EXPLICIT CONSTRUCTIONS OF EXOTIC SPHERES

by

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CONTENTS

INTRODUCTION

The first example of a non-standard differentiable structure on the topological n-sphere was given in 1956 by Milnor [25]* for the case n=7. In the next few years further examples were constructed by Milnor [29] and the structure of the set Θ_n of differentiable structures on the topological n-sphere was investigated, culminating in the classical paper of Kervaire and Milnor [22] proving that Θ_n is a finite abelian group whose structure can be described quite explicitly. I will describe this work in more detail in the first chapter of this paper.

However, Milnor's example for n = 7 was constructed by special means--using the multiplicative properties of quaternions and constructing certain 3-sphere bundles over S^4--and his higher dimensional examples--twisted manifolds obtained by identifying the boundaries of two simple manifolds under a diffeomorphism--are artificial and hard to visualize. Since then, more natural ways have been found of constructing the exotic spheres. It is the purpose of this paper to describe these. Chapter II contains a description of Jänich's work classifying certain G-manifolds (differentiable manifolds on which a Lie group G acts differentiably) in terms of more

* Numbers in brackets refer to the bibliography

algebraic data. Chapter III describes the technique of obtaining manifolds as the boundaries of the manifold with boundary consisting of a number of tangent disc bundles to the sphere "plumbed" together. The generators of the cyclic groups bP_{n+1} (homotopy spheres that bound parallelizable (n+1)-manifolds) are obtainable in this way. Chapter IV gives perhaps the most pleasing and explicit description of exotic spheres as the sets of zeroes of very simple polynomials in the neighbourhood of isolated singularities; all spheres of bP_{n+1} are obtainable in this way. The classification theorem of Chapter II has two uses: to show that two manifolds obtained in different ways (e.g. by plumbing and from algebraic singularities) are in fact diffeomorphic, and to construct manifolds directly (by choosing the algebraic data and using the theorem to construct the corresponding G-manifold). The latter has the drawback that, though we can calculate the homology and find if the manifold is a homotopy sphere, there is no direct way of identifying its differentiable structure.

All of the examples produced lie in the subgroups bP_{n+1} and it would be of considerable interest to give constructions of the other, "very exotic," spheres. None of the three methods described can be easily extended to do so: the classification theorem approach gives no information about differentiable structure, as mentioned; the plumbing technique constructs its examples as boundaries; the manifolds obtainable as the set of zeroes of a complex function cannot be very exotic, for it is a a theorem of Kervaire [20] that the n-spheres that can imbed in codimension two are precisely the elements of bP_{n+1}. Of course, the only sphere that can imbed in codimension 1 is S^n; this seems to indicate that the elements of bP_{n+1} not only differ essentially from very exotic spheres but that this difference is in some way similar to their difference from the standard sphere. This is also suggested by Wu-yi Hsiang's work on the degree of symmetry of the exotic spheres. The degree of symmetry $N(X)$ of a manifold X is the largest dimension of Lie groups that can act effectively on X; by a well-known theorem $N(X^n) \leq \frac{n(n+1)}{2}$ with equality for $X = S^n$ [19]. Then $N(X) \leq \frac{n^2+7}{8}$ for $X \in bP_{n+1} - \{S^n\}$, $n \geq 64$ (see [15]. The bound is sharp), while $N(X) \leq (\frac{n+1}{4})^2 + 3$ for $X \in \Theta_n - bP_{n+1}$, $n \geq 300$ [14]. Our lack of knowledge about very exotic spheres is typified by the best known lower bounds--$N(X) \geq \frac{(n-1)(n-3)}{8}$ for $X \in bP_{n+1}$, $N(X) \geq 3$ for $X \in \Theta_{13}$ ([12] or [15]). One might hope to classify homotopy spheres by defining Θ_n^k as the set of n-spheres that can imbed in codimension k (plainly a subgroup) and expect that the quotients $\Theta_n^{k+1}/\Theta_n^k$ have simple structures or even are cyclic. No approach to these questions is in sight now--the only result I know on the very exotic spheres is that they don't all imbed in codimension 3 (the non-zero element of $\Theta_{16} = Z_2$ doesn't [42]). I mention them only to show how much is left to be done.

None of the work in this paper makes any claim to be original except the computations of signatures in the last chapter. My principal reference was the excellent exposition by Hirzebruch and

Mayer [13].

Finally, I would like to thank my supervisor, Professor M. F. Atiyah of New College, for his help throughout the year. My thanks are also due to Professor E. Brieskorn for a very stimulating series of lectures on the topology of complex singularities which he gave at Oxford in Spring 1966, and to the National Science Foundation for its financial support.

CHAPTER I

General Theory of Exotic Spheres

In the first section of this chapter we will describe the first example of an exotic sphere, given by Milnor in 1956. The remaining two sections describe the structure of the set Γ_n of differentiable structures on the topological n-sphere. By the h-cobordism theorem and its corollary the generalized Poincaré conjecture (in dimensions ≥ 5), proved by Smale in 1962, we can identify Γ_n with the set Θ_n of h-cobordism classes of homotopy n-spheres (or even homology spheres). The latter has a natural abelian group structure given by the connected sum (defined by Seifert in [41]). In section 2 we discuss this structure and prove that the quotient of Θ_n by bP_{n+1} (boundaries of π-manifolds) is a finite group, in fact isomorphic to a subgroup of the quotient of the stable group $\pi_{n+k}(S^k)$ by the image $J(\pi_n(SO_k))$ of the Whitehead homomorphism. In section 3 we show that bP_{n+1} is 0 if n is even and cyclic in any case, and determine its order (within a factor of 2 in some cases).

1. Milnor's original example

The manifold will be the total space of a 3-sphere bundle over S^4. Such bundles are classified by their characteristic maps which are elements of $\pi_3(SO(4))$ [37]. The latter group is Z+Z (an explicit construction assigns to $(m,n) \in Z+Z$ the map $f_{mn}: S^3 \to SO(4)$ given by $f_{mn}(u)\cdot v = u^m v u^n$ for $v \in R^4$, where $u \in S^3$ is thought of as a quaternion). For k an odd integer, let M_k^7 be the bundle with characteristic map $f_{\frac{1+k}{2}\,\frac{1-k}{2}}$. It is obtained by identifying two copies of $D^4 \times S^3$ along their boundaries (this sort of construction, and several similar ones used in this paper, such as taking products of manifolds with boundary or "straightening angles," produce manifolds which have to be smoothed in some way; however, this can always be done, as shown in [8], and I will ignore these questions in the future). Because of this explicit construction, it is very easy to give explicitly a Morse function for M_k with only two (non-degenerate) critical points, proving that M_k is topologically the 7-sphere. To prove that it is not diffeomorphic to S^7 for all k, one defines an invariant of 7-manifolds mod 7 and show that this is 0 for S^7 and non-zero for M_k^7 if $k^2 \not\equiv 1 \pmod 7$.

Any closed oriented 7-manifold M^7 is the boundary of an 8-manifold B^8 [38], under the assumption $H^3(M)=H^4(M)=0$. Choose generators ν and μ of $H_8(B,M)$ and $H_7(M)$ with $\partial\nu=\mu$. Define $\tau(B)$ as the index of the form $\alpha \to \langle \nu, \alpha^2 \rangle$ over the group $H^4(B,M)$/torsion. Let p_1 in $H^4(B)$ be the first Pontrjagin class of the tangent bundle of B. Because H^3 and H^4 of M vanish, the inclusion homomorphism $i: H^4(B,M) \to H^4(B)$ is an isomorphism and we can define $q(B) = \langle \nu, (i^{-1}p_1)^2 \rangle$. Finally our invariant $\lambda(M)$ is defined as $2q(B)-\tau(B) \pmod 7$. To show that this is independent of the choice of B, let B' be another manifold bounded by M and form $C^8 = -B \cup B'$ joined along the common boundary; we must show that $2q(C)-\tau(C) \equiv 0 \pmod 7$ for the unbounded manifold C (it is easy to show that q and τ of C are given by the difference of their values for B' and B). But by [10] or [38], we have $\tau(C) = \langle \nu, \frac{1}{45}(7p_2(C)-p_1^2(C))\rangle$, so that $45\tau(C)+q(C) = 7\langle \nu, p_2(C)\rangle$ equals 0 mod 7, whence $2q(C)-\tau(C)$ also is 0 mod 7.

Finally, to evaluate $\lambda(M_k)$ we represent M_k (the total space of a sphere bundle) as the boundary of the total space of the associated 4-cell bundle. The latter again has an explicit description and one calculates that its index τ is 1 and its Pontrjagin number q is $4k^2$, so that $\lambda(M) \equiv 2q(B)-\tau(B) \equiv 8k^2-1 \equiv k^2-1 \pmod 7$. Since $\lambda(S^7)$ is plainly zero, this proves that M_k is not diffeomorphic to S^7 for $k \not\equiv \pm 1 \pmod 7$.

2. The structure of Θ_n/bP_{n+1}

We quickly sketch the group operation in Θ_n; details can be found in [28] or [21]. The connected sum of two n-manifolds* (without boundary) is obtained by removing a smoothly imbedded disc from each and gluing together along the boundaries formed; the result can be given a differentiable structure and is unique up to diffeomorphy; it is denoted $M \# M'$. Plainly this is a commutative and associative operation with S^n as an identity. The inverse of M is $-M$ (M with orientation reversed); this only works if M is a homotopy sphere and if we are

* All manifolds in this paper are compact, differentiable and orientable and have boundaries, unless the contrary is stated

identifying h-cobordant manifolds (we don't need the h-cobordism theorem for this--or, indeed, at all, now that we're looking at Θ_n rather than Γ_n--since it is quite easy to show directly that h-cobordism is preserved under connected sums).

A manifold is stably parallelizable (S-parallelizable, a π-manifold) if the Whitney sum of its tangent bundle with a trivial line bundle is trivial. It follows from results of Adams that any homotopy sphere is S-parallelizable. If M is connected and $\partial M \neq \emptyset$ then M is parallelizable iff it is stably parallelizable. A submanifold M^n of S^{n+k} (k>n) is S-parallelizable iff its normal bundle is trivial. Using these results, we can study the structure of $\Theta n/bP_{n+1}$.

