Some questions and observations around the mathematics of Seki Takakazu

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Abstract This informal paper, an expanded version of the short talk given by the second author at the conference on the occasion of the 300th anniversary of Seki's death, is slightly non-standard in nature and perhaps requires a short explanatory preamble. The authors are not professional historians of mathematics, and no attempt has been made to interpret the material discussed from a historical viewpoint. Instead, the first section contains several specific mathematical comments, from the point of view of a contemporary professional mathematician (D.Z.), on a few of the problems and solutions of Seki and of his predecessors in China and Japan, pointing out places where the mathematical content is unexpectedly naive or unexpectedly sophisticated, or where particular mathematical features of the problems permit deductions about their authors' methods or views.

The second section concerns the thorny question of possible contacts that Seki or his disciple Takebe Katahiro may have had with European mathematics as a result of the Dutch presence in Dejima and their yearly visit to the Edo court. In particular, we describe the results of a search (by S.W.-Z.) through the archives of the Dutch East India Company for the years in question that yielded details of a meeting between Takebe and the Dutch but show clearly that there was, at least on this occasion, no serious discussion of any scientific or mathematical questions. We also mention a few other arguments militating against the thesis that there was any direct impact of European mathematics .

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1 The mathematics of Seki and his predecessors seen from a modern standpoint

In this section we discuss some of the aspects of the Japanese mathematics of the 17th century, and of the Chinese mathematics on which it is based, which are surprising from a contemporary point of view.

It is of course a commonplace that the nature of scientific progress, like that of any other human activity, is heavily dependent on the culture and social context in which it takes place, and it is in no way surprising that the Japanese mathematics of the Edo period, or the earlier Japanese and Chinese mathematics on which it is based, should often be very different from ours. But there are specific aspects which are startling to a contemporary mathematical eye, either because seemingly simple points are often overlooked or ignored, or, conversely, because the level of sophistication sometimes appears unexpectedly high. Of course any such reaction from a modern mathematician is anachronistic and based entirely on hindsight. Yet mathematics has an intrinsic rhythm and internal "necessariness" which all its practitioners feel, and particular interest therefore attaches to instances where the background or traditions of earlier mathematicians led them to see things in a way strange to us. We will list a number of such cases occurring in the works used or written by Seki, and will formulate a number of specific questions whose answers will doubtless be known to specialists in some cases and unknowable in others, but some of which may perhaps suggest interesting topics for further investigation.

Essentially all of our examples are taken from the book [**?**] by Annick Horiuchi, where they are discussed in their historical context. The English translations of the problems and of passages from this book are taken from [**?**].

1.1 A problem on volumes

Our first example comes from the Introduction to Mathematics [算学啓蒙 *Suanxue Qimeng*] by Zhu Shijie [朱世傑], published in 1299, which was one of the most influential Chinese mathematical texts in the early Edo period and of which a Japanese version with detailed commentary would be published by Takebe in 1690.

Problem 34 of the last chapter Unlocking of Roots [開方釋鎖 *kaifang shisuo*], discussed in [**?**, pp. 74–75], reads as follows in the original Chinese and in English translation:

> 今有立方. 立圓. 平方. 古圓田. 徽圓田. 各一。共積三万三千六百二十二尺二百分尺之三十七。只云 立方面不及立圓径四尺. 多如徽圓径三尺。立圓径如平方面三分之一。古圓周与立方面適等。問五事各幾何。
> 答曰 立方面二十四尺。立圓径二十八尺。平方面八十四尺。古圓周二十四尺。徽圓径二十一尺。

> One now has a cube, a sphere, a square, an old circular field and a Hui circular field. The total accumulation is 33622 *chi*[1] and 37 200ths of a *chi*. It is said only that the side of the cube is 4 *chi* less than the diameter of the sphere and 3 *chi* more than the diameter of the Hui circle, that the diameter of the sphere is equal to a third of the side of the square, and that the circumference of the old circle and the side of the cube are equal. One asks for the values of the five quantities.
> Answer : Side of cube is 24 *chi*, diameter of sphere 28 *chi*, side of square 84 *chi*, circumference of old circle 24 *chi*, diameter of Hui circle 21 *chi*.

Here the word "accumulation" means the sum of all the areas and volumes concerned. The phrase "old circular field" means a field in the shape of a circle with the *lü* [率] (fixed ratio, the Chinese word for π) being taken to be the ancient value of 3, while the "Hui circular field" is one where π is taken to be the value 3.14 ascribed to the third century mathematician Liu Hui [劉徽]. In modern terminology, the problem can therefore be stated as $A+B+C+D+E = 33622\,\frac{37}{200}$ with

$$A = a^3\,, \quad B = t\,b^3, \quad C = c^2\,, \quad D = \frac{1}{4\cdot\pi_{\text{old}}}\,d^2\,, \quad E = \frac{\pi_{\text{Hui}}}{4}\,e^2$$

together with the supplementary conditions $a = b-4 = e+3$, $b = c/3$, $d = a$, where $\pi_{\text{old}} = 3$ and $\pi_{\text{Hui}} = 3.14$ are the "old *lü*" and the "*lü* of Liu Hui" and t is the fixed ratio (=ratio of the volume to the cube of the diameter) for the sphere.

Seen from a modern perspective, the problem and its solution have a number of peculiarities. The most striking, of course, is the use of two different values for π in the same problem, which is quite incomprehensible for us. It is already strange to find Zhu using the "ancient" value of 3 when far more accurate values were already given in the Nine Chapters [九章算術 *Jiuzhang Suanshu*] and commentaries to them (e.g., Liu Hui himself gave the value 3.1416 and not just 3.14 as Zhu assumes; cf. [**?**, pp. 145–148]. But the simultaneous occurrence of two circles with differing values of π indicates a rather nebulous understanding of the relationship between the circumference and area of circles. Of course, this may have been merely an homage to the Ancients, or simply a more poetic way of writing the algebraic problem "$a^3+tb^3+c^2+\frac{1}{12}d^2+\frac{3.14}{4}e^2 = 33622\frac{37}{200}$" in words, in which case it is an entirely legitimate procedure. In any case, this passage suggests our first question:

Q1. How did the idea of π (or of other fixed geometric ratios [定法 *teihō* in Japanese]) evolve in China and Japan? At what stage was it clearly realized that π is a single well-defined number which in principle can be calculated to any desired accuracy?

The next point is the addition of areas and volumes, something that is not only not correct, but also rather unnatural, since figures of different dimensions cannot be juxtaposed, and was, for instance, not a possible operation in classical Greek mathematics, where figures were considered geometrically rather than being expressed in terms of units like the Chinese *chi*. (One can even speculate that this difference of approaches may have actually impeded the development of algebraic notions or notations in Greece, since it was not until Diophantus that polynomials were consid-

[1] *chi* [尺 *shaku* in Japanese] is a unit of length, which is an equivalent of a foot. (editor)

ered, and may perhaps have correspondingly abetted the rather early development of algebraic ideas in China.) Our second question is therefore:

Q2. At what point was it clearly realized that lengths, areas and volumes are incommensurable and must be given in different units (here, *chi*, chi^2 and chi^3), and that expressions involving mixed exponents must be expressed in terms of pure numbers rather than lengths?

