Classification of Hilbert Modular Surfaces

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I. Introduction and Statement of Results

1. 1 In the paper [6] non-singular models for the Hilbert modular surfaces were constructed. In [9] it was investigated how these algebraic surfaces fit into the Enriques-Kodaira rough classification of surfaces ([11], [12]). But this was only done for the surfaces $Y(p)$ belonging to a real quadratic field of prime discriminant. We shall solve the corresponding problem for real quadratic fields of arbitrary discriminant. We shall use the notation of [6] and [9] and refer to these papers very often.

1. 2 Let K be the real quadratic field of discriminant D and $\mathfrak{o}$ its ring of integers. The Hilbert modular group $G=\boldsymbol{SL}_2(\mathfrak{o})/\{1, -1\}$ acts on $\mathfrak{H}\times\mathfrak{H}$ where $\mathfrak{H}$ is the upper half plane. The complex space $\mathfrak{H}^2/G$ can be compactified by finitely many cusps. This gives a compact normal complex space of dimension 2 denoted by $\overline{\mathfrak{H}^2/G}$ which has finitely many singularities (resulting from the cusps and the elliptic fixed points of G). If one resolves these singularities in the canonical minimal way, one gets a non-singular algebraic surface $Y(D)$. Thus for any discriminant D of a real quadratic field (i. e. $D\equiv 1 \bmod 4$ or $D\equiv 0 \bmod 4$, where $D\geqq 5$ and D or $D/4$ respectively is square free) an algebraic surface $Y(D)$ is defined. (Here we have changed the notation of [6] § 4. 5. There $Y(D)$ was called $Y(d)$ where d is the square free part of D.)

1. 3 The rough classification of algebraic surfaces without exceptional curves was recalled in [9] (Chap. I, Theorem ROC). Since the surface $Y(D)$ is regular (see [1] Part I, [2] or [9] Prop. II. 4), it is either *rational* or admits a unique minimal model which is a *K3-surface*, an *honestly elliptic surface* (fibred over the projective line) or a surface of *general type*. Thus there are four distinct possibilities, and we wish to decide for every D which of these four cases happens. It was proved recently that $Y(D)$ is simply-connected ([17] and A. Kas, unpublished). Therefore, the Enriques surface (which is an honestly elliptic surface) cannot occur as minimal model of any $Y(D)$ and the class (rational, blown-up $K3$ surface, blown-up honestly elliptic surface, general type) of $Y(D)$ can be characterized by the Kodaira dimension $\kappa(Y(D))$ (defined as the maximal dimension of the images of $Y(D)$

under the pluricanonical mappings). *Thus* $Y(D)$ *is*

rational	*if and only if*	$\kappa(Y(D)) = -1$,
a blown-up K3 surface	*if and only if*	$\kappa(Y(D)) = 0$,
a blown-up honestly elliptic surface	*if and only if*	$\kappa(Y(D)) = 1$,
of general type	*if and only if*	$\kappa(Y(D)) = 2$.

In Chap. II we shall recall the formulas for the arithmetic genus of $Y(D)$. Since $Y(D)$ is regular, we have $\chi(Y(D))=1+p_g\geqq 1$. It is easy to see that $\chi(Y(D))$ tends to ∞ for $D\to\infty$ and certain estimates (Chap. IV) and explicit calculations will show that

$$\chi(Y(D)) = 1 \Leftrightarrow D = 5, 8, 12, 13, 17, 21, 24, 28, 33, 60. \tag{1}$$

It was proved in [6] § 4. 5 that $Y(D)$ is rational for these values of D. Since the arithmetic genus of any rational surface equals 1, the ten values of D given in (1) are exactly the values for which $Y(D)$ is rational.

The following result was proved in [9]. For convenience we express it in terms of the Kodaira dimension.

If p *is a prime congruent to* 1 mod 4, *then*

$$\begin{aligned} \kappa(Y(p)) &= -1 \Leftrightarrow \chi(Y(p)) = 1 \Leftrightarrow p = 5, 13, 17, \\ \kappa(Y(p)) &= 0 \Leftrightarrow \chi(Y(p)) = 2 \Leftrightarrow p = 29, 37, 41, \\ \kappa(Y(p)) &= 1 \Leftrightarrow \chi(Y(p)) = 3 \Leftrightarrow p = 53, 61, 73, \\ \kappa(Y(p)) &= 2 \Leftrightarrow \chi(Y(p)) \geqq 4 \Leftrightarrow p > 73. \end{aligned} \tag{2}$$

To generalize such results to any discriminant we have to calculate $c_1^2(Y(D))$ which equals $K\cdot K$ where K is a canonical divisor of $Y(D)$. Namely, if $Y(D)$ *is not rational and* $c_1^2(Y(D))>0$, *then* $Y(D)$ *is of general type.* This follows from the rough classification theorem : For the unique minimal model $Y_{\min}(D)$ of $Y(D)$ we have

$$c_1^2(Y_{\min}(D)) \geqq c_1^2(Y(D)) > 0.$$

Therefore, $Y_{\min}(D)$ cannot be a $K3$-surface or an honestly elliptic surface, because for such a surface $c_1^2=K^2=0$. Since $c_1^2(Y(D))$ tends to ∞ for $D\to\infty$, we can reduce the classification to a finite list. This requires certain estimates. In Chap. IV we will prove :

Theorem 1. *If* $D>285$, *then* $c_1^2(Y(D))>0$.

(The proof depends on computer calculations.) There are exactly 50 discriminants with $c_1^2(Y(D))\leqq 0$; they are listed in Chap. IV. We consider these cases by hand and can settle all of them using the methods of [9] (in particular, Proposition I. 8 and I. 9). Many cases are already covered by (1) and (2) above. The result is the following theorem (Chap. V).

Theorem 2. *The Hilbert modular surface* $Y(D)$ *is*

rational	*for*	$D = 5, 8, 12, 13, 17, 21, 24, 28, 33, 60$,
blown-up K3	*for*	$D = 29, 37, 40, 41, 44, 56, 57, 69, 105$,

blown-up honestly elliptic for $D = 53, 61, 65, 73, 76, 77, 85, 88, 92, 93, 120, 140, 165$,
of general type otherwise (i. e. $D = 89$ or $D \geqq 97$, but $D \neq 105, 120, 140, 165$).

1. 4 The Hilbert modular group G belonging to $\boldsymbol{Q}(\sqrt{D})$ acts also on $\mathfrak{H}\times\mathfrak{H}^-$ where $\mathfrak{H}^-$ is the lower half plane. Compactifying $(\mathfrak{H}\times\mathfrak{H}^-)/G$ and resolving all singularities of the compactification $\overline{(\mathfrak{H}\times\mathfrak{H}^-)/G}$ in the minimal way lead to an algebraic surface $Y_-(D)$. Here we get

Theorem 3. *The Hilbert modular surface* $Y_-(D)$ *is*
rational *for* $D = 5, 8, 12, 13, 17$,
blown-up K3 *for* $D = 21, 24, 28, 29, 33, 37, 40, 41$,
blown-up honestly elliptic for $D = 44, 53, 57, 61, 65, 73, 85$,
of general type otherwise (i.e. $D = 56, 60, 69, 76, 77$ or $D \geqq 88$).

1. 5 Let $\mathfrak{b}$ be an ideal in the ring $\mathfrak{o}$ of integers of K. We introduce the group $\boldsymbol{SL}_2(\mathfrak{o}, \mathfrak{b})$ consisting of all matrices $\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} \in \boldsymbol{SL}_2(K)$ such that $\alpha, \delta \in \mathfrak{o}$ and $\beta \in \mathfrak{b}^{-1}, \gamma \in \mathfrak{b}$. The actions of $\boldsymbol{SL}_2(\mathfrak{o}, \mathfrak{b})$ and $\boldsymbol{SL}_2(\mathfrak{o})$ on $\mathfrak{H}^2$ are equivalent (i. e. the groups are conjugate in $\boldsymbol{GL}_2^+(K)$) if $\mathfrak{b}=\lambda\mathfrak{a}^2$ where λ is a totally positive element of K and $\mathfrak{a}$ an ideal in $\mathfrak{o}$ (see [6], 3. 7 (40)). The action of $\boldsymbol{SL}_2(\mathfrak{o})$ on $\mathfrak{H}\times\mathfrak{H}^-$ and the action of $\boldsymbol{SL}_2(\mathfrak{o}, \mathfrak{b})$ on $\mathfrak{H}^2$ are equivalent if $\mathfrak{b}=(\Delta)$ where Δ is an element of $\mathfrak{o}$ of negative norm. The following four conditions on the field K are equivalent:

i) There exists an element Δ of negative norm and an ideal $\mathfrak{a}$ in $\mathfrak{o}$ with $(\Delta) = \mathfrak{a}^2$.
ii) The number -1 is the norm of an element of K.
iii) The discriminant D is a sum of two natural square numbers.
iv) The discriminant D has no prime factor $\equiv 3 \bmod 4$.

If one of these conditions is satisfied, then the actions of $\boldsymbol{SL}_2(\mathfrak{o})$ *on* $\mathfrak{H}^2$ *and* $\mathfrak{H}\times\mathfrak{H}^-$ *are equivalent* under an isomorphism of $\mathfrak{H}^2$ and $\mathfrak{H}\times\mathfrak{H}^-$ given by an element of $\boldsymbol{GL}_2(K)$ whose determinant is positive but has negative norm. The converse is also true (compare 2. 2). For this whole section 1.5 we refer the reader to Hammond [3].