The basic idea is to imbed M^n (a homotopy sphere and hence S-parallelizable) in S^{n+k} (k>n), so that its normal bundle is trivial and we can choose a field φ of normal k-frames. We now use the Pontrjagin-Thom construction(see [38] or the excellent discussion in chapter 6 of [33]) which gives an element of $\pi_{n+k}(S^k)$ (recall that the idea is as follows: to a map $f:M \to S^k$ and a regular point y in S^k one gets a framing on the manifold $f^{-1}(y)$ as $(df)^{-1}(TS^k_y)$; the resulting framed manifold is, up to framed cobordism, independent of y and of the choice of f in its homotopy class and the resulting map of $[M,S^k]$ to h-cobordism classes of framed manifolds is a 1:1 correspondence). Allowing φ to vary we obtain a set p(M) of elements of $\Pi_n = \pi_{n+k}(S^k)$, depending only on the h-cobordism class of M. Now if p(M) contains 0, then some framing φ corresponds to the null map in the Pontrjagin construction, so φ extends over the manifold W which M bounds (M is a π-manifold and so bounds some manifold: see [40]), so W is parallelizable by the remarks in the last paragraph; conversely if $M \in bP_{n+1}$ then $M = \partial W$ with W parallelizable, so the imbedding $M \subseteq S^{n+k}$ extends to $W \subseteq D^{n+k+1}$ and the map associated to the framing on M obtained by restricting that on W is null-homotopic. Finally we note that $p(M)+p(M') \subseteq p(M \# M')$; combining this with the identities $S^n \# S^n = S^n$, $S^n \# M = S^n$, and $M \# (-M) = S^n$ yields that $p(S^n)$ is a subgroup of Π_n and $p(M)$ is a coset of $p(S^n)$, the zero coset iff $M \in bP_{n+1}$. This proves that bP_{n+1} is a subgroup of Θ_n and yields a 1:1 map of the quotient $\Theta_n/bP_{n+1} \longrightarrow \Pi_n/p(S^n)$, proving in particular that this quotient group is finite. Finally, we can identify $p(S^n)$ with $J(\pi_n(SO))$ using Kervaire's interpretation of the Whitehead homomorphism J (see [39]): namely, that for $M: S^n \longrightarrow SO(k)$ representing $\mu \in \pi_n(SO(k))$, ~~then~~ $J(\mu)$ is, within a sign, the homotopy class assigned by the Pontrjagin construction to the field φ of k-frames given at $x \in S^n \subseteq S^{n+k}$ (standard imbedding) as $M(x) \cdot$(standard k-frame of standard imbedding of S^n in S^{n+k}).

3. The order of bP_{n+1}

Since the emphasis of this paper is on the construction of exotic spheres, and since—as discussed in the introduction—none of the "very exotic" spheres (elements of $\Theta_n - bP_{n+1}$) has been constructed, this section is of greater importance to us than the last. For the same reason, it would be out of place to give the details of the proof that bP_{n+1} is the zero group if n is even: we will only comment that the proof is a rather lengthy and intricate application of the technique of killing homotopy groups by a series of spherical modifications. The reader interested in details can find them in the original paper [22]. Even in the case of n odd I will only give the results and a very brief indication of the proof, since what is of importance here is an understanding of the invariants which determine whether two elements of bP_{n+1} are diffeomorphic or not, rather than the number of elements of the group.

The results are as follows: bP_{4k+2} is either 0 or Z_2. It is 0 for k=0,1,3, and 7 and (I believe) for no other known values. ~~Indeed~~ For k>0 it must be Z_2 if k is even [7] and indeed unless k+1 is a power of 2 [6]. The order of the cyclic group bP_{4k}, on the other hand (for k>1) grows more than exponentially with k; its order (after the work of Adams, Kervaire, and Milnor) is $2^{2k-2}(2^{2k-1}-1)\epsilon_k \cdot \text{numerator}\,(4B_k/k)$ (B_k is the k^{th} Bernouilli number) where ϵ_k is 1 or 2 and may well always be 1; the best result known is that it is 1 if k is odd or if k has no odd factor greater than 2^9 (if $k \equiv 0$ or 2 (mod 8)) or than 2^{12} (if $k \equiv 4$ or 6 (mod 8)), in particular if k is a power of 2 [24].

The invariant that distinguishes the elements of bP_{2k} is of quite a different kind for k even and k odd. In the first case, we use the analogue of the invariant $\lambda(M)$ of 7-manifolds described in section 1. Again for an unbounded C^{4k} the results of Hirzebruch [10] give the index $\tau(C)$ as a polynomial L_k in the Pontrjagin classes $p_1,\ldots,p_k$ of C; the coefficient of p_k in this is $s_k=2^{2k}(2^{2k-1}-1)B_k/(2k)!$ and we can use this to show that the invariant mod 1 of an unbounded M^{4k-1} given by $\lambda(M) = (\tau(B)- L_k(j^{-1}p_1,\ldots,j^{-1}p_{k-1},0)[B])/s_k$ where $M = \partial B$ is independent of the choice of B. But in the case where $M\epsilon bP_{4k}$ we can choose B to be parallelizable, so the invariant reduces to $\tau(B)$ (mod s_k) (compare [29]). Moreover, it is a standard result that B can be chosen to be (2k-1)-connected; using this we can show that $\tau(B)$ (plainly an integer since it is the signature of the intersection form of $H_{2k}(B,Z)$) is a multiple of 8. For (see [13], p. 92) any even quadratic form (i. e. f(x,x) an even integer for all x) over a group has a signature divisible by 8, and by Poincaré-Lefschetz duality the evenness of the intersection form can be translated to the vanishing of the operation $Sq^{2k}: H^{2k}(B,M;Z_2) \longrightarrow H^{4k}(W,M;Z_2)$; if this were not null then by the Wu formulas some Stiefel-Whitney class of dimension $\leq$ 2k would be non-zero, a contradiction. Moreover, there is a (4k)-manifold of index 8 that is parallelizable and has a homotopy sphere as its boundary (one will be constructed in chapter III), so the set of values of $\lambda(M) = \tau(B)$ is 8Z. The set of indices of parallelizable (4k)-manifolds bounded by S^{4k-1} is plainly a subgroup. It is non-trivial—i.e. there is a parallelizable 4k-manifold bounded by S^{4k-1} with non-zero index [21]—and if we let t denote its positive generator, then the structure of bP_{4k} as a cyclic group of order t/8 will follow if we show that two elements $M=\partial B$ and $M'=\partial B'$ of bP_{4k} are h-cobordant iff $\tau(B)\equiv\tau(B')$ (mod t). To do so we first extend the connected sum to manifolds with boundary as $(B,\partial B)\#(B',\partial B') = (W,\partial W)$ where W is formed by imbedding half-discs in B and B' and gluing them along their boundaries (so W has the homotopy type of $B\vee B'$ and the boundary $\partial M\#\partial M'$). Thus if $\tau(B)=\tau(B')+\tau(B_o)$ where B_o is parallelizable and bounded by S^{4k-1} (so its index is a multiple of t), then the sum $(-B,-\partial B)\#(B',\partial B')\#(B_o,\partial B_o)$ has boundary $(-M)\#(M')\# S^{4k-1}$ or $(-M)\#(M')$ and index 0; since its index is 0 its homotopy groups can be killed by surgery [31] to give finally an h-cobordism of $(-M)\#(M')$ with the trivial h-cobordism class. Conversely if W is such a cobordism, then gluing it onto $(-B,\partial B)\#(B,\partial B')$ along the common boundary $(-M)\#(M')$ gives a parallelizable manifold with boundary S^{4k-1} and index $-\tau(B)+\tau(B')$, which is therefore a multiple of t.

If k is odd, elements of bP_{2k} are distinguished by a mod 2 invariant called the Arf invariant. After showing that bP_{2k} is zero for k=1,3,7, one constructs in the remaining cases a map $\psi_o\colon H_k(M) \longrightarrow Z_2$ with the properties that $\psi_o(\lambda+\mu)\equiv\psi_o(\lambda)+\psi_o(\mu)+\lambda\cdot\mu$ (mod 2) and $\psi_o(\lambda)=0$ iff an imbedded sphere representing λ has trivial normal bundle. Then one uses the general theory of quadratic forms and Arf invariants ([13], ch. 9) to define c(M): explicitly, $c(M) = \sum_{i=1}^{r} \psi_0(\lambda_i)\psi_0(\mu_i) \in \mathbb{Z}_2$ where $\{\lambda_1,\ldots,\lambda_r,\mu_1,\ldots,\mu_r\}$ is a symplectic basis for $H_k(M)$ (i.e. $\lambda_i\cdot\lambda_j=\mu_i\cdot\mu_j=0$, $\lambda_i\cdot\mu_j=\delta_{ij}$; such a basis can be chosen because since the boundary of M is a (2k-1)-sphere and has no homology in dimensions k, k-1, the intersection pairing in $H_k(M)$ has determinant ±1). One then proves that if c(M)=0, then H_kM can be killed by surgery; it follows that if c(M)=c(M'), the boundaries of M and M' are cobordant. Since c(M) can only take on two values, this proves that bP_{2k} is 0 or Z_2.

CHAPTER II

Classification of G-manifolds-- the manifold $W^{2n-1}(d)$ and knot manifolds

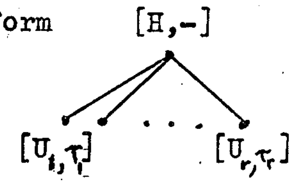
In the first section of this chapter we describe the basic facts about manifolds on which a Lie group G acts differentiably, or G-manifolds (good references are [19] or [13]). The main result is the classification theorem for certain ("special") G-manifolds in terms of fairly algebraic data. More precisely, special G-manifolds with a given orbit structure, or information about what subgroups of G occur as isotropy group of some point, are classified by a manifold with boundary M ($M=X/G$) and a principal bundle over M with a certain structure group that reduces to a smaller one over each component of ∂M (the structure groups in question being defined in terms of the original orbit structure). Later in this paper this will be used to identify manifolds obtained by other means; in this chapter we use it to produce two manifolds. In section 2 we look at O(n)-manifolds with orbit structure O(n-1), O(n-2) and $M = D^2$ and obtain a collection of (2n-1)-manifolds $W^{2n-1}(d)$ corresponding to the S^1-bundles over D^2; we compute their homology to find out for which n and d they are spheres. In section 3 we look at a more complicated example, requiring a more delicate classification theorem, and which produces a series of examples classified, not simply by an integer, but by the set of knots (isomorphism classes of imbeddings $S^1 \longrightarrow S^3$).