The third point is the startling disparity between the very approximate values of π used (even the "lü of Liu Hui" is only a 3-digit approximation) and the very precise value $33622\frac{37}{200}$ given for the total volume. It is clear to us, but evidently not to the Chinese mathematicians of the 13th century, that it makes little sense to give the latter to 8 significant digits when the former is given to only 3- or even 1-digit accuracy, and we will see the same lack of understanding of the meaning of significant figures recurring in some of the other problems we consider below, even by much later mathematicians. This suggests our third question:

Q3. At what point were the rules for calculating with approximate values first understood, viz., that numbers that are being multiplied or divided ought to be specified to the same *relative* precision (number of significant digits), and numbers that are being added or subtracted to the same *absolute* precision (position of the first uncertain digit)? In any case, the author (ostensibly Seki, but probably Takebe Katahiro: cf. [**?**, p. 183]) of the Configurations for the Extraction of Roots [開方算式 *Kaihō Sanshiki*] [**?**, pp. 257–268] understood these rules well and used them absolutely correctly when describing the numerical determination of the root of $11+8x+x^2=0$ to high precision [**?**, p. 184].

The next point again concerns "fixed ratios," but this time for the sphere. Like other Chinese mathematicians of the period, Zhu gives values for the numbers solving his problem, but no indication of how one obtains them, nor any verification that they actually satisfy the terms of the problem. Doing the calculation, we find that they do so only if the ratio t is taken to be 9/16. If we assume the formula $t=\pi/6$ for the ratio of the volume of the sphere to the cube of its diameter, this would correspond to a value of $\pi=(3/2)^3=3.375$, a *lü* which has certainly never occurred in the literature and which is clearly much too big. From this we can see that the relation of the ratios associated to the circle and the sphere was not known at this time, and can deduce that the value for the sphere which is being used is $9/16$ (as opposed to $1/2$ or 0.523, which is what one would get with $t=\pi/6$ if one used the "old" or the "Hui" value for π, respectively). And indeed, this deduction can be confirmed: Problem 53-1 of the famous Jinkōki [塵劫記][2] by Yoshida Mitsuyoshi [吉田光由] gives 48 *chi* as the diameter of a sphere of volume 62208 chi^3, again corresponding to a value of 9/16 of the ratio $t=V/d^3$ (and again illustrating the invalid mixing of approximate and precise numbers). In the commentary to this problem in the modern Japanese edition of the Jinkōki [**?**, p. 149] it is stated that this value is taken from the Unified Foundations of Mathematics [算法統宗 *Suanfa Tongzong*] of Cheng Dawei [程大位], which was published in 1592, 35 years before the Jinkōki but almost 300 years after the Introduction to Mathematics, while according to the

[2] Jinkō [塵劫] is a Buddhist term meaning an eternal time or an extremely large number. (editor)

English edition [**?**, p. 178] it can be found already in the Nine Chapters. Tables of the successive (Occidental or Oriental) values of π can, of course, be found in many places in the literature, but they are always based on the "π" of the circle (defined either as the circumference divided by the diameter or as the area divided by the square of the radius, the equality of these two definitions having been known for a very long time), not for the sphere. This suggests the following questions:

Q4. When was the value 9/16 for t first used, and where did it come from? (It is not a particularly good value, being considerably further from the correct value $\pi/6$ than the simpler fraction 1/2.) When was it first improved?

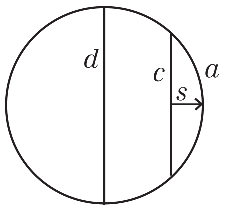
Q5. When was the relation $C = 4\pi/3$ between the constants for the sphere and for the circle first found (the relationship $V = A/3$ between the volume and the surface area of a sphere of radius 1, which follows by considering an inscribed polyhedron with small triangular faces, is much easier and was presumably known earlier), and was it discovered independently in both the East and the West? In any case the correct formulas were known to Takebe [**?**, p. 266].

The final—and perhaps most interesting—aspect of Zhu's problem which we wish to discuss is that a very complicated input, with values of 3 and 3.14 for π and a total "accumulation" of $33622\frac{37}{200}$, leads in the end to a very simple answer in which $a,\dots,e$ are all two-digit whole numbers. This, too, is typical of the ancient Chinese and Japanese mathematical texts: the problems are almost always "fixed" so that, even when the statements are very complicated, the numerical solutions are simple. This is on the one hand nice for the solver, who knows when he has found the right answer, but on the other hand unfair since the author, who usually gives no method of derivation for the solution, starts out knowing the answer and therefore does not actually need to possess any method that would work in general. Moreover, problems "fixed" in this way can also be solved in an easy way, surely not intended: if one knows that the answer is going to come out in small integers, one can find it without really "solving" the problem at all by trial and error, e.g. here by taking one of the unknowns (say a, the side of the cube) as celestial element [天元 *tianyuan*] or chosen independent variable, expressing the other unknowns in terms of this one (here $b,c,d,e = a+4, 3a+12, a, a-3$), and then trying each integer value $a = 4, 5, \dots$ in turn until finding the one for which the total accumulation $a^3 + tb^3 + \dots$ takes on the specified value. From our point of view the whole process seems questionable—the author has "cheated" by working from the answer to the problem rather than inversely, and has given his readers or students the possibility of "cheating" by guessing rather than calculating the answer required—but presumably it did not seem so at the time. In any case, we can ask:

Q6. At what point or by what stages did the shift between problems constructed from known solutions and problems solvable by generally applicable methods occur? Was this distinction ever recognized explicitly by mathematicians of the period?

1.2 The formula for arc length in the Jugairoku

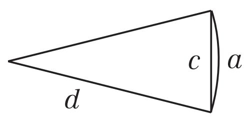
In the Jugairoku [竪亥録][3] by Imamura Tomoaki [今村知商], published in 1639, formulas are given relating the diameter, chord, sagitta ("arrow"), and arc of a circle segment. These formulas, discussed in detail by Horiuchi [**?**, pp. 34–38] are as follows:



$d =$ diameter [径 *kei*]
$c =$ chord [弦 *gen*]
$s =$ sagitta [矢 *shi*]
$a =$ arc [弧 *ko*]

$$d = s + c^2/4s\,,$$
$$c = \sqrt{4s(d-s)}\,,$$
$$s = \bigl(d - \sqrt{d^2 - c^2}\bigr)/2\,,$$
$$a^2 = 4s\bigl(d + s/2\bigr)\,.$$

The first three of these are equivalent to the formula $(c/2)^2 + (d/2 - s)^2 = (d/2)^2$, which follows from Pythagoras's theorem, and are exact, whereas the last is an approximation. (The exact formula is of course transcendental and requires infinite series, as Takebe discovered in 1722.) No derivation for this formula is given. In [**?**], a partial explanation is suggested, namely, that this formula yields the values $a = 0$ for $s = 0$, which is obviously correct, and $a^2 = 5d^2/2$ if $s = d/2$, which is correct if one uses $\pi = \sqrt{10}$, as Imamura did. But this explanation is not sufficient: there are even simpler formulas which would give these two special cases (e.g. $a^2 = 10s^2$), but they would give very poor approximations for other values of s, whereas Imamura's formula is uniformly very good, with an error never exceeding one part in 75. Horiuchi also goes on to say that the Chinese tradition was to use quadratic interpolations, but even then one has a choice of formulas, since any formula of the form $a^2 = ksd + (10 - 2k)s^2$ would give the "right" special values for $s = 0$ and $s = d/2$. One can then ask on what mathematical basis, short of carrying out the much more complicated analysis later given by Takebe and others, one might obtain this simple and very good formula. Here we can offer two explanations, with the hope that experts will be able to say whether one of them might correspond to the procedure Imamura actually used. The first is to consider the asymptotic behavior when s tends to 0, rather than merely the value at $s = 0$.



