}{2})}\, f^+_{\alpha,s} - \frac{2\sqrt{\pi}\,\Gamma(s+\frac12)}{\Gamma(\frac{s+i\alpha}{2})\Gamma(\frac{s-i\alpha}{2})}\, f^-_{\alpha,s}\,, \\ f^L_{s,\alpha} &= \frac{\sqrt{\pi}\,\Gamma(s+\frac12)}{\Gamma(\frac{s+i\alpha+1}{2})\Gamma(\frac{s-i\alpha+1}{2})}\, f^+_{\alpha,s} + \frac{2\sqrt{\pi}\,\Gamma(s+\frac12)}{\Gamma(\frac{s+i\alpha}{2})\Gamma(\frac{s-i\alpha}{2})}\, f^-_{\alpha,s}\,. \end{aligned} \tag{A.20}$$

Thus we see that $f^R_{s,\alpha}$ and $f^L_{s,\alpha}$ extend as elements of $\mathcal{E}_s$, that $f^R_{s,\alpha}$ represents an element of $\mathcal{W}^\omega_s(\mathbb{R}_+)$ with, in the line model, $\rho_s f^R_{s,\alpha}(x) = x^{i\alpha}$, and that $f^L_{s,\alpha}$ represents an element of $\mathcal{W}^\omega_s(\mathbb{R}_-)$ with $\rho_s f^L_{s,\alpha}(x) = (-x)^{i\alpha}$. Inverting the relation in (A.20) one finds, for $s \neq \frac12$, the following expressions for $f^+_{s,\alpha}$ and $f^-_{s,\alpha}$ as a linear combination of $f^R_{s,\alpha}$ and $f^R_{1-s,\alpha}$,

$$\begin{aligned} f^+_{s,\alpha} &= \frac{\sqrt{\pi}\,\Gamma(\frac12 - s)}{\Gamma(\frac{1-s+i\alpha}{2})\Gamma(\frac{1-s-i\alpha}{2})} f^R_{s,\alpha} + \frac{\sqrt{\pi}\,\Gamma(s-\frac12)}{\Gamma(\frac{s+i\alpha}{2})\Gamma(\frac{s-i\alpha}{2})} f^R_{1-s,\alpha}\,, \\ f^-_{s,\alpha} &= \frac{\sqrt{\pi}\,\Gamma(\frac12 - s)}{2\Gamma(1-\frac{s+i\alpha}{2})\Gamma(1-\frac{s-i\alpha}{2})} f^R_{s,\alpha} + \frac{\sqrt{\pi}\,\Gamma(s-\frac12)}{2\Gamma(\frac{s+i\alpha+1}{2})\Gamma(\frac{s-i\alpha+1}{s})} f^R_{1-s,\alpha}\,, \end{aligned} \tag{A.21}$$

and similarly of $f^L_{s,\alpha}$ and $f^L_{1-s,\alpha}$, showing that each of these elements belongs to the direct sums $\mathcal{W}^\omega_s(\mathbb{R}_+) \oplus \mathcal{W}^\omega_{1-s}(\mathbb{R}_+)$ and $\mathcal{W}^\omega_s(\mathbb{R}_-) \oplus \mathcal{W}^\omega_{1-s}(\mathbb{R}_-)$, but not to $\mathcal{W}^\omega_s(I) \oplus \mathcal{W}^\omega_{1-s}(I)$

for any neighborhood I of 0 or ∞ in $\mathbb{P}^1_{\mathbb{R}}$; in other words, just as for the Bessel functions $i_{s,\alpha}$ and $k_{s,\alpha}$, we have a local but not a global boundary splitting.

A.2. **Poisson transforms.** Almost all of the special elements in §A.1 belong to $\mathcal{E}_s$, and hence are the Poisson transform of some hyperfunction by Helgason's Theorem 2.4. Actually in all cases except one the function has polynomial growth, and hence are the Poisson transform of a distribution (Theorem 2.5). In Table 1 and the discussion below we give explicit representations of these eigenfunctions as Poisson transforms of distributions and/or hyperfunctions.