1. The classification theorem

We begin with a quick review of terminology and basic theorems for G-manifolds. For $x \in X$, the isotropy group G_x is the subgroup of G leaving x fixed; the orbit Gx is $\{gx : g \in G\}$; the orbit space is X/G with the quotient topology; the type of an orbit Gx is $\{G_y : y \in Gx\}$ = conjugacy class of G_x in G (since $G_{gx} = gG_xg^{-1}$). Each orbit Gx is a G-manifold and is (i.e. is equivariantly diffeomorphic to) G/G_x, so that Gx and Gy are isomorphic manifolds iff they have the same type. A G-vector bundle is a vector bundle with an action of G on the total space such that g maps a fibre E_x isomorphically to E_{gx}. A G-vector bundle over a homogeneous space G/H is determined by its fibre V at $1H \in G/H$ and is in fact $G \times_H V$. In particular when $H = G_x$, $V = V_x = T_xX/T_xGx$ = normal space to Gx at x, we can identify the "slice bundle" $G \times_{G_x} V_x$ with the normalbundle of Gx in X by $[g,v] \longrightarrow gv$. Applying an equivariant version of the tubular neighbourhood theorem (obtained via Haar measure on G) we obtain the slice theorem--that there is an equivariant diffeomorphism between a neighbourhood of Gx in X and of the zero section in the slice bundle. This is the basic technique in all of G-manifold theory. With it we can prove that if X is compact it has only finitely many orbit types and that for one of these, say (H), the corresponding orbits $X_{(H)}=\{x\in X: (G_x)=(H)\}$ is a submanifold of X. Finally for H a subgroup of G, $P=\{x\in X: G_x=H\}$ is in a natural way a differentiable right principal fibre bundle with structure group $\Gamma=N_H/H$ (N_H=normalizer of H in G) and $X_{(H)}$ is in a canonical way (equivariantly diffeomorphic to) the associated fibre bundle $G/H\times_{\Gamma} P$, where the action of Γ on G/H is given by multiplication (indeed Γ is just the set of equivariant automorphisms of G/H). Thus X is the union of finitely many submanifolds, each of which is the total space of a fibre bundle with the orbit as its fibre. Moreover (at least if X/G is connected) there is a (unique) principal orbit type (H) for which $X_{(H)}$ is open and dense in X, and $X_{(H)}/G$ is connected.

To make the idea of orbit structure precise, we use Jänich's notation of slice diagrams [19]. For a compact Lie group G, and representations σ, σ' of closed subgroups H,H' we say that (H,σ) and (H',σ') represent the same slice type if for some $g\in G$, $H'=gHg^{-1}$ and the representation $\sigma'\circ t$ of H ($th=ghg^{-1}$) is equivalent to the representation σ. The slice types of a G-manifold X are $[G_x,\sigma]$ ($x\in X$) where σ is the "slice representation" $\sigma: G_x \longrightarrow GL(V_x)$ obtained by considering the differential dg of $g\in G_x$ as an automorphism of the normal space V_x. We can partially order slice types by $[U,\tau]\leq[H,\sigma]$ if $[H,\sigma]$ is a slice type of the G-manifold $G\times_U\tau$ (where we use τ to indicate both the representation and the corresponding U-module). Then the slice diagram of X, $\Delta(X)$, is the partially ordered set of slice types of X. A partially ordered set Δ of slice types of G is the slice diagram of a G-manifold iff for each slice type in Δ, all larger slice types are also in Δ. A connected slice diagram of a manifold has an absolute maximum [H,trivial representation] (denoted [H,-]), corresponding to the principal orbit bundle. The word "diagram" refers to the obvious representation of Δ as a graph.

A G-manifold is special if its slice diagram is of the form



where each τ_i is the direct sum of a trivial representation and a transitive representation τ_i' (i.e. $\tau':U \longrightarrow O(k)$ has an image in $O(k)$ which acts transitively on S^{k-1}). If X is a special G-manifold, X/G is a manifold with boundary (having r connected components corresponding to the U_i), for using the slice theorem to represent a neighbourhood of an orbit Gx as $G \times_{G_x} V_x$ we see that a neighbourhood of the point Gx in X/G is $(G \times_{G_x} V_x)/G$ or V_x/G_x, and for $x \in X_{(U_i)}$, $V_x/G_x = \tau_i/U_i$ is a half-space (it is the sum of a Euclidean space and of τ_i'/U_i and the latter is a half-line because $\tau_i'(U_i)$ acts transitively on R^{k_i}). We are now ready to formulate the classification theorem for special G-manifolds; for notational convenience we assume r is 1 and omit the index i. X determines now an n-manifold M with a connected boundary (where U operates via τ on R^{k+n-1}, transitively on $S^{k-1} \subseteq R^k \times \{0\}$ and trivially on $\{0\} \times R^{n-1}$) and splits into a fibre bundle (fibre G/U, group N_U/U) over ∂M (total space $Y = X_{(U)}$) and a fibre bundle (fibre G/H, group N_H/H) over $M-\partial M$ (total space $X-Y$). But this does not determine X unless we know how the total spaces are joined. To do this we define the "difference" $X \odot Y$ (a general construction for a manifold Y imbedded in a manifold X: let N be the normal bundle of Y, SN the sphere bundle of N--defined as N minus the null section factored by R^+; then we can use a tubular neighbourhood theorem to consider N as a neighbourhood of Y in X and can give an obvious differentiable structure to the disjoint union of X-Y and SN). Then $X \odot Y$ is a bounded G-manifold with only a single orbit type, and hence an orbit bundle. The base space is M, the fibre G/H, the structure group is $\Gamma = N_H/H$, the associated right-Γ-principal fibre bundle is $P = \{x \in X \odot Y: G_x = H\}$, and $X \odot Y$ is canonically diffeomorphic to $G/H \times_\Gamma P$ with its boundary SN given by $G/H \times_\Gamma \partial P$. As a last element of structure, the group of the bundle P reduces to a subgroup Ω of Γ over the boundary ∂M. For $\partial P = \{x \in SN: G_x = H\}$; let $p: SN \longrightarrow Y$ be the bundle projection and define Ω as $N_H \cap N_U/H$ (the automorphism group of $G/H \longrightarrow G/U$). Then there is a unique reduction σ (i.e. a section $\sigma: \partial M \longrightarrow \partial P/\Omega$) with $\bigcup_{b \in \partial M} \sigma(b) = \{x \in \partial P: G_{p(x)} = U\}$ by the slice theorem. Now from the principal bundle P and the reduction σ we can construct $X \odot Y$, Y, and SN. To get X, we must use our representation τ of U. Through it, U/H can be identified with S^{k-1}, and hence $G/H = G\times_U U/H$ with $G\times_U S^{k-1}$, so the action of Ω on G/H can be extended to an action on $G\times_U R^k$ mapping fibres linearly onto fibres, making $(G\times_U R^k)_\Omega \times \sigma^*$ into a G-vector bundle over $G/U\times_\Omega \sigma^* = Y$ (σ^* is the total space $\bigcup_{b\in\partial M} \sigma(b) \in \partial P$ of the bundle over ∂M which comes from the principal bundle $\partial P \to \partial P/\Omega$). And this bundle is isomorphic to the normal bundle of Y in X, so we can use it to glue Y back to $X\ominus Y$. Thus given M, P and σ, we define $\tilde{X} = G/H_\Gamma \times P$, $Y = G/U_\Omega \times \sigma^*$, and $N = (G\times_U R^k)_\Omega \times \sigma^*$; choose a metric in N and define X as the gluing of $\tilde{X}$ and DN along their boundary. Then X has a natural G-manifold structure, Y is imbedded in X with normal bundle isomorphic to N and with $X\ominus Y$ isomorphic to $\tilde{X}$, and M, P, and σ are the orbit space and principal bundle with reduction coming from X.

In summary, we have established a 1:1 correspondence between the set $S[U,\tau]$ of equivariant diffeomorphy classes of compact G-manifolds with slice diagram $\left|\begin{matrix}[H,-]\\ [U,\tau]\end{matrix}\right.$ and the set $\mathcal{P}(\Gamma,\Omega)$ (where $\Gamma = N_H/H$, $\Omega = N_H \cap N_U/H$) of isomorphism classes of pairs (P,σ) of right-Γ-principal bundles over an n-dimensional manifold M with boundary and reductions $\sigma: \partial M \to \partial P/\Omega$ (where (P,σ) and (P',σ') are isomorphic if there is an equivariant diffeomorphy from P to P' that takes σ to σ').

2. The homology of $W^{2n-1}(d)$

We now consider O(n)-manifolds with orbit space $M = D^2$ and slice diagram $\left|\begin{matrix}[O(n-2),2]\\ [O(n-1),\rho_{n-1}\oplus 1]\end{matrix}\right.$. Here we have $U = O(n-1)$,

$H = O(n-2)$, $\Gamma = N_H/H = O(2)\times O(n-2)/O(n-2) = O(2)$, $\Omega = O(1)\times O(1)$, $\Gamma/\Omega = O(2)/O(1)\ O(1) = P^1 \cong S^1$. The bundle P must be the trivial bundle $\Gamma\times D^2$ and so the reduction σ is simply a map from $\partial M = S^1$ to $\Gamma/\Omega \cong S^1$, and the isomorphy class of $[P,\sigma]$ in $\mathcal{P}(\Gamma,\Omega)$ is determined by the degree of σ, an integer $d \geq 0$; the corresponding $O(n)$-manifold is denoted $W^{2n-1}(d)$. For example, $W^3(d)$ is diffeomorphic to the lens space $L(d,1)$, as one can show by looking at the details of the construction in the classification theorem.