For the group $\boldsymbol{SL}_2(\mathfrak{o}, \mathfrak{b})$ we consider $\mathfrak{H}^2/\boldsymbol{SL}_2(\mathfrak{o},\mathfrak{b})$, its compactification $\overline{\mathfrak{H}^2/\boldsymbol{SL}_2(\mathfrak{o},\mathfrak{b})}$ and the algebraic surface $Y(D, \mathfrak{b})$ obtained by resolving all singularities (cusps and quotient singularities) of $\overline{\mathfrak{H}^2/\boldsymbol{SL}_2(\mathfrak{o}, \mathfrak{b})}$ in the minimal way. The surface $Y(D,\mathfrak{b})$ is also simply-connected [17]. The surfaces $Y_-(D)$ and $Y(D, \mathfrak{b})$ are isomorphic if $\mathfrak{b}=(\Delta)$, where Δ is an element of $\mathfrak{o}$ of negative norm. *The surfaces* $Y(D)$ *and* $Y_-(D)$ *are isomorphic if one of the above conditions* i)–iv) *is satisfied.*

1. 6 We consider the involution T on $\overline{\mathfrak{H}^2/G}$ induced by $(z_1, z_2) \mapsto (z_2, z_1)$ and study the minimal resolution of $\overline{(\mathfrak{H}^2/G)}/T$. Here we cannot calculate the invariants c_1^2 and χ, because we do not have complete information on the fixed points of $G \cup G{\cdot}T$ in general. However, if $D=p$ is a prime, the fixed points are known [16].

The question, for which primes the surface $\overline{(\mathfrak{H}^2/G)}/T$ is rational, was completely answered in [6]; there are 24 such primes, the largest being 317. For $p>17$ we define in Chap. II a certain non-singular model $Y_T(p)$ of this surface and give its numerical invariants; this is needed to determine how the surface fits into the rough classification scheme. In particular, we need to estimate $c_1^2(Y_T(p))$. In Chap. IV, we prove

Theorem 4. *$c_1^2(Y_T(p))>0$ for $p>821$.*

This reduces the classification question to a finite list. In fact, all cases can be settled here, too, but for this we must refer to [8]. The result was announced in [7].

1.7 For the surfaces $Y(p)$ (and also for the $Y_T(p)$) the arithmetic genus χ surprisingly determines the Kodaira dimension:

$$\kappa = \min\,[2, \chi-2].$$

For arbitrary D this is no longer true:
for $D=85$, 140 and 165 we have $\chi(Y(D))=4$, but the surface is nevertheless a blown-up honestly elliptic surface ($\kappa=1$). In all cases studied up to now, however,

$\chi=1 \Leftrightarrow \kappa=-1$ (rational)
$\chi=2 \Leftrightarrow \kappa=\ \ 0$ (blown-up $K3$)
$\chi=3 \Rightarrow \kappa=\ \ 1$ (blown-up honestly elliptic)
$\chi\geqq 5 \Rightarrow \kappa=\ \ 2$ (general type).

It would be interesting to know whether this is an accident or whether there is some general property of simply-connected algebraic surfaces, valid for all Hilbert modular surfaces, which ensures, for example, that the surface is rational if $\chi=1$ and that it is $K3$ if it is minimal and $\chi=2$. It is known that there exist simply-connected algebraic surfaces with arithmetic genus one which are not rational (I. V. Dolgacev, Dokl. 7(1966)).

II. Numerical Invariants of Hilbert Modular Surfaces

2.1 The basic term for the calculation of invariants of Hilbert modular surfaces is the volume of $\mathfrak{H}^2/G$ with respect to the normalized Euler volume form ([6] § 1 (5))

(1) $$\omega = (2\pi)^{-2} y_1^{-2} y_2^{-2}\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2.$$

If $K=\boldsymbol{Q}(\sqrt{D})$ is the underlying field and G the Hilbert modular group, then

(2) $$\int_{\mathfrak{H}^2/G} \omega = 2\zeta_K(-1),$$

where $\zeta_K(s)$ is the ζ-function of K. We have

(3) $$\zeta_K(-1) = \frac{1}{60} \sum_{\substack{|k|<\sqrt{D} \\ k^2 \equiv D \bmod 4}} \sigma_1\!\left(\frac{D-k^2}{4}\right)$$

(compare [6], § 1 (11), (12)). Here $\sigma_1(n)$ is the sum of the divisors of n.

Let $a_r(G)$ be the number of points in $\mathfrak{H}^2/G$ for which the corresponding points in $\mathfrak{H}^2$ have isotropy groups of order r. Then the Euler number of $\mathfrak{H}^2/G$ is given by the formula ([6] § 1 (21))

$$e(\mathfrak{H}^2/G) = \int_{\mathfrak{H}^2/G} \omega + \sum_{r\geq 2} a_r(G)\frac{r-1}{r}. \tag{4}$$

We restrict ourselves to discriminants $\geqq 13$. Thus we exclude $D=5, 8, 12$. *Then the* $a_r(G)$ *vanish for* $r>3$.

We write

$$a_3(G) = a_3^+(G)+a_3^-(G). \tag{5}$$

Here $a_3^+(G)$ is the number of quotient singularities of $\mathfrak{H}^2/G$ of type $(3\,;1,1)$ whereas $a_3^-(G)$ is the number of quotient singularities of type $(3\,;1,-1)$. Compare [6] § 3 (13). We have complete information [15] on $a_2(G)$, $a_3^+(G)$, $a_3^-(G)$. We will state the result in terms of the discriminant D. It is convenient to introduce also the square free part d of D:

$$\begin{aligned} D &= d && \text{if } d \equiv 1 \mod 4 \\ D &= 4d && \text{if } d \equiv 2 \mod 4 \text{ or } d \equiv 3 \mod 4. \end{aligned}$$

By $h(-N)$ we denote the class number of the imaginary quadratic number field of discriminant $-N$.

We have

$$a_2(G) = \begin{cases} h(-4d) & \text{if } d \equiv 1 \mod 4 \\ 3h(-4d) & \text{if } d \equiv 2 \mod 4 \\ 10h(-d) & \text{if } d \equiv 3 \mod 8 \\ 4h(-d) & \text{if } d \equiv 7 \mod 8 \end{cases} \tag{6}$$

$$a_3^+(G) = \begin{cases} \frac{1}{2}h(-3D) & \text{if } D \not\equiv 0 \mod 3 \\ 4h(-D/3) & \text{if } D \equiv 3 \mod 9 \\ 3h(-D/3) & \text{if } D \equiv 6 \mod 9 \end{cases} \tag{7}$$

$$a_3^-(G) = \begin{cases} \frac{1}{2}h(-3D) & \text{if } D \not\equiv 0 \mod 3 \\ h(-D/3) & \text{if } D \equiv 3 \mod 9 \\ 0 & \text{if } D \equiv 6 \mod 9. \end{cases} \tag{8}$$

The Euler number of $\mathfrak{H}^2/G$ is now calculable. It is not difficult to write a computer program for $\zeta_K(-1)$ as given by formula (3), for the class numbers $h(-N)$ and finally for the Euler number of $\mathfrak{H}^2/G$.

$$\textit{For } D \geqq 13, \quad e(\mathfrak{H}^2/G) = 2\zeta_K(-1)+\frac{1}{2}a_2(G)+\frac{2}{3}a_3(G). \tag{9}$$

The second important invariant of the 4-dimensional rational homology manifold $\mathfrak{H}^2/G$ is the signature. It has no volume contribution. In the formula for sign $\mathfrak{H}^2/G$ only contributions from the quotient singularities of order 3 and from the cusps enter (compare [6] § 3 (43), (44)).

(10) *For* $D \geqq 13$, *we have* $\operatorname{sign} \mathfrak{H}^2/G = 4w - \frac{2}{9}a_3^+(G) + \frac{2}{9}a_3^-(G)$.

Here w is the total parabolic contribution in the sense of Shimizu. According to [4] Theorem 2. 1, it can be expressed in the following form:

(11) $$w = -4\sum h(D_1)h(D_2)u(D_1)^{-1}u(D_2)^{-1}$$

with the summation taken over all decompositions $D = D_1 \cdot D_2$ *with* $D_1 < D_2$ *in which* D_1, D_2 *are discriminants of imaginary quadratic fields.* $u(D_1)$ *and* $u(D_2)$ *respectively are the orders of the groups of units in the corresponding fields.* The parabolic contribution vanishes if and only if D is the sum of two squares, i.e. D contains no prime $\equiv 3 \bmod 4$. Otherwise it is negative ([4] Corollary 2. 2). In particular, it vanishes if the fundamental unit of $K = \boldsymbol{Q}(\sqrt{D})$ has negative norm. (Compare 1. 5.)