	$u \in \mathcal{E}_s$	$\mathsf{P}_s^{-1} u \in \mathcal{V}_s^{-\omega}$	model
A	y^{1-s}	$\delta_{s,\infty} : \varphi^{\mathbb{P}} \mapsto \varphi^{\mathbb{P}}(\infty)$	proj.
B	$\frac{\Gamma(\frac12-s)}{\sqrt{\pi}\,\Gamma(1-s)}\, y^s$	$\mathbf{1}_s$ = integration against 1 for $\mathrm{Re}\, s < \frac12$, with meromorphic continuation	line
C	$-2\frac{\Gamma(\frac32-s)}{\sqrt{\pi}\,\Gamma(1-s)}\ell_s(z)$ ℓ_s as in (A.6)	$\varphi \mapsto \frac{-1}{2}\int_{-\infty}^{\infty}(\varphi(t) - \frac{\varphi_\infty}{\sqrt{1+t^2}})\,dt$ with $\varphi_\infty = \lim_{t\to\infty} \lvert t\rvert^{2s-2}\varphi(t)$	line
D	$R(t;z)^{1-s} \quad (t \in \mathbb{R})$	$\delta_{s,t} : \varphi \mapsto \varphi(t)$	line
E	$\frac{2^{s+\frac12}\lvert\alpha\rvert^{\frac12-s}}{\sqrt{\pi}\,\Gamma(1-s)} k_{s,\alpha}(z)$ $(\alpha \in \mathbb{R} \smallsetminus \{0\})$	integration against $e^{i\alpha t}$ for $\mathrm{Re}\, s < \frac12$, or integration of $-\varphi'$ against $\frac{e^{i\alpha x}}{i\alpha}$ for $0 < \mathrm{Re}\, s < 1$	line
F	$i_{1-s,\alpha}(z)$ $(\alpha \in \mathbb{R} \smallsetminus \{0\})$	support $\{\infty\}$; representative near ∞: $-\frac{i}{2}\tau\,(1+\tau^{-2})^s F(1; 2-2s; i\alpha\tau)$	proj.
G	$\frac{(-1)^n\Gamma(s)}{\Gamma(s+n)}\, p_{s,n}$	$\mathbf{e}_{s,n}$	circle
H	$p_s(w', \cdot)$	$R^{\mathbb{S}}(\,\cdot\,; w')^s$	circle
I	$\frac{\Gamma(1+i\alpha-s)\Gamma(1-i\alpha-s)}{\pi\,\Gamma(2-2s)} f^L_{1-s,\alpha}$	integration against $x^{i\alpha-s}$ on $\mathbb{R}_+$	line
J	$\frac{\Gamma(1+i\alpha-s)\Gamma(1-i\alpha-s)}{\pi\,\Gamma(2-2s)} f^R_{1-s,\alpha}$	integration against $(-x)^{i\alpha-s}$ on $\mathbb{R}_-$	line

TABLE 1. Poisson representation of elements of $\mathcal{E}_s$.

A. In (2.30) we have shown that y^{1-s} is the Poisson transform of the distribution $\delta_{s,\infty}$. See (2.30) for an explicit description of $\delta_{s,\infty}$ as a hyperfunction.

B. The description of y^s as a Poisson transform takes more work. For $\mathrm{Re}\, s < \frac12$ the linear form $\mathbf{1}_s : \varphi \mapsto \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(t)\,dt$ is continuous on $\mathcal{V}^0_{1-s}$, in the line model. Note that the constant function 1 is not in $\mathcal{V}_s^{\omega}$, since it does not satisfy the asymptotic behavior (1.2) at ∞. Application of (2.26) gives the Poisson transform $\mathsf{P}_s\mathbf{1}_s$ indicated in the table.

To describe $\mathbf{1}_s$ as a hyperfunction in the line model (and also to continue it in s) we want to give representatives $g_{\mathbb{R}}$ and g_∞ of $\mathbf{1}_s$ on $\mathbb{R}$ and $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{0\}$, related by $g_\infty(\zeta) = \zeta^{-2s}\, g_{\mathbb{R}}(-1/\zeta)$ up to a holomorphic function on a neighborhood of $\mathbb{R} \smallsetminus \{0\}$.

