CHAPTER 9

Modular Forms with Rational Periods

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INTRODUCTION

Classically, one of the main reasons for the importance of modular forms in number theory was the fact that spaces of modular forms are spanned by forms with rational Fourier coefficients and that these coefficients are often arithmetically interesting functions – one need only think of the numbers of representations of integers by quadratic forms (coefficients of theta series), the sums of powers of divisors of integers (coefficients of Eisenstein series), and the Ramanujan tau-function (coefficients of the discriminant function). The appearance of interesting functions as Fourier coefficients, coupled with the finite dimensionality of the spaces of modular forms, leads to non-trivial identities and congruences with a wide range of applications (asymptotics of numbers of representations by quadratic forms, partition identities, p-adic L-functions, connections with finite simple groups and representations of Lie algebras, examples of non-isometric isospectral Riemannian manifolds, coding theory, etc.).

On the other hand, spaces of modular forms have natural rational structures other than those given by the rationality of Fourier coefficients, namely those defined by the rationality of *periods*. Specifically for $f \in S_{2k} = S_{2k}(SL_2(\mathbb{Z}))$ (we shall for simplicity consider only the full modular group in this paper, and therefore suppress it in the notations) one defines[†] the nth period of f by

$$r_n(f) = \int_0^\infty f(it)t^n dt \qquad (0 \leqslant n \leqslant w := 2k-2);$$

[†] The usual definition of $r_n(f)$ as $\int_0^{i\infty} f(z)z^n dz$ differs from our definition by a factor i^{n+1}; we have preferred a normalization which is real for real f.

then the results of Eichler and Shimura, reviewed in Section 1.1, imply that there exist forms all of whose even or odd periods are rational. More precisely, their work implies that each of the $\mathbb{Q}$-vector spaces

$$\mathfrak{S}_{2k}^{+} = \{f \in S_{2k}\colon\ r_n(f) \in \mathbb{Q} \quad \text{for} \quad 0 \leqslant n \leqslant w, \quad n \quad \text{even}\},$$
$$\mathfrak{S}_{2k}^{-} = \{f \in S_{2k}\colon\ r_n(f) \in \mathbb{Q} \quad \text{for} \quad 0 < n < w, \quad n \quad \text{odd}\}$$

gives a rational structure on S_{2k} (i.e. $\mathfrak{S}_{2k}^{\pm} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} S_{2k}$). Thus S_{2k} has two natural rational structures besides the usual rational structure

$$\mathfrak{S}_{2k}^{0} = \{f \in S_{2k}\colon\ f(z) = \sum_{l=1}^{\infty} a(l) e^{2\pi i l z}, \quad a(l) \in \mathbb{Q} \quad \text{for} \quad l \geqslant 1\}.$$

The purpose of this paper is to give examples of functions belonging to $\mathfrak{S}_{2k}^{+}$ and $\mathfrak{S}_{2k}^{-}$ whose periods are arithmetically interesting expressions – relating to Bernoulli numbers, to binary quadratic forms, to zeta-functions of real quadratic fields, to modular forms of half-integral weight, and to Hilbert modular forms. It is to be hoped that the $\mathbb{Q}$-structures coming from rationality of periods will be a rich source of relations between modular forms and arithmetic, just as the more familiar $\mathbb{Q}$-structure coming from rationality of Fourier coefficients has been in the past.

1. THE EICHLER–SHIMURA ISOMORPHISM AND THE PERIODS OF R_n

1.1. The Eichler–Shimura Theorem

In this section we review the Eichler–Shimura theory in a fair amount of detail. The following notations will be used here and throughout the paper:

Γ is the full modular group $\mathrm{PSL}_2(\mathbb{Z})$, acting in the usual way on the upper half-plane $\mathfrak{H}$; elements of Γ will be denoted $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ rather than $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$$

(thus $S^2 = U^3 = 1$, $U = TS$).

$$\tilde{\Gamma} = \mathrm{PGL}_2(\mathbb{Z}) = \Gamma \cup \varepsilon\Gamma, \qquad \varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

k is an integer greater than or equal to 1; $w = 2k - 2$.

M_{2k} (resp. S_{2k}) is a space of modular (resp. cusp) forms of weight $2k$ on Γ.

$L(f,s)$ $(f \in S_{2k})$ is the L-series of f, i.e., the analytic continuation of

$$\sum_{l=1}^{\infty} a(l)l^{-s}, \quad \text{where} \quad f(z)=\sum_{l=1}^{\infty} a(l)q^{l} \qquad (q=e^{2\pi i z}).$$

$$r_n(f)=\int_0^{\infty} f(it)t^n dt = n!(2\pi)^{-n-1}L(f,n+1) \qquad (0\leqslant n\leqslant w).$$

$$(f,g)=\int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)}\,y^{2k-2}dx\,dy$$

is the Petersson scalar product of f and g.

$\mathbf{V}(K)=\mathbf{V}_{2k-2}(K)=\mathrm{Sym}^w(K\oplus K)=\{$polynomials of degree $\leqslant w$ in one variable with coefficients in $K\}$, where K is any subfield of $\mathbb{C}$. (The letter K will sometimes be omitted from the notation if it is clear which field – usually $\mathbb{C}$ or $\mathbb{Q}$ – is meant.) The space $\mathbf{V}(K)$ is acted on by $\mathrm{PGL}_2(K)$ via

$$(P|\gamma)(X)=(cX+d)^w P\left(\frac{aX+b}{cX+d}\right) \qquad \left(P(X)\in\mathbf{V},\ \gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right).$$

In particular, $\mathbf{V}(K)$ is stable under $\tilde{\Gamma}$ for any K. The element ε acts by $(P|\varepsilon)(X)=P(-X)$ and splits $\mathbf{V}$ up into the direct sum of the spaces $\mathbf{V}^+$ and $\mathbf{V}^-$ of even and odd polynomials, respectively. The action of $\tilde{\Gamma}$ on $\mathbf{V}(K)$ extends to an action of the group algebra $\mathbb{Q}[\tilde{\Gamma}]$; we define

$$\begin{aligned}\mathbf{W}=\mathbf{W}_{2k-2}&=\ker(1+S)\cap\ker(1+U+U^2)\\ &=\{P\in\mathbf{V}: P+P|S=P+P|U+P|U^2=0\}.\end{aligned}$$

From $\varepsilon S\varepsilon=S$ and $\varepsilon U\varepsilon=SU^2S$ it follows that $\mathbf{W}|\varepsilon=\mathbf{W}$, so that $\mathbf{W}=\mathbf{W}^+\oplus\mathbf{W}^-$ ($\mathbf{W}^\pm=\mathbf{W}\cap\mathbf{V}^\pm$). Explicitly, the relations defining $\mathbf{W}$ are

$$\sum_0^w a_n X^n\in\mathbf{W}\Leftrightarrow a_n=(-1)^{n-1}a_{w-n},\ \sum_{m=0}^{w} c_{mn}a_m=0 \qquad (0\leqslant n\leqslant w),$$

where

$$c_{mn}=\begin{cases}\dbinom{m}{n}, & \text{if } m\geqslant n,\\[2ex] \dbinom{w-m}{w-n}, & \text{if } m\leqslant n.\end{cases}$$

Since these have rational coefficients, the space $\mathbf{W}$ is defined over $\mathbb{Q}$; i.e.

$$\mathbf{W}(\mathbb{Q})\otimes_{\mathbb{Q}} K\xrightarrow{\sim}\mathbf{W}(K)=\mathbf{W}(\mathbb{C})\cap\mathbf{V}(K).$$

If $f(z)$ is a cusp form of weight $2k$ on Γ, we define the *period polynomial*

$r(f) \in \mathbf{V}(\mathbb{C})$ by

$$r(f)(X) = \int_0^{i\infty} f(z)(X-z)^w dz = \sum_{n=0}^{w} i^{-n+1}\binom{w}{n} r_n(f) X^{w-n}.$$

We also set

$$r^+(f) = \sum_{\substack{0 \leqslant n \leqslant w \\ n \text{ even}}} (-1)^{n/2}\binom{w}{n} r_n(f) X^{w-n},$$

$$r^-(f) = \sum_{\substack{0 < n < w \\ n \text{ odd}}} (-1)^{(n-1)/2}\binom{w}{n} r_n(f) X^{w-n},$$

so that $r^\pm \in \mathbf{V}^\pm$, $r = ir^+ + r^-$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$\begin{aligned}(r(f)|\gamma)(X) &= \int_0^{i\infty} f(z)[aX + b - z(cX+d)]^w dz \\ &= \int_0^{i\infty} (-cz+a)^{2k} f(z)[X - \gamma^{-1}(z)]^w \frac{dz}{(cz-a)^2} \\ &= \int_{\gamma^{-1}(0)}^{\gamma^{-1}(\infty)} f(z)(X-z)^w dz,\end{aligned}$$

where the final integral is taken over the geodesic (semi-circle or vertical line) joining the cusps $\gamma^{-1}(0)$ and $\gamma^{-1}(\infty)$. Hence

$$r(f)|(1+S) = \int_0^{i\infty} + \int_{i\infty}^{0} = 0,$$
$$r(f)|(1+U+U^2) = \int_0^{i\infty} + \int_1^0 + \int_{i\infty}^1 = 0,$$

so that $r(f)$ belongs to the subspace $\mathbf{W}$. We thus have two maps $r^\pm$: $S_{2k} \to \mathbf{W}^\pm_{2k-2}(\mathbb{C})$. The basic result of Eichler–Shimura is the following:

Theorem (Eichler–Shimura) *The map $r^-: S_{2k} \to \mathbf{W}^-(\mathbb{C})$ is an isomorphism. The map $r^+: S_{2k} \to \mathbf{W}^+(\mathbb{C})$ is an isomorphism onto $\mathbf{W}_0^+(\mathbb{C})$, where $\mathbf{W}_0^+ \subset \mathbf{W}^+$ is a subspace of codimension 1, defined over $\mathbb{Q}$, and not containing the element $p_0(X) = X^w - 1$.*

The injectivity of $r^\pm$ implies that we can define two rational structures $\mathfrak{S}^\pm_{2k}$ on S_{2k} by

$$\begin{aligned}\mathfrak{S}^\pm_{2k}(K) &= (r^\pm)^{-1}(\mathbf{V}(K)) \\ &= \{f \in S_{2k} : r_n(f) \in K \quad \text{for} \quad 0 \leqslant n \leqslant w, (-1)^n = \pm 1\}.\end{aligned}$$

The assertion about the image of $r^{\pm}$ then implies that we have isomorphisms

$$\mathfrak{S}_{2k}^{-}(K) \xrightarrow{r^-} \mathbf{W}_{2k-2}^{-}(K), \qquad \mathfrak{S}_{2k}^{+}(K) \oplus K \xrightarrow{(r^+, p_0)} \mathbf{W}_{2k-2}^{+}(K)$$

(the second map sends (f, c) to $r^+(f)(X) + cp_0(X) \in \mathbf{V}$). This describes the rational structure $\mathfrak{S}^-$ completely – it is isomorphic to $\mathbf{W}^-$ – but does not quite describe $\mathfrak{S}^+$: we know only that there is an exact sequence

$$0 \to \mathfrak{S}_{2k}^{+} \to \mathbf{W}^{+}(\mathbb{Q}) \xrightarrow{\lambda} \mathbb{Q} \to 0$$

with a certain map λ such that $\lambda(p_0) \neq 0$. The theorem defines λ up to a non-zero constant but does not say what it is, i.e. what extra relation is satisfied by even periods of cusp forms besides the relations defining $\mathbf{W}$. We will determine the missing relation λ in Section 4.

The proof of the Eichler–Shimura theorem will not be given in this paper. Good expositions can be found in [10] or [4].

Examples For $2 \leqslant k \leqslant 5$ and $k = 7$ one easily checks that $\mathbf{W}_{2k-2}^{-} = \{0\}$ and that $\mathbf{W}_{2k-2}^{+}$ is spanned by $p_0(X)$, in accordance with the theorem and the fact that $S_{2k} = \{0\}$. For $k = 6$ the space S_{2k} is one-dimensional, with generator

$$\Delta = q \prod_n (1 - q^n)^{24} \in \mathfrak{S}_{12}^{0}.$$

Hence there must exist constants $\omega_{\pm}$ with $\omega_{\pm}^{-1} \Delta \in \mathfrak{S}_{12}^{\pm}$. The space $\mathbf{W}_{10}^{+}$ is spanned by $p_0 = X^{10} - 1$ and $p_1 = X^8 - 3X^6 + 3X^4 - X^2$ and the space $\mathbf{W}_{10}^{-}$ by $p_2 = 4X^9 - 25X^7 + 42X^5 - 25X^3 + 4X$, so we must have $\omega_{+}^{-1} r^{+}(\Delta) \in \mathbb{Q}p_0 + \mathbb{Q}p_1$, $\omega_{-}^{-1} r^{-}(\Delta) \in \mathbb{Q}p_2$. In fact the periods of Δ are given by

n	0 or 10	1 or 9	2 or 8	3 or 7	4 or 6	5
$r_n(\Delta)$	$\frac{192}{691}\omega_+$	$\frac{384}{5}\omega_-$	$\frac{16}{135}\omega_+$	$40\omega_-$	$\frac{8}{105}\omega_+$	$32\omega_-$

with suitable real constants $\omega_+ = 0.021\,446\,0667\ldots$ and $\omega_- = 0.000\,048\,277\,4800\ldots$ (cf. [22]), so

$$\omega_{+}^{-1} r^{+}(\Delta) = -\frac{192}{691} p_0 + \frac{16}{3} p_1, \qquad \omega_{-}^{-1} r^{-}(\Delta) = 192 p_2.$$

The fact that there exist constants $\omega_{\pm}$ such that $\omega_{\pm}^{-1}\Delta$ has rational periods – obvious here because the dimension of S_{12} is one – generalizes in the following way to higher weights:

Theorem (Manin [11]) *There is an action of the Hecke algebra of S_{2k} on $\mathbf{V}_{2k-2}(\mathbb{Q})$ preserving the subspaces $\mathbf{V}^+$, $\mathbf{V}^-$, $\mathbf{W}$, and $\mathbb{Q}p_0$ (explicitly, $p_0|T_l = \sigma_{2k-1}(l)p_0$, where $\sigma_{2k-1}(l) = \sum_{d|l} d^{2k-1}$) and compatible with the period mappings $r^2: S_{2k} \to \mathbf{V}^{\pm}$.*

Corollary *If $f \in S_{2k}$ is a Hecke eigenform, then there exist two constants $\omega_\pm = \omega_\pm(f)$ such that $\omega_\pm^{-1} r_n(f)$ is an algebraic number for $0 \leqslant n \leqslant w$, $(-1)^n = \pm 1$. More precisely, if f is a normalized Hecke eigenform and K_f the number field generated by its coefficients, then there exist real numbers $\omega_\pm(f)$ such that $\omega_{(-1)^n}(f)^{-1} r_n(f) \in K_f$ for $0 \leqslant n \leqslant w$, and these numbers can be chosen so that $\omega_{(-n)^n}(f^\sigma)^{-1} r_n(f^\sigma) = \{\omega_{(-1)^n}(f)^{-1} r_n(f)\}^\sigma$ for all $\sigma \in \mathrm{Aut}\, K_f$.*

Manin's proof of the theorem is entirely explicit: he writes $r_n(f|T_l)$ as a linear combination of periods $r_m(f)$ $(0 \leqslant m \leqslant w,\ m \equiv n \pmod 2)$ with integral coefficients given in terms of certain continued fraction expansions. The theorem also will be a consequence of the results of this paper (in particular, of §1.4). Notice that the corollary implies that

$$\sum_{\sigma \in \mathrm{Gal}(K_f/\mathbb{Q})} \alpha^\sigma \cdot \omega_\pm(f^\sigma)^{-1} f^\sigma(z)$$

belongs to $\mathfrak{S}_{2k}^{\pm}(\mathbb{Q})$ for any $\alpha \in K$; these functions, as f ranges over a set of non-conjugate normalized eigenforms and α over a basis of $K_f/\mathbb{Q}$, give a basis for $\mathfrak{S}_{2k}^{\pm}(\mathbb{Q})$.

We end with a result which follows from a theorem of Rankin [12] that also implies a large part (indeed all, when suitably generalized [22]) of the Eichler–Shimura theorem, even though it antedates it by several years. This theorem and its corollary will play a central role in the paper.

Theorem *The numbers $\omega_+(f)$, $\omega_-(f)$ ($f \in S_{2k}$ a normalized Hecke eigenform) can be chosen in such a way that $\omega_+(f)\omega_-(f) = (f,f)$.*

Proof The theorem of Rankin just mentioned is the identity

$$(f, G_{2n}G_{2k-n}) = \frac{(-1)^n}{2^{2k-1}} r_{2n-1}(f) r_{2k-2}(f)$$

for $k/2 < n \leqslant k-2$ and $f \in S_{2k}$ a normalized Hecke eigenform, where

$$G_{2n}(z) = -\frac{B_{2n}}{4n} + \sum_{l=1}^{\infty} \sigma_{2n-1}(l) q^l \qquad (n \geqslant 2)$$

(B_{2n} is the $2n$th Bernoulli number) is the normalized Eisenstein series of weight $2n$. (The proof of this identity will be reviewed in §1.4.) Since $G_{2n}G_{2k-2n}$ has rational coefficients, the left-hand side equals $\alpha_j(f,f)$ for some $\alpha_j \in K_j$ with $\alpha_{f^\sigma} = (\alpha_f)^\sigma$; moreover $\alpha_f \neq 0$ $(r_m(f) = m!(2\pi)^{-m-1} L(f, m+1) \neq 0$ for

$m \geqslant k$ because the Euler product for $L(f, m+1)$ converges). It follows that $\beta_f = \omega_+(f)\omega_-(f)/(f,f)$ is an element of $K_f^\times$ satisfying $\beta_f^\sigma = \beta_{f^\sigma}$ for all $\sigma \in \mathrm{Gal}\,(K_f/\mathbb{Q})$, and dividing $\omega_+(f)$ or $\omega_-(f)$ by β_f gives a new choice having the desired property.

Corollary *The spaces $\mathfrak{S}_{2k}^+$ and $\mathfrak{S}_{2k}^-$ are dual with respect to the Petersson scalar product*, i.e.

$$F \in \mathfrak{S}_{2k}^\pm \Leftrightarrow (F, G) \in \mathbb{Q} \textit{ for all } G \in \mathfrak{S}_{2k}^\mp.$$

Proof Since $\mathfrak{S}_{2k}^\pm$ are rational structures on the same space S_{2k} and $(\ ,\ )$ is non-degenerate, it suffices to show that (F, G) is rational for $F \in \mathfrak{S}_{2k}^+$, $G \in \mathfrak{S}_{2k}^-$. By the remark following Manin's theorem we have

$$F(z) = \sum_{[f]} \sum_{\sigma} \alpha_f^\sigma \omega_+(f^\sigma)^{-1} f^\sigma(z),$$

$$G(z) = \sum_{[f]} \sum_{\sigma} \beta_f^\sigma \omega_-(f^\sigma)^{-1} f^\sigma(z)$$

for some numbers α_f, $\beta_f \in K_f$, where the outer sums run over a set of representatives f of non-conjugate normalized Hecke eigenforms and the inner sums over $\alpha \in \mathrm{Gal}\,(K_f/\mathbb{Q})$. Since the functions f^σ are pairwise orthogonal and K_f is totally real, we have

$$\begin{aligned}(F, G) &= \sum_{[f]} \sum_{\sigma} \alpha_f^\sigma \beta_f^\sigma \omega_+(f^\sigma)^{-1} \omega_-(f^\sigma)^{-1} (f^\sigma, f^\sigma) \\ &= \sum_{[f]} \mathrm{Tr}_{K_f/\mathbb{Q}}(\alpha_f \beta_f) \in \mathbb{Q}.\end{aligned}$$

1.2. The periods of R_n

We have just seen that a function in S_{2k} whose scalar products with elements of $\mathfrak{S}_{2k}^\pm$ are rational itself belongs to $\mathfrak{S}_{2k}^\mp$. Clearly a spanning set of such functions is given by $\{R_n : 0 \leqslant n \leqslant w, (-1)^n = \pm 1\}$, where R_n is the cusp form characterized by

$$r_n(f) = (f, R_n) \qquad (\forall f \in S_{2k}).$$

Therefore the periods $r_m(R_n)$ for $m \not\equiv n \pmod 2$ are rational numbers. Our object in this section is to compute these numbers, thus obtaining an explicit description of the structure of the spaces $\mathfrak{S}_{2k}^\pm$ and the duality between them.