As was shown in [6] § 3.6, the arithmetic genus $\chi(Y(D))$ of the non-singular model $Y(D)$ of the compactification $\overline{\mathfrak{H}^2/G}$ of $\mathfrak{H}^2/G$ can be calculated in terms of the topological invariants of the non-compact rational homology manifold $\mathfrak{H}^2/G$. We have

(12) $$\chi(Y(D)) = \frac{1}{4}(e(\mathfrak{H}^2/G) + \operatorname{sign}(\mathfrak{H}^2/G)).$$

2. 2 We now consider the action of G on $\mathfrak{H} \times \mathfrak{H}^-$. The rational homology manifold $(\mathfrak{H} \times \mathfrak{H}^-)/G$ admits an orientation reversing homeomorphism onto $\mathfrak{H}^2/G$. Therefore

(13) $$\begin{aligned} e(\mathfrak{H}^2/G) &= e((\mathfrak{H} \times \mathfrak{H}^-)/G), \\ \operatorname{sign}(\mathfrak{H}^2/G) &= -\operatorname{sign}((\mathfrak{H} \times \mathfrak{H}^-)/G). \end{aligned}$$

For the non-singular model $Y_-(D)$ mentioned in the introduction we have

(14) $$\chi(Y_-(D)) = \frac{1}{4}(e(\mathfrak{H}^2/G) - \operatorname{sign}(\mathfrak{H}^2/G)).$$

The formulas (7), (8), (10) imply that $\operatorname{sign}(\mathfrak{H}^2/G)$ is always non-positive. Therefore,

(15) $$\chi(Y_-(D)) \geqq \chi(Y(D)).$$

If we exclude $D = 12$, then sign $(\mathfrak{H}^2/G) = 0$ if and only if D is not divisible by a prime $\equiv 3 \bmod 4$. For $D = 12$ the signature vanishes ([6] § 3. 9). As Hammond showed (see 1. 5), the signature of $\mathfrak{H}^2/G$ vanishes ($D = 12$ again excluded) if and only if the actions of G on $\mathfrak{H}^2$ and on $\mathfrak{H} \times \mathfrak{H}^-$ are equivalent, one direction of this equivalence being clear by the second formula of (13).

2. 3 A "cusp" is described by a pair (M, V) where M is a complete $\boldsymbol{Z}$-module in the real quadratic field K and V a subgroup of finite index in the (infinite cyclic) group of all totally positive units ε with $\varepsilon M = M$. To such a pair (M, V) we associate in a topological way ([6] § 3) a rational number $\delta(M, V)$. Then

(16) $$4w(M, V) = \delta(M, V),$$

where $w(M, V)$ is the Shimizu number of a cusp given by evaluating the L-function

$L(M, V, s)$ for $s=1$ (see [6] 3. 5). The sum of the $w(M, V)$ for all cusps of the Hilbert modular group of the real quadratic field with discriminant D is the number w given in (11). We wish to recall here the expression ([6] 3. 2 Theorem) for $\delta(M, V)$ using the continued fraction describing the resolution of the "cusp singularity" of type (M, V) :

There exists a totally positive number α in K such that

$$\alpha M = \boldsymbol{Z}w_0+\boldsymbol{Z}\cdot 1,$$

where w_0 is reduced, i.e.

$$0 < w_0' < 1 < w_0.$$

Here $x \mapsto x'$ denotes the non-trivial automorphism of K. The number w_0 has a purely periodic continued fraction development

$$w_0 = b_0-\cfrac{1}{b_1-\cfrac{\ddots}{\ddots-\cfrac{1}{b_{r-1}-\cfrac{1}{b_0-\ddots}}}} \qquad (b_i\in \boldsymbol{Z}, b_i\geqq 2)$$

where $((b_0,\dots, b_{r-1}))$ is the primitive period which (up to cyclic permutations) depends only on the strict equivalence class of the module and conversely determines this strict equivalence class. (We recall that by definition the modules M and $\tilde{M}$ are strictly equivalent if and only if there exists an element β of K of positive norm such that $\tilde{M}=\beta M$. They are called equivalent if there exists an element β of K such that $\tilde{M}=\beta M$.) We define

$$\delta(M) = -\frac{1}{3}\sum_{i=0}^{r-1}(b_i-3) \tag{17}$$

and

$$l(M) = r. \tag{18}$$

Thus $l(M)$ is the length of the period which we shall also call the length of the module.

If γ is an element of K with negative norm, then

$$\begin{aligned} \delta(\gamma M) &= -\delta(M), \\ 3\delta(M) &= l(M)-l(\gamma M). \end{aligned} \tag{19}$$

In particular, $\delta(M)=0$ if there exists a unit ε of K with negative norm such that $\varepsilon M=M$.

To prove (19) we observe that w_0 (see above) admits an ordinary continued fraction

$$w_0 = c_0+\cfrac{1}{c_1+\cfrac{1}{c_2+\ddots}} \qquad (c_i\in \boldsymbol{Z},\ c_i\geqq 1 \text{ for } i>0) \tag{20}$$

which is not necessarily purely periodic.

We denote the shortest period of even length by

$$(a_1, \cdots, a_{2s}). \tag{21}$$

Thus it is either the primitive period or twice the primitive period, the latter if and only if the primitive period has odd length. The period $(a_1, \cdots, a_{2s})$ (up to cyclic permutations) depends only on the equivalence class of M and also determines this equivalence class.

A period (21) determines two periods in the sense of continued fractions with minus signs, namely

$$((\underbrace{2, \cdots, 2}_{a_1-1}, a_2+2, \underbrace{2, \cdots, 2}_{a_3-1}, a_4+2, \cdots, \underbrace{2, \cdots, 2}_{a_{2s-1}-1}, a_{2s}+2)) \tag{22}$$

and

$$((\underbrace{2, \cdots, 2}_{a_2-1}, a_3+2, \underbrace{2, \cdots, 2}_{a_4-1}, a_5+2, \cdots, \underbrace{2, \cdots, 2}_{a_{2s}-1}, a_1+2)) \tag{23}$$

(compare [6] 2. 5 (19) and 3. 10). These two periods coincide (up to cyclic permutation) if and only if the period (21) is twice the primitive period of (20), i. e. if the primitive period has odd length. The periods (22), (23) determine the strict equivalence classes contained in the equivalence class of M. There is only one such equivalence class if and only if the primitive period of (20) is odd because this happens if and only if (22) and (23) coincide. Therefore the primitive period of (20) is odd if and only if there exists a unit ε of negative norm with $\varepsilon M = M$. The formulas (19) are an easy consequence of (22), (23). We also observe that $\delta(M)$ is up to sign the alternating sum of the a_i.

If we have a cusp of type (M, V), then V is a subgroup of finite index in the infinite cyclic group U_M^+ of all totally positive units ε with $\varepsilon M = M$, and we have

$$\begin{aligned} \delta(M, V) &= [U_M^+ : V] \cdot \delta(M), \\ l(M, V) &= [U_M^+ : V] \cdot l(M). \end{aligned} \tag{24}$$

For the cusps of the Hilbert modular group the modules M are always strictly equivalent to ideals in the ring $\mathfrak{o}$ of all integers of K. The strict equivalence classes mentioned above correspond to narrow ideal classes, the equivalence classes to ordinary ideal classes. Let C^+ be the group of narrow and C the group of ordinary ideal classes of $\mathfrak{o}$. Then $\mathfrak{a} \mapsto \mathfrak{a}^{-2}$ (where $\mathfrak{a}$ is an ideal in $\mathfrak{o}$) induces homomorphisms $Sq : C \rightarrow C^+$ and $Sq : C^+ \rightarrow C^+$ (see [6] 3. 7 (42)). There are h cusps for the Hilbert modular group $\boldsymbol{SL}_2(\mathfrak{o})/\{1, -1\}$ where h equals $|C|$ and is the class number of K. These cusps are of type $(\mathfrak{a}^{-2}, U^2)$ where U denotes the group of units of $\mathfrak{o}$. Let U^+ be the group of all totally positive units; then $U^+ = U^2$ if and only if there exists a unit of negative norm, otherwise $[U^+ : U^2] = 2$. In the first case $|C| = |C^+| = h$, in the latter $|C^+| = 2 \cdot |C| = 2h$. Let $\varepsilon_0 \in U$ be the fundamental unit ($\varepsilon_0 > 1$). Then

$$\begin{aligned} \delta(\mathfrak{a}^{-2}, U^2) &= 2\delta(\mathfrak{a}^{-2}) & &\text{if } N(\varepsilon_0) = 1, \\ \delta(\mathfrak{a}^{-2}, U^2) &= \delta(\mathfrak{a}^{-2}) = 0 & &\text{if } N(\varepsilon_0) = -1, \\ l(\mathfrak{a}^{-2}, U^2) &= 2l(\mathfrak{a}^{-2}) & &\text{if } N(\varepsilon_0) = 1, \\ l(\mathfrak{a}^{-2}, U^2) &= l(\mathfrak{a}^{-2}) & &\text{if } N(\varepsilon_0) = -1. \end{aligned} \tag{25}$$

The numbers δ and l depend only on the strict module *class*. Therefore, δ and l can be regarded as functions on C^+. For the total parabolic contribution we have in view of (16) and (25)

$$w = \frac{1}{2}\sum_{\mathfrak{a}\in C}\delta(Sq(\mathfrak{a})), \quad (Sq\colon C\to C^+)$$
(26)
$$= \frac{1}{4}\sum_{\mathfrak{a}\in C^+}\delta(Sq(\mathfrak{a})), \quad (Sq\colon C^+\to C^+)$$

which because of (11) is a relation between continued fractions and class numbers of imaginary quadratic fields. (Compare [6] 3. 10 (55)).