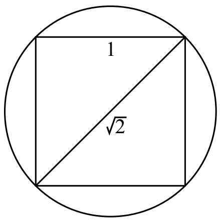
It is obvious from a picture that, when s is very small, a is approximately equal to c, so Imamura's second formula gives $a^2 \approx c^2 \approx 4sd$ and hence $k = 4$.

[3] Jugai [竪亥] is the name of a legendary person in China. (editor)

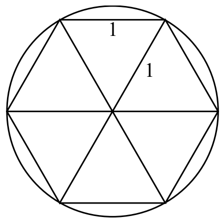
This explanation seems simple and palatable to us, but it is not at all clear that such an argument would have been natural to a Japanese mathematician of the period, and indeed from the talk given by Prof. Morimoto at the Seki memorial conference it seems that even the far more sophisticated Takebe did not employ such asymptotic arguments, but used only precise values based on polygons [**?**, p. 355]. In the case at hand, one can imagine that one simply looks at the pictures for a square and a hexagon, respectively, embedded in a circle.In both cases all the dimensions involved are obvious either *a priori* or from Pythagoras's theorem, and (using Imamura's value $\pi = \sqrt{10}$) one obtains the two further known values

$$\frac{d}{c} = \sqrt{2} \quad \Rightarrow \quad \frac{a}{c} = \frac{\pi\sqrt{2}}{4} = 1.118, \qquad \frac{d}{c} = 2 \quad \Rightarrow \quad \frac{a}{c} = \frac{2\pi}{6} = 1.054.$$

Imamura's formula gives 1.122 and 1.052, respectively, for these two special cases, in excellent agreement with these "exact" values.



$$d = \sqrt{2},\quad c = 1,\quad s = \frac{\sqrt{2}-1}{2},\quad a = \frac{\pi\sqrt{2}}{4},$$



$$d = 2,\quad c = 1,\quad s = 1 - \frac{\sqrt{3}}{2},\quad a = \frac{2\pi}{6}.$$

Of course, it may be that neither of these explanations is correct. Horiuchi mentions that the formula $a^2 = c^2 + 6s^2$, equivalent to Imamura's, is given by Isomura Yoshinori [磯村吉徳] in the Mathematical Methods without Doubts [算法闕疑抄 *Sanpō Ketsugishō*] of 1659 as being based on an unspecified "rule of augmentation" [増術 *zōjutsu*], and also suggests an earlier interpolation method of the Chinese calendarists as a possible, but non-verifiable, source for Imamura's formula. In any case, we can formulate:

Q7. Is there any evidence that could help one decide where the formula for the arc length in the Jugairoku came from, and in particular whether it was based on the considerations of the special cases $c/d = 1/\sqrt{2}$ and $c/d = 1/2$ or on the limiting behavior as c/d tends to 0?

1.3 The value of π in the Sanso

In the Sanso [算組] (or Mathematical Chopping-Board), published in 1663, Muramatsu Shigekiyo [村松茂清] computes π by using polygons of 8, 16, 32, ..., $2^{15} = 32768$ sides to obtain the series of approximations:

$$\begin{aligned}
p_3 &= \underline{3.}061467 \\
p_4 &= \underline{3.}121445 \\
p_5 &= \underline{3.1}36548 \\
&\vdots \\
p_{13} &= \underline{3.141592}5765 \\
p_{14} &= \underline{3.141592}63433 \\
p_{15} &= \underline{3.1415926}487
\end{aligned}$$

The method is fine, although not original (it goes back to the Nine Chapters, and of course in the West to Archimedes), but there are several odd things about Muramatsu's use of it that seem worth commenting on:

(a) Muramatsu calculates to a very high precision, ending up with 8 correct digits of π, but, at least according to the discussion in [**?**], only seems to care about the first three digits, writing that we can neglect the digits 1 and 6 (of 3.1416) but that we do not find any deviation from 3.14 (the value that he ascribes to "Master Meng [孟] from Jin [晋] and Liu Hui from Wei [魏]," although, as already mentioned, Liu Hui actually also gave 3.1416 and Zu Chongzi [祖沖之] had given the much more accurate approximation 355/113 in the 5th century). To a modern mind it seems incredible that one would go to such lengths just to decide between two much older and primitive values, and discard most of what one had calculated with such efforts, bringing us back to the question **Q1** formulated in connection with Zhu Shijie's use of π.

(b) The calculations are carried out to unnecessarily high precision: Muramatsu calculates the sides of each of the successive polygons to 20 decimal digits, even though he could surely have seen after the first few steps that he was gaining only about half a digit of precision with each iteration and therefore, if he was planning to stop with the 2^{15}-gon, was not going to get more than about 8 digits anyway. This is a further instance of the same lack of understanding of the meaning of precision that we already saw in the problem by Zhu Shijie (question **Q3**).

(c) On top of this he was very lucky, because he made three numerical mistakes in his calculation (each of which, of course, persists through all further iterations of the doubling procedure), but each of them occurred only after the 10th digit and therefore did not affect the correctness of the 8 digits which his calculation was in principle capable of producing.

(d) Most importantly, he failed to give any discussion or do any analysis, even a crude one, of the numbers he obtained, not even saying explicitly that they seemed to be converging to a well defined limit or that the persistence of certain digits (those shown underlined in the display above) suggested the correctness of the first 8 digits he had obtained. Had he been the genius that Takebe was, he would have thought of looking at the successive differences (and then perhaps even at the differences of these differences, and then again at the differences of these) and recognized the rules governing these differences, thus obtaining, as Takebe did, many more digits of π without needing to calculate polygons of any larger size than he already had calculated.

1.4 Seki's solution of a problem of the Kokon Sanpōki

In the Hatsubi Sanpō [發微算法] (translatable as something like "Mathematics with Humble Determination"), the only work which Seki Takakazu [関孝和] published during his lifetime, he gave solutions for all of the fifteen bequeathed problems [遺題 *idai*] that had been posed by Sawaguchi Kazuyuki [澤口一之] in the Treatise of Ancient and Modern Mathematics [古今算法記 *Kokon Sanpōki*] of 1671 as a challenge to future mathematicians. We make a few observations here on his solution to the 4th problem, and then in Subsection **??** discuss some peculiar features of the problem itself as well as of Problem No. 14, the most complicated of the fifteen.

Problem 4 is stated in the Treatise of Ancient and Modern Mathematics in the following terms:

> 今有 甲乙丙立方各一。只云 甲積与乙積相併共寸立積十三万七千三百四十坪。又乙積与丙積相併共寸立積十二万千七百五十坪。別甲方面寸為實開平方之見商寸与乙方面寸為實開立方之見商寸及丙方面寸為實開三乗方之見商寸各三和一尺二寸。問甲乙丙方面各幾何。
>
> We have now A, B and C which are each a cube. It is told only that the volume of A and the volume of B together are 137 340 *tsubo*[4], and also that the volume of B and the volume of C together are 121 750 *tsubo*. Furthermore it is told that the quotient in *sun*[5] obtained by placing the side of A as the dividend and opening the square, the quotient in *sun* obtained by placing the side of B as the dividend and opening the cube, and the quotient in *sun* obtained by placing the side of C as the dividend and opening the square multiplied three times, make together 1 *shaku* and 2 *sun*. One asks for the values of the sides of A, B and C.