Theorem 1 *Let m and n be integers of opposite parity, $0 \leqslant m, n \leqslant w$. Then*

$$\begin{aligned}(-1)^k 2^{-w} w!\, r_m(R_n) &= (-1)^{(n-m-1)/2} n!\, \tilde{m}!\, \beta_{n-m} \\ &+ (-1)^{(n+m-1)/2} \tilde{n}!\, \tilde{m}!\, \beta_{\tilde{n}-m} + (-1)^{(n+m+1)/2} n!\, m!\, \beta_{m-\tilde{n}}\end{aligned}$$

$$+(-1)^{(m-n-1)/2}\tilde{n}!m!\beta_{m-n}$$
$$+(-1)^{(n+1)/2}\frac{n!\tilde{n}!\beta_n\beta_{\tilde{n}}}{(w+1)\beta_{w+1}}(\delta_{m,0}+(-1)^k\delta_{m,w})$$
$$+(-1)^{(m+1)/2}\frac{m!\tilde{m}!\beta_m\beta_{\tilde{m}}}{(w+1)\beta_{w+1}}(\delta_{n,0}+(-1)^k\delta_{n,w}),$$

where $\tilde{m}=w-m$, $\tilde{n}=w-n$,

$$\beta_n=\begin{cases}\dfrac{B_{n+1}}{(n+1)!} & \text{if } n\geqslant -1,\quad n \text{ odd},\\ 0 & \text{otherwise},\end{cases}$$

(B_n is the nth Bernoulli number and is equal to the coefficient of $t^n/n!$ in $t/(e^t-1)$), *and* δ_{ij} *is the Kronecker delta symbol.*

Proof The numbers $r_m(R_n)$ have the symmetry properties

$$r_m(R_n)=(R_n,R_m)=(R_m,R_n)=r_n(R_m),$$
$$r_{\tilde{m}}(R_n)=(-1)^k r_m(R_n),\qquad r_m(R_{\tilde{n}})=(-1)^k r_m(R_n),$$

all of which are shared by the formula given in the theorem. Hence it is sufficient to prove the theorem under the restriction $0\leqslant m<n\leqslant\frac{1}{2}w=k-1$; note that under these restrictions the fourth, sixth, seventh and eighth terms of the formula in Theorem 1 always vanish, while the third is non-zero only for $m=k-2$, $n=k-1$ and the fifth only for $m=0$. We will use the following representation of R_n as an infinite series, due to Henri Cohen:

Lemma (Cohen [3]) *For* $0<n<w$, $R_n(z)$ *is given by*

$$R_n(z)=c_{k,n}^{-1}\sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma}(az+b)^{-n-1}(cz+d)^{-\tilde{n}-1},$$

where $c_{k,n}=i^{\tilde{n}+1}2^{-w}\binom{w}{n}\pi.$

Proof Except for details of convergence, this formula is easily checked:

$$\left(f,\sum_{\gamma\in\Gamma}z^{-n-1}|_{2k}\gamma\right)$$
$$=\int_{\mathfrak{H}}f(z)\bar{z}^{-n-1}y^{2k}\frac{dxdy}{y^2}$$
$$=\int_0^\infty y^w\left(\int_{-\infty}^{\infty}f(x+iy)(x+iy-2iy)^{-n-1}dx\right)dy$$

$$= \int_0^\infty y^w\left(\frac{2\pi i}{n!} f^{(n)}(2iy)\right)dy \qquad \text{(Cauchy's theorem)}$$

$$= \frac{2\pi i}{n!}\cdot\frac{w!}{(w-n)!}\left(\frac{i}{2}\right)^n \int_0^\infty y^{w-n} f(2iy)dy$$

(n-fold integration by parts)

$$= 2^{-w} i^{n+1}\binom{w}{n}\pi r_{w-n}(f) = 2^{-w} i^{-\tilde{n}-1}\binom{w}{n}\pi r_n(f).$$

To justify the steps, one notes that

$$\sum_{ad-bc=1} (a^2+b^2)^{-n-1}(c^2+d^2)^{-\tilde{n}-1}$$

converges for $0<n<w$ and therefore the series in the lemma converges uniformly absolutely on sets of the form $y^2 \geqslant c_1 + c_2 x^2$ $(c_1, c_2 > 0)$.

We now prove the theorem, assuming $0 \leqslant m < n \leqslant k-1$. By the lemma, we have

$$c_{k,n} r_m(\mathrm{R}_n) = \int_0^\infty \left(\frac{1}{2}\sum_{ad-bc=1} \frac{t^m}{(ait+b)^{n+1}(cit+d)^{\tilde{n}+1}}\right)dt.$$

The contribution from the terms with $bd=0$, i.e.

$$\int_0^\infty \left(\sum_{l\in\mathbb{Z}} \frac{t^m}{(it)^{n+1}(lit+1)^{\tilde{n}+1}} + \sum_{l\in\mathbb{Z}} \frac{t^m}{(lit+1)^{n+1}(-it)^{\tilde{n}+1}}\right)dt,$$

equals

$$\frac{(-1)^k 2^w}{w!} c_{k,n}\{(-1)^{(n-m-1)/2} n!\tilde{m}!\beta_{\tilde{n}-m}$$
$$+(-1)^{(n+m-1)/2}\tilde{n}!\tilde{m}!\beta_{\tilde{n}-m}\},$$

as one sees easily using Lipschitz's formula,

$$\sum_{l\in\mathbb{Z}} (z+l)^{-v-1} = \frac{(-2\pi i)^{v+1}}{v!}\sum_{l\geqslant 1} l^v e^{2\pi i l z} \qquad (v \geqslant 1, z\in\mathfrak{H}),$$

the integral representation

$$(2\pi)^{-s}\Gamma(s)\zeta(s-v) = \int_0^\infty t^{s-1}\left(\sum_{l\geqslant 1} l^v e^{-2\pi l t}\right)dt \qquad (s > 1+v),$$

and the identity

$$\zeta(n) = (-1)^{n/2} 2^{n-1}\pi^n \beta_{n-1} \qquad (n \geqslant 2 \text{ even}).$$

We have to show that the remaining terms give zero unless $m=0$ or $m+n=w-1$.

The integral over the terms with $bd \neq 0$ we write as $\frac{1}{2}\lim\limits_{\varepsilon\to 0} S_\varepsilon$ with

$$S_\varepsilon = \int_\varepsilon^{1/\varepsilon} \left\{ \sum_{\substack{ad-bc=1\\ bd\neq 0}} \frac{t^m}{(ait+b)^{n+1}(cit+d)^{\tilde{n}+1}} \right\} dt.$$

Here we may interchange the order of summation and integration, since the series converges uniformly absolutely. We also replace a, d and t by their negatives in the terms with $bd < 0$, getting

$$S_\varepsilon = \sum_{\substack{ad-bc=1\\ bd>0}} \left\{ \int_\varepsilon^{1/\varepsilon} + \int_{-1/\varepsilon}^{-\varepsilon} \right\} \frac{t^m\,dt}{(ait+b)^{n+1}(cit+d)^{\tilde{n}+1}}.$$

We write

$$\int_\varepsilon^{1/\varepsilon} + \int_{-1/\varepsilon}^{-\varepsilon} = \int_{-\infty}^{\infty} - \int_{-\varepsilon}^{\varepsilon} - \int_{-\infty}^{-1/\varepsilon} - \int_{1/\varepsilon}^{\infty}$$

and observe that the (convergent) integral from $-\infty$ to $+\infty$ is zero by the residue theorem, because the integrand has only one pole if $ac = 0$, while if $ac \neq 0$ there are two poles, but both lie on the same side of the real axis because of $bd > 0$ and $ad - bc = 1$. In the integrals from $1/\varepsilon$ to ∞ and from $-\infty$ to $-1/\varepsilon$ we replace t by $1/t$. This gives

$$S_\varepsilon = -\sum_{\substack{ad-bc=1\\ bd>0}} \int_{-\varepsilon}^{\varepsilon} \frac{t^m}{(ait+b)^{n+1}(cit+d)^{\tilde{n}+1}}\,dt$$

$$-\sum_{\substack{ad-bc=1\\ ac>0}} \int_{-\varepsilon}^{\varepsilon} \frac{t^{\tilde{m}}}{(ait+b)^{\tilde{n}+1}(cit+d)^{n+1}}\,dt.$$

Since $ad - bc = 1$, we have $ac \cdot bd = ad \cdot bc \geqslant 0$. Hence $bd \geqslant 0$ in the second sum, so that we can write $S_\varepsilon = S'_\varepsilon + S''_\varepsilon + S'''_\varepsilon$ with

$$S'_\varepsilon = -\sum_{\substack{ad-bc=1\\ ac>0,\, bd=0}} \int_{-\varepsilon}^{\varepsilon} \frac{t^{\tilde{m}}}{(ait+b)^{\tilde{n}+1}(cit+d)^{\tilde{n}+1}}\,dt,$$

$$S''_\varepsilon = +\sum_{\substack{ad-bc=1\\ ac=0,\, bd>0}} \int_{-\varepsilon}^{\varepsilon} \frac{t^{\tilde{m}}\,dt}{(ait+b)^{\tilde{n}+1}(cit+d)^{n+1}},$$

$$S'''_\varepsilon = -\sum_{\substack{ad-bc=1\\ bd>0}} \left(\int_{-\varepsilon}^{\varepsilon} \frac{t^m\,dt}{(ait+b)^{n+1}(cit+d)^{\tilde{n}+1}} \right.$$

$$\left. + \int_{-\varepsilon}^{\varepsilon} \frac{t^{\tilde{m}}\,dt}{(ait+b)^{\tilde{n}+1}(cit+d)^{n+1}} \right).$$

The sum S'_ε we write (separating the cases $b = 0$ and $d = 0$, and replacing t by $\pm\,\varepsilon t$) as

$$S'_\varepsilon = 2i^{n+1}\varepsilon^{\tilde{m}-n-1}\left[\varepsilon\sum_{l=1}^{\infty}\int_{-1}^{1}\frac{t^{\tilde{m}-n-1}}{(1+il\varepsilon t)^{\tilde{n}+1}}\,dt\right]$$
$$+2i^{-\tilde{n}+1}\varepsilon^{n-m-1}\left[\varepsilon\sum_{l=1}^{\infty}\int_{-1}^{1}\frac{t^{n-m-1}}{(1+il\varepsilon t)^{n+1}}\,dt\right].$$

The expressions in square brackets are Riemann sums for the integrals

$$\int_0^\infty\int_{-1}^{1}\frac{t^{\tilde{m}-n-1}}{(1+ixt)^{\tilde{n}+1}}\,dt\,dx \quad\text{and}\quad \int_0^\infty\int_{-1}^{1}\frac{t^{n-m-1}}{(1+ixt)^{n+1}}\,dt\,dx,$$

respectively. Hence the limit as $\varepsilon\to 0$ of the first term is zero unless $\tilde{m}=n+1$, i.e. $m=k-2$, $n=k-1$, when it equals

$$2i^k\int_0^\infty\int_{-1}^{1}\frac{1}{(1+ixt)^k}\,dt\,dx=\frac{2\pi i^k}{k-1},$$

and similarly the second term gives $2\pi i^{-\tilde{n}+1}/n$ if $m=n-1$ and 0 otherwise. This gives the third and fourth terms of the formula in Theorem 1. A similar argument shows that (under our assumptions on m and n) $\lim_{\varepsilon\to 0} S''_\varepsilon=0$. Finally, the first term of S'''_ε equals

$$-2\varepsilon^m\sum_{\substack{b,d>0\\(b,d)=1}}\frac{1}{b^{n+1}d^{\tilde{n}+1}}\left[\varepsilon\sum_{l\in\mathbb{Z}+c/d}\int_{-1}^{1}\frac{t^m\,dt}{(1+i\varepsilon t/bd+il\varepsilon t)^{n+1}(1+il\varepsilon t)^{\tilde{n}+1}}\right],$$

where in the inner sum (a,c) denotes a fixed solution of $ad-bc=1$. Again the expression in square brackets is a Riemann sum with a finite limit as $\varepsilon\to 0$, and so this whole expression tends to 0 unless $m=0$, in which case it equals

$$-2\sum_{\substack{b,d>0\\(b,d)=1}}b^{-n-1}d^{-\tilde{n}-1}\int_{-\infty}^{\infty}\int_{-1}^{1}\frac{dt}{(1+ixt)^{2k}}\,dx$$
$$=\frac{\zeta(n+1)\zeta(\tilde{n}+1)}{\zeta(2k)}\,\frac{4\pi}{2k-1}$$
$$=2\cdot\frac{(-1)^k 2^w}{w!}\cdot c_{k,n}\cdot(-1)^{(n+1)/2}\frac{n!\,\tilde{n}!\,\beta_n\beta_{\tilde{n}}}{(w+1)\beta_{w+1}}.$$

The second term in S'''_ε always gives 0 since $m<w$. This completes the proof of Theorem 1.

We would like to thank R. Sczech, who suggested the use of Riemann sums in evaluating $\lim_{\varepsilon\to 0} S_\varepsilon$.

As a numerical check, one can verify that the right-hand side of the formula in Theorem 1 is zero for the 13 pairs (m,n) with $0\leqslant m<n\leqslant k-1\leqslant 4$, $m\not\equiv n$

(mod 2), while for $k=6$ and the nine pairs with $0 \leqslant m < n \leqslant 5$ the values $r_m(R_n)$ equal $r_m(\Delta)r_n(\Delta)/\omega_+(\Delta)\omega_-(\Delta)$ with the values $r_m(\Delta)$ given in §1.1 (note that $\dim S_{12} = 1$ implies $R_n = (\Delta, \Delta)^{-1} r_n(\Delta)\Delta$ in this case).

1.3. Bernoulli polynomials as period polynomials

By an elementary calculation, the result of the last section can be rephrased as a formula for the odd or even period polynomial of the cusp form R_n:

Theorem 1′ *The odd period polynomial of R_n for n even, $0 \leqslant n \leqslant w$, is given by*

$$\begin{aligned}
&(-1)^{k+n/2}\, 2^{-w} r^-(R_n)(X) \\
&= -\frac{1}{n+1} B^0_{n+1}(X) - \frac{X^w}{n+1} B^0_{n+1}\left(\frac{1}{X}\right) \\
&\quad + \frac{1}{\tilde{n}+1} B^0_{\tilde{n}+1}(X) + \frac{X^w}{\tilde{n}+1} B^0_{\tilde{n}+1}\left(\frac{1}{X}\right) \\
&\quad - (\delta_{n,0} - \delta_{\tilde{n},0}) \frac{1}{(w+1)\beta_{w+1}} \sum_{m=-1}^{w+1} \beta_m \beta_{\tilde{m}} X^m.
\end{aligned}$$

The even period polynomial of R_n for n odd, $0 < n < w$, is given by

$$\begin{aligned}
&(-1)^{k+(n-1)/2} 2^{-w} r^+(R_n)(X) \\
&= \frac{1}{n+1} B^0_{n+1}(X) - \frac{X^w}{n+1} B^0_{n+1}\left(\frac{1}{X}\right) + \frac{1}{\tilde{n}+1} B^0_{\tilde{n}+1}(X) \\
&\quad - \frac{X^w}{\tilde{n}+1} B^0_{\tilde{n}+1}\left(\frac{1}{X}\right) - \frac{2k}{B_{2k}} \cdot \frac{B_{n+1}}{n+1} \cdot \frac{B_{\tilde{n}+1}}{\tilde{n}+1} p_0(X).
\end{aligned}$$

Here $\tilde{n}$, $\tilde{m}$, β_m have the same meaning as in Theorem 1, $p_0(X) = X^w - 1$, and $B^0_n(X)$ is the nth Bernoulli polynomial without its B_1-term:

$$B^0_n(X) = \sum_{\substack{i=0 \\ i \neq 1}}^{n} \binom{n}{i} B_i X^{n-i} = \sum_{\substack{0 \leqslant i \leqslant n \\ i \text{ even}}} \binom{n}{i} B_i X^{n-i}.$$

The first equation needs a bit of explanation if $n=0$ or $n=w$, since then not all terms on the right are in $\mathbf{V}_{2k-2}$. If, say, n equals w, then the first term $(-1/(w+1))\,B^0_{w+1}(X)$ has a leading coefficient $-X^{2k-1}/(2k-1)$ of too large a degree for an element of $\mathbf{V}$, while the second term $(-X^w/(w+1))B^0_{w+1}(1/X)$ contains a negative power $-X^{-1}/(2k-1)$. These two terms are cancelled, however, by the end terms $m=-1$ and $m=w+1$ of the folowing sum. Similar remarks apply if $n=0$.

In this section we will check directly that the expressions on the right of the formulae in Theorem 1′ belong to $\mathbf{W}^{\pm}$, at the same time getting a better understanding for the structure of period polynomials.

Assume first that $0 < n < w$, to avoid the difficulties with the powers X^{-1}, X^{2k-1} just mentioned. Since we know that $p_0(X) \in \mathbf{W}^+$, we must check that the polynomial

$$\pm \frac{1}{n+1}\left[B_{n+1}^0(X) \mp X^w B_{n+1}^0\left(\frac{1}{X}\right)\right]$$
$$+ \frac{1}{\tilde{n}+1}\left[B_{\tilde{n}+1}^0(X) \mp X^w B_{\tilde{n}+1}^0\left(\frac{1}{X}\right)\right]$$

belongs to $\mathbf{W}^\pm$, where $(-1)^n = \mp 1$. Since $f(X) \mapsto X^w f(1/X)$ is the action of εS on $\mathbf{V}$ (notations as in §1.1) and $B_{\tilde{n}+1}^0$, $B_{\tilde{n}+1}^0$ are eigenfunctions of ε with eigenvalue $(-1)^{n+1} = \pm 1$, this polynomial can be rewritten as

$$\left[\pm \frac{1}{n+1} B_{n+1}^0 + \frac{1}{\tilde{n}+1} B_{\tilde{n}+1}^0 \right] \Bigg| (1-S),$$

from which it is clear (since $S^2 = 1$) that it is annihilated by $1 + S$. We must check that it is annihilated by $1 + U + U^2$ or (equivalently) that it is in the image of $1 - U$.