Formula (1.26) gives a representative $g^{\mathbb{P}}$ in the projective model:

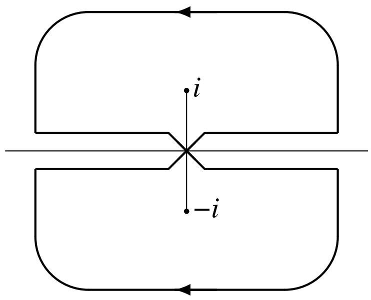
$$g^{\mathbb{P}}(\zeta) \;=\; \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\zeta + i}{t - \zeta}\,(t-i)^s (t+i)^{s-1}\,dt \qquad (\zeta \in \mathbb{P}^1_{\mathbb{C}} \smallsetminus \mathbb{P}^1_{\mathbb{R}})\,.$$

(The factor $(t^2+1)^{s-1}$ comes from passage between models.) This function extends both from $\mathfrak{H}$ and from $\mathfrak{H}^-$ across the real axis. An application of Cauchy's formula shows that the difference of both extension is given by $(\zeta^2+1)^s$, corresponding to the function 1 in the line model. See (1.5).

To get a representative near ∞ we write

$$(1 + e^{2\pi i s})\, g^{\mathbb{P}}(\zeta) \;=\; \frac{1}{2\pi i}\int_C \frac{\zeta + i}{(z-\zeta)(z+i)}\,(z^2+1)^s\,dz\,, \tag{A.22}$$

where C is the contour shown below. The factor $(z^2+1)^s$ is multivalued on the contour and is fixed by choosing $\arg(z^2+1) \in [0, 2\pi)$. On the part of the contour just above $(0,\infty)$ the argument of z^2+1 is approximately zero, and just below $(0,\infty)$ the argument is approximately 2π. Near $(-\infty, 0)$ the argument is approximately 2π just above the real line and approximately 0 below the real line. We take the contour so large that $\zeta \in \mathfrak{H} \cup \mathfrak{H}^-$ is inside one of the loops of C. If we let the contour grow, the arcs in the upper and lower half-planes give a contribution $o(1)$. In the limit, for $\operatorname{Re} s < \frac{1}{2}$, we are left with twice the integral along $(0,\infty)$ and along $(-\infty, 0)$, both once with the standard value and and once with $e^{2\pi i s}$ times the standard value. This gives the equality (A.22), and the continuation of $g^{\mathbb{P}}$ as a meromorphic function of s.



Now consider $\zeta \in \mathfrak{H}^\pm$ with $|\zeta| > 1$. Moving the path of integration across ζ, we obtain with Cauchy's theorem that $(1 + e^{2\pi i s})\, g^{\mathbb{P}}(\zeta)$ is equal to $\pm(\zeta^2+1)^s$ plus a holomorphic function of ζ on a neighborhood of ∞. The term $\pm(\zeta^2+1)^s$ obeys the choice of the argument discussed above. To bring it back to the standard choice of arguments in $(-\pi, \pi]$, we write it as $\zeta^{2s}(1+\zeta^{-2})^s$ for $\zeta \in \mathfrak{H}$ and as $-(-\zeta)^{2s}(1+\zeta^{-2})^s$ for $\zeta \in \mathfrak{H}^-$. The factor $(1+\zeta^{-2})^s$ is what we need to go back to the line model with (1.5). Thus we arrive at the following representatives in the line model.

$$g_{\mathbb{R}}(\zeta) \;=\; \begin{cases} 1 & \text{on } \mathfrak{H}\,, \\ 0 & \text{on } \mathfrak{H}^-\,; \end{cases} \qquad g_\infty(\zeta) \;=\; \pm\zeta^{-2s}\,(1 + e^{\mp 2\pi i s})^{-1} \text{ on } \mathfrak{H}^\pm\,. \tag{A.23}$$

Finally one checks that $g_{\mathbb{R}}(\zeta) - (\zeta^2)^{-s} g_\infty(-1/\zeta)$ extends holomorphically across both $\mathbb{R}_+$ and $\mathbb{R}_-$, thus showing that the pair $(g_{\mathbb{R}}, g_\infty)$ determines the hyperfunction $\mathbf{1}_s$. These representatives also show that $\mathbf{1}_s$ extends meromorphically in s, giving $\mathbf{1}_s \in \mathcal{V}_s^{-\omega}$ for all $s \neq \frac{1}{2}$ with $0 < \operatorname{Re} s < 1$.