Stated in modern notation, the problem asks us to find (numerically) the values of three numbers a, b and c satisfying the simultaneous equations

$$a^3 + b^3 = 137340\,, \qquad b^3 + c^3 = 121750\,, \qquad \sqrt[2]{a} + \sqrt[3]{b} + \sqrt[4]{c} = 12\,. \qquad (1)$$

Seki's solution, discussed in detail in [**?**, Chapter 6] (whose notations we follow), is truly amazing. He first replaces the three concrete numbers 137340, 121750 and 12 by letters, say N, N' and N''. He then chooses as *tianyuan* [天元] or basic independent variable the quantity $x = \sqrt[4]{c}$ and defines

$$m = N'' - x\,, \qquad n = N' - x^{12}\,, \qquad o = N - n$$

so that $m = \sqrt[2]{a} + \sqrt[3]{b}$, $n = b^3$, $o = a^3$ and the problem has been reduced to finding x such that $m = \sqrt[6]{o} + \sqrt[9]{n}$. So far, nothing very surprising. But now he introduces the six new quantities

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[4] *tsubo* [坪] is a unit of area and volume. *A tsubo* means normally about $4m^2$ but it is used here to mean 1 sun^3. (editor)

[5] *sun* [寸 *cun* in Chinese] is a unit of length equal to 0.1 *shaku*≒ 3cm. (editor)

$$\begin{aligned} p &= 36m^{14} + 9m^2o^2\,, \\ q &= 252m^5n + 126m^8o\,, \\ r &= 126m^{10}o\,, \\ s &= 9m^{16} + 72m^7n + 18mno + 36m^4o^2\,, \\ t &= m^{18} + 84m^6o^2 + n^2\,, \\ u &= 2m^9n + 168m^3no + 84m^{12}o + o^3 \end{aligned}$$

and then gives the answer to the problem in the form of the remarkable formula

$$\begin{aligned} &o^2p^3 + 3o^2pq^2 + 3opru + 3opst + 3oqrt + 3oqsu + 3ors^2 + or^3 + t^3 + 3tu^2 \\ &= 3o^2p^2q + o^2q^3 + 3oprt + 3opsu + 3oqru + 3oqst + 3or^2s + os^3 + 3t^2u + u^3 \end{aligned}$$

(here some misprints in [**?**] have been corrected), which when multiplied out gives an equation of degree 108 for x. Of course he gives no justification for the correctness of this solution, let alone any indication of how one should go about finding it—this was done later by Takebe Katahiro [建部賢弘] in the *Hatsubi Sanpō Endan Genkai* [發微算法演段諺解] or "Commentaries in the Vernacular on the *Hatsubi Sanpō*," published in 1685—but it is indeed correct. The following aspects strike us particularly:

(a) The solution is incredibly complicated. Even verifying its correctness is tedious and would be seen by most mathematicians today as something that cannot reasonably be done without using an electronic computer, while to find this solution from scratch would appear to a modern mathematician to require modern algebraic tools like Gröbner bases.

(b) Seki doesn't actually write out the full equation of degree 108 for x, but merely says that one can do so if so desired and then solve it by the standard Chinese method. One can wonder—and the question is posed explicitly in [**?**]—whether he himself wrote out or solved this equation. The answer is surely negative, for at least three reasons: First of all, the calculation is so complicated that it would surely have defied even his extraordinary computational abilities, since the polynomial equation in question has the form

$$\begin{aligned} &x^{108} - 9x^{102} + 648x^{101} - 19440x^{100} + 312030x^{99} - 2835000x^{98} + \cdots \\ &\cdots + 812692048405755416419319591620935800302655616965774072340 48x \\ &- 1795038440510576073588274626088072882897678805742137464 3904 = 0\,. \end{aligned}$$

Secondly, no matter how modest he was, he would surely have given some indication of having done such a mammoth calculation if he had actually performed it. Thirdly, the solution as he wrote it out contains one or two minor inaccuracies (one coefficient and one exponent are written incorrectly), so that if he had actually multiplied everything out and found the value of x numerically "by the Chinese method" he would have discovered that it failed to satisfy the conditions of the original problem.

(c) Less trivial is the remark that it would in any case have been superfluous to do this numerical computation, and that for the same reason Seki's entire solution, brilliant though it is, is totally unnecessary if one merely wants to solve the given problem numerically, which is ostensibly his goal (and anyway is the best one can hope for, since, as we know today, polynomial equations of high degree cannot in general be solved exactly in closed form). Indeed, suppose that one were to make the effort and write out the polynomial $P(x)$ of degree 108 completely. The "Chinese method" consists of either binary interpolation or some form of Newton's method, so in its crudest (but numerically quite sufficient) form would consist in finding two values of x for which $P(x)$ takes on opposite signs and then repeatedly bisecting the interval they define and retaining only the half which includes a sign change. But this method works perfectly well for the original problem, without the necessity of any elimination theory at all! Indeed, taking the "*tianyuan*" to be (say) $x = \sqrt[2]{a}$ (here Seki's choice $x = \sqrt[4]{c}$ would serve equally well), we can rewrite the original problem as $f(x) = 12$, where $f(x) := x + \sqrt[9]{137340 - x^6} + \sqrt[12]{x^6 - 15590}$. The algebraic function $f(x)$ could be calculated numerically just as easily as the polynomial function $P(x)$, since the four arithmetic operations and the numerical extraction of square and cube roots with the use of counting rods [算木 *sangi*] were familiar procedures at Seki's time, and by calculating successively the values

$$\begin{array}{lll} f(5) \approx 10.02 & f(6.0) \approx 11.9228 & f(6.06) \approx 11.9877 \\ f(6) \approx 11.92 & f(6.1) \approx 12.0300 & f(6.07) \approx 11.9983 \\ f(7) \approx 12.61 & f(6.2) \approx 12.1327 & f(6.08) \approx 12.0089 \\ \vdots & \vdots & \vdots \end{array}$$

one would obtain after just a few steps an accurate numerical value for x and hence also for a, b and c:

$$\begin{aligned} x &= 6.07158517504163027357\cdots \\ a &= 36.86414653778530412872\cdots \\ b &= 44.35167618766745730349\cdots \\ c &= 32.55638211638110312958\cdots \end{aligned}$$

In other words, it is just as easy to solve the original problem as it is to solve the "simpler" one to which Seki reduces it, and by exactly the same method. Whether Seki was aware of this, of course, must remain moot. We can nevertheless ask:

Q8. Were interpolation methods or "Newton's method" ever used by *wasan* mathematicians for the numerical solution of (non-polynomial) algebraic equations?