Let

$$f(X) = \frac{(-1)^{n+1}}{n+1} B_{n+1}^0(X) + \frac{1}{\tilde{n}+1} B_{\tilde{n}+1}^0(X).$$

The polynomial $B_{n+1}^0(X)$ differs by only the monomial $\frac{1}{2}(n+1)X^n$ from the usual Bernoulli polynomial $B_{n+1}(X)$, whose most important property (the one that led Bernoulli to introduce it) is

$$B_{n+1}(X+1) - B_{n+1}(X) = (n+1)X^n.$$

Hence

$$B_{n+1}^0(X) - B_{n+1}^0(X-1) = \left[B_{n+1}(X) + \frac{n+1}{2} X^n \right]$$
$$- \left[B_{n+1}(X) - \frac{n+1}{2}(X-1)^n \right] = \frac{n+1}{2}[X^n + (X-1)^n],$$

so

$$f(X) - f(X-1) = \frac{(-1)^{n+1}}{2}[X^n + (X-1)^n] + \tfrac{1}{2}[X^{\tilde{n}} + (X-1)^{\tilde{n}}].$$

Combining this with the identity

$$(X-1)^n = (X-1)^w \left(\frac{1}{X-1} \right)^{\tilde{n}} = (-1)^n X^{\tilde{n}} | U^{-1}$$

gives

$$f|(1-T^{-1})=\frac{(-1)^{n+1}}{2}X^n-\tfrac{1}{2}X^{\tilde{n}}|U^{-1}$$
$$+\tfrac{1}{2}X^{\tilde{n}}+\frac{(-1)^n}{2}X^n|U^{-1}=g|(1-U^{-1}),$$

where $g(X)=\frac{1}{2}(X^{\tilde{n}}+(-1)^{n+1}X^n)$. The fact that our original polynomial $f|(1-S)$ is in $\mathrm{Im}(1-U)$ is now a formal calculation:

$$\begin{aligned}f|(1-S)&=f|(1-U)+f|(1-T^{-1})|U\\&=f|(1-U)+g|(1-U^{-1})|U=(f-g)|(1-U).\end{aligned}$$

This proves what we wanted. Furthermore, the last calculation generalizes immediately to give

Theorem 2 *Let f, g be two polynomials in $\mathbf{V}$ satisfying the identity*

$$f|(1-T^{-1})=g|(1-U^{-1}). \qquad (*)$$

Then $f|(1-S)\in\mathbf{W}$. Conversely, given any $h\in\mathbf{W}$ there exist $f, g\in\mathbf{V}$ satisfying $(*)$ and $f|(1-S)=h$; the element g can be chosen to satisfy $g|S=-g$.

Proof The first statement follows from the formal calculation just given, which shows that

$$f|(1-S)=(f-g)|(1-U)\in\mathrm{Im}(1-S)\cap\mathrm{Im}(1-U)=\mathbf{W}.$$

For the second, suppose that $h\in\mathbf{W}$ and set $f=\frac{1}{2}h$, $g=\frac{1}{6}h|(U^2-U)$. Then

$$f|(1-T^{-1})=\tfrac{1}{2}h|(1-SU^{-1})=\tfrac{1}{2}h|(1+U^2)=-\tfrac{1}{2}h|U,$$
$$g|(1-U^{-1})=\tfrac{1}{6}h|(U^2-2U+1)=-\tfrac{1}{2}h|U,$$

and clearly $f|(1-S)=h$. The last statement follows if we note that we can subtract from f and g any polynomial invariant under S without changing $f|(1-S)=h$ or affecting the relation $(*)$; applying this remark with the polynomial $\frac{1}{2}g|(1+S)$ leads to f and g with $g|S=-g$.

Notice that the essential element of the pair (f,g) in Theorem 2 is g, since f is determined by g up to a constant (the kernel of $1-T^{-1}$ consists of periodic polynomials, hence constants) and changing f by a constant changes $h=f|(1-S)$ only by a multiple of $p_0(X)=X^w-1$, the 'trivial' element of $\mathbf{W}_{2k-2}$. Conversely, the only condition on g in order that there should be an $f\in\mathbf{V}$ satisfying $(*)$ is that $g|(1-U^{-1})$ have degree less than w. If we assume – as we may by the last statement of the theorem – that $g|S=-g$, then $g|(1-U^{-1})=g|(1+T^{-1})$, which has the same degree as g. Hence the requirements on g become: (i) $g|S=-g$, (ii) $\deg(g)<w$, and we have a map from the set of such g to $\mathbf{W}/\langle p_0\rangle$ defined by

$$g \mapsto f|(1-S), \quad \text{where} \quad f|(1-T^{-1}) = g|(1+T^{-1}).$$

A basis for polynomials g satisfying (i) and (ii) is clearly given by $\{g = \frac{1}{2}(X^{\tilde{n}} + (-1)^{n+1}X^n) : 0 < n < w\}$, and the calculations preceding Theorem 2 show that the corresponding elements of $\mathbf{W}/\langle p_0 \rangle$ are just the period polynomials of the R_n, $0 < n < w$.

It remains to discuss the anomalous cases $n = 0$, w. By symmetry we may suppose that $n = 0$. The formula for $(-1)^k 2^{-w} r^-(R_0)(X)$ can be written

$$\left[-B_1^0(X) + \frac{1}{w+1} B_{w+1}^0(X) \right] \Big| (1-S) - \frac{1}{(w+1)\beta_{w+1}} c_k(X),$$

where

$$c_k(X) = \sum_{m=-1}^{2k-1} \beta_m \beta_{2k-2-m} X^m \qquad (k \geqslant 0).$$

The first term $[-B_1^0 + (w+1)^{-1} B_{w+1}^0] | (1-S)$ is not a polynomial, but it is annihilated by $1+S$ and $1+U+U^2$ by the same calculation as for $0 < n < w$. The function $c_k(X)$ is also clearly annihilated by $1+S$ (since it is odd and symmetric), so to check that the whole expression is in $\mathbf{W}_{2k-2}$ we must check that c_k is annihilated by $1+U+U^2$, i.e.

$$c_k(X) + X^w c_k\left(\frac{X-1}{X}\right) + (X-1)^w c_k\left(\frac{1}{1-X}\right) = 0 \qquad (k > 0).$$

The numbers β_n $(n \geqslant -1)$ are the Laurent coefficients of $\frac{1}{2}\coth\frac{1}{2}t$ around $t = 0$, so we have the generating series

$$\sum_{k=0}^{\infty} c_k(X) t^{2k-2} = \sum_{m,n \geqslant -1} \beta_m \beta_n X^m t^{m+n} = \tfrac{1}{4}\coth\tfrac{1}{2}t \coth\tfrac{1}{2}Xt.$$

Hence

$$\sum_{k=0}^{\infty} \left[c_k(X) + X^{2k-2} c_k\left(\frac{X-1}{X}\right) + (X-1)^{2k-2} c_k\left(\frac{1}{1-X}\right) \right] t^{2k-2}$$
$$= \frac{1}{4}\left[\coth\frac{t}{2}\coth\frac{Xt}{2} + \coth\frac{Xt}{2}\coth\frac{(X-1)t}{2} + \coth\frac{(X-1)t}{2}\coth\frac{-t}{2} \right] = \frac{1}{4},$$

and the assertion follows.

1.4. Rankin's method

By the result of Manin quoted in §1.1, we know that the Hecke operators T_l $(l \geqslant 1)$ preserve $\mathfrak{S}_{2k}^{\pm}(\mathbb{Q})$ and hence that the numbers $r_m(R_n|T_l)$ $(m \not\equiv n \pmod 2)$

are rational. Knowing these numbers will give an explicit description of the spaces $\mathfrak{S}_{2k}^{\pm}$ (and hence also of S_{2k}) as modules over the Hecke algebra. By the self-adjointness of T_l, we have

$$\begin{aligned} r_m(R_n \mid T_l) &= (R_n \mid T_l, R_m) = (R_m, R_n \mid T_l) \\ &= (R_m \mid T_l, T_n) = r_n(R_m \mid T_l), \end{aligned}$$

i.e. the numbers $r_m(R_n \mid T_l)$ are symmetric in m and n. To compute them, we could use the same method as in the case $l=1$, simply noting that $R_n \mid T_l$ is given by a formula like that in the lemma in §1.2 but with the sum taken over matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant l. The calculation then proceeds much as before except that there are extra terms coming from the fact that $ad-bc=l$ no longer implies that $ad \cdot bc \geqslant 0$ (as for $l=1$), and the terms with $ad \cdot bc < 0$ must be treated separately. We prefer, however, to give a completely different proof, based on Rankin's method, since this will also permit the introduction of several ideas used in the sequel.

We begin with a property which is a formal consequence of the definitions of the rational structures $\mathfrak{S}_{2k}^{\pm}$ and $\mathfrak{S}_{2k}^{0}$.

Proposition *Let n be an integer satisfying $0 \leqslant n \leqslant w$ and let $\rho_n : S_{2k} \to S_{2k}$ be the linear map defined by the property: $\rho_n(f) = r_n(f) f$ iff f is a normalized Hecke eigenform. Then ρ_n maps $\mathfrak{S}_{2k}^{\pm}$ to $\mathfrak{S}_{2k}^{0}$, where $(-1)^n = \pm 1$.*

Proof This is essentially a restatement of Manin's theorem, since by linearity we see that

$$\text{coefficient of } q^l \text{ in } \rho_n(f) = r_n(f \mid T_l) \qquad (l = 1, 2, 3, \ldots)$$

for any $f \in S_{2k}$, and for $f \in \mathfrak{S}_{2k}^{\pm}$ the numbers on the right are all rational. More explicitly, an arbitrary function in $\mathfrak{S}_{2k}^{\pm}$ can be written as $\sum_f \alpha_f \cdot \omega_{\pm}(f)^{-1} f(z)$, where the sum is over all normalized Hecke eigenforms and the α_f are algebraic numbers satisfying $(\alpha_f)^\sigma = \alpha_{f^\sigma}$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$; the image of such an element under ρ_n will be $\sum_f \beta_f f(z)$ with $\beta_f \; (= \alpha_f \cdot r_n(f)/\omega_{\pm}(f)) \in \bar{\mathbb{Q}}, \; (\beta_f)^\sigma = \beta_{f^\sigma}$, and this is the typical form of an element in $\mathfrak{S}_{2k}^{0}$.

By the identity above, the number $r_n(R_m \mid T_l)$ is the lth Fourier coefficient of $\rho_n(R_m)$. From the definition of R_m, we have the eigenfunction expansion

$$R_m(z) = \sum_f r_m(f) \cdot (f, f)^{-1} \cdot f(z),$$

where as usual $\sum_f$ denotes a sum over normalized Hecke eigenforms and (f, f) is the Petersson scalar product. Hence

$$\rho_n(R_m) = \sum_f r_m(f) r_n(f) (f, f)^{-1} f,$$

i.e. $\rho_n(R_m)$ is the cusp form characterized by the property $(f, \rho_n(R_m)) = r_m(f) r_n(f)$ (f a normalized Hecke eigenform). If we can construct a modular form with this property, then its Fourier coefficients will be the numbers we are looking for.

To construct this function, we use Rankin's identity – already quoted and used in §1.1:

$$(f, G_{2n} G_{2k-2n}) = \frac{(-1)^n}{2^{2k-1}} r_{2k-2}(f) r_{2n-1}(f),$$

where f is a Hecke eigenform and $k/2 < n \leqslant k-2$. The proof is simple: for $n \geqslant 2$ we have, writing $\Gamma_\infty = \langle T \rangle \subset \Gamma$,

$$G_{2n}(z) = -\frac{B_{2n}}{4n} \sum_{\left(\begin{smallmatrix} a & b \\ c & c \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} (cz+d)^{-2n}$$

and therefore

$$\begin{aligned}
&-\frac{4n}{B_{2n}} (f, G_{2n} G_{2k-2n}) \\
&= \int_{\Gamma \backslash \mathfrak{H}} f(z) \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} (c\bar{z}+d)^{-2n} \overline{G_{2k-2n}(z)}\, y^{2k} \frac{dxdy}{y^2} \\
&= \int_{\Gamma \backslash \mathfrak{H}} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} f(\gamma z) \overline{G_{2k-2n}(\gamma z)}\, \mathrm{Im}(\gamma z)^{2k} \frac{dxdy}{y^2} \\
&= \int_{\Gamma_\infty \backslash \mathfrak{H}} f(z) \overline{G_{2k-2n}(z)}\, y^{2k} \frac{dxdy}{y^2} \\
&= \int_0^\infty \sum_{l=1}^\infty a(l) \sigma_{2k-2n-1}(l) e^{-4\pi l y} y^{2k-2}\, dy \\
&= \frac{(2k-2)!}{(4\pi)^{2k-1}} \sum_{l=1}^\infty \frac{a(l) \sigma_{2k-2n-1}(l)}{l^{2k-1}} \\
&= \frac{(2k-2)!}{(4\pi)^{2k-1}} \zeta(2n)^{-1} L(f, 2k-1) L(f, 2n),
\end{aligned}$$

where $\{a_l\}$ are the Fourier coefficients of f, $L(f,s) = \sum_1^\infty a(l) l^{-s}$ its L-series, and the last line of the calculation (the only one that requires f to be an eigenform) follows from the properties of the Euler product of $L(f,s)$. The conditions $n > \frac{1}{2}k$ and $n \leqslant k-2$ are needed to make the series absolutely convergent and to make G_{2k-2n} a modular form, respectively. The relations $r_n(f) = n!(2\pi)^{-n-1} L(f, n+1)$ and $\zeta(2n) = (-1)^{n-1} B_{2n} (2\pi)^{2n} / 2(2n)!$ now complete the proof.

Rankin's identity actually remains true for $n = \frac{1}{2}k$ (although the above proof breaks down), and then by the symmetry property $r_m = (-1)^k r_{w-m}$ for all $2 \leqslant n \leqslant k-2$; it is true also for $n=1$ or $n = k-1$ if the function $G_{2n} G_{2k-2n}$

is replaced by

$$G_2G_{2k-2}+\frac{1}{8\pi i(k-1)}G'_{2k-2},$$

where $G_2=-\frac{1}{24}+\sum\sigma_1(l)q^l$. (The function G_2 is not a modular form, but instead satisfies

$$G_2\left(\frac{az+b}{cz+d}\right)=(cz+d)^2G_2(z)+\frac{12c}{2\pi i}(cz+d)$$

for $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma$, from which it easily follows that $G_2f+(1/4\pi ih)f'$ is a modular form of weight $h+2$ for any $f\in M_h(\Gamma)$.) We will need a further generalization of Rankin's identity, proved in [22] and which we shall quote without proof. To state it, we need certain differential operators F_m which were introduced by Cohen [2].

For $m\in\mathbb{Z}_{\geqslant 0}$ and $a,b\in\mathbb{R}$, let $F_m^{(a,b)}$ be the $(-1)^m$-symmetric bilinear form on smooth functions defined by

$$F_m^{(a,b)}(f,g)=(2\pi i)^{-m}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\cdot\frac{\Gamma(a+m)\,\Gamma(b+m)}{\Gamma(a+i)\Gamma(b+m-i)}f^{(i)}g^{(m-i)},$$

where $f^{(n)}$ denotes the nth derivative of f. Then Cohen shows the following:

(i) If f,g are modular forms of weights a and b on some group $\Gamma'\subset PSL_2(\mathbb{R})$, then $F_m(f,g):=F_m^{(a,b)}(f,g)$ is a modular form of weight $a+b+2m$ on Γ' and is a cusp form if $m>0$ ($F_0(f,g)$ is simply the product fg).

(ii) If f is a modular form of weight a on some group $\Gamma'\subset\Gamma$, then

$$F_m(f,G_2):=F_m^{(a,2)}(f,G_2)+\frac{m!}{2(a+m)}(2\pi i)^{-m-1}f^{(m+1)}$$

is a modular form of weight $a+2+2m$ on Γ' and a cusp form if $m>0$.

(iii)
$$F_m(G_2,G_2):=F_m^{(2,2)}(G_2,G_2)+(1+(-1)^m)\frac{m!}{2(2+m)}(2\pi i)^{-m-1}G_2^{(m+1)}$$

is a modular form of weight $4+2m$ on Γ and a cusp form if $m>0$.

The generalization of Rankin's identity proved in [22] is then that the scalar product of a Hecke eigenform f with a function $F_\nu(G_{2n_1},G_{2n_2})$ of the same weight is up to a simple factor the product of two periods of f (Rankin's identity was the case $v=0$). More precisely, if m and n are integers of opposite parity with $0 \leqslant m < n \leqslant k-1$, then the function

$$X_{m,n} = (-1)^{k+(n-m+1)/2} 2^{2k-1} \frac{(w-m)!}{w!} \cdot \left[F_m(G_{2k-m-n-1}, G_{n-m+1}) + \delta_{m,0} \frac{k}{B_{2k}} \frac{B_{n+1}}{n+1} \frac{B_{2k-n-1}}{2k-n-1} G_{2k} \right]$$

has the property

$$(f, X_{m,n}) = r_m(f) r_n(f)$$

for all normalized Hecke eigenforms $f \in S_{2k}$. Also, $X_{m,n}$ is itself a cusp form (this is the reason for subtracting a multiple of G_{2k} when $m=0$, which does not affect the scalar product with $f \in S_{2k}$ but makes $X_{0,n}$ vanish at infinity) and has rational Fourier coefficients (by the definitions of G_{2r} and of the operators F_m). By the remarks at the beginning of this section, we must have $X_{m,n} = \rho_n(R_m) = \rho_m(R_n)$. This proves

Theorem 3 *For* $0 \leqslant m < n \leqslant k-1$, $m \not\equiv n \pmod 2$, *let* $X_{m,n} \in \mathfrak{S}_{2k}^{0}$ *be the function defined above and let* l *be a positive integer. Then* $r_m(R_n \mid T_l)$ *is the* lth *Fourier coefficient of* $X_{m,n}$.

Looking at the explicit formulae for the Fourier coefficients of $F_m(G_{2n_1}, G_{2n_2})$, we easily see that the case $l=1$ of this gives Theorem 1, while for $l>1$ one obtains equally explicit formulae. In particular, for $m=0$, $3 \leqslant n \leqslant k-1$ odd (i.e. the case of Rankin's identity) we obtain

$$\begin{aligned} &(-1)^{k+(n+1)/2} 2^{-2k+1} r_0(R_n \mid T_l) \\ &= \text{coefficient of } q^l \text{ in } G_{n+1} G_{\tilde{n}+1} + \frac{k}{B_{2k}} \frac{B_{n+1}}{n+1} \frac{B_{\tilde{n}+1}}{\tilde{n}+1} G_{2k} \\ &= \sum_{h=1}^{l-1} \sigma_n(h) \sigma_{\tilde{n}}(l-h) \\ &\quad - \frac{B_{\tilde{n}+1}}{2(\tilde{n}+1)} \sigma_n(l) - \frac{B_{n+1}}{2(n+1)} \sigma_{\tilde{n}}(l) \\ &\quad + \frac{k}{B_{2k}} \frac{B_{n+1}}{n+1} \frac{B_{\tilde{n}+1}}{\tilde{n}+1} \sigma_{2k-1}(l) \end{aligned}$$

for $l \geqslant 1$, where $\tilde{n} = w-n$ as in §1.2. The formulae for $r_m(R_n \mid T_l)$ are similar, but the numbers $\sigma_n(h)\sigma_{\tilde{n}}(l-h)$ are multiplied by a coefficient which is a homogeneous polynomial of weight m in l and h with rational coefficients.

Notice that our explicit formula for $X_{m,n}$ proves that $\rho_n(R_m)$ has rational

Fourier coefficients and hence implies the proposition at the beginning of this section and (working backwards) Manin's theorem on the invariance of $\mathfrak{S}_{2k}^{\pm}$ under the Hecke algebra.

2. THE PERIODS OF $f_{k,D}$

2.1. Modular forms of half-integral weight

Let $D>0$ be a discriminant (i.e. an integer congruent to 0 or 1 modulo 4) and k a positive even integer. The function

$$f_{k,D}(z)=\frac{D^{k-1/2}}{\pi\binom{2k-2}{k-1}}\cdot\frac{1}{2}\sum_{b^2-4ac=D}(az^2+bz+c)^{-k}\qquad(z\in\mathfrak{H}),$$

where the sum is over all triples $(a,b,c)\in\mathbb{Z}^3$ with $b^2-4ac=D$, belongs to S_{2k}. This function was introduced in [19, Appendix 2] in connection with the Doi–Naganuma lifting from elliptic to Hilbert modular forms, and shown in [8], [7] to be the Dth Fourier coefficient of the holomorphic kernel function for the Shimura–Shintani correspondence between modular forms of integral and half-integral weight. In this section we will use this property of $f_{k,D}$ to show that it belongs to $\mathfrak{S}_{2k}^{+}$ and compute its even periods.