For the relation between the cases **A** and **B**, we use (2.25) to get

$$\mathsf{P}_s I_{1-s}\delta_{1-s,\infty}(z) \;=\; \mathsf{P}_{1-s}\delta_{1-s,\infty}(z) \;=\; y^s\,.$$

The fact that the Poisson transformation is an isomorphism $\mathcal{V}_s^{-\omega} \to \mathcal{E}^s$ implies

$$\mathbf{1}_s \;=\; \frac{\Gamma(\frac12 - s)}{\sqrt{\pi}\,\Gamma(1-s)}\, I_s\delta_{s,\infty}\,. \tag{A.24}$$

C. For $\mathrm{Re}\, s < \frac12$ we have

$$\langle \varphi, \mathbf{1}_s\rangle \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\Bigl(\varphi(t) - \varphi^{\mathbb{P}}(\infty)(1+t^2)^{s-1}\Bigr)\,dx + \frac{\Gamma(\frac12 - s)}{\sqrt{\pi}\,\Gamma(1-s)}\delta_{s,\infty}(\varphi^{\mathbb{P}})\,.$$

So the distribution L_s given by

$$L_s : \varphi \mapsto \frac{1}{\pi}\int_{-\infty}^{\infty}(\varphi(t) - \varphi^{\mathbb{P}}(\infty)(1+t^2)^{s-1})\,dt\,,$$

which is well defined for $\mathrm{Re}\, s < 1$, is equal to $\mathbf{1}_s - \frac{\Gamma(\frac12 - s)}{\sqrt{\pi}\,\Gamma(1-s)}\,\delta_{s,\infty}$ for $\mathrm{Re}\, s < \frac12$. The results of the cases **A** and **B** give the expression of $\mathsf{P}_s L_s$ as a multiple of ℓ_s defined in (A.6). Going over to the line model, we obtain the statement in the table.

D. This is simplify the definition of the Poisson transformation in (2.22) and (2.23) applied to the delta distribution at t. It also follows from Case **A**, using the G-equivariance.

The latter method involves a transition between the models. We explain some of the steps to be taken. In the projective model, $\delta^{\mathbb{P}}_{s,t} : \varphi^{\mathbb{P}} \mapsto (1+t^2)^{s-1}\,\varphi^{\mathbb{P}}(t)$. We have

$$\langle \delta^{\mathbb{P}}_{s,t}\big|_{2s}\begin{bmatrix} t & -1\\ 1 & 0\end{bmatrix}, \varphi^{\mathbb{P}}\rangle \;=\; \langle \delta^{\mathbb{P}}_{s,t}, \varphi^{\mathbb{P}}\big|_{2-2s}\begin{bmatrix} 0 & 1\\ -1 & t\end{bmatrix}\rangle \;=\; \cdots \;=\; \delta^{\mathbb{P}}_{s,\infty}(\varphi^{\mathbb{P}})\,.$$

Hence

$$\mathsf{P}_s(\delta_{s,t})(z) \;=\; \mathsf{P}_s\Bigl(\delta_{s,\infty}\Big|\begin{bmatrix} 0 & 1\\ -1 & t\end{bmatrix}\Bigr)(z) \;=\; (\mathsf{P}_s\delta_{s,\infty})(1/(t-z)) \;=\; \Bigl(\frac{y}{|t-z|^2}\Bigr)^{1-s}\,.$$

E. For $\alpha \neq 0$ we need no complicated contour integration. When $\mathrm{Re}\, s < \frac12$ the distribution $\varphi \mapsto \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(t)e^{i\alpha t}\,dt$ in the line model, is equal to $\varphi \mapsto \frac{-1}{\pi i\alpha}\int_{-\infty}^{\infty}\varphi'(t)e^{i\alpha t}\,dt$. The latter integral converges absolutely for $\mathrm{Re}\, s < 1$.