The pair (M, V) determines a singularity whose minimal resolution is cyclic. The number of curves in this resolution equals $l(M, V)$ (see [6] 2. 5 Theorem). The Hilbert modular surface $\mathfrak{H}^2/G$ for the field K of discriminant D is compactified by h points. They are singularities in the compactification $\overline{\mathfrak{H}^2/G}$ which when resolved minimally give rise to h cycles of curves. The number of all these curves will be denoted by $l_0(D)$. We have

$$(27) \qquad l_0(D) = \begin{cases} \sum_{\mathfrak{a}\in C} l(Sq(\mathfrak{a})) & \text{if } N\varepsilon_0 = -1 \\ 2\sum_{\mathfrak{a}\in C} l(Sq(\mathfrak{a})) & \text{if } N\varepsilon_0 = 1 \end{cases}$$

or equivalently

$$(28) \qquad l_0(D) = \sum_{\mathfrak{a}\in C^+} l(Sq(\mathfrak{a})) \qquad (Sq : C^+\to C^+).$$

The Hilbert modular surface $(\mathfrak{H}\times\mathfrak{H}^-)/G$ is also compactified by h points. These cusps are of type $(\gamma\mathfrak{a}^{-2}, U^2)$ where γ is an element of K of negative norm. We denote by $l_0^-(D)$ the number of curves needed to resolve all these cusp singularities minimally. Then

$$l_0^-(D) = \sum_{\mathfrak{a}\in C^+} l(\gamma Sq(\mathfrak{a}))$$

and by (19) and (26)

$$(29) \qquad l_0(D) - l_0^-(D) = 12w.$$

2. 4 Let $Y(D)$ be the surface obtained from $\overline{\mathfrak{H}^2/G}$ by minimal resolutions of all the singular points (see Chap. I). If we assume $D\geqq 13$, we have only quotient singularities of order 2 or 3. Those of order 2 are resolved by one curve; those of order 3 by one or two curves depending on whether the type is $(3; 1, 1)$ or $(3; 1, -1)$. As in [9] (Proposition II. 2 and (7)) we conclude

$$(30) \qquad \begin{aligned} e(Y(D)) &= e(\mathfrak{H}^2/G) + a_2(G) + a_3^+(G) + 2a_3^-(G) + l_0(D) \\ &= 2\zeta_K(-1) + \frac{3}{2}a_2(G) + \frac{5}{3}a_3^+(G) + \frac{8}{3}a_3^-(G) + l_0(D) \end{aligned}$$

for $D \geqq 13$.

Noether's formula states that $c_1^2(Y(D)) + e(Y(D)) = 12\chi(Y(D))$. Using (9), (10), (12), (29), (30) we obtain

$$(31)' \qquad c_1^2(Y(D)) = 4\zeta_K(-1) - l_0^-(D) - \frac{a_3^+(G)}{3}.$$

If we consider the action of G on $\mathfrak{H}\times\mathfrak{H}^-$ instead of $\mathfrak{H}\times\mathfrak{H}$, then $a_3^+(G)$, $a_3^-(G)$ interchange their role. The same is true for $l_0(D)$, $l_0^-(D)$. This implies

$$(32) \qquad c_1^2(Y_-(D)) = 4\zeta_K(-1) - l_0(D) - \frac{a_3^-(G)}{3}.$$

We have

$$(33) \qquad c_1^2(Y_-(D)) \geqq c_1^2(Y(D)), \qquad e(Y_-(D)) \geqq e(Y(D))$$

and in fact

$$\begin{aligned} c_1^2(Y_-(D)) - c_1^2(Y(D)) &= l_0^-(D) - l_0(D) && \text{if } D \not\equiv 0(3) \\ &= l_0^-(D) - l_0(D) + h\left(-\frac{D}{3}\right) && \text{if } D \equiv 0(3) \\ e(Y_-(D)) - e(Y(D)) &= l_0^-(D) - l_0(D) && \text{if } D \not\equiv 0(3) \\ &= l_0^-(D) - l_0(D) + 3h\left(-\frac{D}{3}\right) && \text{if } D \equiv 0(3). \end{aligned}$$

The corresponding inequality for the arithmetic genus was mentioned before (15).

2. 5 As mentioned in Chapter I, the surfaces $Y(D)/T$ will be investigated for prime discriminants in a later paper [8]. However the necessary estimates for c_1^2 will be done in this paper.

Let p be a prime $\equiv 1 \bmod 4$. The surface $Y(p)$ has some exceptional curves which can be blown down to give a surface $Y^0(p)$. We always assume $p > 17$ to ensure that $Y(p)$ is not rational and exceptional curves do not meet. (For details see [6] § 5 and [9]). The involution $(z_1, z_2) \mapsto (z_2, z_1)$ induces an involution T on $Y^0(p)$ which has no isolated fixed points. The fixed point set is a non-singular curve F_p^0. We have

$$(34) \qquad e(Y^0(p)/T) = \frac{1}{2}(e(Y^0(p)) + e(F_p^0)).$$

The Euler number $e(F_p^0)$ is given by a classical formula. Namely, the curve F_p^0 is the compact non-singular model of $\mathfrak{H}/\Gamma_0^*(p)$ where $\Gamma_0^*(p)$ is the normal extension of $\Gamma_0(p)$ by the element $\begin{bmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{bmatrix}$. This element induces an involution on $\overline{\mathfrak{H}/\Gamma_0(p)}$ which has $h(-4p)$ fixed points according to Fricke (loc. cit in [6]). Put $\varepsilon = 1$ if $p \equiv 1 \bmod 3$ and $\varepsilon = 0$ if $p \equiv 2 \bmod 3$. Then $\Gamma_0(p)$ has 2ε fixed points of order 3 and 2 fixed points of order 2 and two cusps. Therefore

$$e(\overline{\mathfrak{H}/\Gamma_0(p)}) = -\frac{p+1}{6} + \frac{4}{3}\varepsilon + 3$$

and

$$(35) \qquad e(F_p^0) = \frac{1}{2}\left(-\frac{p+1}{6} + \frac{4}{3}\varepsilon + 3 + h(-4p)\right).$$

Put $\delta=1$ if $p\equiv 1 \bmod 8$ and $\delta=0$ if $p\equiv 5 \bmod 8$. Then $Y^0(p)$ was obtained from $Y(p)$ by blowing down $4+2\delta+\varepsilon$ curves. By (30) we get

$$(36)\quad e(Y^0(p)) = 2\zeta_K(-1)+\frac{3}{2}h(-4p)+\frac{13}{6}h(-3p)+l_0(p)-4-2\delta-\varepsilon$$

and by (34) and (35)

$$e(Y^0(p)/T) = \zeta_K(-1)+h(-4p)+\frac{13}{12}h(-3p)+\frac{1}{2}l_0(p) -\frac{p+1}{24}-\frac{5}{4}-\frac{\varepsilon}{6}-\delta.$$

For the arithmetic genera of $Y^0(p)$ and $Y^0(p)/T$ we have the following formulas (cf. [6] 5. 6 (20), (21))

$$(37)\qquad \chi(Y^0(p)) = \frac{1}{2}\zeta_K(-1)+\frac{h(-4p)}{8}+\frac{1}{6}h(-3p)$$

$$(38)\qquad \chi(Y^0(p)/T) = \frac{1}{2}\left(\chi(Y^0(p))-\frac{p+1}{24}+\frac{\varepsilon}{3}+\frac{5}{4}+\frac{\delta}{2}\right).$$

By Noether's formula

$$c_1^2(Y^0(p)/T) = 12\chi(Y^0(p)/T)-e(Y^0(p)/T)$$

which yields

$$(39)\qquad c_1^2(Y^0(p)/T) = 2\zeta_K(-1)-\frac{h(-4p)}{4}-\frac{1}{12}h(-3p)-\frac{1}{2}l_0(p) -\frac{5p}{24}+\frac{13}{6}\varepsilon+4\delta+8+\frac{13}{24}.$$

Since $K=\boldsymbol{Q}(\sqrt{p})$ has a unit of negative norm, $l_0(p)$ and $l_0^-(p)$ coincide. The class number $h(p)$ is odd. Thus $Sq: C\to C$ is bijective and $l_0(p)$ equals the number of *all* reduced quadratic irrationalities of discriminant p which was denoted in [9] by $l(p)$. In [6] it was shown that many curves on $Y^0(p)/T$ can be blown down. The "tail" of the resolution of the principal cusp (see [6] 5. 8) admits $\left[\frac{\sqrt{p}-1}{2}\right]$ blow-downs (for $p>17$). If we use the basic configuration of curves on $Y^0(p)$ (see [6] 5. 4 (8)) we get on $Y^0(p)/T$ exceptional curves which come from the $h(-3p)/2$ "crosses" and the $h(-4p)/2$ curves of self-intersection number -2 on $Y^0(p)$. (The "crosses" were denoted in [9] p. 18 by C_i, C_i', the (-2)-curves by D_i.) We have $T(C_i)=C_i'$ and $T(D_i)=D_i$. The images of C_i and D_i are the exceptional curves in $Y^0(p)/T$ we are looking for.