To state the result, we will need the number-theoretical function $H_k(D)$ defined by H. Cohen [2]. This is defined either as the Dth Fourier coefficient of an appropriate Eisenstein series of weight $k+\frac{1}{2}$ or as a special value of a Dirichlet L-series: if D is a fundamental discriminant (i.e. either 1 or the discriminant of a real quadratic field), then

$$H_k(D)=L\left(1-k,\left(\frac{D}{\cdot}\right)\right),$$

while if D is an arbitrary discriminant we write D as D_0f^2 with D_0 fundamental and $f\in\mathbb{N}$ and set

$$H_k(D)=L\left(1-k,\left(\frac{D_0}{\cdot}\right)\right)\cdot\sum_{d\mid f}\mu(d)\left(\frac{D_0}{d}\right)d^{k-1}\sigma_{2k-1}\left(\frac{f}{d}\right).$$

The polynomials $p_0(=X^w-1)$ and B_n^0 have the same meaning as in Theorem 1. Then we have

Theorem 4 *Let k and D be positive integers, k even. Then*

$$r^{+}(f_{k,D})(X)=\sum_{\substack{a,b,c\in\mathbb{Z}\\ a<0<c\\ b^2-4ac=D}}(aX^2+bX+c)^{k-1}+\frac{B_k}{B_{2k}}H_k(D)p_0(X)$$

$$\begin{cases} \frac{1}{k}(B_k^0(mX) - X^w B_k^0(mX^{-1})) & \text{if} \quad D = m^2, \\ 0 & \text{if} \quad D \neq \text{square}. \end{cases}$$

In particular, $f_{k,D}$ *belongs to* $\mathfrak{S}_{2k}^+$.

Notice that the sum on the right is finite, since $0 > ac > -D/4$, $|b| < \sqrt{D}$. Notice, too, that the theorem is vacuous if D is not a discriminant, i.e. if $D \equiv 2$ or $3 \pmod 4$, because then $f_{k,D}$, $H_k(D)$ and the sum in the theorem all vanish.

If $k = 2$ or 4, then $S_{2k} = \{0\}$ and consequently $f_{k,D} \equiv 0$. Restricting to the case that D is the discriminant of a real quadratic field, and computing the coefficient of X^w on the right of Theorem 4, we obtain

Corollary *Let* K *be a real quadratic field,* D *its discriminant,* $\zeta_K(s)$ *the Dedekind zeta-function of* k. *Then* $\zeta_K(-1)$ *and* $\zeta_K(-3)$ *are given by the formula*

$$\zeta_K(1-k) = \frac{B_{2k}}{k} \sum_b \sigma_{k-1}\left(\frac{D-b^2}{4}\right) \qquad (k = 2, 4),$$

where the sum is over integers b *satisfying* $|b| < \sqrt{D}$, $b \equiv D \pmod 2$ *and* $\sigma_{k-1}(n)$ *as usual denotes* $\sum_{\substack{c|n \\ c>0}} c^{k-1}$.

This identity was proved by Siegel [13] (in a somewhat different form; for the above formula see [1] or [20]) by studying the restriction to the diagonal of the Eisenstein series of weight k for the Hilbert modular group of K; the relation of this method to ours will be discussed briefly in §2.4.

We observe that the Eichler–Shimura theorem implies that the 0th period can be expressed in terms of the periods r_n $(0 < n < w, n$ even). Hence Theorem 4 implies the existence of a formula of the type

$$\zeta_K(1-k) = \sum_b \sum_{j=0}^{[(b-1)/2]} g_{k,j}(b^2, D)\sigma_{k-1-2j}\left(\frac{D-b^2}{4}\right)$$

for *every* k, not just $k = 2$ or 4, where $\sum_b$ has the same meaning as in the Corollary and the $g_{k,j}$ are universal homogeneous polynomials of degree j in two variables. For instance, for $k = 6$ we have

$$\zeta_K(-5) = \sum_b \left[\frac{691}{16380}\sigma_5\left(\frac{D-b^2}{4}\right) + \frac{45}{364}(9b^2 - D)\sigma_3\left(\frac{D-b^2}{4}\right)\right]$$

or

$$\zeta_K(-5) = \sum_b \left[\frac{691}{16380} \sigma_5\left(\frac{D-b^2}{4}\right) - \frac{5}{52}(21b^4 - 14b^2D + D^2)\sigma_1\left(\frac{D-b^2}{4}\right)\right].$$

However, to give canonical and explicit formulae of this type for general k we need to have a description of the map λ discussed in §1.1; this will be given in Section 4.

Proof of Theorem 4 Let $S_{k+1/2}$ denote the space of cusp forms of weight $k+\frac{1}{2}$ on $\Gamma_0(4)$ whose nth Fourier coefficients for $n \not\equiv 0, 1 \pmod 4$ vanish. Then the following facts are known:

(i) The map $\mathscr{S}_1$ which sends $\sum_{n\geqslant 1} c(n)q^n$ to $\sum_{n\geqslant 1}(\sum_{d|n} d^{k-1}c(n^2/d^2))q^n$ maps $S_{k+1/2}$ to S_{2k}([6]; the map $\mathscr{S}_1$ is a slight modification of the lifting map defined by Shimura in [16]).

(ii) the function

$$\Omega_k(z,\tau) = (-1)^{k/2}2^{3k-1}\sum_{D>0} f_{k,D}(z)e^{2\pi i D\tau} \qquad (z,\tau\in\mathfrak{H})$$

belongs to $S_{k+1/2}$ as a function of τ [8, Theorem 2] and is the holomorphic kernel function of $\mathscr{S}_1$ [7]; i.e.

$$(\mathscr{S}_1 g)(z) = \langle g, \Omega_k(-\bar{z},\cdot)\rangle = \frac{1}{6}\int_{\Gamma_0(4)\backslash\mathfrak{H}} g(\tau)\overline{\Omega_k(-\bar{z},\tau)}\, v^{k-3/2}\,du\,dv$$

$$(g\in S_{k+1/2}, \tau = u+iv\in\mathfrak{H}).$$

It follows that Ω_k is also the holomorphic kernel function for the adjoint map $\mathscr{S}_1^*: S_{2k}\to S_{k+1/2}$; in particular

$$(-1)^{k/2}2^{-3k+1}\mathscr{S}_1^*(R_n) = \sum_{D>0}\langle R_n, f_{k,D}\rangle e^{2\pi i D\tau}$$

$$= \sum_{D>0}\overline{r_n(f_{k,D})}\, e^{2\pi i D\tau}$$

is in $S_{k+1/2}$. So to prove the theorem we have to identify the function $\mathscr{S}_1^*(R_n)$. By definition, it is the unique cusp form in $S_{k+1/2}$ satisfying

$$\langle g, \mathscr{S}_1^*(R_n)\rangle = r_n(\mathscr{S}_1 g) \qquad \forall g\in S_{k+1/2}.$$

Using the fact that $r_n(f)$ is essentially equal to the L-series of f at $s = 2k-1-n$, we see that $r_n(\mathscr{S}_1 g)$ is, up to a simple factor, the value of $\sum_{m\geqslant 1} c(m^2)m^{-s}$ at

$s=2k-1-n$, i.e. the convolution of the L-series of g and of $\Theta(\tau)=\sum_m e^{2\pi i m^2\tau}$ at $s=\frac{1}{2}(2k-1-n)$. By Rankin's method and its generalization, as already used in the proof of Theorem 3, this number, for n even and less than k, is essentially the scalar product of g with $F_{n/2}(\Theta, G_{k-n})$, so that $\mathscr{S}_1^*(R_n)$ is a multiple of the cuspidal part of $F_{n/2}(\Theta, G_{k-n})$. The exact computation gives

$$\mathscr{S}_1^*(R_n)=2^{3k-1}i^{k-1}\frac{(w-n)!}{w!}\frac{(w/2)!}{(w/2-n/2)!}$$
$$\cdot\left[F_{n/2}(\Theta(\tau), G_{k-n}(4\tau))-\delta_{n0}\frac{B_k}{B_{2k}}\mathscr{H}_{k+1/2}(\tau)\right]$$
$$(n \text{ even}, 0\leqslant n\leqslant k-2),$$

where

$$\mathscr{H}_{k+1/2}(\tau)=-\frac{B_{2k}}{2k}+\sum_{D>0}H_k(D)e^{2\pi i D\tau}$$

is the Eisenstein series of weight $k+\frac{1}{2}$ introduced by Cohen [2]. Comparing the coefficients of $e^{2\pi i D\tau}$ on the two sides of this equation gives the theorem.

Let us discuss the result and the relationship between S_{2k} and $S_{k+1/2}$ in a little more detail. It is known [6] that there is a natural action of the Hecke algebra on $S_{k+1/2}$ and that (i) the map $\mathscr{S}_1: S_{k+1/2}\rightarrow S_{2k}$ is Hecke-equivariant; (ii) S_{2k} and $S_{k+1/2}$ are isomorphic as Hecke modules. (It is not known whether $\mathscr{S}_1$ gives the isomorphism; this is equivalent to the non-vanishing of $L(f, k)$ for all eigenforms $f\in S_{2k}$.) Hence there exists a basis g_i $(1\leqslant j\leqslant r)$ of $S_{k+1/2}$ corresponding to the basis f_j $(1\leqslant j\leqslant r)$ of normalized Hecke eigenforms in S_{2k}, where $r=\dim S_{2k}=\dim S_{k+1/2}$. The g_j can be chosen to have algebraic Fourier coefficients (more precisely, Fourier coefficients in the same field as f_j). If we write

$$g_j(z)=\sum_{\substack{D>0\\ D\equiv 0,1 (\mathrm{mod}\, 4)}} c_j(D)q^D,$$

then $\mathscr{S}_1(g_j)=c_j(1)f_j$. The g_j are mutually orthogonal with respect to the Petersson scalar product, so that the kernel function Ω_k of $\mathscr{S}_1$ is given by

$$\Omega_k(z,\tau)=\sum_{j=1}^{r}(g_j, g_j)^{-1}g_j(\tau)\cdot c_j(1)f_j(z),$$

and therefore

$$\mathscr{S}_1^*(R_n)=\sum_j (g_j, g_j)^{-1}r_n(f_j)c_j(1)g_j(\tau)$$

or (taking the Dth Fourier coefficients on both sides)

$$(-1)^{k/2}2^{3k-1}r_n(f_{k,D})=\sum_j (g_j, g_j)^{-1}c_j(1)c_j(D)r_n(f_j).$$

The fact that this is rational for m even means (since $c_j(1)$ and $c_j(D)$ are algebraic numbers and transform appropriately under $\mathrm{Gal}(\mathbb{Q}/\mathbb{Q})$) that the scalar product (g_j, g_j) is an algebraic multiple of $\omega_+(f_j)$, the algebraic factor transforming in the usual way under $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (i.e. mapping to the corresponding factor for $g_{j'}$, where $f_j^\sigma = f_{j'}$). This fact was proved previously by Shimura [17] and the authors [8]. One could also study the map $\mathscr{S}_{D'}: S_{k+1/2}\to S_{2k}$ defined by

$$\sum_{n>0} c(n)q^n \mapsto \sum_{n>0}\left\{\sum_{d|n}\left(\frac{D'}{d}\right)d^{k-1}c\left(\frac{n^2}{d^2}D'\right)\right\}q^n,$$

where $D'>0$ is a second (say, fundamental) discriminant; this would replace the numbers $c_j(1)c_j(D)$ in the above discussion by $c_j(D)c_j(D')$ and the function $F_{n/2}(\Theta(\tau), G_{k-n}(4\tau))$ in the proof of Theorem 4 by

$$\mathrm{Tr}^{\Gamma_0(4)}_{\Gamma_0(4D')}(F_n(\Theta(D'\tau), G_{k-n}(\tau)))$$

(cf. the computations in [8]), while the Eisenstein series discussed in §2.4 would have to be replaced by Eisenstein series associated to a non-trivial genus character; we do not carry out any of this.

2.2. Sums of powers of reduced quadratic forms as period polynomials

This section is the analogue of § 1.2: we shall check directly that the polynomial occurring on the right-hand side of Theorem 4 belongs to the space $\mathbf{W}^+_{2k-2}$. For convenience we avoid the case that D is a square; then (since p_0 belongs to $\mathbf{W}^+$) we must check that the polynomial

$$P_{k,D}(X) = \sum_{\substack{a,b,c\in\mathbb{Z}\\ b^2-4ac=D\\ a>0>c}} (aX^2+bX+c)^{k-1}$$

belongs to $\mathbf{W}^+_{2k-2}$.

Recall that an indefinite binary quadratic form $[a,b,c]=aX^2+bXY+cY^2$ is said to be *reduced* if

$$a>0,\quad c>0,\quad b>a+c.$$

There are only finitely many reduced quadratic forms of discriminant D, and each Γ-equivalence class of forms of discriminant D contains at least one reduced form (the reduced forms in a given class naturally form a cycle, corresponding to the period of a continued fraction; this will be discussed further is §3.1). Let

$$Q_{k,D}(X) = \sum_{\substack{[a,b,c]\text{reduced}\\ b^2-4ac=D}} (aX^2-bX+c)^{k-1},$$

an element of $\mathbf{V}_{2k-2}$. Now one checks easily that

$$\begin{aligned}&\{(a,b,c)\colon a>0>c\}\\ &\quad=\{-(a-b+c,-2a+b,\ a)\colon[a,b,c]\ \text{reduced}\}\\ &\qquad\cup\{(c,\ b-2c,\ a-b+c)\colon[a,b,c]\ \text{reduced}\},\end{aligned}$$

the two sets on the right being disjoint. Applying this to the triples with discriminant D and summing the $(k-1)$th powers of the corresponding quadratic forms gives

$$\begin{aligned}P_{k,D}(X)&=\sum_{\substack{\operatorname{disc}Q=D\\ Q\ \text{reduced}}}\{-Q(X-1,-X)^{k-1}+Q(1,X-1)^{k-1}\}\\ &=Q_{k,D}(X)|(-U+U^2).\end{aligned}$$

Hence $P_{k,D}|(1+U+U^2)=0$. Since, clearly, $P_{k,D}|\varepsilon=P_{k,D}$ (replace b by $-b$ in the definition of $P_{k,D}$) and $P_{k,D}|S=-P_{k,D}$ (replace (a,b,c) by $(-c,b,-a)$), it follows that $P_{k,D}\in\mathbf{W}^+_{2k-2}$ as desired.

We also find that

$$\begin{aligned}P_{k,D}|(1-T^{-1})&=P_{k,D}|(1-SU^{-1})\\ &=P_{k,D}|(1+U^{-1})\\ &=Q_{k,D}|(-U+U^2)(1+U^{-1})\\ &=-Q_{k,D}|(1-U^{-1})\end{aligned}$$

and $P|(1+S)=2P$, so that the pair $f=\frac{1}{2}P_{k,D}$, $g=-\frac{1}{2}Q_{k,D}$ exhibits $P_{k,D}$ in the form described in §1.2 (Theorem 2).

2.3. Periods of $f_{k,D,\mathscr{A}}$

The fact that $f_{k,D}$ is a modular form comes from the invariance under Γ of the set of binary quadratic forms of discriminant D. By the same argument, the function defined by

$$f_{k,D,\mathscr{A}}(z)=\frac{D^{k-1/2}}{\pi\binom{2k-2}{k-1}}\cdot\frac{1}{2}\sum_{[a,b,c]\in\mathscr{A}}(az^2+bz+c)^{-k},$$

where $\mathscr{A}$ is a Γ-equivalence class of quadratic forms of discriminant D and k any positive integer (not necessarily even), also belongs to S_{2k}. Clearly $f_{k,D}=\sum_{\mathscr{A}}f_{k,D,\mathscr{A}}$, where the sum is over the finitely many equivalent classes of forms of discriminant D. In this section we will refine the results of §2.1 by computing the periods of the individual $f_{k,D,\mathscr{A}}$; the proof will be considerably harder because we no longer have available the interpretation of these functions as the coefficients of the kernel function for the lifting to half-integral weight. Actually, what we calculate is not the nth period of $f_{k,D,\mathscr{A}}$, but rather of

$$f^{+}_{k,D,\mathscr{A}}=f_{k,D,\mathscr{A}}+f_{k,D,\mathscr{A}'} \quad \text{or} \quad f^{-}_{k,D,\mathscr{A}}=i(f_{k,D,\mathscr{A}}-f_{k,D,\mathscr{A}'}),$$

(depending whether n is even or odd), where

$$\mathscr{A}'=\{[a,-b,c]:[a,b,c]\in\mathscr{A}\}$$

is the image of $\mathscr{A}$ under ε. (Under the correspondence between equivalence classes of binary quadratic forms and ideal classes in $\mathbb{Q}(\sqrt{D})$, $\mathscr{A}$ and $\mathscr{A}'$ correspond to conjugate ideal classes.) These periods turn out to be rational, so that $f^{\pm}_{k,D,\mathscr{A}}$ belongs to $\mathfrak{S}^{\pm}_{2k}$.

To state the precise result, we introduce the polynomial

$$Q_{k,D,\mathscr{A}}(X)=\sum_{\substack{Q\in\mathscr{A}\\ Q\,\text{reduced}}} Q(X,-1)^{k-1}\in\mathbf{V}(\mathbb{Q})$$

(so that $\sum_{\mathscr{A}} Q_{k,D,\mathscr{A}}$ is the function $Q_{k,D}$ defined in §2.2) and the zeta-function defined by meromorphic continuation of

$$\zeta_{\mathscr{A}}(s)=\sum_{\substack{(m,n)\in\mathbb{Z}^2/\Gamma_{Q_0}\\ Q_0(m,n)>0}} Q_0(m,n)^{-s} \qquad (\mathrm{Re}(s)>1),$$

where Q_0 is any quadratic form in $\mathscr{A}$ and Γ_{Q_0} its stabilizer in Γ. If $D>1$ is a fundamental discriminant, so that $\mathscr{A}$ corresponds to an ideal class in a real quadratic field, then $\zeta_{\mathscr{A}}$ is the usual zeta-function of that ideal class; in any case $\zeta_{\mathscr{A}}(s)$ extends to a meromorphic function of s (holomorphic except at $s=1$) and satisfies

$$\sum_{\mathscr{A}}\zeta_{\mathscr{A}}(s)=\zeta(s)L_D(s),$$

where the sum is over all $\mathscr{A}$ of discriminant D and $L_D(s)$ is the L-function defined for arbitrary $D\in\mathbb{Z}$ in [22], the value of which at $s=1-k$ is the number $H_k(D)$ which occurred in §2.2. Then we have

Theorem 5 *Let $k\geqslant 2$ and $\mathscr{A}$ a Γ-equivalence class of binary quadratic forms of discriminant $D>0$, D not a square. Let $\mathscr{A}^*$ denote the class of forms*

$$\mathscr{A}^*=\{[-a,b,-c]:[a,b,c]\in\mathscr{A}\}=\{-Q:Q\in\mathscr{A}'\}.$$

Then

$$r^{+}(f^{+}_{k,D,\mathscr{A}})+r^{-}(f^{-}_{k,D,\mathscr{A}}) = (Q_{k,D,\mathscr{A}}+Q_{k,D,\mathscr{A}^*})|(-U+U^2)+\frac{\zeta_{\mathscr{A}}(1-k)}{\zeta(1-2k)}p_0.$$

The formulae for $r^{+}(f^{+}_{k,D,\mathscr{A}})$ and $r^{-}(f^{-}_{k,D,\mathscr{A}})$ separately can be obtained by looking at the even or odd terms on the right, respectively. That Theorem 5 really generalizes Theorem 4 follows from the formulae in §2.2 and the identity

$$\sum_{\mathscr{A}} \zeta_{\mathscr{A}}(1-k) = -\frac{B_k}{2k} H_k(D) \qquad (k>0 \text{ even}).$$

Theorem 4 was stated only for even k because $f_{k,D}$ vanishes for k odd ($f_{k,D,\mathscr{A}}$ and $f_{k,D,\mathscr{A}^*}$ cancel), but Theorem 5 is valid for all $k \geqslant 2$. One can also give $r^{\pm}(f^{\pm}_{k,D,\mathscr{A}})$ when D is a square, but this case is less interesting and we have omitted it. As with Theorem 4, Theorem 5 gives explicit formulae for $\zeta_{\mathscr{A}}(1-k)$ if $S_{2k}=\{0\}$ (i.e. for $k=2,3,4,5$ or 7), since then the expression on the right must vanish. One such identity, obtained by looking at the constant term in Theorem 5, is

Corollary *For $k \in \{2,3,4,5,7\}$ and $\mathscr{A}$ a class of quadratic forms of positive, non-square discriminant,*

$$\frac{2k}{B_{2k}} \zeta_{\mathscr{A}}(1-k) = \sum_{\substack{[a,b,c]\in\mathscr{A}\\ \text{reduced}}} \{(a-b+c)^{k-1} - a^{k-1}\}$$
$$+(-1)^k \sum_{\substack{[a,b,c]\in\mathscr{A}^*\\ \text{reduced}}} \{(a-b+c)^{k-1} - c^{k-1}\}.$$

As with the corollary to Theorem 4, this could also be proved by Siegel's method of restricting Hecke–Eisenstein series (cf. §2.4).