F. Since $i_{s,\alpha}$ has exponential growth we really need a hyperfunction. The representative in the table does not behave well near 0. However it is holomorphic on a deleted neighborhood of ∞, and represents a hyperfunction on $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{0\}$ in the projective model. We extend it by zero to obtain a hyperfunction on $\mathbb{P}^1_{\mathbb{R}}$.

The path of integration $\int_{C_+} - \int_{C_-}$ can be deformed into a large circle $|\tau| = R$, such that we can replace τ by $\tau - x$ in the integration. We obtain

$$\begin{aligned}&\frac{1}{\pi}\int_{|\tau|=R}\frac{i\tau}{-2}(1+\tau^{-2})^s F(1;2-2s;i\alpha\tau)\Bigl(\frac{y(1+\tau^2)}{(\tau-z)(\tau-\bar z)}\Bigr)^{s-1}\frac{d\tau}{1+\tau^2}\\ &\quad=\; \frac{1}{2\pi i}y^{1-s}\int_{|\tau|=R}(1+\frac{x}{\tau})^{1-2s}(1+\frac{y^2}{\tau^2})^{s-1}F(1;2-2s;i\alpha(\tau+x))\,\frac{d\tau}{\tau}\,.\end{aligned}$$

Expand the factors $(1+\frac{x}{\tau})^{1-2s}$ and $(1+\frac{y^2}{\tau^2})^{s-1}$ and the hypergeometric function into power series and carry out the integration term by term. In the resulting sum we recognize the power series of $e^{i\alpha x}$, and after some standard manipulations with gamma factors, also the expansion of the modified Bessel function $I_{1/2-s}(|\alpha|y)$.

G. See the discussion after Theorem 2.4.

H. See (2.31).

I and **J .** Integration against $x \mapsto x^{i\alpha-s}$ on $(0,\infty)$ and against $x \mapsto (-x)^{i\alpha-s}$, in the line model, defines distributions. For $\rho e^{i\phi} \in \mathfrak{H}$, the Poisson integral leads to

$$\frac{\rho^{i\alpha}(\sin\phi)^{1-s}}{\pi}\int_0^\infty t^{i\alpha-s}(t^2+1+2Ct)^{s-1}\,dt\,,$$

with $C = \mp\cos\phi$. Let us consider this for small values of C, i.e., for points near $i\mathbb{R}_+$ in $\mathfrak{H}$. Expanding the integrand in powers of C gives a series in which one may separate the even and odd terms, and arrive at

$$\frac{\rho^{i\alpha}\sqrt{1-C^2}^{\,1-s}}{2\pi\Gamma(1-s)}\Bigl(\Gamma(\tfrac{1-i\alpha-s}{2})\Gamma(\tfrac{1+i\alpha-s}{2})F(\tfrac{1-i\alpha-s}{2},\tfrac{1+i\alpha-s}{2};\tfrac12;C^2)$$
$$-2C\,\Gamma(1-\tfrac{i\alpha+s}{2})\Gamma(1+\tfrac{i\alpha-s}{2})F(1-\tfrac{i\alpha+s}{2},1+\tfrac{i\alpha-s}{2};\tfrac32;C^2)\Bigr).$$

Now take $C=-\cos\phi$, respectively $C=\cos\phi$, and conclude that we have a multiple of $f^L_{1-s,\alpha}$, respectively $f^R_{1-s,\alpha}$.

A.3. **Transverse Poisson transforms.** In Table 2 we give examples of pairs $u = \mathsf{P}^\dagger_s\varphi$, $\varphi = \rho_s u$, where $\varphi \in \mathcal{V}^\omega_s(I)$ for some $I \subset \partial\mathbb{H}$.