The surface obtained from $Y^0(p)/T$ by these blow-downs will be denoted by $Y_T(p)$. We have (for $p>17$)

$$(40)\qquad c_1^2(Y_T(p)) = 2\zeta_K(-1)+\frac{h(-4p)}{4}+\frac{5}{12}h(-3p)-\frac{1}{2}l_0(p)-\frac{5p}{24} +\left[\frac{\sqrt{p}-1}{2}\right]+\frac{13}{6}\varepsilon+4\delta+8+\frac{13}{24}.$$

III. The Hurwitz-Maaß Extension of the Hilbert Modular Group, Skew-Hermitian Curves on $Y(D)$

3. 1 Let $K=\boldsymbol{Q}(\sqrt{D})$ be as before a real quadratic field and $\mathfrak{o}$ its ring of integers. We consider the matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with entries in $\mathfrak{o}$ such that $ad-bc$ is a totally positive unit of $\mathfrak{o}$. These matrices constitute a group which we divide by its center $\left\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \middle| a\in U\right\}$ where U is again the group of all units of $\mathfrak{o}$ (cf. introduction). We get the extended Hilbert modular group G_e. We have $G_e/G\simeq U^+/U^2$. It is a group of order 1 or 2.

Now we take the matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with entries in $\mathfrak{o}$ such that $w=ad-bc$ is totally positive and $a/\sqrt{w}$, $b/\sqrt{w}$, $c/\sqrt{w}$, $d/\sqrt{w}$ are algebraic integers not necessarily in $\mathfrak{o}$. The group of all these matrices has to be divided by its center $\left\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \middle| a\in \mathfrak{o}\right\}$. We get a group G_m which is a normal extension of G. It was introduced and studied by Hurwitz [10] § 3 and Maaß [13]. Obviously, the square of every element of G_m/G is the identity element. If we associate to $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the ideal $(\sqrt{w})$ of $\mathfrak{o}$ (consisting of all elements $x\in\mathfrak{o}$ such that $x/\sqrt{w}$ is an algebraic integer) we get a homomorphism π : $G_m/G\to C$ which maps G_m/G onto the kernel of $Sq: C\to C^+$. The group G_e/G is the kernel of π. Thus $[G_m : G]$ equals the order of the kernel of $Sq: C^+\to C^+$ which is 2^{t-1} where t is the number of primes dividing the discriminant D.

We remark that every line and column of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ generates the ideal $(\sqrt{w})$ in $\mathfrak{o}$.

3. 2 The group G_m/G operates on $\mathfrak{H}^2/G$ and also on the compactification $\overline{\mathfrak{H}^2/G}$. The cusps (considered as singular points of $\overline{\mathfrak{H}^2/G}$) are in one-to-one correspondence with C if one associates to a point $m/n\in \boldsymbol{P}_1(K)$ with $m, n\in\mathfrak{o}$ the ideal (m, n), see [6] 3. 7. Then $g\in G_m/G$ operates on C by multiplication with $\pi(g)$. This is easy to check. Two cusps represented by ideals $\mathfrak{a}$, $\mathfrak{b}$ are in the same orbit of the G_m/G-action if and only if $\mathfrak{a}^{-2}$, $\mathfrak{b}^{-2}$ represent the same element of C^+. This is true if and only if they have the same cycle of curves in their resolution. The group G_m/G operates also on $Y(D)$. The subgroup G_e/G keeps the cusps invariant, but is on each cycle the identity or the "covering translation" of order 2 depending on whether $|G_e/G| =|U^+/U^2|$ equals 1 or 2. The latter case is true if and only if there is no unit of negative norm. In this case the resolution cycle is twice the primitive cycle belonging to the module $\mathfrak{a}^{-2}$. Thus the group G_m/G of order 2^{t-1} operates freely on the union of the h cuspidal cycles of curves of $Y(D)$ (where $h=|C|$). *Each primitive cycle belonging to an element in the image of $Sq: C\to C^+$ occurs in $Y(D)$ exactly 2^{t-1} times (a twofold cycle counts as twice the primitive cycle).* The union of these 2^{t-1} primitive cycles is an orbit of the G_m/G action on $Y(D)$.

3. 3 We shall discuss the curves on the Hilbert modular surfaces defined by skew-hermitian matrices. By a *skew-hermitian matrix* we mean a matrix of the form

(1) $$\begin{bmatrix} a_1\sqrt{D} & \lambda \\ -\lambda' & a_2\sqrt{D} \end{bmatrix} \quad \text{where } \lambda \in \mathfrak{o} \text{ and } a_1, a_2 \in \boldsymbol{Z}.$$

Its determinant is

(2) $$N = a_1a_2D+\lambda\lambda'.$$

The matrix (1) is called primitive if there is no natural number>1 dividing a_1, a_2, λ. For a given natural number N the curve F_N in $\mathfrak{H}^2/G$ is defined to be the set of all points of $\mathfrak{H}^2/G$ which have representatives $(z_1, z_2)\in\mathfrak{H}^2$ for which there exists a primitive skew-hermitian matrix of determinant N such that

(3) $$a_1\sqrt{D}z_1z_2-\lambda' z_1+\lambda z_2+a_2\sqrt{D} = 0.$$

It can be shown that F_N defines a curve in $\overline{\mathfrak{H}^2/G}$ and in $Y(D)$, also to be denoted by F_N. The curve F_N is not necessarily irreducible. By (2) *the curve F_N is non-empty if and only if the residue class of N modulo D can be represented by a norm in* $\mathfrak{o}$. If N is prime to D, this condition can be expressed in terms of values of "genus characters" of N; see [5] Satz 141. The group G_m/G operates on F_N and on the set of its irreducible components. The component of F_N defined by (3) passes through a cusp if and only if there exists an element $x\in K\cup\infty=\boldsymbol{P}_1(K)$ such that

(4) $$a_1\sqrt{D}xx'-\lambda' x+\lambda x'+a_2\sqrt{D} = 0.$$

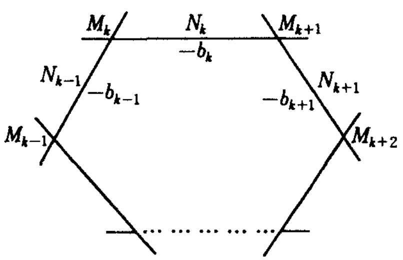
Since the matrix (1) can be diagonalized over the field K, this holds if and only if N is a norm in K. This is a condition only on N, so either all components of F_N pass through a cusp or none of them do.

The reduced quadratic irrationalities of discriminant D are of the form

$$w = \frac{M+\sqrt{D}}{2N}$$

where M and N are natural numbers, $0<w'<1<w$, and $M^2-D\equiv 0 \pmod{4N}$. There are only finitely many. Their continued fractions are *purely periodic.* Thus the reduced quadratic irrationalities of discriminant D are arranged in cycles which correspond bijectively to the elements of C^+ (see [6] 2. 6 and 4. 1 (5)). For a given cycle we index the reduced quadratic irrationalities as $w_k=\dfrac{M_k+\sqrt{D}}{2N_k}$ where k runs through $\boldsymbol{Z}/l\boldsymbol{Z}$ with l being the length of the cycle. We illustrate such a cycle as follows:

(5)



$$\left(w_k = b_k - \frac{1}{w_{k+1}}, \quad b_k = (M_k + M_{k+1})/2N_k\right).$$

The k-th line of (5) represents the rational curve S_k of self-intersection number $-b_k$ in the resolution of the corresponding cyclic singularity. To the k^{th} corner of (5) we associate the quadratic form $p^2 N_{k-1} + pqM_k + q^2 N_k$ of discriminant D. In the resolution of the cusps of $\overline{\mathfrak{H}^2/G}$ we have exactly the cycles associated to the squares in C^+, i.e. to the image of $Sq: C^+ \to C^+$. In $Y(D)$ there are 2^{t-1} cycles of curves (see 3. 2) belonging to a given element of $Sq(C^+)$; in each cycle we have a local coordinate system (u_k, v_k) centered at the k^{th} corner of the cycle. (The curve S_k is given by $v_k = 0$ and S_{k-1} by $u_k = 0$.) To relate this coordinate system as in [6] 2. 3 (11) to the coordinates (z_1, z_2) of $\mathfrak{H}^2$ by equations

$$\begin{aligned} 2\pi i z_1 &= A_{k-1} \log u_k + A_k \log v_k \\ 2\pi i z_2 &= A'_{k-1} \log u_k + A'_k \log v_k \end{aligned}$$

we must transform the cusp to ∞. This is done by an isomorphism between $Y(D)$ and $Y(D, \mathfrak{b}_0)$. (see 1. 5) where $\mathfrak{b}_0$ is an ideal in $\mathfrak{o}$ such that $\mathfrak{b}_0^{-1}$ represents the element of $Sq(C^+)$ corresponding to the cycle (compare [6] 3. 7). Two such isomorphisms differ by an element of G_m/G. For integers $p, q \geq 0$ (not both 0) we consider the local curve $u_k^q = v_k^p$. It has (p, q) branches

$$u_k^{q/(p,q)} = \zeta v_k^{p/(p,q)} \quad \text{with} \quad \zeta^{(p,q)} = 1 \tag{6}$$

of which $\varphi((p, q))$ are primitive, i.e. ζ is a (p, q)-th primitive root of unity.

As can be checked, the $\varphi((p, q))$ primitive branches (6) belong to F_N, where

$$N = p^2 N_{k-1} + pqM_k + q^2 N_k. \tag{7}$$

We identify the triples $(k|0, 1)$ and $(k+1|1, 0)$. For any triple $(k|p, q)$ belonging to an element of $Sq(C^+)$ we have 2^{t-1} local curves $u_k^p = v_k^q$ in $Y(D)$. They are transformed to each other under G_m/G. It is not difficult to prove the following lemma.