We would like to look at the homology of $W^{2n-1}(d)$ to find out for what values of n and d it is a homology sphere. For this let π: $W(d) \longrightarrow D^2$ be the projection and define $A_\pm = \pi^{-1}(D^2_\pm)$, where $D_\pm$ are the closed upper and lower half discs. A_+ and A_- are equivariantly diffeomorphic to $S^{n-1}\times D^n$ with the action $A(x,y) = (Ax,Ay)$ of $O(n)$; their intersection is the common boundary $S^{n-1}\times S^{n-1}$. Thus $W(d) = A_+ \cup_{\varphi_d} A_-$ where $\varphi_d: S^{n-1}\times S^{n-1} \longrightarrow S^{n-1}\times S^{n-1}$ is an equivariant diffeomorphism. A_+, A_-, and $A_+\cap A_-$ have trivial fundamental group and therefore so does $W(d)$ by Van-Kampen's theorem ($n\geq 3$). To determine the homology of $W(d)$ we look at the Mayer-Vietoris sequence. $H_*(A_+) = H_*(A_-) = H_*(S^{n-1}\times D^n) = H_*(S^{n-1})$; $H_*(A_+\cap A_-) = H_*(S^{n-1}\times S^{n-1})$. Hence the sequence yields $H_k(W(d)) = 0$ for $k \neq 0, n-1, n, 2n-1$. Naturally H_0 and H_{2n-1} are Z, and by duality H_{n-1} and H_n are isomorphic (W is unbounded). The critical part of the Mayer-Vietoris sequence is ($n\geq 3$)

$$0 \longrightarrow H_n(W) \longrightarrow H_{n-1}(S^{n-1}\times S^{n-1}) \xrightarrow[j_{A_+*}\oplus j_{A_-*}]{} H_{n-1}(S^{n-1})\oplus H_{n-1}(S^{n-1})$$

$$\longrightarrow H_{n-1}(W) \longrightarrow 0,$$ and the middle two groups are both

$Z+Z$, so that the two homology groups of interest are the kernel and cokernel of a map $f: Z+Z \longrightarrow Z+Z$. Now j_{A_+} is the inclusion of $S^{n-1} \times S^{n-1}$ in $S^{n-1} \times D^n$ while j_{A_-} is the composition of φ_d with this inclusion, so (if $p_1: S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$ is projection on the first factor) $j_{A_+ *} = p_{1*}$ and $j_{A_- *} = p_{1*} \cdot \varphi_{d*}$, and f is the matrix $\begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$ with $a = p_{1*}\varphi_{d*}(1,0)$ and $b = p_{1*}\varphi_{d*}(0,1)$. These can be calculated directly by determining explicitly the action of φ_d (this is done in [3], where the manifolds $W^{2n-1}(d)$ were first introduced). A neater way, due to Jänich [19] is to note that $S^{n-1} \times S^{n-1}$ has an $O(n)$-manifold structure preserved by φ_d, as follows: it splits into the principal orbit bundle and a singular orbit bundle $X_{(O(n-1))}$ containing only two orbits--the diagonal D' and counterdiagonal D'' of $S^{n-1} \times S^{n-1}$, so that φ_d must be a diffeomorphism of $D' \cup D''$ onto itself. We can replace φ_d by $\psi\varphi_d$ without changing its diffeomorphy type, where ψ is the restriction of a diffeomorphism of $S^{n-1} \times D^n$ onto itself. Since $(x,y) \mapsto (x,-y)$ is such a ψ and interchanges D' and D'', we can assume that φ_d takes each of D' and D'' onto itself. On each it can only be the identity or antipodal map since these are the only maps of S^{n-1} onto itself as an $O(n)$-manifold; since $(x,y) \mapsto (-x,-y)$ is a ψ of the type mentioned we can assume that φ_d is the identity on D' and is $\pm$ the identity on D''. Now the counterdiagonal D'' represents $(1,1)$ or $(1,-1)$ in $Z+Z = H_{n-1}(S^{n-1} \times S^{n-1})$, depending whether $x \mapsto -x$ preserves or reverses the orientation of S^{n-1}, i.e. whether n is even or odd. In the latter case (the only one accessible to this approach), we therefore can calculate the form of $f:Z+Z \to Z+Z$ knowing only the action of φ_d on D' and D" (using the matrix representation above). Thus (1) if φ_d is the identity on D' and D", then $\varphi_{d*} = id$ so f is represented by the matrix $\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$ and so $H_n(W)$ and $H_{n-1}(W)$, the kernel and cokernel of f, are both Z; (2) if φ_d is the identity on D' and the antipodal map on D", then φ_{d*} takes (1,1) to (1,1) and (1,-1) to (-1,1) so in our matrix formula $f = \begin{vmatrix} 1 & 0 \\ a & b \end{vmatrix}$ we have a=0, b=1, f = id, and so $H_n(W)$ and $H_{n-1}(W)$ are 0. Now consider $Q = \{x \in W(d): G_x = 0(n-1)\}$, the principal bundle with group $N_U/U = 0(1)$ over $\partial D^2 = S^1$ associated to the singular orbitbundle of W(d). Then $Y = 0(n)/0(n-1)_{0(1)} \times Q$. As in the proof of the classification theorem, Q is determined by the reducing map σ (of degree d), and one easily finds that Q is trivial iff σ has ~~even~~ winding number, i.e. if d is even. But Q is a subset of the singular set of $S^{n-1} \times D^n \cup_{\varphi_d} S^{n-1} \times D^n$ and so is determined by $\varphi_d|_{D' \cup D''}$; thus Q is trivial iff φ_d is the identity on $D' \cup D''$, i.e. in case (1) above.

Summarising, if n>1 is odd, then $W^{2n-1}(d)$ is a homology sphere iff d is odd; if d is even its n^{th} and $(n-1)^{th}$ homology groups are Z.

To identify the differentiable structure on W(d), in the cases when it is a homology sphere, we will have to identify it with manifolds constructed by other methods.

3. Knot manifolds

To define knot manifolds we need an extension of the classification theorem to manifolds with three orbittypes. An (M,M_0) manifold (where M is a bounded manifold and M_0 a closed submanifold of its boundary with codimension 2) is an $O(n)$-manifold such that

(1) the slice diagram $\Delta(X)$ is a subdiagram of $[O(n),\rho_n+\rho_n+\text{triv}]$ (ρ_n is the action of $O(n)$ on R^n),

(2) $(X/O(n),F)$ is diffeomorphic to (M,M_0) (F=fixpoint set)

(3) The principal orbitbundle is trivial (over $M-\partial M$)

(4) The normal bundle of F in X is a trivial G-vector bundle.

Then X-F is a special $O(n)$-manifold over $M-M_0$ with $\Gamma/\Omega = S^1$ so the classifying reductions σ are maps $\partial M-M_0 \longrightarrow S^1$ or elements of $H^1(\partial M-M_0)$. Using this, we find the following classification theorem: equivariant diffeomorphy classes of (M,M_0) manifolds are in 1:1 correspondence with $S(M,M_0)/Z_2 \times Diff(M,M_0)$, where $S(M,M_0)$ is a subset of $H^1(\partial M-M_0)/2j^*H^1(M)$ ($j:\partial M-M_0 \longrightarrow M$ is inclusion) with two elements for each connected component of M_0. In the case $(M,M_0) = (D^4,k)$, therefore, where k is a knot (smoothly imbedded 1-sphere in S^3), $S(M,M_0)$ has two elements and so all (D^4,k) manifolds are equivariantly diffeomorphic. The corresponding "knot manifold" $\Phi_n(k)$ is $(2n+1)$-dimensional splits into a 1-dimensional manifold k (the fixpoints) with orbit type $(O(n))$; the remaining space is a special $O(n)$-manifold whose orbits are $(2n-3)$-dimensional Stiefel manifolds $O(n)/O(n-2)$ and $(n-1)$-dimensional spheres $O(n)/O(n-1)$.

To look at the structure of $\Phi_n(k)$ we must define certain knot invariants. We also define $\Phi_1(k)$ as the 2-fold branched covering of S^3 along the knot k (plainly an O(1)-manifold). Its first integral homology group is of finite, odd order, equal up to sign to the determinant of the knot, which equals $\Delta(-1)$ up to sign ($\Delta(t)$ is the Alexander polynomial of k). The Robertello invariant of k, c(k), is the residue of $\frac{\Delta(-1)^2-1}{8}$ mod 2 and is a cobordism invariant mod 2 of k. Another cobordism invariant of k is the signature $\tau(k)$: if $\hat{X}$ is an infinite cyclic covering of S^3-k, then $H^* = H^*(\hat{X},\partial\hat{X};R)$ satisfies Poincaré duality so the pairing $\cup: H^1 \otimes H^1 \longrightarrow H^2 = R$ is non-degenerate and so the pairing

$\langle a,b\rangle = a \cup t^*b + b \cup t^*a$ ($a,b \in H^1$, t a generator for the group of covering transformations of $\hat{X}$) is symmetric and non-degenerate and therefore has a signature, denoted $\tau(k)$. Then using an equivariant handlebody construction starting out from a Seifert surface spanned in the knot k, we get:

> For $n \geq 1$, $\Phi_n(k)$ bounds a parallelizable manifold. For n even it is the element of $bP_{2n+2} = Z_2$ with Arf invariant c(k); for n odd it is a sphere iff the determinant of the knot k is ± 1; in this case it is the element of bP_{2n+2} with signature $\pm\tau(k)$.

The discussion of (M,M_0)-manifolds is due to Jänich [19]; the results on the homology, Arf invariants, and signatures of the knot-manifolds $\Phi_n(k)$ can be found in Hirzebruch [12].

The result says that every element of bP_{n+1} is a knot manifold and the more knotted the knot, the more exotic the corresponding sphere.

CHAPTER III

Equivariant plumbing-- the tree manifold $M^{2n-1}(T)$

We now describe the construction which leads to the most direct calculation of homology groups and signatures. We start from very simple bounded 2n-manifolds (the disc bundle over S^n and related bundles), "plumb" them together by a standard construction (making sure that this is done in an equivariant manner if we have a Lie group acting on the original manifolds), and consider the boundary of the resulting manifold.