	$u = \mathsf{P}^\dagger_s\varphi \in \mathcal{W}^\omega_s(I)$	$\varphi = \rho_s u \in \mathcal{V}^\omega_s(I)$	I	model
a	$Q_{s,n}$	$(-1)^n \frac{\sqrt{\pi}\,\Gamma(s+n)}{\Gamma(s+\frac12)}\,\mathbf{e}_{s,n}$	$\mathbb{S}^1$	circle
b	$q_s(\,\cdot\,,w')$	$\frac{\sqrt{\pi}\,\Gamma(s)}{2^{2s}\,\Gamma(s+1/2)}\,\frac{(1-\lvert w'\rvert^2)^s}{\lvert\xi-w'\rvert^{2s}}$	$\mathbb{S}^1$	circle
c	y^s	1	$\mathbb{R}$	line
d	$R(t;z)^s \quad (t\in\mathbb{R})$	$\lvert t-x\rvert^{-2s}$	$\mathbb{R}\smallsetminus\{t\}$	line
e	$R(\zeta;z)^s \quad (\zeta\in\mathbb{C}\smallsetminus\mathbb{R})$	$(\zeta-x)^{2s}$ (multivalued)	$\mathbb{R}$	line
f	$i_{s,\alpha}$	$e^{i\alpha x}$	$\mathbb{R}$	line
g	$f^R_{s,\alpha}$	$x^{i\alpha-s}$	$(0,\infty)$	line
h	$f^L_{s,\alpha}$	$(-x)^{i\alpha-s}$	$(-\infty,0)$	line

TABLE 2. Transverse Poisson representations of boundary germs

In Cases **c**, **d**, **f**, **g** and **h** in the table the eigenfunction u is in $\mathcal{E}_s = \mathcal{E}_s(\mathbb{H})$, hence it is also a Poisson transform. If we write $u = \mathsf{P}_{1-s}\alpha$, then entries **A**, **D**, **F**, **J** and **I**, respectively, in Table 1 (with the s replaced by $1-s$ in most cases) show that the support of α is the complement of the set I in $\partial\mathbb{H}$ for each of these cases, illustrating Theorem 5.4.

A.4. **Potentials for Green's forms.** If $u, v \in \mathcal{E}_s(U)$ for some $U \subset \mathbb{H}$, then the Green's forms $\{u, v\}$ and $[u, v]$ are closed. So if U is simply connected there are well defined potentials of $[u, v]$ and $\{u, v\}$ in $C^\omega(U)$, related according to (2.13). We list some examples of potentials F of $\{u, v\}$ in Table 3. Then $\frac{1}{2i}F + \frac{1}{2}uv$ is a potential of the other Green's form $[u, v]$. We found most of these potentials by writing down $\{u, v\}$, guessing