(d) Finally, we pose a question concerning an algebraic aspect of Seki's solution. To perform his elimination, he needs to repeatedly transform algebraic equations of the form $Q(x, \sqrt{y}) = 0$ or $Q(x, \sqrt[3]{y}) = 0$, where y is a polynomial in x, into purely polynomial equations for x. The method to do this (stated by Takebe in the *Endan* to the *Hatsubi Sanpō* and mentioned by Prof. Komatsu in his talk at this conference [**?**]) is based on the two algebraic identities

$$A+B=0 \ \Rightarrow\ A^2-B^2=0\,, \tag{2}$$
$$A+B+C=0 \ \Rightarrow\ A^3+B^3+C^3-3ABC=0\,. \tag{3}$$

Then in the first case we can split up the polynomial $Q(x,\eta)$ into even and odd powers of $\eta=\sqrt{y}$ to rewrite the given equation as $Q_0(x,\eta^2)+\eta Q_1(x,\eta^2)=0$ and use (2) to replace this by the purely polynomial equation $Q_0(x,y)^2-yQ_1(x,y)^2=0$, and similarly in the second case split up $Q(x,\eta)$ according to the values modulo 3 of the exponents of $\eta=\sqrt[3]{y}$ to rewrite the equation as $Q_0(x,\eta^3)+\eta Q_1(x,\eta^3)+\eta^2 Q_2(x,\eta^3)=0$ and use (3) to replace this by the polynomial equation $Q_0(x,y)^3+yQ_1(x,y)^3+y^2Q_2(x,y)^3-3yQ_0(x,y)Q_1(x,y)Q_2(x,y)=0$. The identities (2) and (3) can be verified easily by using the factorizations $A^2-B^2=(A+B)(A-B)$ and $A^3+B^3+C^3-3ABC=(A+B+C)(A^2+B^2+C^2-AB-AC-BC)$, but the second of these is not obvious and in any case does not explain on what basis the expression $A^3+B^3+C^3-3ABC$ was found originally. More synthetic proofs of (2) and (3), not requiring one to know the answers in advance, can be given using substitution rather than factorization:

$$\begin{aligned} B=-A \quad &\Rightarrow \quad B^2=(-A)^2=A^2\,,\\ C=-A-B \quad &\Rightarrow \quad C^3=(-A-B)^3=-A^3-3A^2B-3AB^2-B^3\\ &\qquad\quad =-A^3-B^3+3ABC\,. \end{aligned}$$

Q9. Can one determine whether the equations (2) and (3) for elimination of square and cube roots were first obtained by *wasan* mathematicians by factorization, by substitution, or by some other method? Cf. [**?**, p. 150].

1.5 On Problems 4 and 14 of the Kokon Sanpōki

In the last subsection we discussed Seki's solution of Problem 4 of the *Kokon Sanpō ki*, but not where the problem itself comes from. We observed above that the Chinese and Japanese tradition usually involved "fixing" problems in advance so that, even if the data in the problem was complicated (like the number $33622\frac{37}{200}$ in Zhu's problem), the answer came out in simple integers (like 24, 28, … in that problem). But here the situation is different: the two numbers 137340 and 121750 in equation (1) are just as complicated as the number occurring in Zhu's problem, but the solution, as we saw above, is very far from being integral. Why, then, did Sawaguchi choose the specific numbers 137340 and 121750? The following simple numerical experiment suggests that they were not random choices. We look at the decompositions of 12 as a sum of three positive integers (there are only 55 of them, so this is easily done), and in each case define a, b and c as the square, cube, and fourth power, respectively, of the three summands. Then in one case, viz., $a=49$, $b=27$, $c=16$, we find that two of the sums of their cubes differ by only a few units from the numbers in Sawaguchi's problem. True, the sums of cubes in question are the wrong ones, since for these values of a, b and c one has

$$a^3 + b^3 = 137332\,, \qquad a^3 + c^3 = 121745\,, \qquad \sqrt[2]{a} + \sqrt[3]{b} + \sqrt[4]{c} = 12 \qquad (4)$$

instead of (1), with a rather than b in the second equation. But the probability of this near-coincidence being accidental is astronomically small, so we can ask:

Q10. Was Problem 4 in the *Kokon Sanpōki* in some way based on equation (4)?

Obviously, there is no way to decide this for sure (Sawaguchi, like everybody else in this business, kept his secrets well), but if we take as a working hypothesis that the two equations are in fact connected and try to imagine how the connection might look, we may be led to relationships that can be tested in others of Sawaguchi's problems. We first observe that, if Sawaguchi had simply started out (*à la* Zhu) with the integral values $\sqrt[2]{a} = 7$, $\sqrt[3]{b} = 3$, $\sqrt[4]{c} = 2$ to obtain the three relations (4), the problem obtained would have been much too simple for him to include in his collection of 15 *idai*: a potential solver would only need to calculate $\sqrt[3]{121745} = 49.5\cdots$ and then try subtracting 49^3 from 137332 and 121745 to discover that the differences were a 9th and a 12th power, respectively, making the solution immediate. So we can imagine that Sawaguchi might have changed the problem a little to make it non-trivial, in one of two ways:

(a) replace the numbers "137332" and "121745" in (4) by very similar numbers, say 137340 and 121750;

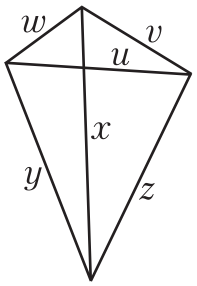
(b) and/or replace the left-hand side of the second equation in (4) by $b^3 + c^3$.

If he had done merely (a), this would have had the advantage that he could be fairly sure that the new problem, even if he did not know how to solve it analytically, was non-defective (i.e., had a unique real solution). Doing (b) as well, to give the problem as actually stated in the *Kokon Sanpōki*, would not have this property, and it is quite possible that Sawaguchi simply made a mistake in transcription. In that case, the problem he meant to give (namely, (1) with $a^3 + c^3$ in the middle equation) would have had a solution in numbers very close to integers:

$$\sqrt[2]{a} = 7.00007044\cdots\,, \quad \sqrt[3]{b} = 3.00001517\cdots\,, \quad \sqrt[4]{c} = 1.99991437\cdots\,.$$

If this hypothesis is true, then we would expect some others of his problems to have the same property, and indeed, we will now see that this is the case.

Problem 14, discussed in detail at the lecture by Professor Komatsu [**?**] at this conference, is the most complicated of the 15 *idai* in the *Sanpōki*, since it leads (as Seki showed) to a polynomial equation of degree $2 \times 3^6 = 1458$. The problem requires finding four points in the plane such that the differences of the cubes of the six distances x, y, z, u, v, w between them have given values:



$$\begin{aligned} x^3 - y^3 &= 271\,, \\ y^3 - z^3 &= 217\,, \\ z^3 - u^3 &= 60.8\,, \\ u^3 - v^3 &= 326.2\,, \\ v^3 - w^3 &= 61\,. \end{aligned} \tag{5}$$

These five equations alone would lead to an indeterminate problem in the six unknowns and must be supplemented by a sixth equation

$$\begin{aligned} P(x,\dots,w) := {} & (u^2+v^2-w^2)(x^2y^2+w^2z^2) + (v^2+w^2-u^2)(y^2z^2+u^2x^2) \\ & + (w^2+u^2-v^2)(z^2x^2+v^2y^2) - u^2x^4 - v^2y^4 - w^2z^4 - u^2v^2w^2 = 0 \end{aligned}$$

expressing the fact that the four vertices are coplanar. (The polynomial P, divided by 144, gives the square of the volume of a tetrahedron with edges of length $x,\dots,w$ and hence vanishes if the vertices lie in a plane.) This last equation, as discussed in [**?**], is contained in the *Sanso* and hence was known to both Sawaguchi and Seki.