If E_1 and E_2 are two n-disc bundles over S^n (in the applications below they will be bundles ${}_kDS^n$, the bundle of vectors of length ≤ 1 in the vector bundle whose characteristic map is the k^{th} power of that of the tangent bundle of S^n), we "plumb" them by choosing imbeddings of D^n in the base spaces S^n, identifying the locally trivial subbundles of the E_i over the image of D^n with $D^n x D^n$, and identifying the two copies of $D^n x D^n$ under $(x,y) \longrightarrow (y,x)$. The resulting space can be given a differentiable structure by the usual process of straightening angles (see comment on p. 6). If a Lie group G acts on E_1 and E_2, the plumbing is equivariant if the imbeddings $f_i : D^n \longrightarrow S^n_i$ (i=1,2) and the trivializations are all equivariant. G will be O(n) or O(n-1) and will act on $D^n x D^n$ by $A(x,y) = (Ax,Ay)$. The group O(n) acts on ${}_0DS^n = S^n x D^n$ and on

$_1DS^n$. The bundle $_kDS^n$ is formed (using a characteristic map) by identifying parts of two trivial bundles, and the condition that this identification is compatible with the operations of G turns out to imply that $G \subset O(n-1)$ unless k=0,1 (see [13] for details of the actions).

Now given a valued tree (a tree together with an even integer at each vertex, i.e. with a map $\lambda: T \longrightarrow 2Z$) we define $\bar{M}^{2n}(T)$ as the result of putting $_{\lambda(a)/2}DS^n$ at the vertex a of T and plumbing together any two manifolds corresponding to vertices connected by an edge. The resulting manifold is unique up to diffeomorphy (independent of the order of the plumbings); we denote its boundary $M^{2n-1}(T)$. If G acts on each $_kDS^n$ with $2k \in \lambda(T)$, and the plumbing is equivariant, $M^{2n-1}(T)$ is a G-manifold, not necessarily unique up to equivariant diffeomorphy (see fig., p. 31).

If G is O(n) and $M^{2n-1}(T)$ is to be a G-manifold, then not only must we have $\lambda(T) \subset \{0,2\}$ but also T must be a graph A_k (a linear graph of k vertices; notation from Lie group theory). For the imbeddings $f_i: D^n \longrightarrow S^n_i$ (i=1,2) of the plumbing must be equivariant and hence f(0) must be in the fixpoint set of S^n under G. With G=O(n) this fixpoint set contains only two points so we can plumb at most two manifolds to any given one.

1. The homology of $M^{2n-1}(T)$

First $\bar{M}^{2n}(T)$ has the homotopy type of $S^n \vee \ldots \vee S^n$ (h copies, where h is the number of vertices of T) because it can be retracted to the zero-section of the plumbed disc-bundles (details in [2]).

From the exact sequence of $(\bar{M},\partial\bar{M})$ and Poincaré-Lefschetz duality (whence $H_i(\bar{M},\partial\bar{M})=H^{2n-i}(\bar{M})=\mathrm{Hom}(H_{2n-i}(\bar{M}),Z)=0$ for $i\neq n,2n$) we obtain that $H_i(\partial\bar{M})$ is 0 for $i\neq 1,n-1,n,2n-1$, and that H_n and H_{n-1} of $\partial\bar{M}$ are the kernel and cokernel of $f:H_n(\bar{M})\longrightarrow H^n(\bar{M})$ (f is defined as $P\circ i_*$ where P is the Poincare-Lefschetz isomorphism and i_* is induced by $i:\bar{M}\subset(\bar{M},\partial\bar{M})$). The intersection form $S:H_n(\bar{M})\longrightarrow H_n(\bar{M})$ is defined by $S(x,y)=\langle f(x),y\rangle$; it is symmetric for n even and skew-symmetric for n odd. Therefore $M^{2n-1}(T)$ or $\partial\bar{M}$ is a homotopy sphere iff S is invertible, that is iff $\det S=\pm 1$.

Following [13] (as we are doing in this whole discussion) we define forms corresponding to a tree with vertices $1,\ldots,h$ (and corresponding generators $e_1,\ldots,e_h$ of $H_n(\bar{M}^{2n}(T))$) by:

$$S_T(e_i,e_j)=\begin{cases}\lambda(i) & i=j\\ 1 & i\neq j,\ i \text{ and } j \text{ joined by an edge}\\ 0 & \text{otherwise}\end{cases}\qquad S_T^0(e_i,e_j)=\begin{cases}0 & i=j\\ 1 & i\neq j,\ i \text{ and } j \text{ joined by an edge}\\ 0 & \text{otherwise}\end{cases}$$

If n is even, the intersection form for $\bar{M}^{2n}(T)$ is just S_T, because the cycles involved have intersection number +1 if the corresponding vertices are joined by an edge (the orientations were chosen consistently) while the self-intersection of e_i is the Euler class of the bundle ${}_kTS^n$ $(k=\lambda(i)/2)$ which is k times that of TS^n, or k times the Euler number of S^n, or $2k$ (since n is even). If n is odd the Euler number of S^n is 0 so the self-intersection numbers are 0, while without orientation we can only assert that S is a skew-symmetric matrix whose entries are ± 1 at (ij) if i and j are joined by a vertex and 0 otherwise.

Thus if n is even, $M^{2n-1}(T)$ is a homotopy sphere iff the determinant of the matrix S_T is ± 1. In the case when n is odd, arguments about the matrices which I do not reproduce (see [13]) yield that $M^{2n-1}(T)$ is a sphere if any of the following equivalent conditions hold:

det $S = \pm 1$ (which we had obtained); det $S \neq 0$; det $S^0_T = \pm 1$; det $S^0_T \neq 0$; det S_T odd; the vertices of T can be numbered $e_1, f_1, \ldots, e_p, f_p$ (2p vertices) so that e_i and f_i are joined by an edge.

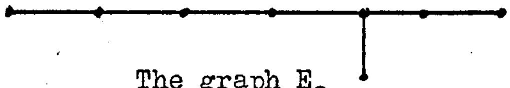
2. The differentiable structure of $M^{2n-1}(T)$

First we comment that it is quite easy to show that $\bar{M}^{2n}(T)$ is stably parallelizable ([13] p. 58) so that in the cases when $M^{2n-1}(T)$ is a sphere, it is in bP_{2n}. If n is even, then this happens iff the hxh matrix S_T is invertible (has determinant ± 1), and the signature which identifies the differentiable structure on $M^{2n-1}(T)$ is just the signature of the form S_T. In the case when n is odd, a discussion of Arf invariants yields finally that the Arf invariant of $M^{2n-1}(T)$ is 0 or 1 (mod 2) according to the parity of det S_T (mod 8) (det S_T must be odd by the conditions at the end of the first section): it is 0 if det S_T is ± 1 (mod 8) and 1 if det S_T is ± 3 (mod 8). For a tree T satisfying the conditions at the end of section 1, if $\lambda(T) = 0$, then $M^{2n-1}(T)$ is the standard sphere in bP_{2n} [16].

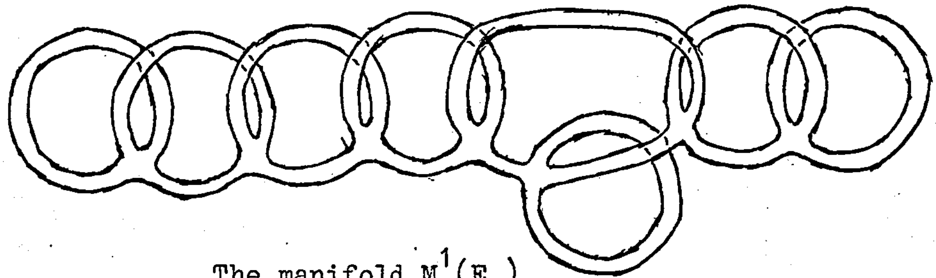
To connect these with the results of the last chapter, we must identify some of the tree manifolds with those constructed using G-manifold theory. If G is $O(n)$, the tree must be A_k and the manifolds that are plumbed must all be $S^n x D^n$ or ${}_1DS^n$ (see introduction to this chapter). If they are all ${}_1DS^n$, that is if $\lambda(a)=2$ for every vertex a of A_k, then S_{A_k} is the matrix with 2's on the main diagonal, 1's on the diagonals above and below this one, and 0's elsewhere, and its determinant is k+1 (easy induction). Since this is never ± 1, $M^{2n-1}(A_k)$ is never a sphere if n is even. If n is odd, it is a sphere iff k+1 is odd and the Kervaire sphere iff $k+1 \equiv \pm 3 \pmod 8$. (Indeed with k=1, n odd, the construction of $M^{2n-1}(A_2)$--i.e. the plumbing of two tangent disc manifolds to S^n--was the first construction of the Kervaire sphere). The interest of this lies in the fact that as an $O(n)$-manifold it is a special $O(n)$-manifold over D^2 with orbit types $(O(n-1))$ and $(O(n-2))$ and so is a $W^{2n-1}(d)$. The fixpoint set of $W^{2n-1}(d)$ under $O(n-2)$ is $W^3(d)$ which is the lens space $L(d)$ and has first homology group Z_d. Therefore the integer d classifying $M^{2n-1}(A_k)$ is the order of the first homology group of the three-dimensional manifold $M^3(A_k)$ which is its fixpoint set under $O(n-2)$. But (see p. 27) $H_1(M^3(A_k))$ is the cokernel of $f:H_2(\bar{M}^4(A_k)) \longrightarrow H^2(\bar{M}^4(A_k))$ and so its order is just the determinant of S_{A_k} since this is non-zero. Thus the invariant d is the determinant of S_{A_k} or k+1 and so $W^{2n-1}(d) = M^{2n-1}(A_{d-1})$ is a sphere iff n and d are odd, and is then the standard or Kervaire sphere according as d is ± 1 or $\pm 3 \pmod 8$.

This simultaneously identifies the differentiable structure on $W^{2n-1}(d)$ when it is a sphere and proves that $M^{2n-1}(A_k)$ is well-defined up to equivariant diffeomorphy as an $O(n)$-manifold.