Theorem 5 was used by D. Kramer [9] to show that the functions $f_{k,D,\mathscr{A}}$ (or even $f^{\pm}_{k,D,\mathscr{A}}$ with a fixed choice of sign) generate S_{2k} as D runs over all discriminants and $\mathscr{A}$ over all classes of forms of discriminant D. In fact, it is sufficient to restrict to D of the form $D_0 f^2$ with D fixed. It is also conjectured, but not known, that for k even the functions $f_{k,D}$ span S_{2k}; this is equivalent to the question mentioned in §2.1 whether $\mathscr{S}_1$ is an isomorphism.

Proof of Theorem 5 The method is similar to that used in §1.1. We set

$$c_{k,D} = \binom{2k-2}{k-1} D^{1/2-k}\pi,$$

so that

$$2i^{-n^2} c_{k,D} r_n(f^{\pm}_{k,D,\mathscr{A}}) = \int_0^\infty \left(\sum_{[a,b,c]\in\mathscr{A}} \pm \sum_{[a,b,c]\in\mathscr{A}'} \right) \frac{t^n\,dt}{\{a(it)^2+bit+c\}^k}$$

and write the integral as $\lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon}$. On the compact interval $[\varepsilon,\varepsilon^{-1}]$ we may interchange the order of summation and integration, since the series converges absolutely uniformly. If $(-1)^n = \pm 1$, we therefore have

$$2i^{-n^2} c_{k,D} r_n(f^{\pm}_{k,D,\mathscr{A}}) = \lim_{\varepsilon\to 0} S_\varepsilon,$$

$$S_\varepsilon = \sum_{[a,b,c]\in\mathscr{A}} \left(\int_\varepsilon^{1/\varepsilon} + \int_{-1/\varepsilon}^{-\varepsilon} \right) \frac{t^n\,dt}{(-at^2+bit+c)^k}.$$

Write

$$\int_{\varepsilon}^{1/\varepsilon}+\int_{-1/\varepsilon}^{-\varepsilon} \quad \text{as} \quad \int_{-\infty}^{\infty}-\int_{-\infty}^{-1/\varepsilon}-\int_{-\varepsilon}^{\varepsilon}-\int_{1/\varepsilon}^{\infty}.$$

In the integrals from $-\infty$ to $-1/\varepsilon$ and $1/\varepsilon$ to ∞ we replace t by $1/t$ and $[c,-b,a]$ by $[a,b,c]$ (which is in the same class $\mathscr{A}$ since $S\in\Gamma$). In the integral from $-\infty$ to ∞ the only terms which contribute are the finitely many ones with $ac<0$, since if $ac>0$ the two poles of the integral lie on the same side of the real axis ($ac=0$ cannot occur since D is not a square). Hence $S_\varepsilon = S_1 + S'_\varepsilon$ with

$$S_1 = \sum_{\substack{[a,b,c]\in\mathscr{A}\\ ac<0}} \int_{-\infty}^{\infty} \frac{t^n dt}{(-at^2+bit+c)^k},$$

$$S'_\varepsilon = -\sum_{[a,b,c]\in\mathscr{A}} \int_{-\varepsilon}^{\varepsilon} \frac{t^n+(-1)^k t^{\tilde n}}{(-at^2+bit+c)^k}dt$$

($\tilde n = w-n$ as in §1.1). We claim that the integral in S_1 equals

$$2i^{-n}c_{k,D}\binom{w}{n}^{-1}\operatorname{sgn}(a)\,d_{k,n}(c,-b,a),$$

where $d_{k,n}(a,b,c)$ denotes the coefficient of X^n in $(aX^2+bX+c)^{k-1}$. Assuming this, we find the contribution of S_1 to $r_n(f^{\pm}_{k,D,\mathscr{A}})$ to be

$$(-1)^{[n/2]}\binom{w}{n}^{-1}\Bigg(\sum_{\substack{[a,b,c]\in\mathscr{A}\\ a>0>c}} d_{k,n}(c,-b,a) - \sum_{\substack{[a,b,c]\in\mathscr{A}\\ a<0<c}} d_{k,n}(c,-b,a)\Bigg),$$

and the contribution to $r^+(f^+_{k,D,\mathscr{A}})+r^-(f^-_{k,D,\mathscr{A}})$ therefore

$$\sum_{\substack{[a,b,c]\in\mathscr{A}\\ a>0>c}} (aX^2-bX+c)^{k-1} - \sum_{\substack{[a,b,c]\in\mathscr{A}\\ a<0<c}} (aX^2-bX+c)^{k-1},$$

which equals $(Q_{k,D,\mathscr{A}}+Q_{k,D,\mathscr{A}^*})|(-U+U^2)$ by the same calculation as in §2.2. To check the claim we use the residue theorem:

$$\int_{-\infty}^{\infty}\frac{t^n dt}{(-at^2+bit+c)^k}$$
$$=2\pi i\frac{\operatorname{sgn} a}{a^k}\cdot\frac{1}{(k-1)!}\cdot\frac{d^{k-1}}{dt^{k-1}}\frac{t^n}{\{t-\frac12 i(b-\sqrt{D})/a\}^k}\Bigg|_{t=i(b+\sqrt{D})/2a}$$
$$=\frac{2\pi i}{(k-1)!}\cdot\frac{\operatorname{sgn} a}{a^k}\cdot\binom{w}{n}^{-1}$$
$$\cdot\text{coefficient of } X^{w-n} \text{ in } \Phi(X, \tfrac12 i(b+\sqrt{D})/a)),$$

where

$$\Phi(X,t)=\frac{\partial^{k-1}}{\partial t^{k-1}}\frac{(X+t)^w}{\{t-\frac{1}{2}i(b-\sqrt{D})/a\}^k}\ .$$

Expanding

$$(X+t)^w=\left[\left(X+i\frac{b-\sqrt{D}}{2a}\right)+\left(t-i\frac{b-\sqrt{D}}{2a}\right)\right]^w$$

by the binomial theorem, we find

$$\begin{aligned}\Phi(X,t)&=\frac{\partial^{k-1}}{\partial t^{k-1}}\sum_{j=0}^{w}\binom{w}{j}\left\{X+i\frac{b-\sqrt{D}}{2a}\right\}^{w-j}\left\{t-i\frac{b-\sqrt{D}}{2a}\right\}^{j-k}\\&=(-1)^{k-1}\sum_{j=0}^{k-1}\binom{w}{j}\frac{(w-j)!}{(k-1-j)!}\left\{X+i\frac{b-\sqrt{D}}{2a}\right\}^{w-j}\\&\quad\cdot\left\{t-i\frac{b-\sqrt{D}}{2a}\right\}^{j-2k+1}\\&=\frac{(-1)^{k-1}}{(k-1)!}\left\{X+i\frac{b-\sqrt{D}}{2a}\right\}^{k-1}\\&\quad\cdot\left\{t-i\frac{b-\sqrt{D}}{2a}\right\}^{-2k+1}(X+t)^{k-1},\end{aligned}$$

$$\Phi\left(X,i\frac{b+\sqrt{D}}{2a}\right)=\frac{(-1)^{k-1}w!}{(k-1)!}\left(\frac{i\sqrt{D}}{a}\right)^{-2k+1}\cdot\left(\frac{aX^2+ibX-c}{a}\right)^{k-1}$$

and hence

$$\int_{-\infty}^{\infty}\frac{t^n dt}{(-at^2+bit+c)^k}=2\pi i^{-n}\operatorname{sgn}(a)\binom{w}{k-1}\binom{w}{n}^{-1}D^{-k+1/2}d_{k,n}(c,-b,a),$$

whence the claim.

We still have to study the term S_ε'. We write

$$S_\varepsilon'=-\left\{\sum_{\substack{[a,b,c]\in\mathscr{A}\\ c>0}}+(-1)^k\sum_{\substack{[a,b,c]\in\mathscr{A}'^*\\ c>0}}\right\}\int_{-\varepsilon}^{\varepsilon}\frac{t^n+(-1)^k t^{\tilde n}}{(-at^2+bit+c)^k}dt$$

and apply the following lemma, whose proof will be given at the end.

Lemma *Let $\mathscr{C}$ be a Γ-equivalence class of forms of non-square discriminant $D>0$. Then for $0 \leqslant m \leqslant 2k-2$ one has*

$$\lim_{\varepsilon\to 0} \sum_{\substack{[a,b,c]\in\mathscr{C}\\ c>0}} \int_{-\varepsilon}^{\varepsilon} \frac{t^m dt}{(-at^2+bit+c)^k} = \delta_{m,0} \frac{2\pi}{2k-1} \frac{\zeta_{\mathscr{C}}(k)}{\zeta(2k)}.$$

Taking into account the identity $\zeta_{\mathscr{A}'}(s) = \zeta_{\mathscr{A}}(s)$ and the functional equations

$$\pi^{-s} D^{s/2} \Gamma\left(\frac{s}{2}\right)^2 \{\zeta_{\mathscr{A}}(s) + \zeta_{\mathscr{A}^*}(s)\} = (\text{same with } s \to 1-s),$$

$$\pi^{-s} D^{s/2} \Gamma\left(\frac{s+1}{2}\right)^2 \{\zeta_{\mathscr{A}}(s) - \zeta_{\mathscr{A}^*}(s)\} = (\text{same with } s \to 1-s),$$

we deduce

$$\begin{aligned}\lim_{\varepsilon\to 0} S'_\varepsilon &= (\delta_{n,w} - \delta_{n,0}) \frac{2\pi}{(2k-1)\zeta(2k)} \{\zeta_{\mathscr{A}}(k) + (-1)^k \zeta_{\mathscr{A}^*}(k)\} \\ &= (\delta_{n,w} - \delta_{n,0}) \cdot 2\pi \binom{2k-2}{k-1} D^{1/2-k} \frac{\zeta_{\mathscr{A}}(1-k)}{\zeta(1-2k)},\end{aligned}$$

giving the last term of the formula in Theorem 5. It remains to prove the lemma, i.e. to compute $\lim_{\varepsilon\to 0} L_\varepsilon$, where

$$\begin{aligned} L_\varepsilon &= \varepsilon^{m+1} \sum_{\substack{[a,b,c]\in\mathscr{C}\\ c>0}} \int_{-1}^{1} \frac{t^m dt}{\{a(i\varepsilon t)^2 + bi\varepsilon t + c\}^k} \\ &= \varepsilon^m \sum_{\substack{[a,b,c]\in\mathscr{C}\\ c>0,\, b (\mathrm{mod}\ 2c)}} c^{-k} \left[\varepsilon \sum_{n\in\mathbb{Z}+b/2c} \int_{-1}^{1} \frac{t^m dt}{\{D\varepsilon^2/4c^2 + (1+n\varepsilon it)^2\}^k} \right].\end{aligned}$$

The inner sum is a Riemann sum for the integral

$$\int_{-\infty}^{\infty} \int_{-1}^{1} \frac{t^m dt}{(1+ixt)^{2k}} dx.$$

Hence $\lim_{\varepsilon\to 0} L_\varepsilon = 0$ unless $m=0$, when it equals

$$\frac{2\pi}{2k-1} \sum_{\substack{[a,b,c]\in\mathscr{C}\\ c>0,\, b(\mathrm{mod}\, 2c)}} c^{-k}$$

(the integral has already occurred in the proof of Theorem 1). The sum equals $\zeta_{\mathscr{C}}(k)/\zeta(2k)$ by the argument given in [22, p. 131], namely

$$\begin{aligned}
\sum_{\substack{[a,b,c]\in\mathscr{C}\\ c>0,\,b(\mathrm{mod}\,2c)}} c^{-k} &= \sum_{\substack{Q\in\mathscr{C}/\langle(\begin{smallmatrix}1&0\\1&1\end{smallmatrix})\rangle\\ Q(0,1)>0}} Q(0,1)^{-k}\\
&= \sum_{A\in\Gamma_{Q_0}\backslash\Gamma/\langle(\begin{smallmatrix}1&0\\1&1\end{smallmatrix})\rangle} (Q_0\circ A)(0,1)^{-k} \qquad (Q_0\in\mathscr{C}\quad\text{fixed})\\
&= \sum_{\substack{(b,d)\in\mathbb{Z}^2/\Gamma_{Q_0}\\ Q_0(b,d)>0\\ (b,d)=1}} Q_0(b,d)^{-k} \qquad \left(A=\begin{pmatrix}\cdot & b\\ \cdot & d\end{pmatrix}\right)\\
&= \zeta(2k)^{-1}\zeta_{\mathscr{C}}(k).
\end{aligned}$$

This completes the proof.

2.4. Restrictions of Hecke–Eisenstein series

Let $\mathscr{A}$ be a narrow ideal class of a real quadratic field K, corresponding to a Γ-equivalence class (also denoted $\mathscr{A}$) of binary quadratic forms of discriminant $D=\operatorname{disc}K$. For an integer $k\geqslant 3$ define the *Hecke–Eisenstein series of weight* k associated to $\mathscr{A}$ by

$$G_k^{K,\mathscr{A}}(z,z')=\frac{(k-1)!^2}{(2\pi)^{2k}}D^{k-1/2}N(\mathfrak{a})^k$$

$$\times\sum_{\substack{(\lambda,\mu)\in\mathfrak{a}\times\mathfrak{a}/U^+\\ (\lambda,\mu)\neq(0,0)}}\frac{1}{(\lambda z+\mu)^k(\lambda' z'+\mu')^k}\qquad (z,z'\in\mathfrak{H}),$$

where $\mathfrak{a}$ is any ideal in the class $\mathscr{A}$ (the definition does not depend on the choice) and U^+ is the group of totally positive units of K. Denote by

$$G_{k,D,\mathscr{A}}(z)=G_k^{K,\mathscr{A}}(z,z)\in M_{2k}(\Gamma)$$

its restriction to the diagonal. These restrictions (or at least the sums $G_{k,D}=\sum_{\mathscr{A}}G_{k,D,\mathscr{A}}$) were studied by Siegel [13] in order to obtain formulae for the zeta-values $\zeta_{\mathscr{A}}(1-k)$ (or $\zeta_K(1-k)$). Specifically, the fact that $G_{k,D,\mathscr{A}}(z)$ has rational (actually, integral) Fourier coefficients and constant term $\frac{1}{2}\zeta_{\mathscr{A}}(1-k)$ implies the rationality of, and explicit formulae for, $\zeta_{\mathscr{A}}(1-k)$. On the other hand, we have obtained formulae of the same sort for $\zeta_{\mathscr{A}}(1-k)$ by studying the functions $f_{k,D,\mathscr{A}}$. The relation between the two approaches is given by

Theorem 6 *For $k, D, \mathscr{A}$ as above we have*

$$G_{k,D,\mathscr{A}}(z) = \frac{\zeta_{\mathscr{A}}(1-k)}{\zeta(1-2k)} G_{2k}(z) + \rho_{2k-2}(f^+_{k,D,\mathscr{A}}(z)),$$

where $\rho_{2k-2}: S_{2k} \to S_{2k}$ is the map introduced in §1.4.

Since ρ_{2k-2} maps $\mathfrak{S}^+_{2k}$ to $\mathfrak{S}^0_{2k}$, this relates the fact used by us that $f^+_{k,D,\mathscr{A}}$ has rational even periods to the fact used by Siegel that $G_{k,D,\mathscr{A}}$ has rational Fourier coefficients. Using the known formula for the coefficients of $G_{k,D,\mathscr{A}}$, we also can deduce (by comparing coefficients of q^m in Theorem 6) a formula for $r_{2k-2}(f^+_{k,D,\mathscr{A}} | T_m)$; the case $m=1$ generalizes the special case $n=0$ of the result for $r_n(f^+_{k,D,\mathscr{A}})$ proved in §2.3.

Proof of Theorem 6 From the definition, we have

$$G_{k,D,\mathscr{A}}(z) = \frac{(k-1)!^2}{(2\pi)^{2k}} D^{k-1/2} N(\mathfrak{a})^k$$

$$\sum_{(\lambda,\mu)\in\mathfrak{a}\setminus\{0,0\}/U^+} \{N(\lambda)z^2 + \mathrm{Tr}(\lambda'\mu)z + N(\mu)\}^{-k}$$

where λ' is the conjugate of λ over $\mathbb{Q}$. Choose an oriented $\mathbb{Z}$-basis α, β for $\mathfrak{a}$ (i.e. one with $\alpha'\beta - \beta'\alpha > 0$) and let

$$Q_0(x,y) = N(\alpha x + \beta y)/N(\mathfrak{a})$$

be the corresponding quadratic form. Then $Q_0 \in \mathscr{A}$ and Γ_{Q_0} can be identified with U^+. Each $(\lambda,\mu)\in\mathfrak{a}\times\mathfrak{a}$ can be represented as $(\alpha,\beta)M$ for some unique $M\in\Gamma_{Q_0}\backslash M_2(\mathbb{Z})\backslash\{0\}$, and under this correspondence

$$N(\mathfrak{a})^{-1}[N(\lambda)z^2 + \mathrm{Tr}(\lambda'\mu)z + N(\mu)] = (Q_0\circ M)(z,1).$$

Therefore

$$\frac{(2\pi)^{2k} D^{1/2-k}}{(k-1)!^2} G_{k,D,\mathscr{A}}(z) = \sum_{M\in\Gamma_{Q_0}\backslash M_2(\mathbb{Z})\setminus\{0\}} (Q_0\circ M)(z,1)^{-k}$$

$$= \sum_{m\in\mathbb{Z}} G^{(m)}_{k,D,\mathscr{A}}(z),$$

where

$$G^{(m)}_{k,D,\mathscr{A}}(z) = \sum_{M\in\Gamma_{Q_0}\backslash\Delta_m} (Q_0\circ M)(z,1)^{-k}$$

with $\Delta_m = \{M\in M_2(\mathbb{Z}): \det M = m,\ M \neq 0\}$. For $m \geqslant 1$ clearly

$$G^{(m)}_{k,D,\mathscr{A}}(z) + G^{(-m)}_{k,D,\mathscr{A}}(z)$$

$$= \sum_{\Gamma_{Q_0}\backslash\Delta_m} (Q_0\circ M)(z,1)^{-k} + \sum_{\Gamma_{Q_0}\backslash\Delta_m} (Q_0'\circ M)(z,1)^{-k}$$

$$= m^{-2k+1}\cdot 2c_{k,D}(f_{k,D,\mathscr{A}}(z) + f_{k,D,\mathscr{A}'}(z))|T(m),$$

where $Q_0' = Q_0\circ\varepsilon$ and $T(m)$ denotes the mth Hecke operator in S_{2k}, defined by

$$f|T(m) = m^{k-1}\sum_{M\in\Gamma\backslash\Delta_m} f|_{2k}M;$$

the constant $c_{k,D}$ is defined as in the proof of Theorem 5.