	u	v	F such that $dF = \{u, v\}$	domain
1	y^s	y^{1-s}	$(2s-1)x$	$\mathfrak{H}$
2	$y^{1/2}$	$y^{1/2}\log y$	$-x$	$\mathfrak{H}$
3	$R(t;z)^s$ $t \in \mathbb{R}$	$R(p;z)^{1-s}$ $p \in \mathbb{R} \setminus \{t\}$	$\frac{(t-x)(p-x)+y^2}{y(p-t)} R(t;z)^s R(p;z)^{1-s}$	$\mathfrak{H}$
4	$R(t;z)^s$ $t \in \mathbb{R}$	$R(t;z)^{1-s}$	$\frac{s}{t-z} + \frac{s-1}{t-\bar z} - iR(t;z)$	$\mathfrak{H}$
5	y^s	$R(t;z)^{1-s}$ $t \in \mathbb{R}$	$-(iy^s + (t-z)y^{s-1})\, R(t;z)^{1-s}$	$\mathfrak{H}$
6	y^s	$R(t;z)^s$	$-F_s((x-t)/y)$, F_s as in (A.25)	$\mathfrak{H}$
7	$R(t;z)^s$ $t \in \mathbb{R}$	$R(p;z)^s$ $p \in \mathbb{R} \setminus \{t\}$	$((p-t)^2)^{-s} F_s\big(\frac{(p-x)(t-x)+y^2}{y(p-t)}\big)$	$\mathfrak{H}$
8	$k_{s,\alpha}$	$i_{s,\alpha}$	$\frac{i\,\Gamma(s+1/2)}{2^{3/2-s}\alpha\lvert\alpha\rvert^{s-1/2}}\, e^{2i\alpha x}$	$\mathfrak{H}$
9	$P_{s,0}(re^{i\theta})$	$Q_{s,0}(re^{i\theta})$	$-\theta$	$U \subset \mathbb{D} \setminus \{0\}$ simply connected
10	$P_{s,n}(re^{i\theta})$ $n \in \mathbb{Z} \setminus \{0\}$	$Q_{s,n}(re^{i\theta})$	$-\frac{(-1)^n\,\Gamma(s+n)}{2in\,\Gamma(s-n)}\, e^{2in\theta}$	$\mathbb{D} \setminus \{0\}$
11	$P_{s,-n}(re^{i\theta})$ $n \in \mathbb{Z} \setminus \{0\}$	$Q_{s,n}(re^{i\theta)}$	$-2in \int P_{s,-m}(r)\, Q_{s,n}(r)\, \frac{dr}{r}$ $-(-1)^n\theta$	$U \subset \mathbb{D} \setminus \{0\}$ simply connected
12	$P_{s,m}(re^{i\theta})$ $m \in \mathbb{Z}$	$Q_{s,n}(re^{i\theta})$ $n \in \mathbb{Z} \setminus \{-m\}$	$e^{i(m+n)\theta)} r\,(Q_{s,n}(r)\,\partial_r P_{s,m}(r)$ $-P_{s,m}(r)\,\partial_r Q_{s,n}(r))/i(m+n)$	$\mathbb{D} \setminus \{0\}$

TABLE 3. Potentials for Green's forms

F, and checking our guess.

Case **3** is essentially (2.16). In Case **6** we needed the following function:

$$F_s(r) = 2s \int_r^\infty (1+q^2)^{-s-1}\, dq\,, \tag{A.25}$$

For **7** we have used that $(\operatorname{Im} gz)^s = R(t;z)^s$ and $R(0;gz)^s = |p-t|^{2s} R(p;z)^s$ with $g = \left[\begin{smallmatrix} \frac{-1}{p-t} & \frac{p}{p-t} \\ -1 & t \end{smallmatrix}\right]$ with $t, p \in \mathbb{R}$. So **6** leads to the potential in **7** if $p \neq t$ are real. We write

$((p-t)^2)^{-s}$ and not $|p-t|^{-2s}$ to allow holomorphic continuation in p and t. For Case **8** we use that if $u(z) = e^{i\alpha x} f(y)$ and $v(z) = e^{i\alpha x} g(y)$, then

$$\{u, v\} = e^{2i\alpha x}(f'g - fg')\, dx,$$

and that the Wronskian $fg' - f'g$ is constant if $u, v \in \mathcal{E}_s$. Cases **9**–**12** are obtained in a similar way. In **9** and **11** the potentials are multivalued if U is not simply connected.

Cases **3**–**5** are valid on $\mathfrak{H}$ if t and p are real. Otherwise $\{u, v\}$ and F are multivalued with branch points at t, and at p in **3**. We have to chose the same branch in $\{u, v\}$ and F. Also in **7** the branches have to be chosen consistently. In **4** there are singularities at $t = z$ and $t = \bar{z}$, but $\{u, v\}$ and F are univalued.

A.5. **Action of the Lie algebra.** The real Lie algebra of G has $\mathbf{H} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\mathbf{V} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\mathbf{W} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ as a basis. Any $\mathbf{Y}$ in the Lie algebra acts on $\mathcal{V}_s^\infty$ by $f|_{2s}\mathbf{Y} = \partial_t f|_{2s} e^{t\mathbf{Y}}\big|_{t=0}$. Note that for right actions we have $f \,|\, [\mathbf{Y}_1, \mathbf{Y}_2] = (f \,|\, \mathbf{Y}_2) \,|\, \mathbf{Y}_1 - (f \,|\, \mathbf{Y}_1) \,|\, \mathbf{Y}_2$.