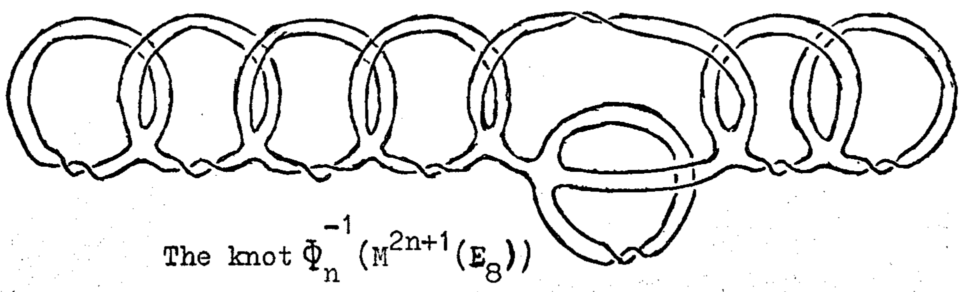
When the group G is $O(n-1)$ we can have any tree with any (even) values. The $O(n-1)$-manifold structure on $M^{2n-1}(T)$ consists of the fixpoint set F (orbit type $(O(n-1))$) and the special $O(n-1)$-manifold M-F (orbit types $(O(n-2))$ and $(O(n-3))$). The orbit space is a 4-dimensional manifold with boundary. If it is D^4, we have a knot manifold. In the case $T = E_8$ with constant values $\lambda(a)=2$, $S_T(x,x) = 2x_1^2+\ldots+2x_8^2+2x_1x_2+\ldots+2x_6x_7+2x_5x_8$; S_{E_8} has determinant +1 (in general det $S_{E_k} = 9-k$ if the tree is valued with 2) and signature 8, so that for n even, $M^{2n-1}(E_8)$ is the generator of bP_{2n} (if n is odd it is the standard sphere because det S_{E_8} is ± 1 mod 8). This turns out to be a knot-manifold corresponding to the knot shown on the next page. A little unravelling will convince the reader that this is the torus knot (3,5). Further information on the connection between knot and tree manifolds can be found in the work of Hirzebruch and Jänich.



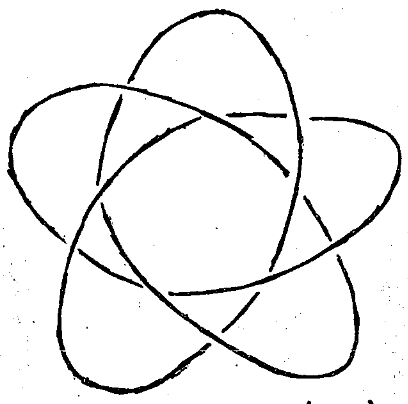
The graph E_8



The manifold $M^1(E_8)$



The knot $\Phi_n^{-1}(M^{2n+1}(E_8))$



The torus knot $t(3,5)$

CHAPTER IV

Singularities of algebraic functions-- the manifold $\Sigma(a_1,\dots,a_n)$

The most explicit description of a manifold that is possible is as the set of zeroes of a polynomial in the neighbourhood of a point. Let $f(z_1,\dots,z_{n+1})$ be a non-constant polynomial, V its set of zeroes, $K = V\cap S_\varepsilon$ where S_ε and D_ε are the sphere and disc of radius ε centred at $\vec{0}$. The topology of $V\cap D_\varepsilon$ is closely related to that of K; indeed, for small ε, $(D_\varepsilon, V\cap D_\varepsilon)$ is homeomorphic to (cone over S_ε, cone over K). For small ε, ϕ: $S_\varepsilon - K \longrightarrow S^1$ defined by $\phi(\vec{z}) = f(\vec{z})/|f(\vec{z})|$ is the map of a smooth fibre bundle whose fibre $F_\Theta = \phi^{-1}(e^{i\Theta})$ is a parallelizable 2n-manifold. If $\vec{0}$ is a regular point of f, then K is an unknotted sphere in S_ε and the fibre bundle is the trivial one $S^1 \times C^n$. If $\vec{0}$ is an isolated singularity, the topology of the fibre can be given very precisely; F_Θ has the homotopy type of $S^n \vee \dots \vee S^n$ (m copies, where m is the degree of the mapping $S_\varepsilon \longrightarrow S_1$ given by $\vec{z} \longmapsto (\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_{n+1}})/|\text{same}|$), and indeed its closure is diffeomorphic to a handlebody obtained from D^{2n} on the addition of handles of index n. F_Θ is the interior of a smooth manifold-with-boundary $F_\Theta \cup K$. Moreover, K is (n-2)-connected and (2n-1)-dimensional, so it is a topological sphere iff the intersection pairing $s: H_n F_\Theta \longrightarrow H_n F_\Theta$ has determinant ± 1. This condition can also be stated $\Delta(1) = \pm 1$, where $\Delta(t)$, the generalized

Alexander polynomial (a topological invariant of $S_\epsilon - K$) is the characteristic polynomial $\det(tI_* - h_*)$ of the characteristic homomorphism $h: H_*F_0 \longrightarrow H_*F_0$ of the fibre F_0. If n is even, the differentiable structure on K (if it is a sphere) is given by the signature of the pairing s; if n is odd then its Arf invariant (if it is a sphere) is 0 or 1 according as $\Delta(-1)$ is ± 1 or ± 3 (mod 8) [23]. (A general reference for the above is [34]).

In the special case when $f(\vec{z}) = z_1^{a_1} + \ldots + z_{n+1}^{a_{n+1}}$, the intersection K of the set of zeroes with the unit sphere is denoted $\Sigma(a_1, \ldots, a_{n+1})$. In this case its homology and (if it is a sphere) differentiable structure have been calculated, by Pham, Brieskorn, and Hirzebruch; a sketch of these calculations is given in the first two sections of this chapter.

1. The homology of $\Sigma(a_1, \ldots, a_n)$

Let $a = (a_1, \ldots, a_n)$ be a fixed n-tuple of integers > 1. $\xi_k = e^{\frac{2i\pi}{a_k}}$. $S^{2n-1} = \{\vec{z} \in C^n: |z_1|^2 + \ldots + |z_n|^2 = 1\}$. X is the set of zeroes of $f(\vec{z}) = z_1^{a_1} + \ldots + z_n^{a_n}$. $\Sigma = X \cap S^{2n-1}$. $\Xi(t) = f^{-1}(t)$; $\Xi = \Xi(1)$. ω_k is the automorphism of Ξ consisting of multiplying the k^{th} coordinate by ξ_k; the ω_k generate a group $\Omega = \prod_k Z_{a_k}$. J is the integer group ring of Ω and I the ideal of J generated by elements $1 + \omega_k + \ldots + \omega_k^{a_k - 1}$. G is the graph with n vertices, labelled by the a_k, which has two vertices connected by an edge iff the corresponding a_k have a common factor other than 1; it will be used for a neat formulation of results.

Now the set $\underline{e} = \{\vec{z} \in \Xi : z_k \text{ real}, \geq 0\}$ is plainly homeomorphic to the standard simplex $\{\vec{x} \in R^n : x_k \geq 0,\ x_1 + \ldots + x_n = 1\}$. $\underline{E}$ is the simplicial complex whose simplices are $\underline{e}$ and its sides and their images under the transformations of Ω. The homology $H_{n-1}(\Xi)$ (all homology with integer coefficients) is generated by the cycle $e = (1-\omega_1)\ldots(1-\omega_n)\underline{e}$ as a J-module and hence is J/I (since e is annulled exactly by I); this follows from a calculation of the simplicial homology of $\underline{E}$ and the easily constructed deformation retraction of Ξ onto $\underline{E}$. Moreover it is easy to show that for $n \geq 3$, $\underline{E}$ is simply connected, so the same holds for Ξ. To compute the homology of Σ, we use Alexander duality to consider instead its complement $S^{2n-1} - \Sigma$; the latter is a deformation retract of $Y = C^n - X$. The map $f: Y \to C^* = C - \{0\}$ is the projection map of a fibre bundle with fibre Ξ. The homotopy of Y follows immediately from the exact homotopy sequence of the fibration: $\pi_1(Y) = Z$, $\pi_{n-1}(Y) = J/I$. The homology of Y is calculated from the spectral sequence of the fibre bundle with E^2-term $E^2_{p,q} = H_p(C^*, H_q(\Xi(t)))$. $\pi_1(C^*)$ acts on $H_{n-1}(\Xi)$ (as $t = e^{i\theta}$ goes around a cycle in C^*, we get an isotopy $\Xi \to \Xi(t)$; when t becomes 1 again the resulting automorphism of Ξ is $\omega = \prod_k \omega_k$), so we get that $E^{\infty}_{p,q} = E^2_{p,q}$ and is Z if (p,q) is (0,0) or (1,0), $\ker(1-\omega)$ if (p,q) is (0,n-1), $\operatorname{coker}(1-\omega)$ if (p,q) is (1,n-1), and 0 otherwise. Therefore $H_i(Y)$ is 0 for $i \neq 0, 1, n-1, n$ and is 0 for $i = n-1, n$ iff $1-\omega: J/I \to J/I$ is an isomorphism, that is if $\Delta(1) = \pm 1$ where $\Delta(t) = \det(t1 - \omega)$. This translates back into the information that Σ is (n-3)-connected ($n \geq 4$) and has the homology of a sphere iff $\Delta(1) = \pm 1$ (we have proved this completely rather than quote the general result for singularities mentioned in the introduction to the chapter). To compute $\Delta(t)$, we consider J/I as the tensor product of the Z-modules V_k spanned by the powers of ω_k, so $\omega: J/I \longrightarrow J/I$ is $\omega_1 \otimes \ldots \otimes \omega_n$ (where $\omega_k : V_k \longrightarrow V_k$ is multiplication by ω_k) so tensoring everything with C, we find that for each a_k^{th} root of unity $x_k = \xi_k^{i_k}$ the vector $1 + x_k \omega_k + \ldots + (x_k \omega_k)^{a_k - 1}$ is an eigenvector of ω_k with eigenvalue x_k^{-1}, so the eigenvalues of ω are all numbers $x_1^{-1} \ldots x_n^{-1}$. Therefore

$$(1) \quad \Delta(t) = \prod_{0 < i_k < a_k} (t - \xi_1^{i_1} \ldots \xi_n^{i_n}).$$

In particular $\Delta(1)$ has positive real part, so:

THEOREM: $\Sigma(a_1, \ldots, a_n)$ is a homology sphere ($n \geq 4$) of dimension 2n-3 if and only if $\Delta(1) = 1$, where $\Delta(t)$ is given by (1).

This can be reformulated in terms of a condition on the graph G which permits one to say instantly of an n-tuple of numbers whether the corresponding Brieskorn manifold Σ is a sphere (homology sphere $\Rightarrow$ sphere since $2n-3 \geq 5$):

THEOREM: $\Sigma(a_1, \ldots, a_n)$ is a sphere ($n \geq 4$) iff the graph G either has more than one isolated point or has a single isolated point and also a component with an odd number of points any two of which have largest common factor exactly 2.