In his talk, Professor Komatsu cited the numerical solution

$$\begin{aligned} & x = 10.0000056403\,, && y = 9.0000069815\,, && z = 8.0000083910\,, \\ & u = 7.6699093899\,, && v = 5.0000228360\,, && w = 4.0000359240\,. \end{aligned}$$

calculated by Dr. Kinji Kimura (cf. [**?**]) using Gröbner bases and a numerical computation library. These approximate values certainly fit in with the prediction of being, in five cases out of six, very close to integers. Let us suppose that Sawaguchi started with the arbitrarily chosen simple integral values

$$x = 10, \quad y = 9, \quad z = 8, \quad v = 5, \quad w = 4 \tag{6}$$

for five of the unknowns. Then substituting them into the equation $P(x,\dots,w) = 0$ he would have ended up with the quadratic equation

$$100\,u^4 - 6065\,u^2 + 10726 = P(10, 9, 8, u, 5, 4) = 0$$

for u^2, giving the numerical value

$$u = \sqrt{\frac{1213 + 69\sqrt{273}}{40}} = 7.6698551213\cdots \tag{7}$$

for the sixth unknown u. These numbers would have led in turn to the values

$$\begin{aligned} x^3 - y^3 = 271\,, \quad y^3 - z^3 = 217\,, \quad z^3 - u^3 = 60.80790567\cdots\,, \\ u^3 - v^3 = 326.19209432\cdots\,, \quad v^3 - w^3 = 61 \end{aligned}$$

for the data of the problem. These are strikingly similar to the numbers in (5), the only difference, apart from the degree of precision chosen for the non-integral numbers, being that one has 60.8 instead of 60.08. This suggests:

Q11. Was the number 60.08 in Sawaguchi's problem originally 60.8 ? Was the intended solution the one given by (6) and (7)?

Here we have a definite prediction that can be checked against the documentary evidence. In the facsimile copy of the *Hatsubi Sanpō* [**?**], the number in question is given as 60.08, but of course Seki might easily have made an error of transcription in copying from the *Kokon Sanpōki*. To check this, Prof. Komatsu kindly asked the director of a *wasan* exhibition at Tokyo University of Science that was being held concurrently with the conference to show us an original edition of the *Sanpōki* that was on display there. Rather disappointingly, the number there also had a clear "0" [零 *rei*] before the "8". However, later we noticed that the problem as cited in 1914 by Smith and Mikami [**?**, p. 101] indeed gave the relevant number as "60.8," which suggests that there must be different original editions of the *Sanpōki* and that there was a copying error at some stage. Furthermore, the numerical solution of the problem as given in (5) is

$$x = 9.9977624076\,, \qquad y = 8.9972373104\,, \qquad z = 7.9965030158\,,$$
$$u = 7.6660952225\,, \qquad v = 4.9910355706\,, \qquad w = 3.9859690172\,,$$

while the solution for the problem with 60.08 replaced by 60.8 is

$$x = 10.0000057162\,, \quad y = 9.0000070570\,, \qquad z = 8.0000089315\,,$$
$$u = 7.6699096344\,, \qquad v = 5.0000228647\,, \qquad w = 4.0000357259\,.$$

The latter numbers are very much closer to the ones given by Kimura, suggesting that he, too, must have used a source in which the third number in Sawaguchi's problem appeared as 60.8 rather than 60.08.

We have given this numerical analysis at some length because the details of the various numerical coincidences give concrete and rather convincing support to hypotheses which would otherwise by their nature be somewhat speculative. To go further, of course, one should search for more cases of the same phenomenon. Our next question is therefore:

Q12. Are there other problems, by Sawaguchi or other authors of the time, whose solutions involve numbers very close to integers and which thus may have been constructed starting from integral solutions?

We mention one final point. If the above reconstruction is correct and Sawaguchi really intended his problem to have the solution given by equations (5) and (6), then one can wonder whether he might not have attempted to give a version having a solution consisting *entirely* of integers. To do this, he would have had to find a solution in integers of the indeterminate problem $P(x,\ldots,w) = 0$, and could then simply have given the values of the differences $x^3 - y^3, \ldots, v^3 - w^3$ as the data in the other five equations of his problem. In fact the equation $P(x,\ldots,w) = 0$ does have various integral solutions, examples being $(x,y,z,u,v,w) = (17,13,8,9,11,5)$,

$(22,17,19,4,8,6)$, or, with $x > y > z > u > v > w$, $(27,24,17,13,11,9)$. Of course, these are not particularly easy to find without a computer, although they are easy enough to verify numerically, but there are many other problems (in particular, those concerning Pythagorean triples, i.e., right triangles with sides of integral length) whose formulations in terms of geometry that was familiar at this epoch would have been quite natural and which can be solved by hand, either by systematic analysis or by trial and error. This suggests our final question:

Q13. Were any problems of what we now call Diophantine analysis ever considered by Japanese mathematicians during the Edo period?

2 Did Seki or Takebe learn any mathematics from the Dutch?

The mathematics of Seki and Takebe, seen from the point of view of the twenty-first century, is a "quantum jump" beyond of that of their Chinese or Japanese predecessors, and it is of course a very natural question whether this progress was due entirely to the natural genius of these two mathematicians or whether they were directly or indirectly inspired by the mathematics done in Europe during the preceding decades. Western authors in the early twentieth century tended to say that there of course must have been such an inspiration. But they give little evidence, and a modern reader cannot help feeling that this is often simply a reflection of their *a priori* belief that the miracles of Western science could not have been discovered by the members of such a different culture.

For instance, George Sansom, in his well-known book *Japan, A Short Cultural History* [**?**], quotes Engelbert Kaempfer, who came to Nagasaki in 1690 as a physician for the Dutch, as having said that the Japanese "had little taste for speculative philosophy, which ... they thought an amusement proper for lazy monks" and then goes on to say

> The crowning triumph of the Western intellect, the great gift which at that time Europe might have made to them, they were either unprepared or unwilling to receive, for, to quote the same authority, they knew "nothing of mathematics, more especially of its deeper and speculative parts" [**?**, p. 421].

In a footnote [**?**, p. 428], he adds that—whereas the Chinese seemed to have made few advances after their discoveries of the 12th and 13th centuries—

> The Japanese worked out an original method of the differential calculus from hints coming through the Dutch, and in general they appear to have displayed remarkable ingenuity in application of a limited knowledge; but Kaempfer's judgement as to their backwardness in theory seems to have been correct.

But he gives no evidence at all for his assertion that the progress made by the Japanese was based on "hints coming from the Dutch."

Similarly, in the well-known history of Japanese mathematics by Smith and Mikami, we find a somewhat vague reference to "others whose names are not now remembered" who "might have formed a possible medium of communication with the West in the time of Seki," followed by the more specific passage

> . . . we have the record of two men who were in touch with Western mathematics. These men were Hayashi Kichizaemon and his disciple Kobayashi Yoshinobu, both of them interpreters in the open port of Nagasaki. Each of these men knew the Dutch language, and each was interested in the sciences, the latter being well versed in the astronomy of the West. . . . While it is probable that these men did not know much of the European mathematics of the time, it is inconceivable that they were unaware of the general trend of the science, and that they should fail to give to inquirers some hint as to the nature of this work. [**?**, pp. 140–141]

and then, in summary, the following:

> The conclusion appears from present evidence to be that some knowledge of European mathematics began to find its way into Japan in the seventeenth century; that we have no definite information as to the nature of this work beyond the fact that mathematical astronomy was part of it; that there is no evidence that Seki or his school borrowed their methods from the West; but that Japanese mathematicians of that time might very well have known the general trend of the science and the general nature of the results attained in European countries [**?**, pp. 141–142].