For $m=0$ we write

$$G^{(0)}_{k,D,\mathscr{A}}(z) = \sum_{M\in\Gamma\backslash\Delta_0}\ \sum_{\gamma\in\Gamma_{Q_0}\backslash\Gamma/\Gamma_M} (Q_0\circ\gamma\circ M)(z,1)^{-k},$$

where γ runs over a set of representatives of Γ/Γ_M which are inequivalent under left multiplication by the generator of Γ_{Q_0} and $\Gamma_M = \{N\in\Gamma : NM = M\}$. It is easily seen that a set of representatives for $\Gamma\backslash\Delta_0$ is

$$\left\{\begin{pmatrix} m & n\\ 0 & 0\end{pmatrix} : m>0,\quad n\in\mathbb{Z}_0 \quad\text{or}\quad m=0,\quad n\geqslant 0\right\}$$

and the isotropy group Γ_M of any M in this set is $\Gamma_\infty = \langle T\rangle$. Therefore (denoting as usual by $\sum'$ a sum with the zero term omitted)

$$\begin{aligned}
G^{(0)}_{k,D,\mathscr{A}}(z) &= \sum_{m,n\in\mathbb{Z}}{}'\ \sum_{\gamma\in\Gamma_{Q_0}\backslash\Gamma\ \Gamma_\infty} \left\{Q_0\circ\gamma\circ\begin{pmatrix} m & n\\ 0 & 0\end{pmatrix}\right\}(z,1)^{-k}\\
&= \frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \left\{\sum_{Q\in\mathscr{A}/\Gamma_\infty} + (-1)^k \sum_{Q\in\mathscr{A}'^*/\Gamma_\infty}\right\} Q(mz+n,0)^{-k}\\
&= \left\{\frac{1}{2}\sum_{m,n}{}'(mz+n)^{-2k}\right\}\\
&\qquad\cdot\left\{\sum_{\substack{[a,b,c]\in\mathscr{A}\\ a>0\\ b\,(\mathrm{mod}\,2a)}} + (-1)^k \sum_{\substack{[a,b,c]\in\mathscr{A}'^*\\ a>0\\ b\,(\mathrm{mod}\,2a)}}\right\} a^{-k}\\
&= \{\zeta_{\mathscr{A}}(k) + (-1)^k\zeta_{\mathscr{A}^*}(k)\}\,E_{2k}(z),
\end{aligned}$$

where $E_{2k}(z) = \{\tfrac{1}{2}\zeta(1-2k)\}^{-1}G_{2k}(z) = 1+\cdots$ denotes the normalized Eisenstein series and the last line follows from the identity

$$\frac{1}{2}\sum_{m,n}{}'(mz+n)^{-2k} = \zeta(2k)\cdot E_{2k}(z)$$

and the formula for $\zeta_{\mathscr{A}}(k)/\zeta(2k)$ given at the end of §2.3.

Putting together the formulae for the various $G^{(m)}_{k,D,\mathscr{A}}$ and using the identity

$$\zeta_{\mathscr{A}}(k)+(-1)^k\zeta_{\mathscr{A}^*}(k)=\frac{\pi^{2k}2^{2k-1}}{(k-1)!^2}D^{-k+1/2}\zeta_{\mathscr{A}}(1-k),$$

we find that

$$G_{k,D,\mathscr{A}}(z)=\tfrac{1}{2}\zeta_{\mathscr{A}}(1-k)E_{2k}(z) + \frac{(2k-2)!}{(2\pi)^{2k-1}}\sum_{m=1}^{\infty} m^{-2k+1}\{f_{k,D,\mathscr{A}}+(-1)^k f_{k,D,\mathscr{A}^*}\}\,|\,T_m$$

(notice that $f_{k,D,\mathscr{A}'}=(-1)^k f_{k,D,\mathscr{A}^*}$). To identify the cuspidal part of this, we compute the Petersson scalar product with a normalized Hecke eigenform $f=\sum a(m)q^m\in S_{2k}$:

$$\begin{aligned}\frac{(2\pi)^{2k-1}}{(2k-2)!}(f,G_{k,D,\mathscr{A}}) &= \sum_{m=1}^{\infty} m^{-2k+1}(f,(f_{k,D,\mathscr{A}}+f_{k,D,\mathscr{A}'})|T_m)\\ &= \sum_{m=1}^{\infty} m^{-2k+1}(f\,|\,T_m,f^{+}_{k,D,\mathscr{A}})\\ &= \sum_{m=1}^{\infty} a_m m^{-2k+1}(f,f^{+}_{k,D,\mathscr{A}})\\ &= L(f,2k-1)(f,f^{+}_{k,D,\mathscr{A}})\\ &= \frac{(2\pi)^{2k-1}}{(2k-2)!}r_{2k-2}(f)\cdot(f,f^{+}_{k,D,\mathscr{A}}).\end{aligned}$$

Therefore if $f^{+}_{k,D,\mathscr{A}}(z)=\sum_f \alpha_f f(z)$ (sum over normalized Hecke eigenforms), then the cuspidal part of $G_{k,D,\mathscr{A}}$ is

$$\sum_f \alpha_f r_{2k-2}(f)f(z)=\rho_{2k-2}(f^{+}_{k,D,\mathscr{A}}).$$

The theorem follows.

3. HYPERBOLIC PERIODS

3.1. Periods around closed geodesics

The periods $r_n(f)$ of a cusp form $f\in S_{2k}$ are integrals of f along a certain geodesic in $\mathfrak{H}$, namely the one joining 0 to $i\infty$. All geodesics in $\mathfrak{H}$ have the form of a semi-circle from α to β, where α,β are distinct points in $\mathbb{R}\cup\{\infty\}$ and the semi-circle degenerates to a vertical line if α or β is infinite. If α and β are rational, then there is a sequence $\alpha_0=\alpha,\alpha_1,\dots,\alpha_N=\beta$ of elements of $\mathbb{Q}\cup\{\infty\}$ such that the geodesic from α_{i-1} to α_i is Γ-equivalent to the geodesic from 0 to $i\infty$, so integrals of f from α to β can be expressed as linear combinations of the numbers $r_m(f)$. (This is the basis of Manin's proof of the formula for $r_n(f\,|\,T_l)$.) If the pair $\{\alpha,\beta\}$ is not defined over $\mathbb{Q}$, then the image of the geodesic from α to β is dense in $\Gamma\backslash\mathfrak{H}$, and the integral of f along this geodesic makes no sense. There remains the case that α and β are not individually rational but that $\{\alpha,\beta\}$ is defined over $\mathbb{Q}$ (i.e. Gal $(\mathbb{C}/\mathbb{Q})$-invariant); this occurs when α and β are conjugate quadratic irrationalities, i.e. when α and β are the roots of an equation

$$ax^2+bx+c=0 \qquad (a,b,c\in\mathbb{Z}, \qquad (a,b,c)=1).$$

In this case the geodesic C joining α and β is given by the equation

$$a|z|^2+bx+c=0 \qquad (z=x+iy\in\mathfrak{H}).$$

Since α and β are real, we must have $D=b^2-4ac>0$; moreover, D should be a non-square since we want α and β to be irrational. We write $C=C_Q$, where Q is the quadratic form $[a,b,c]$. There is an infinite cyclic subgroup Γ_Q of Γ, corresponding to the group of totally positive units of $K=\mathbb{Q}(\sqrt{D})$, preserving Q and hence C_Q; this group is generated by the matrix

$$\gamma_Q=\begin{pmatrix}\frac{1}{2}(t-bu) & -cu\\ au & \frac{1}{2}(t+bu)\end{pmatrix}\in\Gamma,$$

where (t,u) is the smallest positive solution of Pell's equation $t^2-Du^2=4$. One checks that the expression

$$f(z)(az^2+bz+c)^{k-1}\,dz$$

is invariant under γ_Q, so the number

$$r_Q(f)=\int_{\Gamma_Q\backslash C_Q} f(z)(az^2+bz+c)^{k-1}\,dz$$

makes sense. (There is a slight question of orientation; we orient the geodesic from $(-b-\sqrt{D})/2a$ to $(-b+\sqrt{D})/2a$, i.e. from left to right if $a>0$ and from right to left if $a<0$; then the oriented integral will go from z_0 to $\gamma_Q z_0$, where z_0 is any point of C_Q.) Replacing Q by a Γ-equivalent form replaces C_Q by a Γ-translate and Γ_Q by a conjugate group and does not change either the curve $\Gamma_Q\backslash C_Q\subset\Gamma\backslash\mathfrak{H}$ or the number $r_Q(f)$. We therefore also write $C_{\mathscr{A}}$ for $\Gamma_Q\backslash C_Q$ (a closed geodesic in $\Gamma\backslash\mathfrak{H}$) and $r_{\mathscr{A}}(f)$ for $r_Q(f)$, where $\mathscr{A}$ denotes the Γ-equivalence class of the form Q.

The relation of the 'periods around closed geodesics' $r_{\mathscr{A}}(f)$ and the cusp forms $f_{k,D,\mathscr{A}}(z)$ studied in Section 2 is given by the following proposition ([14], [7]):

Proposition *Let $\mathscr{A}$ be a Γ-equivalence class of primitive binary quadratic* *forms of discriminant $D>0$, D not a square. Then for any $f\in S_{2k}$ we have*

$$r_{\mathscr{A}}(f)=2^{2k-2}(f,f_{k,D,\mathscr{A}}).$$

Proof By the usual unfolding argument, we have

$$\begin{aligned}c_{k,D}(f,f_{k,D,\mathscr{A}})&=\int_{\Gamma\backslash\mathfrak{H}}\sum_{\gamma\in\Gamma_Q\backslash\Gamma}(Q\circ\gamma)(\bar z,1)^{-k}f(z)y^{2k-2}dxdy\\&=\int_{\Gamma_Q\backslash\mathfrak{H}}f(z)(a\bar z^2+b\bar z+c)^{-k}y^{2k-2}dxdy,\end{aligned}$$

where $Q=[a,b,c]$ is any element of $\mathscr{A}$ and $c_{k,D}$ has the same meaning as in §2.3. Let

$$\theta=\arg\left(\frac{z-\beta}{z-\alpha}\right),$$

where α and β are the roots of $az^2+bz+c=0$, with $\alpha<\beta$. Then $0<\theta<\pi$ and θ is invariant under Γ_Q (replacing z by $\gamma_Q z$ multiplies $(z-\alpha)/(z-\beta)$ by $[\frac{1}{2}(t+u\sqrt{D})]^{-2}$). Also

$$\begin{aligned}d\theta&=d\,\mathrm{Im}\,[\log(z-\beta)-\log(z-\alpha)]\\&=\mathrm{Im}\left[\left(\frac{1}{z-\beta}-\frac{1}{z-\alpha}\right)dz\right]\\&=\sqrt{D}\,\mathrm{Im}\left[\frac{dz}{az^2+bz+c}\right],\end{aligned}$$

so

$$dz\,d\theta=\frac{\sqrt{D}}{a\bar z^2+b\bar z+c}dx\,dy,$$

and

$$\frac{y^2}{|az^2+bz+c|^2}=\frac{1}{D}\sin^2\theta.$$

Therefore

$$\begin{aligned}&f(z)(a\bar z^2+b\bar z+c)^{-k}y^{2k-2}dxdy\\&\quad=D^{1/2-k}f(z)(az^2+bz+c)^{k-1}\sin^{2k-2}\theta\,dz d\theta.\end{aligned}$$

For each $\theta\in(0,\pi)$ the integral of $f(z)\,(az^2+bz+c)^{k-1}dz$ from z_0 to $\gamma_Q z_0$, where $z_0\in\mathfrak{H}$ is any point with $\arg\{(z_0-\beta)/(z_0-\alpha)\}=\theta$, is $r_{\mathscr{A}}(f)$, independent of θ. Hence

$$c_{k,D}(f,f_{k,D,\mathscr{A}})=D^{1/2-k}r_{\mathscr{A}}(f)\int_0^\pi\sin^{2k-2}\theta\,d\theta,$$

and the theorem follows.

Now Theorem 5 (§2.3) tells us that $f^{\pm}_{k,D,\mathscr{A}}$ belongs to $\mathfrak{S}^{\pm}_{2k}$ and therefore has rational scalar product with any $f\in\mathfrak{S}^{\mp}_{2k}$. By the above proposition, this scalar product is a rational multiple of $r_{\mathscr{A}}(f)+r_{\mathscr{A}'}(f)$ or $i(r_{\mathscr{A}}(f)-r_{\mathscr{A}'}(f))$. Hence it should be possible to write $r_{\mathscr{A}}(f)$ as a rational linear combination of odd periods of f plus i times a rational linear combination of even periods. This is the content of the following theorem:

Theorem 7 *Let $\mathscr{A}$, D be as in the proposition, $k>1, f\in S_{2k}$. Then*

$$r_{\mathscr{A}}(f)=\sum_{n=0}^{w} i^{-n+1}q^{(n)}_{k,D,\mathscr{A}}r_n(f),$$

where $q^{(n)}_{k,D,\mathscr{A}}$ is the coefficient of X^n in the polynomial $Q_{k,D,\mathscr{A}}(X)$ ($Q_{k,D,\mathscr{A}}$ as in §2.3).

Proof We may assume that the form $Q=[a,b,c]\in\mathscr{A}$ in the definition of $r_{\mathscr{A}}(f)$ is a reduced form, since every class of forms contains a reduced representative. The reduced forms in $\mathscr{A}$ form a cycle $Q_0=Q, Q_1,\dots,Q_r=Q_0$, where each Q_j is related to its predecessor by $Q_j=Q_{j-1}\circ M_j$ with

$$M_j=\begin{pmatrix} m_j & 1\\ -1 & 0\end{pmatrix}$$

for some integer $m_j\geqslant 2$; the m_j are determined by the continued fraction expansion

$$\frac{b+\sqrt{D}}{2a}=m_1-\cfrac{1}{m_2-\cfrac{1}{m_3-\ddots}},$$

which is pure periodic ($m_j=m_{j+r}$) because Q is reduced [23].

Choose $z_0\in\mathfrak{H}$ and set $z_j=M_1\dots M_jz_0$, so that $z_r=\gamma_Qz_0$. Then

$$\begin{aligned} r_{\mathscr{A}}(f)&=\int_{z_0}^{\gamma_Qz_0}f(z)Q(z,1)^{k-1}\,dz\\ &=\sum_{j=1}^{r}\int_{z_{j-1}}^{z_j}f(z)Q_0(z,1)^{k-1}\,dz\\ &=\sum_{j=1}^{r}\int_{z_0}^{M_jz_0}f(z)Q_{j-1}(z,1)^{k-1}\,dz,\end{aligned}$$

where in the last line we have acted on z by $M_1\dots M_{j-1}$. Now $M_jz_0=-m_j-1/z_0$. Since the last formula is true for any $z_0\in\mathfrak{H}$, we can let z_0 tend to 0, so $M_jz_0\to i\infty$. Then

$$r_{\mathscr{A}}(f)=\sum_{j=1}^{r}\int_0^{i\infty}f(z)Q_{j-1}(z,1)^{k-1}\,dz$$

$$= \int_0^{i\infty} f(z) Q_{k,D,\mathscr{A}}(-z)\,dz,$$

$$= \sum_{n=0}^{w} (-1)^n q_{k,D,\mathscr{A}}^{(n)} \int_0^{i\infty} f(z) z^n\,dz,$$

as was to be shown.

In view of the proposition preceding the theorem, an equivalent formulation is

$$f_{k,D,\mathscr{A}}(z) = 2^{-w} \sum_{n=0}^{w} i^{1-n} q_{k,D,\mathscr{A}}^{(n)} R_n(z).$$

Replacing $\mathscr{A}$ by $\mathscr{A}'$ replaces $f_{k,D,\mathscr{A}}(z)$ by $\overline{f_{k,D,\mathscr{A}}(-\bar{z})}$. Since $\overline{R_n(-\bar{z})} = R_n(z)$, we find that

$$f_{k,D,\mathscr{A}'}(z) = 2^{-w} \sum_{n=0}^{w} i^{n-1} q_{k,D,\mathscr{A}}^{(n)} R_n(z)$$

and therefore

$$f_{k,D,\mathscr{A}}^{\pm}(z) = 2^{3-2k} \sum_{\substack{0 \leqslant n \leqslant w \\ (-1)^n = \mp 1}} (-1)^{[(n-1)/2]} q_{k,D,\mathscr{A}}^{(n)} R_n(z),$$

which gives $f_{k,D,\mathscr{A}}^{\pm}$ as an explicit rational linear combination of the functions R_n belonging to $\mathfrak{S}_{2k}^{\pm}$.

3.2. The scalar product of $f_{k,D,\mathscr{A}_1}^{+}$ and $f_{k,D_2,\mathscr{A}_2}^{-}$

We have proved in §2.3 and again in §3.1 that the cusp form $f_{k,D,\mathscr{A}}^{\pm}$ belongs to $\mathfrak{S}_{2k}^{\pm}$. Since $\mathfrak{S}_{2k}^{+}$ and $\mathfrak{S}_{2k}^{-}$ are dual $\mathbb{Q}$-vector spaces, it follows that the scalar product $(f_{k,D_1,\mathscr{A}_1}^{+}, f_{k,D_2,\mathscr{A}_2}^{-})$ is rational for any classes $\mathscr{A}_1, \mathscr{A}_2$ of quadratic forms of discriminant D_1, D_2. The various formulae proved so far in this paper would permit us to give various expressions for this number – for example, we could express $f_{k,D_1,\mathscr{A}_1}^{+}$ as a rational linear combination of R_n (n odd) by the results of §3.1 and then compute $(R_n, f_{k,D_2,\mathscr{A}_2}^{-})$ by the formulae in §2.3, or we could express both $f_{k,D_i,\mathscr{A}_i}$ in terms of the R_n and use the results of Section 1. The most natural formula, however, is obtained by writing the scalar product as $2^{1-2k}(r_{\mathscr{A}_1} + r_{\mathscr{A}_1'})(f_{k,D_2,\mathscr{A}_2}^{-})$ and computing the integral around the geodesic. This was done by S. Katok in her thesis [5]. The calculations are somewhat analogous to those in §§1.4 and 2.3 of this paper. One first uses an 'unfolding argument' to write the integral as a sum of integrals parametrized by the set of double cosets $\Gamma_{Q_1}\backslash\Gamma/\Gamma_{Q_2}$, where Q_i is a form in the class $\mathscr{A}_i$. All but finitely many of these integrals are zero because their integrands are rational functions all of whose poles lie on the same side of the path of integration. The remaining double cosets are those corresponding to the intersections of the geodesics $C_{\mathscr{A}_1} \cup C_{\mathscr{A}'_1}$ and $C_{\mathscr{A}_2}$ in $\Gamma \backslash \mathfrak{H}$. The final result, which we do not state in detail, expresses the scalar product as a sum of local contributions from these intersection points, each given as a simple multiple of $P_{k-1}(\cos\theta)$, where θ is the angle between the two geodesics at their intersection point and P_{k-1} denotes the $(k-1)$th Legendre polynomial.