In the projective model:

$$\begin{aligned} f|_{2s}\mathbf{H}(\tau) &= \left(2s\frac{1-\tau^2}{1+\tau^2} + 2\tau\partial_\tau\right) f(\tau), \\ f|_{2s}\mathbf{V}(\tau) &= \left(-4s\frac{\tau}{1+\tau^2} + (1-\tau^2)\partial_\tau\right) f(\tau), \\ f|_{2s}\mathbf{W}(\tau) &= (1+\tau^2)\partial_\tau f(\tau). \end{aligned} \tag{A.26}$$

For the elements $\mathbf{E}^+ = \mathbf{H} + i\mathbf{V}$ and $\mathbf{E}^- = \mathbf{H} - i\mathbf{V}$ in the complexified Lie algebra we find:

$$\begin{aligned} f|_{2s}\mathbf{E}^+(\tau) &= \left(-2s\frac{\tau+i}{\tau-i} - i(\tau+i)^2\partial_\tau\right) f(\tau), \\ f|_{2s}\mathbf{E}^-(\tau) &= \left(-2s\frac{\tau-i}{\tau+i} + i(\tau-i)^2\partial_\tau\right) f(\tau). \end{aligned} \tag{A.27}$$

In particular

$$\mathbf{e}_{s,n}|_{2s}\mathbf{W} = 2in\,\mathbf{e}_{s,n}, \qquad \mathbf{e}_{s,n}|_{2s}\mathbf{E}^\pm = -2(s \mp n)\,\mathbf{e}_{s,n\mp 1}. \tag{A.28}$$

By transposition these formulas are also valid on hyperfunctions.

The Lie algebra generates the universal enveloping algebra, which also acts on $\mathcal{V}_s^\infty$. The center of this algebra is generated by the Casimir operator $\omega = -\frac{1}{4}\mathbf{E}^+\mathbf{E}^- + \frac{1}{4}\mathbf{W}^2 - \frac{i}{2}\mathbf{W}$. It acts on $\mathcal{V}_s$ as multiplication by $s(1-s)$.

For the action of G by left translation on functions on $\mathfrak{H}$:

$$\begin{aligned} \mathbf{W} &= (1+z^2)\partial_z + (1+\bar{z}^2)\partial_{\bar{z}}, \qquad \mathbf{E}^\pm = \mp i(z \pm i)^2\partial_z \mp i(\bar{z} \pm i)^2\partial_{\bar{z}}, \\ \omega &= (z-\bar{z})^2\partial_z\partial_{\bar{z}} = \Delta, \end{aligned} \tag{A.29}$$

and on $\mathbb{D}$:

$$\begin{aligned} \mathbf{W} &= 2iw\partial_w - 2i\bar{w}\partial_{\bar{w}}, \qquad \mathbf{E}^+ = 2\partial_w - 2\bar{w}^2\partial_{\bar{w}}, \\ \mathbf{E}^- &= -2w^2\partial_w + 2\partial_{\bar{w}}, \qquad \omega = -(1-|w|^2)^2\,\partial_w\partial_{\bar{w}}. \end{aligned} \tag{A.30}$$

A counterpart of (A.28) is

$$\begin{aligned} P_{s,n} \,|\, \mathbf{W} &= 2in\,P_{s,n}, \qquad P_{s,n} \,|\, \mathbf{E}^+ = 2(s-n)(s+n-1)\,P_{s,n-1}, \\ P_{s,n} \,|\, \mathbf{E}^- &= 2\,P_{s,n+1}, \qquad Q_{s,n} \,|\, \mathbf{E}^+ = 2(s-n)(s+n-1)\,Q_{s,n-1}, \\ Q_{s,n} \,|\, \mathbf{E}^- &= 2\,Q_{s,n+1}, \qquad Q_{s,n} \,|\, \mathbf{W} = 2in\,Q_{s,n}. \end{aligned} \tag{A.31}$$

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