But the authors give little evidence, either here or later, to substantiate the last of these assertions, and in general it has to be said that this book, despite the eminence of the first author as a historian of Western mathematics and the preeminence of the second author as a historian of Japanese mathematics, is marred in many places by the Western prejudices of its first author (cf. [**?**, p. XXVI, footnote 3]).

The question about Western influence obviously cannot be resolved easily, at least without further evidence coming to light, and certainly not by the present authors. But we would like to add a few intrinsic arguments concerning the various "hints from the Dutch" theses, and then give some concrete documentary evidence supporting the opposing viewpoint, that Seki and Takebe had not had any direct contact with the mathematics which had been done in Europe between the closing of Japan in 1639 and the lifting of the partial ban on foreign books in 1720.

The first and most obvious argument is that at least some of the discoveries of Seki and Takebe cannot have come from Western sources simply because they predated them. The most notable of these is of course Seki's independent discovery of determinants, which were found also by Leibniz in 1693 but published only after his (and *a fortiori* after Seki's) death, and which were treated in much greater generality by Seki; here even Smith and Mikami speak of a "marked proof of Seki's genius" and concede that "Seki was not only the discoverer but that he had a much broader idea than that of his great German contemporary" [**?**, pp. 124–125]. A further example are the Bernoulli numbers, which Seki discovered independently of Jacob Bernoulli and actually published earlier (the publications by Seki and Bernoulli, both posthumous, are from 1712 and 1713, respectively). This example is in some ways even more striking than the first, since determinants are an essential mathematical tool which must inevitably be discovered when one studies systems of linear equations (as had been done in the East for centuries), while Bernoulli numbers constitute a far more specific discovery that is not an inevitable consequence of any particular "general trend in the science." Seki's theory of elimination, too, although undoubtedly based in part on Chinese precedents, goes much further than anything that European mathematicians could have done at the time, or indeed for the next hundred years. Knowing that in these cases the discoveries were his, why should we doubt that other discoveries that Seki made, even when these occurred later than in the West, were wholly his own?

Secondly, and no less importantly, the level of the mathematical advances we are speaking of is so high that it is hard to imagine how they could have been transmitted by general osmosis or "hints." Before the lifting of the ban, these hints would have had to come via the Dutch. But, as Goodman, who studied the Dutch-Japanese interaction more intensively than any other historian, says,

> Further, the Dutch who came to Japan were hardly trained scientists and were no more able to respond to scientific inquiries abroad than they would have been at home … the overwhelming proportion of Japanese contact with Hollanders was with the average employees of the Dutch East India Company … [**?**, pp. 64–65].

And even if one imagines, despite the inherent unlikeliness and the lack of any evidence to this effect, that at some point there was a member of the Dutch contingent who was versed in the state-of-the-art mathematics of the day and had Japanese interlocutors capable of and interested in learning this material, it would still have been impossible for the former to transmit his knowledge to the latter because of the linguistic barriers involved. One must remember that, while in the 16th century there were a number of Europeans (principally Jesuit missionaries) who had spent years in Japan and had mastered the language, and of Japanese who had become fluent in various European languages, the situation was completely different during the first half of the Edo period, when the *bakufu* followed a deliberate policy of preventing linguistic competence on either side, demanding that members of the Dutch contingent (in particular, the *opperhoofd* or ship's captain) be replaced every year so that they could not learn too much about Japan and themselves providing Japanese interpreters with a very inadequate knowledge of the Dutch language. There was no serious training of interpreters before the lifting of the ban, and no Dutch-Japanese dictionary until 1745. That sophisticated abstract mathematical ideas like interpolation or the series development of functions could have been discussed in such a context seems very unlikely.

In a related vein, one must remember that not only the language, but also the backgrounds, the styles of presentation and above all the aims of the research itself were so different in the two cultures that it is by no means clear that a Japanese mathematician, exposed in an unsystematic way to a piece of Western mathematics, would have been able to appreciate it (just as, of course, a Western mathematician would not have been able to appreciate Japanese mathematics without long exposure to it). And indeed, even as late as 1811 a Japanese mathematician who *had* seen Western mathematics was able to write with obvious conviction that "foreign mathematics is not on so high a plane as the mathematics of our own country," (quoted in [**?**, p. 172]).

Of course, European mathematical knowledge transmitted in *written* form might have been understood by a qualified mathematician working imaginatively against the language barrier. But despite the vague comments of Sansom or of Smith–Mikami, there seems to be no evidence that such texts were available before 1726, the date of the importation of the Encyclopedia of Calendrical Mathematics 曆算全書 *Lisuan Quanshu*] by Mei Wending [梅文鼎], into which Takebe had the marks necessary for Japanese to read the Chinese text inserted. Thus, when Smith (we assume that it was he) challenges the credit given to Takebe for his infinite series for the square of the arc, writing

> The series seems, however, to have been given by Pierre Jartoux, a Jesuit missionary, resident in Peking ... There is a tradition that Jartoux gave nine series, of which three were transmitted to Japan, and it seems a reasonable conjecture that Western learning was responsible for his work, that he was responsible for Takebe's series, and that Takebe explained the series as best he could [**?**, pp. 154–155],

he gives a few internal arguments for this thesis, such as the somewhat awkward and unconvincing presentation of the series in Takebe's text, but again no documentation that Takebe had seen the series that were transmitted to Japan. (In fact, a footnote to the passage says that the three series in question appeared in a book by Mei Kucheng [梅瑴成] with unknown date and *without* evidence that they reached Japan in this period, i.e., before Takebe's *Fukyū Tetsujutsu* [不休綴術] containing his infinite series was published.) In the book by Horiuchi, as well as a much more careful analysis of the Jartoux issue [**?**, pp. 296–298], we find a different possibility for the path that might have led Takebe to infinite series:

> ...Seki's treatise on faulty problems bristles with ideas, more or less well-mastered, on the relations between the coefficients and the roots. It is also the place where Seki proceeds for the first time to the extraction of a literal magnitude in an equation with literal coefficients. So we are in the presence here of a new extension of the use of literal notation, an extension that will later inspire Takebe to express the square root of a quadratic equation in the form of an infinite series [**?**, p. 182].

Later, there is a careful analysis of Seki's procedure of the arc, giving a coherent background on which Takebe's series could have arisen without any Western input, and ending with the remarks

> Here again, the methods brought into play by Seki to treat the problem are incomparable with those of his predecessors ... We thus see that Seki extended the use of this algebraic tool to contexts where it was assumed a priori that there existed a functional relation between two geometric quantities even though the expression for this relation was not yet known. Here we touch an essential feature of Seki's works, that of having considerably enlarged the domain of use of algebraic techniques. Mikami very early stressed the historical importance of the solution proposed by Seki, which he considered more important than the exact solution later obtained by Takebe ... Mikami here forcefully asserts that, if Seki had not introduced algebraic techniques into this domain, Takebe's solution would never have seen the light of day [**?**, p. 251].