3.3. Applications to zeta-functions of real quadratic fields

Another way to combine the formulae proved so far is to compute (via the results of Sections 1 and 2) the periods of both sides of the identity

$$f^{\pm}_{k,D,\mathscr{A}} = 2^{-2k+1} \sum_{\substack{0 \leqslant n \leqslant w \\ (-1)^n = \mp 1}} (-1)^{[(n-1)/2]} q^{(n)}_{k,D,\mathscr{A}} R_n$$

proved in §3.1. In particular, taking the 0th period of $f^{+}_{k,D,\mathscr{A}}$, we find after a short calculation:

Theorem 8 *Let $\mathscr{A}$ be an equivalence class of quadratic forms of non-square discriminant D and $\zeta_{\mathscr{A}}(s)$ the corresponding zeta-function. Then for $k \geqslant 2$ we have*

$$\zeta_{\mathscr{A}}(1-k) = -\sum_{\substack{[a,b,c]\in\mathscr{A} \\ \text{reduced}}} \left\{ \frac{B_{2k}}{k}(a^{k-1} + (b-a-c)^{k-1}) + \sum_{\substack{n=1 \\ n\,\text{odd}}}^{2k-3} \left[\frac{B_{2k}}{2k}\left(\frac{B_{n+1}}{n+1} + \frac{B_{\tilde{n}+1}}{\tilde{n}+1} + \delta_{1,n}\frac{1+\delta_{k,2}}{2k-2} \right) + \frac{B_{n+1}}{n+1}\frac{B_{\tilde{n}+1}}{\tilde{n}+1} \right] d_{k,n}(a,b,c) \right\},$$

where $d_{k,n}(a,b,c)$ denotes the coefficient of X^n in $(aX^2+bX+c)^{k-1}$.

As an example, we have

$$\zeta_{\mathscr{A}}(-5) = \frac{1}{2^2\cdot 3^2\cdot 5\cdot 7\cdot 13} \sum_{\substack{[a,b,c]\in\mathscr{A} \\ \text{reduced}}} [691a^5 + 691(b-a-c)^5 + 3(24bc^4 - 50abc^3 - 25b^3c^2 + 30a^2bc^2 + 20ab^3c + b^5)].$$

By applying other periods than r_0, we obtain other identities among the coefficients of $Q(X,Y)^{k-1}$, where Q runs over the reduced polynomials in the class $\mathscr{A}$. We do not give these explicitly, since the reader can easily work them out if he so desires.

A formula for $\zeta_{\mathscr{A}}(1-k)$ of the same type as Theorem 8 was proved by D.

Kramer in his thesis [9]. His proof was based on the method of [21], in which formulae for $\zeta_{\mathscr{A}}(1-k)$ in terms of *rational* (rather than polynomial) functions of the coefficients of reduced polynomials were given. This method did not use the theory of modular forms or their periods but was based instead on a certain decomposition of the zeta-function $\zeta_{\mathscr{A}}(s)$ into simpler Dirichlet series; this decomposition, first given in [18], was generalized by Shintani to arbitrary totally real number fields and the special values of their zeta-functions [15].

Finally, we can use Theorem 7 to obtain results on modular forms of half-integral weight. By the results quoted in §2.1 we have

$$(-1)^{k/2}2^{3k-1}f_{k,D}(z)=\sum_j (g_j,g_j)^{-1}c_j(1)c_j(D)f_j(z),$$

where the f_j are the Hecke eigenforms in S_{2k} and

$$g_j=\sum_{\substack{n>0\\ n\equiv 0,1\ (\mathrm{mod}\ 4)}} c_j(n)q^n$$

the corresponding eigenforms in $S_{k+1/2}$. Thus we have

$$(-1)^{k/2}2^{3k-1}(f,f_{k,D})=\frac{(f,f)}{(g,g)}c(1)c(D)$$

for any Hecke eigenform f, where we have omitted the index j. On the other hand, by the results of §3.1 we have

$$\begin{aligned}2^{2k-1}(f,f_{k,D})&=\sum_{\mathscr{A}} r_{\mathscr{A}}(f)\\ &=\sum_{\mathscr{A}}\sum_{\substack{n=0\\ n\ \mathrm{odd}}}^{w}(-1)^{(n-1)/2}q^{(n)}_{k,D,\mathscr{A}}r_n(f)\end{aligned}$$

(the terms with n even drop out when we combine the classes $\mathscr{A}$ and $\mathscr{A}'$.) Hence we find

$$c(1)g(z)=(-1)^{k/2}2^k\frac{(g,g)}{(f,f)}\sum_{\substack{n=0\\ n\ \mathrm{odd}}}^{w}(-1)^{(n-1)/2}r_n(f)\Psi_n(z)$$

where

$$\Psi_n(z)=\sum_{\substack{D>0\\ D\equiv 0,1\ (\mathrm{mod}\ 4)}}\left(\sum_{\substack{\mathrm{disc}\,Q=D\\ Q\ \mathrm{reduced}}} d_{k,n}(Q)\right)e^{2\pi i D z}$$

($d_{k,n}(Q)=$ coefficient of X^n in $Q(X,1)^{k-1}$, as usual). Since the numbers $r_n(f)$ $(0<n<w, n$ odd) are not linearly independent, it does not follow from this formula that the functions $\Psi_n(z)$ belong to $S_{k+1/2}$, and indeed by looking at examples one sees that they do not. On the other hand, the formula shows that

all functions in $S_{k+1/2}$ are linear combinations of the Ψ_n, so these functions – which can be thought of as theta series with respect to the indefinite ternary form b^2-4ac and the spherical polynomials $d_{k,n}(a,b,c)$ – are very related to forms of half-integral weight; it might be of interest to study them further.

4. COMPLEMENTS

4.1 Reinterpretation of formulae and extension to non-cusp forms

As pointed out in §1.1, the Eichler–Shimura theorem gives natural isomorphisms

$$\mathfrak{S}_{2k}^{-} \xrightarrow[r]{\sim} \mathbf{W}^{-}(\mathbb{Q}), \qquad \mathfrak{S}_{2k}^{+}\oplus\mathbb{Q} \xrightarrow[(r^+,p_0)]{\sim} \mathbf{W}^{+}(\mathbb{Q})$$

rather than simply $\mathfrak{S}_{2k}^{\pm}\simeq \mathbf{W}^{\pm}(\mathbb{Q})$. Thus one might expect that to any naturally occurring example of a cusp form $f\in\mathfrak{S}_{2k}^{+}$ there is associated a rational constant c such that the polynomial $r^+(f)(X)+cp_0(X)$ is a simpler polynomial than $r^+(f)(X)$ itself, i.e. that the formulae for $r_n(f)$ involve correction terms for $n=0$ and $n=w$. This is indeed what we found for the various special functions treated in this paper: the even period polynomial of $(-1)^{(\tilde{n}-1)/2}2^{-w}R_n(z)$ $(0<n<w, n$ odd, $\tilde{n}=w-n)$ was

$$\left(\frac{1}{n+1}B_{n+1}^{0}+\frac{1}{\tilde{n}+1}B_{\tilde{n}+1}^{0}\right)\Bigg|(1-S)\;-\frac{2k}{B_{2k}}\frac{B_{n+1}}{n+1}\frac{B_{\tilde{n}+1}}{\tilde{n}+1}p_0(X)$$

(Theorem 1′) and that of $f_{k,D}(z)$ (D the discriminant of a real quadratic field K) was

$$-P_{k,D}(X)-\frac{k}{B_{2k}}\zeta_K(1-k)p_0(X)$$

(Theorem 4), while for $f_{k,D,\mathscr{A}}^{+}(z)$ there was a similar result with ζ_K replaced by the corresponding partial zeta function $\zeta_{\mathscr{A}}$ (Theorem 5). Thus if we define a map

$$\tilde{r}^{+}:\mathfrak{S}_{2k}^{+}\oplus\mathbb{Q}\xrightarrow{\sim}\mathbf{W}_{2k-2}^{+}(\mathbb{Q})$$

by

$$(f,c)\mapsto r^{+}(f)(X)+\frac{2k}{B_{2k}}cp_0(X),$$

where the factor $2k/B_{2k}$ has been included to simplify the formulae, then we have

$$\left((-1)^{(\tilde{n}-1)/2}2^{-w}R_n,\frac{B_{n+1}}{n+1}\frac{B_{\tilde{n}+1}}{\tilde{n}+1}\right)\overset{\tilde{r}^{+}}{\mapsto}$$

$$\left(\frac{1}{n+1}B^0_{n+1}+\frac{1}{\tilde n+1}B^0_{\tilde n+1}\right)\bigg|(1-S),$$

$$(f_{k,D},\tfrac12\zeta_K(1-k))\overset{\tilde r^+}{\longmapsto}-P_{k,D}(X)$$

and, more generally (Theorem 5),

$$(f^+_{k,D,\mathscr{A}},\zeta_{\mathscr{A}}(1-k))\overset{\tilde r^+}{\longmapsto}(Q_{k,D,\mathscr{A}}+Q_{k,D,\mathscr{A}^*})|(-U+U^2)|(1+\varepsilon)/2$$

In other words, the formulae for the period polynomials force us to 'augment' the modular forms $R_n, f_{k,D}$ and $f^+_{k,D,\mathscr{A}}$ by the constants

$$(-1)^{(\tilde n-1)/2}2^w\frac{B_{n+1}}{n+1}\frac{B_{\tilde n+1}}{\tilde n+1},\qquad \tfrac12\zeta_K(1-k)\quad\text{and}\quad\zeta_{\mathscr{A}}(1-k),$$

respectively. In the same spirit, we notice that the formulae obtained for the images of our special functions in $\mathfrak{S}^+_{2k}$ under the maps ρ_m defined in the proposition in §1.4 involved an extra multiple of $G_{2k}(z)$ for $m=0$ and $m=w$. For example, the function $\rho_m(R_n)$ $(0\leqslant m<n\leqslant\frac12 w)$ was shown in §1.4 to be a multiple of $F_m(G_{2k-m-n-1},G_{n-m+1})$ for $m\neq 0$ but a linear combination of $G_{2k-n+1}G_{n+1}$ and G_{2k} for $m=0$, and similarly $\rho_0(f^+_{k,D,\mathscr{A}})$ in §2.4 turned out to be a linear combination of $G_{k,D,\mathscr{A}}$ and G_{2k}. In each case the coefficient of G_{2k} in $\rho_0(f)$ was the same as the multiple of p_0 occurring in $r^+(f)$. Thus as well as the augmented period map $\tilde r^+$ we have an augmented version of ρ_0 given by

$$\tilde\rho_0:\mathfrak{S}^+_{2k}\oplus\mathbb{Q}\xrightarrow{\sim}\mathfrak{M}^0_{2k}(\mathbb{Q}),\quad (f,c)\mapsto\rho_0(f)-\frac{2k}{B_{2k}}cG_{2k}(z)$$

($\mathfrak{M}^0_{2k}$ is the space of modular forms with rational Fourier coefficients); then we have the commutative diagram

$$\begin{tikzcd}
& \mathfrak{M}^0_{2k}(\mathbb{Q}) \arrow[dl, "\text{constant term}"'] \arrow[dr, "\text{coefficient of } q^1"] & \\
\mathbb{Q} & \mathfrak{S}^+_{2k}\oplus\mathbb{Q} \arrow[l, "-\frac12 pr_2"'] \arrow[r, "\left(r_0,\,-\frac{2k}{B_{2k}}\right)"'] \arrow[u, "\tilde\rho_0"', "\wr"] \arrow[d, "\tilde r^+"', "\wr"] & \mathbb{Q} \\
& \mathbf{W}^+(\mathbb{Q}) \arrow[ul, dashed, "\lambda"] \arrow[ur, "\text{constant term}"'] &
\end{tikzcd}$$

(where the map λ, defined by the diagram, will be given explicitly in §4.2) and our two basic examples become

$$\begin{tikzcd}
& -2G_{n+1}(z)G_{\tilde{n}+1}(z) \arrow[dl] \arrow[dr] & \\
-\frac{1}{2}\frac{B_{n+1}}{n+1}\frac{B_{\tilde{n}+1}}{\tilde{n}+1} & \left((-1)^{(n-1)/2}2^{-w}R_n(z), \frac{B_{n+1}}{n+1}\frac{B_{\tilde{n}+1}}{\tilde{n}+1}\right) \arrow[l, mapsto] \arrow[r, mapsto] \arrow[u, mapsto] \arrow[d, mapsto] & \frac{B_{n+1}}{n+1}+\frac{B_{\tilde{n}+1}}{\tilde{n}+1} \\
& \left(\frac{1}{n+1}B^0_{n+1}(X)+\frac{1}{\tilde{n}+1}B^0_{\tilde{n}+1}(X)\right)\Big|(1-S) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \frac{1}{2}G_{k,D}(z) \arrow[dl] \arrow[dr] & \\
-\frac{1}{4}\zeta_K(1-k) & \left(f_{k,D}(z), \frac{1}{2}\zeta_K(1-k)\right) \arrow[l, mapsto] \arrow[r, mapsto] \arrow[u, mapsto] \arrow[d, mapsto] & \sum_{\substack{|b|<\sqrt{D} \\ b\equiv D(2)}} \sigma_{k-1}\left(\frac{D-b^2}{4}\right) \\
& -P_{k,D}(X) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

(and similarly for $G_{k,D,\mathscr{A}}$).

However, augmenting $(-1)^{(\tilde{n}-1)/2}2^{-w}R_n$ and $f_{k,D}$ by the constants $B_{n+1}B_{\tilde{n}+1}/(n+1)(\tilde{n}+1)$ and $\frac{1}{2}\zeta_K(1-k)$ in this way is a purely *ad hoc* construction, based on the forms of the formulae for their period polynomials. To see in a natural way where these constants come from, we give a different and more natural interpretation of the somewhat artificial space $\mathfrak{S}^+_{2k}\oplus\mathbb{Q}$. By the last theorem of §1.1, there is a natural identification of $\mathfrak{S}^+_{2k}$ with $\mathrm{Hom}_{\mathbb{Q}}(\mathfrak{S}^-_{2k},\mathbb{Q})$ given by the Petersson scalar product; we will extend this to an identification of $\mathfrak{S}^+_{2k}\oplus\mathbb{Q}$ with $\mathrm{Hom}_{\mathbb{Q}}(\mathfrak{M}^-_{2k},\mathbb{Q})$, where $\mathfrak{M}^-_{2k}$ is a rational structure on M_{2k} extending $\mathfrak{S}^-_{2k}$. To do this we notice that the periods $r_n(f)$ $(0\leqslant n\leqslant w)$ can be defined for any $f=\sum_0^\infty a(l)q^l\in M_{2k}$, not just cusp forms, by the formula

$$r_n(f)=\frac{n!}{(2\pi)^{n+1}}L(f,n+1),$$

where $L(f,s)$ is the meromorphic continuation of the series $\sum_1^\infty a(l)l^{-s}$. For $f\in S_{2k}$ this definition agrees with the definition of $r_n(f)$ as a period integral, so we only have to calculate the new periods for $f=G_{2k}$. We have

$$L(G_{2k},s)=\sum_1^\infty\sigma_{2k-1}(l)l^{-s}=\zeta(s)\zeta(s-2k+1),$$

so that

$$r_n(G_{2k})=\begin{cases}\dfrac{n!}{(2\pi)^{n+1}}\zeta(n+1)\zeta(-\tilde{n}) & (0<n\leqslant w, \tilde{n}=w-n), \\ \dfrac{1}{2\pi}\zeta'(-2k+2) & (n=0),\end{cases}$$

or – using the functional equation and special values of $\zeta(s)$ –

$$r_0(G_{2k}) = \frac{(-1)^{k-1}w!}{2^{2k}\pi^{2k-1}}\zeta(2k-1) = \alpha, \qquad \text{say,}$$

$$r_w(G_{2k}) = -\frac{1}{2}\frac{w!}{(2\pi)^{2k-1}}\zeta(2k-1) = (-1)^k\alpha,$$

$$r_n(G_{2k}) = 0 \quad (0 < n < w, n \text{ even}),$$

$$r_n(G_{2k}) = \frac{(-1)^{(n+1)/2}}{2}\frac{B_{n+1}}{n+1}\frac{B_{\tilde{n}+1}}{\tilde{n}+1} \quad (0 < n < w, n \text{ odd}).$$

Thus if we define $\mathfrak{M}_{2k}^{+}$ as the space of modular forms $f \in M_{2k}$ such that $r_n(f)$ is rational for all n with $(-1)^n = \pm 1$, then

$$\mathfrak{M}_{2k}^{-} = \mathfrak{S}_{2k}^{-} \oplus \mathbb{Q}\cdot G_{2k}, \quad \mathfrak{M}_{2k}^{+} = \mathfrak{S}_{2k}^{+} \oplus \mathbb{Q}\cdot\alpha^{-1}G_{2k}.$$

(As an amusing sidelight, note that these formulae and the expected duality between the plus and minus spaces suggests that (G_{2k}, G_{2k}) should be a rational multiple of α for any reasonable extension of the Petersson scalar product to non-cusp forms, and this is indeed true for the extension given in [24, pp. 434–5].) It is now clear in what sense the cusp form R_n and the number $B_{n+1}/(n+1)$. $B_{n+1}/(\tilde{n}+1)$ are related: the former describes the action of r_n on S_{2k} and the latter on G_{2k}. More precisely, if we define an isomorphism

$$\iota : \mathfrak{S}_{2k}^{+} \oplus \mathbb{Q} \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Q}}(\mathfrak{M}_{2k}^{-}, \mathbb{Q})$$

by

$$\iota(f, c)(g + c'G_{2k}) = 2^w(f, g) + \tfrac{1}{2}(-1)^{k-1}cc'$$
$$(f \in \mathfrak{S}_{2k}^{+}, g \in \mathfrak{S}_{2k}^{-}, c, c' \in (\mathbb{Q}),$$

then the pair

$$\left(2^{-w}R_n, (-1)^{(n-1)/2}\frac{B_{n+1}}{n+1}\frac{B_{\tilde{n}+1}}{\tilde{n}+1}\right) \in \mathfrak{S}_{2k}^{+} \oplus \mathbb{Q} \quad (n \text{ odd}),$$

which occurred somewhat unnaturally before, simply corresponds to the map $r_n : \mathfrak{M}_{2k}^{-} \to \mathbb{Q}$. (The factors 2^w and $(-1)^{k-1}/2$ in the definition of ι were included to make this simple statement true.) The other pair we encountered, namely $(f_{k,D,\mathscr{A}}^{+}, \zeta_{\mathscr{A}}(1-k))$, corresponds under ι to the integral-around-a-geodesic map $r_{\mathscr{A}} + r_{\mathscr{A}'}$ studied in Section 3. Indeed, for cusp forms this is the content of the Proposition in §3.1, while for the Eisenstein series we have the following calculation:

$$r_{\mathscr{A}}(G_{2k}) = \int_{\Gamma_Q \backslash C_Q} \frac{(-1)^k(2k-1)!}{(2\pi)^{2k}} \sum_{\mathbb{Z}^2/\{\pm 1\}}{}' \frac{1}{(mz+n)^{2k}} \cdot (az^2 + bz + c)^{k-1}dz$$

$$= \frac{(-1)^k(2k-1)!}{(2\pi)^{2k}} \sum_{\mathbb{Z}^2/\Gamma_Q}{}' \int_{C_Q} \frac{(az^2+bz+c)^{k-1}}{(mz+n)^{2k}}dz$$

(this is the usual 'unfolding trick')

$$=\frac{(-1)^k(2k-1)!}{(2\pi)^{2k}}\sum_{\mathbb{Z}^2/\Gamma_Q}{}' \frac{(k-1)!^2}{(2k-1)!}\frac{D^{k-1/2}}{(an^2-bmn+cm^2)^k}$$

(the integration is performed by making the substitution $t=1/(mz+n)$ if $n\neq 0$ to get an integral of the form $\int_\alpha^\beta (t-\alpha)^{k-1}(t-\beta)^{k-1}dt$, which after the substitution $t=\alpha+(\beta-\alpha)x$ becomes a standard beta integral)

$$=\frac{(-1)^k(k-1)!^2D^{k-1/2}}{(2\pi)^{2k}}(\zeta_{\mathscr{A}}(k)+(-1)^k\zeta_{\mathscr{A}^*}(k))$$
$$=\tfrac{1}{2}(-1)^k\zeta_{\mathscr{A}}(1-k).$$

4.2. Explicit description of the map λ

The main result of this section is the following theorem:

Theorem 9 *Define rational numbers $\lambda_{k,n}$ ($k\geqslant 2$, $0\leqslant n\leqslant 2k-2$, n even) by*

$$\lambda_{k,n}=B_{2k}\left[1+\binom{2k-1}{n}-\binom{2k-1}{n+1}\right]+2\sum_{r=1}^{k}\binom{2r-1}{n}\binom{2k}{2r}B_{2r}B_{2k-2r}.$$

Then

(i) $\lambda_{k,n}=-\lambda_{k,2k-2-n}$;

(ii) $\sum_{\substack{n=0\\ n\text{ even}}}^{2k-2}(-1)^{n/2}\lambda_{k,n}r_n(f)=0$ *for all* $f\in S_{2k}$.