Lemma. *For given N the union of all the primitive branches* (6) *satisfying* (7) *(restricted to a sufficiently small neighborhood of all the resolved cusps of $Y(D)$) equals the intersection of F_N with this neighborhood.*

The equation (3) defines a curve in $\mathfrak{H}^2$ which is the graph of the fractional linear transformation

$$z_1 \to z_2 = \frac{\lambda' z_1 - a_2\sqrt{D}}{a_1\sqrt{D} z_1 + \lambda} \tag{8}$$

from $\mathfrak{H}$ to $\mathfrak{H}$. Thus the curve (3) can be identified in a specific way with $\mathfrak{H}$. Then the irreducible component of F_N (given by (3)) has $\overline{\mathfrak{H}/\Gamma}$ as its non-singular model where Γ is the subgroup of the Hilbert modular group G consisting of all elements of G which map the curve (3) to itself. The non-singular compact curve $\overline{\mathfrak{H}/\Gamma}$ is obtained from $\mathfrak{H}/\Gamma$ by "adding" finitely many cusps. Their number will be denoted by $\sigma(\Gamma)$. The non-singular model of F_N is a disjoint union of finitely many curves $\overline{\mathfrak{H}/\Gamma_j}$. The sum of the $\sigma(\Gamma_j)$ is by definition the number $\sigma(F_N)$ of cusps of F_N. For given

$\mathfrak{B}\in C^+$ ($\mathfrak{B}^{-1}$ representing a cycle (5)) the triples $(k|p, q)$, where $(k|0, 1)$ is to be identified with $(k+1|1, 0)$, are in one-to-one correspondence with the (integral) ideals $\mathfrak{b}\in\mathfrak{B}$ (see [6] 4. 1). We have $\mathfrak{b}\mathfrak{b}'=(N)$ with N as in (7) and $(p, q)=n(\mathfrak{b})$ where $n(\mathfrak{b})$ is the greatest natural number such that $\mathfrak{b}/n(\mathfrak{b})$ is an integral ideal. The set of all ideals $\mathfrak{b}\subset\mathfrak{o}$ which belong to an ideal class $\mathfrak{B}\in Sq(C^+)$ is the principal genus $\mathfrak{P}$. By the lemma we have

$$\sigma(F_N) = 2^{t-1} \sum_{\substack{\mathfrak{b}\in\mathfrak{P} \\ \mathfrak{b}\mathfrak{b}'=(N)}} \varphi(n(\mathfrak{b})). \tag{9}$$

The curve F_N has a cusp ($\sigma(F_N)\geqq 1$) if and only if N is a norm in K (see (4)). Thus (9) is in agreement with the well-known fact that a natural number is a norm in K if and only if it is the norm of an ideal in the principal genus. If N is a norm in K, then the sum in (9) can be taken over all integral ideals $\mathfrak{b}$ with $\mathfrak{b}\mathfrak{b}'=(N)$. They are automatically in the principal genus. In some cases (9) gives information on the number of components of F_N.

First we need a definition. N is called *admissible* if it is the norm of an ideal $\mathfrak{b}$ in the principal genus which is primitive, i.e. $n(\mathfrak{b})=1$. This happens if and only if N is a norm in K and every prime factor of N decomposes or ramifies in $\mathfrak{o}$, the ramifying prime factors having exponent 1 in N.

Proposition. *If N is admissible and not divisible by the square free part d of D, then F_N has 2^{t-1-r} components where r is the number of primes dividing (D, N). If N is admissible and divisible by d, then F_N has 2^{t-r} (thus 1 or 2) components. The group G_m/G operates transitively on the set of components.*

We indicate the proof. If $(p, q)=1$ then (6) can be represented by the "diagonal" in $\mathfrak{H}^2/SL_2(\mathfrak{o}, \mathfrak{b})$ where $\mathfrak{b}$ is the primitive ideal with norm N corresponding to $(k|p, q)$. Compare [6] 4. 1. Therefore, in this case, the non-singular model of the component of F_N represented by (6) is $\overline{\mathfrak{H}/\Gamma}$ where $\Gamma=\Gamma_0(N)/\{1,-1\}$ or where Γ is a certain extension of index 2 of $\Gamma_0(N)/\{1,-1\}$. The latter case happens if and only if N is divisible by d. As is well-known, the cusps of $\overline{\mathfrak{H}/\Gamma}$ can be represented by rational numbers a/c with $(a, c)=1$, $c>0$ and $c|N$. For any divisor c of N we have $\varphi((c, N/c))$ cusps. If $d|N$ and Γ is an extension of index 2 of $\Gamma_0(N)/\{1,-1\}$, then a cusp with denominator c is identified with a cusp of denominator $cd/(c, d)^2$. The given equation (6) from which we started is a description of the embedding of $\overline{\mathfrak{H}/\Gamma}$ in $Y(D)$ near the cusp of $\overline{\mathfrak{H}/\Gamma}$ given by $c=N$. For a given divisor c of N it can be shown that $\overline{\mathfrak{H}/\Gamma}$ near a cusp with denominator c is imbedded in $Y(D)$ by an equation (6) where $(k|p, q)$ corresponds to the ideal $\tilde{\mathfrak{b}}=(\mathfrak{b}\cdot c)/(\mathfrak{b}, c)^2$ which has norm N and for which $(p, q)=n(\tilde{\mathfrak{b}})=(c, N/c)$. All ideals with norm N are obtained in this way. As can be checked, we get for given c for the various cusps with denominator c all the $\varphi((p, q))$ primitive roots of unity in (6). We conclude that all components of F_N are equivalent under G_m/G. The number of cusps of $\overline{\mathfrak{H}/\Gamma_0(N)}$ equals

$$\sigma(\Gamma_0(N)) = \sum_{c|N}\varphi((c, N/c)).$$

Formula (9) now implies the proposition if N is not divisible by d. If N is divisible by d, then all components have $\frac{1}{2}\sigma(\Gamma_0(N))$ cusps. Again (9) implies the result.

3. 4 Suppose we have two different skew-hermitian curves in $\mathfrak{H}^2$, one given by (3) with determinant N and the second one by

$$b_1\sqrt{D}z_1z_2-\mu' z_1+\mu z_2+b_2\sqrt{D}=0$$

with determinant M. They intersect in $\mathfrak{H}^2$ if and only if the matrix

$$B=\begin{bmatrix}\mu & b_2\sqrt{D}\\ -b_1\sqrt{D} & \mu'\end{bmatrix}\begin{bmatrix}\lambda' & -a_2\sqrt{D}\\ a_1\sqrt{D} & \lambda\end{bmatrix}$$

has a fixed point in $\mathfrak{H}$ (compare (8)) which happens if and only if

$$4NM-\operatorname{tr}(B)^2>0.$$

It is easy to check that $\operatorname{tr}(B)^2-4NM$ is divisible by D and its quotient by D is a discriminant (i.e. $\equiv 0$ or 1 mod 4). *Therefore, if the two curves intersect in $\mathfrak{H}^2$, then the following condition holds.*

(10) There exists $x\in\boldsymbol{Z}$ such that $|x|<\sqrt{4NM}$ and $4NM-x^2\equiv 0 \bmod D$ with $(x^2-4NM)/D\equiv 0$ or 1 mod 4.

If (10) *is not satisfied for $N\neq M$, then F_N and F_M do not intersect in $\mathfrak{H}^2/G$.*

Lemma. *If* (10) *is not satisfied for $M=N$, then two different components of F_N do not intersect in $\mathfrak{H}^2/G$ and moreover F_N is non-singular in $Y(D)$ outside the resolved cusps.*

Proof. Assume that (10) is not satisfied for $M=N$. If a component of F_N is given by (8) with $\mathfrak{H}/\Gamma$ as its non-singular model, then the isotropy group of the Hilbert modular group G at a point x of $\mathfrak{H}^2$ satisfying (8) is contained in Γ. It also follows that there is only one skew-hermitian curve of determinant N in $\mathfrak{H}^2$ passing through x. If the isotropy group of G at x is trivial, then F_N is non-singular in the point of $Y(D)$ represented by x. If the isotropy group is of order r, then it is of type $(r;1,1)$. This follows from (8). (For $D>12$ we have $r=2$ or 3; see [15].) The curve F_N passes in $Y(D)$ transversally through the curve of self-intersection number $-r$ which gives the resolution of the quotient singularity. (Condition (10) and the lemma were suggested to us by P. Hahnel and H.-P. Kraft.)

The necessary and sufficient condition that F_N be non-singular in the neighborhood of a resolved cusp given by a cycle (5) is that for all p, q satisfying (7) one of the exponents $p/(p,q)$ or $q/(p,q)$ in (6) be equal to 1. Thus:

If (10) *is not satisfied for $M=N$ and if in the lemma in* 3. 3 *all pairs p, q are such that $p|q$ or $q|p$, then F_N is non-singular in $Y(D)$.*

In particular F_1 is non-singular in $Y(D)$ and has 2^{t-1} components.