2. The differentiable structure of $\Sigma(a_1, \ldots, a_n)$

That Σ bounds a parallelizable manifold follows from the description of K for general singularities, given in the introduction to this chapter, as the boundary of the parallelizable $F_\theta \cup K$. We can get explicitly a manifold which Σ bounds by noting that $\Sigma(t)=\Xi(t)\cap S^{2n-1}$ bounds $M(t)=\Xi(t)\cap D^{2n}$, and that for $|t|$ small enough, $\Sigma(t)$ is diffeomorphic to Σ (by Ehresmann's theorem), while $M(t)$ has a trivial normal bundle in C^n and therefore is stably parallelizable and so parallelizable (see p. 8). Choose $|t|$ small enough that this all holds; then $M=M(t)$ is a bounded parallelizable manifold whose boundary is diffeomorphic to Σ. Moreover, $M-\partial M$ is diffeomorphic to $\Xi(t)$ because $|z_1|^2+\ldots+|z_n|^2$ on $\Xi(t)$ has no critical points outside $M(t)$. This permits us to calculate the invariants of M. In particular when n is even, so that (if Σ is a sphere) the differentiable structure of $\Sigma \in bP_{2n-2}$ is determined by an Arf invariant, this invariant is 0 or 1 (mod 2) according as $\Delta(-1)$ is ± 1 or ± 3 (mod 8) by Levine's theorem. Again Brieskorn in [5] has reformulated this condition on the polynomial (t) in a form concerning the graph G which permits us to immediately identify the structure on for given $a_1,\ldots,a_n$:

THEOREM: $\Sigma(a_1,\ldots,a_n)$ is the Kervaire sphere (in the case $n\geq 4$ and even, Σ a sphere) if and only if G consists of exactly two components, one of which is an isolated point $a_k \equiv \pm 3$ (mod 8).

In particular if $(a_1,\ldots,a_n)=(2,\ldots,2,d)$ with n and d odd, the corresponding Brieskorn manifold is a sphere, and is the Kervaire sphere iff d is ± 3 (mod 8). This is exactly the manifold $W^{2n-1}(d)$ constructed in section 1 of Chapter II using

G-manifold theory, and which was identified with $M^{2n-1}(A_{d-1})$ on page 29. To see this directly (rather than using the fact that $M^{2n-1}(A_{d-1})$ when n and d are odd is the element of bP_{2n} with Arf invariant equal to $(d^2-1)/8$ mod 2, and that we have just proved that $\Sigma^{2n-1}(d,2,\ldots,2)$ also is) we equip $\Sigma(d,2,\ldots,2)$ with an $O(n)$-manifold structure by $A(z_0,z_1,\ldots,z_n) = (z_0,A(z_1,\ldots,z_n))$ ($A\in O(n)$ operates on $\vec{z}=\vec{x}+i\vec{y}\in C^n$ by $A\vec{z}=A\vec{x}+iA\vec{y}$). The details of the proof that the orbit space is D^2, that the structure is that of a special $O(n)$-manifold, and that the orbit types are $(O(n-1))$ and $(O(n-2))$ can be found on pp. 31-34 of [13].

If n is odd, the differentiable structure on $\Sigma(a_1,\ldots,a_n)$ (when it is a sphere) is determined by a signature. The calculation of the terms of the intersection matrix of Ξ was carried out by Pham [35]. As a basis of $H_{n-1}(\Xi) = J/I\otimes C$, use the eigen-vectors introduced on p. 35, namely

$$v_j = \prod_{k=1}^{n}(1+x_k\omega_k+\ldots+(x_k\omega_k)^{a_k-1}) \quad (x_k = e^{2\pi i j_k/a_k}),$$

where $j=(j_1,\ldots,j_k)$ is an n-tuple of integers with $0<j_k<a_k$).
The result is that the intersection numbers are

$$(2)\ \langle v_i,v_j\rangle = (-1)^{(n-1)(n-2)/2}(1-x_1\ldots x_k)\prod_{k=1}^{n}(1-x_k^{-1})(1+x_ky_k+\ldots+(x_ky_k)^{a_k-1}).$$

This is 0 unless $i+j=a$ ($i_k+j_k=a_k$ all k). Therefore the vectors v_j+v_{a-j} and $i(v_j-v_{a-j})$ give a basis of $J/I\otimes R$ with respect to which the intersection matrix is diagonal; its entries are given by

$$\langle v_j+v_{a-j},v_j+v_{a-j}\rangle = \langle i(v_j-v_{a-j}),i(v_j-v_{a-j})\rangle = 2\langle v_j,v_{a-j}\rangle.$$

These entries are real and, using (2), we see that they are positive exactly when $0 < \sum_{k=1}^{n} j_k/a_k < 1$ (mod 2). For we have:

$$\langle v_j, v_{a-j}\rangle = (-1)^{(n-1)/2}(\prod_k a_k)(\prod_k (1-x_k^{-1}) + \prod_k (1-x_k^{-1}))$$

$$= \text{Re } (-1)^{(n-1)/2}(\prod_k a_k)(\prod_k (1-x_k))$$

$$= \text{Re } (-1)^{(n-1)/2} \prod_k (-2ia_k\, e^{\pi i j_k/a_k} \sin(\pi j_k/a_k))$$

$$= \text{Re } (-\exp(\pi i/2 + \pi i \sum_k j_k/a_k)\cdot \prod_k 2a_k \sin(\pi j_k/a_k))$$

Since $\prod_k (2a_k \sin \pi j_k/a_k)$ is real and positive, we find as stated that $\langle v_j, v_{a-j}\rangle$ is positive exactly when $\text{Re}(e^{\pi i(\frac{1}{2}+\sum_k j_k/a_k)})$ is negative, i.e. when $0 < \sum j_k/a_k < 1 \pmod 2$. Hence the signature of M, which is the same as that of the diffeomorphic manifold Σ, is $\tau_+ - \tau_-$ where $\tau_\pm$ is the number of n-tuples j of integers (with $0<j_k<a_k$) for which $\sum_{k=1}^{n} j_k/a_k$ reduced mod 2 lies between 0 and ± 1. Reformulating this somewhat,

THEOREM: $\Sigma(a_1,\ldots,a_n)$, where $n \geq 5$ is odd and Σ is a sphere, is the element of bP_{2n-2} identified by the signature

$$(3) \qquad \sum_{0<i_k<a_k} (-1)^{[i_1/a_1+\ldots+i_n/a_n]} \; ,$$

where $[x]$ denotes the greatest integer $\leq$ x.

In the case when $(a_1,\ldots,a_n) = (p,q,2,\ldots,2)$, $n\geq 5$ odd, the Brieskorn manifold is a sphere exactly when p and q are relatively prime odd integers. This is just the knot manifold (of dimension 2n-3) corresponding to the torus knot t(p,q) [12].

3. Computing the signature of $\Sigma(a_1,\ldots,a_n)$

The theorems of this chapter allow us to immediately recognize whether a given $\Sigma(a_1,\ldots,a_n)$ is a sphere and, if n is even, which element of bP_{2n-2} it is. However, formula (3) for the signature of the sphere $\Sigma(a)$ when n is odd is very awkward for calculation. In this section we give certain means to calculate it when n is small. We write i,j,a for the various n-tuples and j/a for $i_1/j_1+\dots+i_n/j_n$, and define for any even n the function of n-tuples of integers

$$\tau(a_1,\dots,a_n) = \sum_{0<j<a} (-1)^{[j/a]}.$$

Note also that $\tau(a_1,\dots,a_n,2,2) = -\tau(a_1,\dots,a_n)$, so we never need more than one 2. In the important case $a = (p,q,2)$ (p,q odd and prime to one another) Hirzebruch slightly recast formula (3) to obtain $\tau(p,q,2) = \frac{1}{2}(p-1)(q-1) + n(p,q) + n(q,p)$, where $n(p,q)$ denotes the number of elements of $\{q, 2q, \dots, \frac{1}{2}(p-1)q\}$ whose smallest (in absolute value) residue mod p is negative, but even this is cumbersome if p and q are at all large.

To calculate $\tau(a)$ easily, we work out the effect of some simple changes of variables. In the case $a = (p,q,2)$,

(1) $\tau(p,q,2) = \tau(q,p,2)$ [immediate]

(2) $\tau(p,1,2) = 0$ [from the formula. Of course there is no corresponding Brieskorn manifold]

(3) $\tau(-p,q,2) = -\tau(p,q,2)$ [indeed changing the sign of one a_k always changes the sign of $\tau(a)$, for if $a=(p,b)$,

$$\tau(-p,b) = \sum_{i=1}^{-p-1} \sum_{0<j<b} (-1)^{[\frac{i}{-p} + \frac{j}{b}]} = \sum_{0<j<b} \Big(- \sum_{i=-p}^{0} (-1)^{[\frac{i}{p} + \frac{j}{b}]} \Big)$$

$$= - \sum_{0<j<b} \Big((-1)^{[\frac{j}{b}]} + (-1)^{[\frac{j}{b}+1]} + \sum_{i=1}^{p-1} (-1)^{[\frac{i}{p} + \frac{j}{b}]} \Big) = -\tau(p,b) \text{]}$$

(4) $\tau(p,q+2mp,2) = \tau(p,q,2) + m(p^2-1)$ [proved below].

But these suffice to define $\tau(p,q,2)$ for (and only for) p and q odd and relatively prime. For we can use (1) to ensure $q>p$, then use (4) (with negative m) to reduce q to a value between $-p$ and $+p$, use (3) if necessary to make p and q positive, and use (1) to get $q>p$, thus reducing the norm $\max(p,q)$. Since q and p are odd and relatively prime, and since none of the operations on the variables in (1) to (4) change this circumstance, one eventually gets down to $q=1$ and can use (2). Thus the pair $(p,q)=(115,73)$ becomes in succession $(73,115)$, $(73,-31),(31,73),(31,11), (9,11),(11,9),(9,7),(7,5),(5,3),(3,1)$ and so $(115,73,2) = (73^2-1)-(31^2-1)-(11^2-1)-(9^2-1)+(7^2-1)$ $-(5^2-1)+(3^2-1) = 4200$, so $\Sigma(115,73,2,\ldots,2)$ is the $(-1)^m \cdot \frac{4200}{8}$ th element of bP_{4m+4}. Note that p^2-1 is also a multiple of 8 for p odd, so that from the properties (1) to (4) and the algorithm just demonstrated we have a proof directly that $\tau(p,q,2) \equiv 0 \ (8)$.