Since $\lambda_{k,0}=-\lambda_{k,2k-2}=-3(2k-1)B_{2k}\neq 0$, this theorem yields a relation among the coefficients of period polynomials which is not satisfied by the polynomial $p_0(X)$ and therefore exhibits the map $\lambda:\mathbf{W}_{2k-2}\to\mathbb{Q}$ discussed in §§1.1 and 4.1:

$$\lambda:\sum_{\substack{n=0\\ n\text{ even}}}^{2k-2}\binom{2k-2}{n}a_nX^n\mapsto\frac{1}{24k(2k-1)}\sum_n\lambda_{k,n}a_n.$$

The first few coefficients $\lambda_{k,n}$ are given in Table 1, where for convenience a common denominator of the $\lambda_{k,n}$ has been chosen for each k (thus $\lambda_{4,0}=7/10$, $\lambda_{4,2}=31/18$). As a numerical check of part (ii) of the theorem, we have (using

Table 1 The coefficients λ_{kn}

k	Denominator	$n=0$	$n=2$	$n=4$	$n=6$	$n=8$
1	1	0				
2	10	3	−3			
3	−14	5	0	−5		
4	90	63	155	−155	−63	
5	−66	135	854	0	−854	−135
6	2730	22 803	263 781	327 166	−327 166	−263 781
7	−90	4095	74 404	212 325	0	−212 325
8	1530	488 295	12 754 911	62 018 627	55 137 531	−55 137 531
9	−3990	11 186 085	396 185 430	2 880 943 650	5 467 320 254	0
10	6930	209 009 367	9 625 959 997	97 060 379 284	298 093 976 908	217 739 243 986

the values for $r_n(\Delta)$ given in §1.1)

$$\sum_{\substack{n=0\\ n\,\text{even}}}^{10} (-1)^{n/2}\lambda_{6,n}r_n(\Delta)$$
$$=2\cdot\frac{\omega_+}{2730}\left(22\,803\cdot\frac{192}{691}-263\,781\cdot\frac{16}{135}+327\,166\cdot\frac{8}{105}\right)=0.$$

Proof of Theorem 9 For the proof of part (i), and for later purposes, it is convenient to define the $\lambda_{k,n}$ for all k and n satisfying $0\leqslant n\leqslant 2k-2$ by

$$\lambda_{k,n}=B_{2k}\left[(-1)^n+\binom{2k-1}{n}-\binom{2k-1}{n+1}\right]$$
$$+2\sum_{r=1}^{k}\binom{2k}{2r}\binom{2r-1}{n}B_{2r}B_{2k-2r}-\binom{2k}{n+1}B_{n+1}B_{\tilde{n}+1},$$

where $\tilde{n}=2k-2-n$ as usual. Then

$$\tfrac{1}{2}(\lambda_{k,n}+\lambda_{k,\tilde{n}})=(-1)^nB_{2k}+\sum_{r=1}^{k}\binom{2k}{2r}\left[\binom{2r-1}{n}\right.$$
$$\left.+\binom{2r-1}{\tilde{n}}\right]B_{2r}B_{2k-2r}-\binom{2k}{n+1}B_{n+1}B_{\tilde{n}+1},$$

so

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(2k)!}\sum_{n=0}^{2k-2}(\lambda_{k,n}+\lambda_{k,2k-2-n})x^ny^{2k-2-n}$$
$$=\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\frac{x^{2k-1}+y^{2k-1}}{x+y}+$$

$$+\sum_{\substack{r\geqslant 1\\ s\geqslant 0}}\frac{B_{2r}}{(2r)!}\frac{B_{2s}}{(2s)!}\Big\{y^{2s-1}(x+y)^{2r-1}-\delta_{s,0}\,x^{2r-1}/y$$
$$+x^{2s-1}(x+y)^{2r-1}-\delta_{s,0}y^{2r-1}/x\Big\}$$
$$-\sum_{\substack{n\geqslant 1\\ n,\tilde n\ \text{odd}}}\sum_{\tilde n\geqslant 1}\frac{B_{n+1}}{(n+1)!}\frac{B_{\tilde n+1}}{(\tilde n+1)!}x^n y^{\tilde n}-\frac{1}{4}$$
$$=\frac{1}{2}\frac{1}{x+y}\left(\coth\frac{x}{2}-\frac{2}{x}+\coth\frac{y}{2}-\frac{2}{y}\right)$$
$$+\frac{1}{4}\coth\frac{y}{2}\left(\coth\frac{x+y}{2}-\frac{2}{x+y}\right)-\frac{1}{2y}\left(\coth\frac{x}{2}-\frac{2}{x}\right)$$
$$+\frac{1}{4}\coth\frac{x}{2}\left(\coth\frac{x+y}{2}-\frac{2}{x+y}\right)-\frac{1}{2x}\left(\coth\frac{y}{2}-\frac{2}{y}\right)$$
$$-\frac{1}{4}\left(\coth\frac{x}{2}-\frac{2}{x}\right)\left(\coth\frac{y}{2}-\frac{2}{y}\right)-\frac{1}{4}$$
$$=0.$$

We remark that an identity similar to (i) was proved by D. Kramer in his thesis [9, proof of Theorem 4].

For (ii) we give two proofs. The first is based on an identity of Haberland's expressing the Petersson scalar product of two cusp forms in terms of their periods. The second is direct but rather computational.

Haberland's identity, proved in Section 7 of [4], is

$$(f,g)=\frac{1}{3\cdot 2^{2k-1}}\sum_{\substack{0\leqslant m<n\leqslant w\\ m\not\equiv n\,(\mathrm{mod}\,2)}}(-1)^{(n+1-m)/2}\binom{w}{n}\binom{n}{m}r_n(f)\overline{r_{w-m}(g)}$$
$$(f,g\in S_{2k})$$

(actually, he states this only for $f=g$ a Hecke eigenform; his formula must also be corrected by a factor $w!^2/i$). His proof uses the language of group cohomology but can be given purely in terms of the period polynomials. We will do this here, at the same time generalizing the formula by allowing one of the forms, say g, to be non-cuspidal. Then taking g to be G_{2k}, which is orthogonal to cusp forms, will give a non-trivial relation among the periods $r_n(f)$.

It is convenient to introduce the pairing

$$\left\langle\sum_0^w a_n X^n,\sum_0^w b_n x^n\right\rangle=\sum_0^w(-1)^n\binom{w}{n}^{-1}a_n b_{w-n}$$

on $\mathbf{V}_{2k-2}$. This pairing is easily checked to be symmetric, non-degenerate and Γ-invariant (i.e. $\langle F|\gamma, G|\gamma\rangle = \langle F, G\rangle$ for $F, G \in \mathbf{V}$, $\gamma \in \Gamma$). Then Haberland's formula can be written

$$-6(2i)^{2k-1}(f,g) = \langle r(f)|(T-T^{-1}), \overline{r(g)}\rangle,$$

where

$$\overline{r(g)}(X) = \overline{r(g)(\bar{X})} = \sum_m i^{m-1}\binom{w}{m}\overline{r_m(g)}\, X^{w-m}.$$

To prove it, we define

$$F(z) = \int_z^{i\infty} f(u)(u-\bar{z})^w du,$$

so that $\partial F(z)/\partial z = -f(z)(z-\bar{z})^w$. Then, denoting by $\mathscr{D}$ the standard fundamental domain for $\Gamma\backslash\mathfrak{H}$, we have

$$\begin{aligned}(2i)^{2k-1}(f,g) &= -\int_{\mathscr{D}} f(z)\overline{g(z)}(z-\bar{z})^w dz\,d\bar{z}\\ &= \int_{\mathscr{D}} d[F(z)\overline{g(z)\,dz}]\\ &= \int_{\partial\mathscr{D}} F(z)\overline{g(z)dz}\end{aligned}$$

by Stokes' theorem. Since F and g are periodic, the integrals along the vertical sides of $\partial\mathscr{D}$ cancel and we can replace $\partial\mathscr{D}$ by its bottom side C, an arc going from ρ^2 to ρ ($\rho = e^{\pi i/3} = (1+i\sqrt{3})/2$). Also, S maps C to itself with the orientation reversed, so

$$\begin{aligned}2(2i)^{2k-1}(f,g) &= \int_C [F(z)\overline{g(z)dz} - F(Sz)\overline{g(Sz)d(Sz)}]\\ &= \int_C (F(z) - \bar{z}^w F(-1/z))\overline{g(z)dz}.\end{aligned}$$

But

$$F(z) - \bar{z}^w F(-1/z) = \int_0^{i\infty} f(u)(u-\bar{z})^w du = r(f)(\bar{z}),$$

so the right-hand side equals $\langle r(f), \bar{H}\rangle$ with

$$\begin{aligned}H(X) &= \sum_{n=0}^{w}(-1)^n\binom{w}{n}X^{w-n}\int_C z^n g(z)dz\\ &= \int_{\rho^2}^{\rho} g(z)(z-X)^w dz \in \mathbf{V},\end{aligned}$$

where now the path of integration from ρ^2 to ρ can be chosen arbitrarily, since the integrand is holomorphic. If g as well as f is a cusp form, then we can write $H = H_{\rho^2} - H_\rho$, where

$$H_{z_0}(X) = \int_{z_0}^{i\infty} g(z)(z-X)^w dz \qquad (z_0 \in \mathfrak{H}).$$

Then

$$H_{z_0}|\gamma = \int_{\gamma^{-1}(z_0)}^{\gamma^{-1}(i\infty)} g(z)(z-X)^w dz \quad \text{for} \quad \gamma \in \Gamma,$$

so

$$H_\rho|(1-U) = \left(\int_\rho^{i\infty} + \int_0^\rho\right) g(z)(z-X)^w dz = r(g)(X)$$

and $H_\rho|T = H_{\rho^2}$. On the other hand, using the period relations, we have

$$\begin{aligned} r(f)|(1-T^{-1}) &= r(f)|(1+ST^{-1}) = r(f)|(1+U^2) \\ &= \tfrac{1}{3} r(f)|(U^2-U)|(1-U^{-1}), \end{aligned}$$

so

$$\begin{aligned} \langle r(f), \bar{H}\rangle &= \langle r(f), \bar{H}_\rho|(T-1)\rangle = \langle r(f)|(T^{-1}-1), \bar{H}_\rho\rangle \\ &= \tfrac{1}{3}\langle r(f)|(U-U^2), \bar{H}_\rho|(1-U)\rangle \\ &= \tfrac{1}{3}\langle r(f)|(TS-ST^{-1}), \overline{r(g)}\rangle \\ &= -\tfrac{1}{3}\langle r(f)|(T-T^{-1}), \overline{r(g)}\rangle, \end{aligned}$$

completing the proof of Haberland's formula (the proof being, as we said, his, but with the terminology of cohomology of groups removed). If g is not a cusp form, but has instead a Fourier development $\sum_{l=0}^\infty a_l q^l$, then the analytic continuation of the L-series of g is given by

$$\begin{aligned} (2\pi)^{-s}\Gamma(s)L(g,s) &= \int_{t_0}^\infty (g(it)-a_0)t^{s-1}dt \\ &\quad + (-1)^k \int_{t_0^{-1}}^\infty (g(it)-a_0)t^{2k-s-1}dt \\ &\quad - a_0\left(\frac{t_0^s}{s} + (-1)^k \frac{t_0^{2k-s}}{2k-s}\right) \qquad (\text{any } t_0 \in \mathbb{R}), \end{aligned}$$

so the period polynomial as defined in §4.1 is given by

$$r(g) = H_{z_0}(X) - H_{Sz_0}(X)|S \qquad (\text{any } z_0 \in \mathfrak{H}),$$

where now

$$H_{z_0}(X) = \int_{z_0}^{i\infty} (g(z) - a_0)(z - X)^w dz$$
$$+ a_0 \frac{(X - z_0)^{w+1} - X^{w+1}}{w+1}.$$

In particular $r(g) = H_\rho - H_{\rho^2}|S = H_{\rho^2} - H_\rho|S$ as before, and we still have $H = H_{\rho^2} - H_\rho$; the difference is that now $H_{\rho^2} = H_\rho|T + a_0 E$, where

$$E(X) = \frac{(X+1)^{w+1} - X^{w+1}}{w+1}.$$

A calculation similar to the one for cusp forms now gives

$$-6(2i)^{2k-1}(f, g) = \langle r(f)|(T - T^{-1}), \overline{r(g)} \rangle - 2\bar{a}_0 \langle r(f), E_1 \rangle,$$

where

$$E_1 = E|(1 + T^{-1}) = \frac{(X+1)^{w+1} - (X-1)^{w+1}}{w+1}.$$

In particular, for $g = G_{2k}$ we obtain

$$\left\langle r(f), r(G_{2k})|(T - T^{-1}) - \frac{B_{2k}}{2k} E_1 \right\rangle = 0,$$

and in view of the formulae for $r_n(G_{2k})$ in §4.1 this is equivalent to the identity (ii).

We now turn to the second proof of (ii). By Theorem 1, we have

$$(-1)^{n/2} r_n\left((-1)^{k+(m+1)/2} 2^{-w} \binom{w}{m} R_m \right)$$
$$= c(n) - c(\tilde{n}) \qquad (m \text{ odd}, n \text{ even}),$$

where

$$c(n) = c(k, m; n) = \frac{n!}{m!} \beta_{n-m} + \frac{n!}{\tilde{m}!} \beta_{n-\tilde{m}} - \delta_{n,w} \frac{\beta_m \beta_{\tilde{m}}}{(2k-1)\beta_{2k}}.$$

Since the functions R_m with m odd span S_{2k}, and in view of part (i) of the theorem, it suffices to show that $\sum_n \lambda_{k,n} c(n) = 0$ for each odd m. We have

$$\sum_{\substack{n=0 \\ n \text{ even}}}^{w} \lambda_{k,n} c(n) = t_{k,m} + t_{k,\tilde{m}} - 3\binom{2k}{m+1} B_{m+1} B_{\tilde{m}+1},$$

where

$$t_{k,m} = \frac{1}{m!} \sum_n n! \beta_{n-m} \lambda_{k,n}.$$

Using the identity

$$\sum_{j=0}^{(N-1)/2} \binom{N}{2j} B_{2j} = \tfrac{1}{2}(N + \delta_{N,1}) \qquad (N \geqslant 1 \text{ odd})$$

we find that

$$\begin{aligned} t_{k,m} = B_{2k}\Bigg[&\frac{1}{m}\sum_{j=0}^{(\tilde{m}+1)/2}\binom{2j+m-1}{2j}B_{2j} + \frac{1}{2}\binom{2k-1}{m} \\ &-\binom{2k-1}{m}\sum_{j=0}^{(\tilde{m}+1)/2}\binom{2k-m-1}{2j}\frac{B_{2j}}{m+2j}\Bigg] \\ &+\sum_{r=(m+1)/2}^{k}\binom{2k}{2r}\binom{2r-1}{m}B_{2r}B_{2k-2r} \\ &+\binom{2k}{m+1}B_{m+1}B_{\tilde{m}+1}. \end{aligned}$$

The following lemma (with $N = 2k-1$) tells us that the expression in square brackets equals

$$\frac{1}{2}\left[-1+\binom{2k-1}{m}-\binom{2k-1}{m+1}\right],$$

so that

$$t_{k,m} = \tfrac{1}{2}\lambda_{k,m} + \frac{3}{2}\binom{2k}{m+1}B_{m+1}B_{\tilde{m}+1}$$

with $\lambda_{k,m}$ (m odd) defined as in the proof of (i); the antisymmetry of $\lambda_{k,m}$ under $m \mapsto \tilde{m}$ then gives the desired result.

Lemma *Let m, N be integers satisfying $1 \leqslant m \leqslant N-1$. Then*

$$\sum_{0 \leqslant j \leqslant (N-m)/2}\left[\frac{1}{m}\binom{2j+m-1}{2j} - \frac{1}{m+2j}\binom{N}{m}\binom{N-m}{2j}\right]B_{2j} = -\frac{1}{2}\left[\binom{N}{m+1}+1\right].$$

Proof Fix $N \geqslant 2$ and let

$$\alpha(x) = \sum_{m=1}^{N}\sum_{0 \leqslant j \leqslant (N-m)/2}\frac{1}{m}\binom{2j+m-1}{2j}B_{2j}x^m,$$

$$\beta(x) = \sum_{m=1}^{N}\sum_{0 \leqslant j \leqslant (N-m)/2}\frac{1}{m+2j}\binom{N}{m}\binom{N-m}{2j}B_{2j}x^m,$$

$$\gamma(x) = \frac{1}{2}\sum_{m=0}^{N-1}\left[\binom{N}{m+1}+1\right]x^m.$$

We want to show that the polynomials $\beta(x)-\alpha(x)$ and $\gamma(x)$ differ by a constant. For this, it suffices to show that their difference is periodic. Now

$$\alpha(x)=\sum_{l=1}^{N}\frac{1}{l}\sum_{j=0}^{(l-1)/2}\binom{l}{2j}B_{2j}x^{l-2j}=\sum_{l=1}^{N}\frac{1}{l}B_l^0(x)+\text{constant},$$

so

$$\begin{aligned}\alpha(x+1)-\alpha(x)&=\sum_{l=1}^{N}\tfrac{1}{2}[(x+1)^{l-1}+x^{l-1}]\\&=\frac{(x+1)^N-1}{2x}+\frac{x^N-1}{2(x-1)}\end{aligned}$$

by a property of B_l^0 mentioned in §1.2. Similarly,

$$\begin{aligned}\beta(x)&=\sum_{0\leqslant j<N/2}\binom{N}{2j}B_{2j}\sum_{1\leqslant m\leqslant N-2j}\binom{N-2j}{m}\frac{x^m}{m+2j}\\&=\sum_{0\leqslant j<N/2}\binom{N}{2j}B_{2j}\int_0^1 t^{2j-1}\{(1+xt)^{N-2j}-1\}\,dt\\&=\int_0^1 t^{N-1}\left\{B_N^0\left(x+\frac{1}{t}\right)-B_N^0\left(\frac{1}{t}\right)\right\}dt,\end{aligned}$$

and so, by the same property of B_N^0 as before,

$$\begin{aligned}\beta(x+1)-\beta(x)&=N\int_0^1\frac{\{(x+1)t+1\}^{N-1}+(xt+1)^{N-1}}{2}\,dt\\&=\frac{(x+2)^N-1}{2(x+1)}+\frac{(x+1)^N-1}{2x}.\end{aligned}$$

Finally,

$$\gamma(x)=\frac{(x+1)^N-1}{2x}+\frac{x^N-1}{2(x-1)},$$

so that

$$\begin{aligned}\gamma(x+1)-\gamma(x)&=\frac{(x+2)^N-1}{2(x+1)}-\frac{x^N-1}{2x}\\&=\beta(x+1)-\beta(x)-\alpha(x+1)+\alpha(x).\end{aligned}$$

This proves the lemma and completes the proof of Theorem 9.

By combining Theorem 9, which expresses the 0th and wth periods of a cusp form in terms of the even periods r_n with $0\leqslant n\leqslant w$, with Theorem 4 (resp. Theorem 5) for the periods of $f_{k,D}$ (resp. $f_{k,D,\mathscr{A}}$) we obtain a formula – similar to Theorem 8 and to the formulae in [9] – for the zeta-values $\zeta_K(1-k)$ (resp. $\zeta_{\mathscr{A}}(1-k)$) as sums of polynomials in the coefficients of reduced quadratic forms.

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