3. 5 If N is a prime, then the curve F_N is non-empty if and only if N is a norm in K, and N is a norm in K if and only if the t characters χ_i $(i=1,\cdots,t)$ do not take a value -1 at N. Here we define the χ_i as follows. We write D as product of prime discriminants

$$D = \prod_{i=1}^{t} D_i,$$

for example $60=(-3)\cdot(-4)\cdot 5$. Then

$$\chi_i(N) = \left(\frac{D_i}{N}\right) \qquad \text{for } N \text{ odd}$$

and

$$\chi_i(2) = \left(\frac{D_i}{2}\right) = \begin{cases} 0 & \text{for } D_i \equiv 0 \quad (4) \\ 1 & \text{for } D_i \equiv 1 \quad (8) \\ -1 & \text{for } D_i \equiv 5 \quad (8). \end{cases}$$

Using the proposition in 3. 3, we get

Proposition. *If N is a prime and $D \neq N$, $4N$, then the number of components of F_N equals*

$$\frac{1}{2}\prod_{i=1}^{t}(1+\chi_i(N)). \tag{11}$$

3. 6 Let N be a prime. We wish to study the curve F_{N^2} in $Y(D)$. If N decomposes in $\mathfrak{o}$, i.e. $(D/N)=1$, then N^2 is admissible and we have the proposition in 3. 3; the curve F_{N^2} has 2^{t-1} components. By (9), F_{N^2} has $2^{t-1}(N-1)$ cusps if $(D/N)\neq 1$ and $2^{t-1}(N+1)$ cusps if $(D/N)=1$.

If $\mathfrak{o}=\boldsymbol{Z}w_0+\boldsymbol{Z}$ where w_0 is reduced, then one of the local coordinate systems for the cusp at ∞ is given by

$$\begin{aligned} 2\pi i z_1 &= w_0 \log u_0 + \log v_0 \\ 2\pi i z_2 &= w_0' \log u_0 + \log v_0. \end{aligned} \tag{12}$$

There are $N-1$ cusps of F_{N^2} corresponding to

$$u_0 = \zeta \quad \text{where} \quad \zeta^N = 1, \quad \zeta \neq 1 \tag{13}$$

(compare (6), (7); we have $N_0=1, p=0, q=N$, and the N in (7) has to be replaced here by N^2) or to skew-hermitian forms

$$Nz_1 - Nz_2 = r(w_0-w_0') \text{ where } (r, N) = 1, \ (w_0-w_0' = \sqrt{D}). \tag{14}$$

The curve S_0 (given by $v_0=0$) intersects the $N-1$ branches (13) of F_{N^2} transversally.

The component of F_{N^2} given by (14) has $\overline{\mathfrak{H}/\Gamma}$ as model where the subgroup Γ of G carrying (14) to itself has to be determined. The result is independent of r. We list it and give also the number of cusps $\sigma(\Gamma)$ which is well-known for the groups in question:

$$\text{If}\left(\frac{D}{N}\right) = -1, \text{ then } \Gamma = \Gamma'(N)/\{1, -1\} \text{ and } \sigma(\Gamma) = N-1. \tag{15}$$

Here $\Gamma'(N)$ is defined as follows. Consider the multiplicative group of the field $\boldsymbol{F}_{N^2}$

as subgroup of $GL_2(F_N)$, take the intersection with $SL_2(F_N)$. Its inverse image in $SL_2(Z)$ is $\Gamma'(N)$.

$$
(16)\quad
\begin{array}{ll}
\text{If } N|D,\ N \neq 2 \text{ and } D \neq N, 4N, \text{ then } \Gamma = \Gamma_1(N)/\{1,-1\} & \\
 & \text{and } \sigma(\Gamma) = N-1. \\
\text{If } D \text{ even, } N=2,\ D \neq 8, \text{ then } \Gamma = \Gamma_1(2)/\{1,-1\} = \Gamma_0(2)/\{1,-1\} & \\
 & \text{and } \sigma(\Gamma) = 2. \\
\text{If } D = N \text{ or } D = 4N\ (N \neq 2), \text{ then } \Gamma = \Gamma_1^*(N)/\{1,-1\} & \\
 & \text{and } \sigma(\Gamma) = \dfrac{N-1}{2}. \\
\text{If } D = 8,\ N = 2, \text{ then } \Gamma = \Gamma_1^*(2)/\{1,-1\} & \text{and } \sigma(\Gamma) = 1.
\end{array}
$$

The group $\Gamma_1(N)$ consists of those matrices in $SL_2(Z)$ which are of the form $\pm\begin{bmatrix}1 & *\\ 0 & 1\end{bmatrix}$ modulo N and $\Gamma_1^*(N)$ is an extension of index 2 of $\Gamma_1(N)$. The proof of (15) and (16) is carried out by applying the method of [6] p. 270 to equation (14). To bring Γ into the above form one must conjugate in $GL_2(K)$. Using (9), (15), (16) (and the Proposition in 3. 3 for the case $(D/N)=1$) we get

Proposition. *If N is an odd prime, then the curve F_{N^2} has 2^{t-1} components, except in the case $D=N$ or $D=4N$ where it has 2 or 4 components respectively. If $N=2$, then F_4 has 2^{t-1} components if D is odd or if $D=8$. If D is even $(D\neq 8)$ then F_4 has 2^{t-2} components.*

Remark. For $N \nmid D$ the skew-hermitian curves (14) all belong to the same component of F_{N^2} and G_m/G operates transitively on the set of components. If $N|D$ $(N\neq 2)$, then two skew-hermitian curves (14) belong to the same component if and only if the two values of (r/N) are both equal to $+1$ or both equal to -1. In [7] § 3 it was stated that the curve $F_N((N/p)\neq -1)$ on $Y(p)$ (p prime) is irreducible. This has to be corrected as pointed out by Hammond. It will be shown in a forthcoming dissertation by Hans-Georg Franke (Bonn) that F_N is irreducible if $N\not\equiv 0\ (p^2)$. If $N\equiv 0\ (p^2)$ then F_N has exactly two components.

3. 7 An exceptional curve on an algebraic surface is a non-singular rational curve of self-intersection number -1. If the surface is regular and not rational, then any two exceptional curves are disjoint and can be blown down simultaneously. *In this section we assume that $Y(D)$ is not rational.* Thus we exclude 10 discriminants (Chap. I (1)). How many exceptional curves can be found on $Y(D)$ using skew-hermitian curves ?

For a discrete subgroup Γ of $PL_2^+(R)$ with $\mathfrak{H}/\Gamma$ of finite volume the number

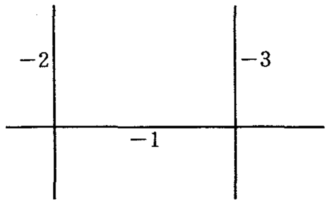
$$
(17)\qquad c_1(\Gamma) = 2e(\overline{\mathfrak{H}/\Gamma}) - \sum_{r\geq 2} a_r(\Gamma) - \sigma(\Gamma)
$$

was introduced in [6] 4. 3. We recall that e denotes the Euler number, $a_r(\Gamma)$ the number of Γ-equivalence classes of fixed points of order r of Γ and $\sigma(\Gamma)$ the number of cusps. If a component E of a skew-hermitian curve in $Y(D)$ has the non-singular model $\overline{\mathfrak{H}/\Gamma}$, then

$$c_1 \cdot E \geqq c_1(\Gamma)$$

where $c_1 \cdot E$ denotes the value of the first Chern class of $Y(D)$ on E. Since $Y(D)$ is not rational, $c_1 \cdot E \geqq 1$ implies that $c_1 \cdot E = 1$ and E is an exceptional curve (see [6] 4. 4 Corollary I). The curve F_1 has 2^{t-1} components (see the Proposition in 3. 3). Each component passes through a quotient singularity of order 2 and one of order 3 on $\mathfrak{H}^2/G$ and is on $Y(D)$ an exceptional curve which gives rise to a configuration

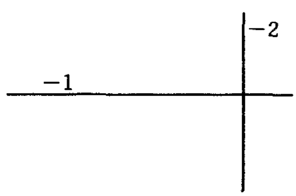
(18)



of non singular rational curves. These configurations are disjoint to each other. *Using* F_1 *we have found on* $Y(D)$ *in this way* $3 \cdot 2^{t-1}$ *curves which can be blown down.*

The groups Γ which occur for the components of F_2, F_3 are $\Gamma_0(2)$ and $\Gamma_0(3)$ respectively (to be divided by $\{1, -1\}$). For the components of F_4 we have $\Gamma = \Gamma_0(4)$ or $\Gamma = \Gamma'(2)$ if D is odd (to be divided by $\{1, -1\}$). (These groups were treated in [6] 5. 5 if D is a prime.) If D is even, the group for F_4 is $\Gamma_0(2)/\{1, -1\}$. If $3|D$ $(D \neq 12)$, then the components of F_9 have the group $\Gamma_1(3) = \Gamma_0(3)$ (to be divided by $\{1, -1\}$). For these groups Γ (namely $\Gamma_0(2)$, $\Gamma_0(3)$, $\Gamma'(2)$, $\Gamma_0(4)$, always divided by $\{1, -1\}$) the value of $c_1(\Gamma)$ equals 1. Since $Y(D)$ is supposed to be not rational, *all components of* F_2, F_3, F_4 *and* (if $3|D$) F_9 *give exceptional curves.* Each component of F_2 passes through a quotient singularity of order 2 on $\mathfrak{H}^2/G$ and gives on $Y(D)$ a configuration.