For more than two of the a_k greater than two, we can set up the analogues of (1) to (4). Indeed τ is plainly symmetric in its variables and vanishes if any a_k is 1, while its properties under changes of sign in the a_k were proved in (3) above. As to (4), we can only say that if one a_k changes by a multiple of all the others, the effect on τ is independent of a_k. For we have

$$\sum_{i=1}^{a-1} (-1)^{[\frac{i}{a}+x]} = (-1)^{[x]} \Big(\sum_{\substack{0<i<a \\ 0<\frac{i}{a}+x-[x]<1}} (1) + \sum_{\substack{0<i<a \\ 1\le\frac{i}{a}+x-[x]<2}} (-1) \Big)$$

$$= (-1)^{[x]} \big(a-1-[(x-[x])a] - [(x-[x])a] \big)$$

$$= (-1)^{[x]} \big(a-1+2a[x]-2[ax] \big)$$

for any integer a and real x, and therefore if x is rational with denominator d, so that xd is an integer, we have

$$\sum_{0<i<a+md} (-1)^{\left[\frac{i}{a+md}+x\right]} = (-1)^{[x]}\left(a-1+2a[x]-2[ax]+md(1+2[x]-2x)\right)$$

$$= \sum_{0<i<a} (-1)^{\left[\frac{i}{a}+x\right]} + md\cdot(-1)^{[x]}(1+2[x]-2x).$$

If we put $x = i_1/a_1+\dots+i_n/a_n$ and sum over n-tuples $(i_1,\dots,i_n)$, we obtain

$$\tau(a+N,a_1,\dots,a_n) = \tau(a,a_1,\dots,a_n) + Nf(a_1,\dots,a_n),$$

where N is any multiple of $a_1,\dots,a_n$ and

$$f(a) = \sum_{0<i<a}(-1)^{[i/a]}(1+2[i/a]-2i/a).$$

When n is 2, we find

$$f(p,q) = \sum_{0<i<p}\left(\sum_{0<j<q\left(\frac{p-i}{p}\right)}(1-2j/q-2i/p) - \sum_{q\left(\frac{p-i}{p}\right)<j<q}(1+2-2j/q-2i/p)\right)$$

$$= \sum_{0<i<p}\Big((1-2\tfrac{i}{p})\left[\tfrac{p-i}{p}q\right] - \tfrac{1}{q}\left[\tfrac{p-i}{p}q\right]\left[\tfrac{p-i}{p}q+1\right]$$

$$- (3-2\tfrac{i}{p})\left[\tfrac{i}{p}q\right] + q-1 - \tfrac{1}{q}\left[\tfrac{p-i}{p}q\right]\left[\tfrac{p-i}{p}q+1\right]\Big).$$

Although it is not possible to sum this directly, we can deduce from it that the function $h(p,q) = pqf(p,q)$ satisfies

$$h(q+mp,p) = h(q,p) + 2mpq(p^2-1)/3 + m^2p^2(p^2-1)/3;$$

the calculation is very tedious and I omit it. Moreover $f(p,q)$ is symmetric, vanishes when p or q is 1, and changes sign if p or q does (from the corresponding properies of τ), so $h(p,q)$ is symmetric, vanishes if p or q is 1, and is unchanged if p or q changes sign. But all of these properties are shared by the function $\bar{h}(p,q) = (p^2-1)(q^2-1)/3$, and as before a function

$h(p,q)$ is determined for all relatively prime p,q if its behaviour is known under $(p,q) \mapsto (q,p)$, $(-p,q)$, $(p,1)$, and $(p,q+mp)$. Therefore $h(p,q)=\bar{h}(p,q)$ and so

$$\tau(p,q,r+mpq) = \tau(p,q,r) + m(p^2-1)(q^2-1)/3.$$

It is to be noted that since p and q are relatively prime, they cannot both be multiples of 3 and at least one is odd, so $(p^2-1)(q^2-1)/3$ is an integer and a multiple of 8. If q is 2, we have proved $f(p,2) = (p^2-1)/2p$, assertion (4) above.

Using this, we have made a table of $\tau(p,q,r)$ for certain small values of p and q (and all r) on the following page. If $\Sigma(p,q,r,2,\dots,2)$ is to be a sphere then p,q, and r must all be relatively prime by the theorems of section one.

As far as the interpretation of these formulas, we note that symmetry under interchange of the a_k's is trivial. The relation $\tau(1,a_1,\dots,a_n)=0$ probably has only a formal significance. The relation $\tau(-a,a_1,\dots,a_n)=-\tau(a,a_1,\dots,a_n)$ seems to indicate that $\Sigma(-a,a_1,\dots,a_n)$ is a manifold, diffeomorphic to $\Sigma(a,a_1,\dots,a_n)$ with its orientation preserved. This is false, however. The equations $z^{-a}+\dots+z_n^{a_n}=0$, $|z|^2+\dots+|z_n|^2=1$ define a set which avoids $z=0$ and might be expected to be a manifold, but if we look at the corresponding Jacobian, namely

$$\begin{bmatrix} -az^{-a-1} & a_1z_1^{a_1} & \dots & a_nz_n^{a_n-1} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -a\bar{z}^{-a-1} & -a_1\bar{z}_1^{+a_1-1} & \dots & a_n\bar{z}_n^{-a_n-1} \\ \bar{z} & \bar{z}_1 & \dots & \bar{z}_n & z & z_1 & \dots & z_n \end{bmatrix}$$

TABLE OF SIGNATURES OF BRIESKORN MANIFOLDS

$$f = -\tau(p,q,r)/8$$

p	q	r	f
2	3	6k±1	k
2	5	10k±1	3k
		10k±3	3k±1
2	7	14k±1	6k
		14k±3	6k±1
		14k±5	6k±2
2	9	18k±1	10k
		18k±5	10k±3
		18k±7	10k±4
2	11	22k±1	15k
		22k±3	15k±2
		22k±5	15k±3
		22k±7	15k±5
		22k±9	15k±6
2	13	26k±1	21k
		26k±3	21k±2
		26k±5	21k±4
		26k±7	21k±6
		26k±9	21k±7
		26k±11	21k±9
2	15	30k±1	28k
		30k±7	28k±6
		30k±11	28k±10
		30k±13	28k±12
2	17	34k±1	36k
		34k±3	36k±3
		34k±5	36k±5
		34k±7	36k±7
		34k±9	36k±10
		34k±11	36k±12
		34k±13	36k±14
		34k±15	36k±16
2	19	38k±1	45k
		38k±3	45k±3
		38k±5	45k±6
		38k±7	45k±8
		38k±9	45k±10
		38k±11	45k±13
		38k±13	45k±15
		38k±15	45k±18
		38k±17	45k±20
2	21	42k±1	55k
		42k±5	55k±6
		42k±11	55k±15
		42k±13	55k±17
		42k±17	55k±22
		42k±19	55k±25
2	23	46k±1	66k
		46k±3	66k±4
		46k±5	66k±7
		46k±7	66k±10
		46k±9	66k±13
		46k±11	66k±15
		46k±13	66k±19
		46k±15	66k±22
		46k±17	66k±24
		46k±19	66k±27
		46k±21	66k±30
3	4	12k±1	5k
		12k±5	5k±2
3	5	15k±1	8k
		15k±2	8k±1
		15k±4	8k±2
		15k±7	8k±4
3	7	21k±1	16k
		21k±2	16k±1
		21k±4	16k±3
		21k±5	16k±4
		21k±8	16k±6
		21k±10	16k±8
3	8	24k±1	21k
		24k±5	21k±4
		24k±7	21k±6
		24k±11	21k±10
3	10	30k±1	33k
		30k±7	33k±8
		30k±11	33k±12
		30k±13	33k±14
4	5	20k±1	15k
		20k±3	15k±2
		20k±7	15k±5
		20k±9	15k±7
4	7	28k±1	30k
		28k±3	30k±3
		28k±5	30k±5
		28k±9	30k±10
		28k±11	30k±12
		28k±13	30k±14
4	9	36k±1	50k
		36k±5	50k±7
		36k±7	50k±10
		36k±11	50k±15
		36k±13	50k±18
		36k±17	50k±24
5	6	30k±1	35k
		30k±7	35k±8
		30k±11	35k±13
		30k±13	35k±15

The condition that Σ fail to be a manifold is that at some point on it, we have $\bar{z} = -a\alpha z^{-a-1}$, $\bar{z}_k = \alpha a_k z_k^{a_k-1}$ (all k) for some non-zero complex α, so that the above matrix has rank only two. Combining this with the equations defining Σ yields

$$0 = z^{-a} + z_1^{a_1} + \dots + z_n^{a_n}$$
$$= \frac{1}{\alpha}\left(-\frac{|z|^2}{a} + \frac{|z_1|^2}{a_1} + \dots + \frac{|z_n|^2}{a_n}\right).$$

When all the exponents were positive, the corresponding condition contradicted the fact that $\vec{0} \notin \Sigma$, but now there is a subset of Σ of codimension 1 on which the condition obtains, so that on this subset Σ fails to be a manifold.

The other remarkable properties of the signature, namely that $\tau(a_1,\dots,a_n,2,2) = -\tau(a_1,\dots,a_n)$ and that $\tau(p,q,r+pq,2,\dots,2) = \tau(p,q,r,2,\dots,2) \pm (p^2-1)(q^2-1)/3$, are no easier to explain. The first probably indicates that the addition of two more complex variables with exponent two is some sort of algebraic suspension operation. The second suggests that $\Sigma(p,q,r,2,\dots,2)$ may add (connected sum) to a manifold that is a product of two manifolds M_p and M_q (depending only on p and q) to form $\tau(p,q,r+pq,2,\dots,2)$. In general the algebraic operation of adding to one exponent a multiple of the others may correspond to adding handles of some sort that only depend on the other exponents. But there seems to be no algebraic interpretation of the very non-algebraic connected sum operation.

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