One final argument—admittedly of a more subjective nature—is on the level of mathematical style and of the personalities of the protagonists. Written mathematics is characterized not only by its contents, but by the way in which the author sees and presents the discoveries that he is expounding. Seki's approach to algebraic problems, as indicated in the passage just quoted, was extraordinarily original and innovative, but Seki himself, who was steeped in and deeply respectful of the Chinese tradition, typically couched his exposition in the context of this tradition, and indeed may well have believed that he was working within it even when in fact he was doing something very new. We saw one example of this in the first part of this paper, where he formulated his solution of a problem of Sawaguchi as if he were presenting an algorithm for the numerical determination of the sought-for root, but in fact showed no interest in actually finding this root but instead describes a purely algebraic procedure for expressing the solution of the problem as the root of a polynomial—a modern mathematician *malgré lui*. He presents the elimination theory in the traditional language of *sangi* [算木] but uses these in a new way, very different from his predecessors in Japan or China, and of course even more different from the way in which any European mathematician, had he achieved the same results, would have formulated them. If Seki had been exposed, even tangentially, to Western mathematical thinking, then surely one can speculate that this might have affected the presentation, as well as the contents, of what he wrote. As to Takebe, it can be mentioned that he was particularly noted for his honesty, that he was far more inclined to speak of his own weaknesses in mathematics than to take unearned credit for things he had not done, and that he explicitly expressed his enthusiasm about ideas in astronomy coming from Europe when he learned about them after the lifting of the ban [**?**, pp. 225–226]. There seems to be no reason to think that he would have kept silent about recent beautiful mathematics done by Europeans if he had known about or made use of it.

We now turn to the documentary evidence. We have already mentioned that the *bakufu* [幕府], fearful of the Hollanders' acquiring too much knowledge about Japan, required that the ship's captain be replaced every year. The East India Company, which was just as interested in ensuring that precisely this knowledge should be available, therefore had each *opperhoofd* keep a very detailed diary and made sure that several transcriptions of it should be made, so that at least one copy would survive the perilous return trip to Holland. These diaries, preserved in their entirety in the Dutch National Archives in The Hague, constitute a miraculous record, of a dimension perhaps unparalleled by any other historical document, of the relations between the Dutch contingent and the Japanese during the entire Edo period. A search through the diaries of the relevant years failed to turn up any evidence of an actual exchange of information on any kind of mathematical question whatsoever. There was one fully documented personal encounter between Takebe and the Hollanders in 1727 in which Takebe put detailed questions to the Dutch. These questions concerned such issues as the way the Dutch put out fires, how they named their sons, whether they used matches, fans, or ear-picks, and whether they were acquainted with black magic (the complete list of questions is given in the Appendix), but absolutely nothing specifically scientific, let alone mathematical. Indeed, there is no indication anywhere in the diaries that the Dutch were even aware that Takebe (whom they knew only as an ambassador of the Shogun and consistently referred to as "the Imperial Minion") was a mathematician at all. The only thing indicating that they had recognized Takebe's ability is a diary entry a few pages later in which the captain tells how a defective Dutch watch was said to have been repaired by Takebe and that he "gladly believed this" because, on the occasion when Takebe had asked him the thirty questions, he had observed "that nature had not spared in dispensing knowledge to him, but that he is versed in different sciences and is a fine erudite man" ("in verschejde wetenschappen ervaaren en een fijn doorsleepen man is"). But this is still very far from a technical discussion of mathematics! And indeed, in the entry for the very day on which Takebe put his thirty questions, the captain describes how at the end of their meeting the Japanese continued with various "mathematical, astronomical and geometric propositions, … to which ⟨I⟩ responded never to have learned the said Sciences, with which the interrogation came to an end." This agrees well with the comment of Goodman cited above.

In summary, the documentary evidence, though admittedly inconclusive, seems to us to give more support to the view that the Japanese did *not* learn any mathematically interesting facts from the Dutch than to the view that they did do so.

Appendix: Takebe's questions to the Dutch in Dejima

We give here a translation of the part of the diary entry for March 25, 1727, that enumerates the questions put by Takebe. The rest of the text, describing the sea-bass that the Japanese presented to the Dutch and wished to watch them to eat, a meeting with the Imperial watchmaker, and the captain's inability to respond to the "mathematical, astronomical and geometric propositions" put forward by the Japanese delegation, is omitted.

Tuesday 25th

In the morning around ten o'clock came the Senior Interpreter Kizits with Lord Takebe Fiko Sira Sama accompanied by three of the First Servants of his Imperial Majesty named [Master] Faomi Foukan, Maayeda Kioriso-o and Ito ga au, arrived, mutually having exchanged some compliments, ⟨they⟩ sat down, ⟨and⟩ the following questions were asked, such as

Firstly — the four elements, whether they are known to us, and how they were named;
Secondly — the Elements, whether we don't apply them to the human body;
Thirdly — the twelve Zodiac signs, whether we know them, how they are named, and whether these are applied to anything;
Fourthly — compasses, what about the compass which points wrongly, whether this by some means can be discovered and demonstrated;
Fifthly — whether without compass, East can be shown and by what means;
Sixthly — whether East and West are known;
Seventhly — whether we have lanterns in use like the Japanese: if so, how and from which kind of material they are made;
Eighthly — whether witchcraft, or black magic, is known to us and is in use;
Ninthly — on which day we rest, and on which ones we work, ⟨and⟩how these are called;
Tenthly — Sunday, how long it has been in use, and why so named;
Eleventhly — the measure of an *ikken* [1.92 m.], and whether others are in use and how they are named;
Twelfthly — ⟨whether⟩ the *gantang* [measure used for rice and pepper, ca. 8 1/2 liters] is used by us, or whether ⟨there are⟩ others and how these are then named;

Thirteenthly — the *dai ching* [steelyard balance], whether it is used for the money that is paid out daily, or for what;
Fourteenthly — Dutch houses, whether they are like the Japanese, then how they are built;
Fifteenthly — warehouses, how they are built by the Hollanders;
Sixteenthly — the Lords' or big manor houses, whether they aren't distinguishable from ordinary men's houses;
Seventeenthly — whether when there is a fire in Holland, whole streets burn down at once like in Japan;
Eighteenthly — how and by what a fire is extinguished;
Nineteenthly — rice, wheat, barley, buckwheat and other grains, how they are presented;
Twentiethly — rice as well as other grains, whether they can be stored in Holland and on Jakarta longer than a year;
Twenty-firstly — the names which are born by somebody for a thousand years or less, whether the descendants may continue to bear them;
Twenty-secondly — whether the Daimyo or other great ⟨persons⟩ don't change their names when they get another function;
Twenty-thirdly — whether the Hollanders like the Japanese have two names in use, i.e. how one names each other daily and the other in writing, that is when one signs one's name;
Twenty-fourthly — when to the father or to the master of a house a son is born, whether he himself gives the name or whether the name is given by Dutch priests, and whether like with the Japanese the name then is used for a signature seal;
Twenty-fifthly — the sulfur-match, whether it is used by us Hollanders, and from what it is made;
Twenty-sixthly — fire, when one wants to keep it, how this is done;
Twenty-seventhly — ink, from what substance it is made;
Twenty-eighthly — quills, from what they are made;
Twenty-ninthly — ear-picks, whether they are used by us;
Thirtiethly — fans, whether they are in use by the Hollanders.
Which above-mentioned questions ⟨I⟩ answered according to my knowledge of science shortly and concisely, . . .

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