(19)

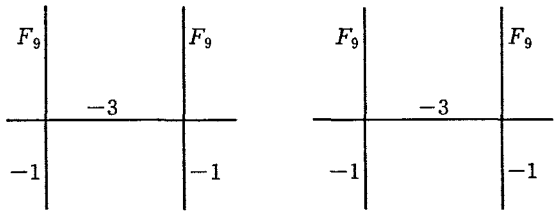


Every component of F_2 gives two curves which can be blown down. The curve F_4 has 2^{t-1} or 2^{t-2} components. In the latter case we have a configuration (19) for each component because the group is $\Gamma_0(2)$ (see (16)). Therefore F_4 gives always 2^{t-1} curves to blow down, F_1 *and* F_4 *together give* 2^{t+1} *curves to blow down.* For F_3 and F_9 the corresponding group has no fixed point of order 2. There is no configuration (19). No additional blow-downs occur in this way. For $D = 105$ a special situation occurs. We have

$$\frac{4 \cdot 9 \cdot 9 - 3^2}{105} = 3.$$

Thus condition (10) is satisfied. In fact, it can be checked that the 4 components of F_9 meet in quotient singularities of order 3 on $\mathfrak{H}^2/G$ and this leads on $Y(105)$ to a configuration like this

(20)



$D=105$

which gives two extra curves to blow down. In fact, for $D=105$ there are six quotient singularities of order 3 on $\mathfrak{H}^2/G$, all of type $(3\,;\,1,\,1)$. Four of them lie on the 4 components of F_1. The two others give rise to the two curves of self-intersection number -3 in (20). Two intersecting components of F_9 never occur for other D (with $3|D$) as can be checked by condition (10).

By the propositions in 3. 5 and 3. 6 we know the number of components of F_2, F_3, F_4, F_9, hence we can eollect the information on exceptional curves in the following theorem.

Theorem. *Suppose $Y(D)$ is not rational. Then $\beta(D)$ curves on $Y(D)$ can be blown down where*

$$\beta(D) = 2^{t+1} + \prod_{i=1}^{t}(1+\chi_i(2)) + \frac{1}{2}\prod_{i=1}^{t}(1+\chi_i(3)) + 2^{t-1}\left(1-\left(\frac{D}{3}\right)^2\right) + \begin{cases} 2 & \text{for } D = 105 \\ 0 & \text{for } D \neq 105. \end{cases} \tag{21}$$

We call $Y^0(D)$ the surface obtained from $Y(D)$ by blowing down these $\beta(D)$ curves. We define $Y^0(D)$ only if $Y(D)$ is not rational. Clearly

$$c_1^2(Y^0(D)) = c_1^2(Y(D)) + \beta(D). \tag{22}$$

We conjecture that $Y^0(D)$ is the minimal model. For D equal to a prime, this was conjectured in [9]. In fact, van der Geer and van de Ven have checked the conjecture for several prime values of D where $Y^0(D)$ is of general type. When $Y^0(D)$ is not of general type, then the conjecture holds because $c_1^2(Y^0(D))=0$ as we shall see.

3. 8 For the surface $Y_-(D)$ introduced in Chapter I similar considerations hold. We have a curve F_N given by all primitive equations (3) with $a_1a_2D+\lambda\lambda'= -N$. This curve passes through a cusp if and only if $-N$ is a norm in K. If -1 is a norm in K, then $Y(D)$ and $Y_-(D)$ are isomorphic. In this case (provided $Y(D)$

is not rational) we can blow down $\beta(D)$ curves on $Y_-(D)$; the resulting surface $Y_-^0(D)$ is then isomorphic to $Y^0(D)$. If condition (10) is not satisfied for N, M $(N\neq M)$, then F_N and F_M do not meet on $(\mathfrak{H}\times\mathfrak{H}^-)/G$. If condition (10) is not satisfied for $M=N$, then F_N is non-singular on $Y_-(D)$ outside the resolved cusps. The lemma in 3. 3 holds in the same way except that one has to take the triples $(k|p,q)$ belonging to an element of $(\sqrt{D})Sq(C^+)$ where $(\sqrt{D})$ denotes here the element of C^+ represented by the ideal $(\sqrt{D})$. The natural number N is called *admissible* for $Y_-(D)$ if $-N$ is a norm in K and all primes dividing N decompose or ramify in $\mathfrak{o}$, but where a prime which ramifies occurs in N only with exponent 1. The number of components of F_N(N admissible for $Y_-(D)$) is as in the proposition in 3. 3. If -1 is not a norm in K, then F_1, F_4, F_9 are empty on $Y_-(D)$, so we can only blow down F_2 and F_3, and this gives (*if* $Y_-(D)$ *is not rational*)

$$\beta_-(D)=\prod_{i=1}^{t}(1+\chi_i(-2))+\frac{1}{2}\prod_{i=1}^{t}(1+\chi_i(-3)) \tag{23}$$

blow-downs. (Note that $\chi_i(-N)=(\text{sign } D_i)\ \chi_i(N)$.) Again we conjecture that the surface $Y_-^0(D)$ obtained by these blow-downs is minimal.

IV. Estimates of the Numerical Invariants

4. 1 The purpose of this chapter is to prove the facts

$$\begin{aligned}
&\chi(Y(D))=1 \Leftrightarrow D=5, 8, 12, 13, 17, 21, 24, 28, 33, 60,\\
&\chi(Y_-(D))=1 \Leftrightarrow D=5, 8, 12, 13, 17,\\
&c_1^2(Y(D))\leq 0 \Rightarrow D\leq 285,\\
&c_1^2(Y_-(D))\leq 0 \Rightarrow D\leq 136,\\
&c_1^2(Y_T(p))\leq 0 \Rightarrow p\leq 821 \qquad (p\equiv 1 (\text{mod } 4) \text{ prime})
\end{aligned}$$

(compare Chapter I), thus reducing the problem of classifying all Hilbert modular surfaces to the consideration of a finite list. Since all of the invariants have been calculated (by computer) up to at least $D=1500$, it will suffice to prove

(1) $D>1500\Rightarrow\chi(Y(D))>1,\ \chi(Y_-(D))>1,\ c_1^2(Y(D))>0,\ c_1^2(Y_-(D))>0,$

(2) $p>1500\Rightarrow c_1^2(Y_T(p))>0.$

There are precisely 50 discriminants for which the four inequalities of (1) are not all satisfied ; complete numerical data on these discriminants is given in section 4. 5.

4. 2 As explained in 2. 1, the dominant term in the formula for all of these numerical invariants is

$$\zeta_K(-1)=\frac{1}{60}\sum_{\substack{k^2<D\\ k^2\equiv D(\text{mod }4)}}\sigma_1\left(\frac{D-k^2}{4}\right). \tag{3}$$

From $\sigma_1(n)\geq n+1$ we deduce easily that

$$\zeta_K(-1)>D^{3/2}/360\ ; \tag{4}$$

this result can also be obtained by writing

$$\zeta_K(2)=\zeta(2)\prod\left(1-\left(\frac{D}{p}\right)p^{-2}\right)^{-1}>\zeta(2)\prod(1+p^{-2})^{-1}=\zeta(4)=\frac{\pi^4}{90}$$

and applying the functional equation of $\zeta_K(s)$.

From 2. 4 (31) and 2. 1 (9), (10), (12) we have

$$c_1^2(Y(D))=4\zeta_K(-1)-l_0^-(D)-\frac{1}{3}a_3^+(G)$$

and

$$\chi(Y(D))=\frac{1}{2}\zeta_K(-1)+\frac{1}{8}a_2(G)+\frac{1}{9}a_3^+(G)+\frac{2}{9}a_3^-(G)+w\,;$$

moreover, by 2. 3 (29),

$$w=\frac{1}{12}(l_0(D)-l_0^-(D))>-\frac{1}{12}l_0^-(D),$$

and hence

$$\chi(Y(D))>\frac{1}{2}\zeta_K(-1)+w>\frac{1}{6}\zeta_K(-1)+\frac{1}{12}(4\zeta_K(-1)-l_0^-(D))$$
$$>\frac{1}{2160}D^{3/2}+\frac{1}{12}c_1^2(Y(D))>\frac{c_1^2(Y(D))}{12}+1 \qquad (D>200).$$

Hence the inequality $\chi(\hat{Y}(D))>1$ in (1) will follow once we have proved that $c_1^2(Y(D))$ is positive; since (by 2. 2 (15) and 2. 4 (33)) the values of χ and c_1^2 for $Y_-(D)$ are at least as large as for $Y(D)$, the remaining two inequalities in (1) will also follow. Thus to prove (1) we have to show

(5) $$4\zeta_K(-1)-l_0^-(D)-a_3^+(G)/3>0 \qquad (D>1500),$$

while for (2) the inequality

(6) $$2\zeta_K(-1)-\frac{1}{2}l_0(p)-\frac{5p}{24}+\frac{\sqrt{p}}{2}>0 \qquad (p>1500)$$

will certainly suffice (equation 2. 5 (40)).

We introduce a new invariant

(7) $$l(D)=\sum_{\mathfrak{a}\in C^+}l(\mathfrak{a})$$

(notation as in 2. 3). If $D=p$ is a prime, then $Sq: C^+\to C^+$ is an isomorphism and

(8) $$l_0(p)=l_0^-(p)=l(p)\,;$$

if, however, D has t distinct prime factors, then Sq has a kernel of order 2^{t-1} and so

(9) $$l_0^-(D)=\sum_{\mathfrak{a}\in C^+}l(\gamma Sq(\mathfrak{a}))=2^{t-1}\sum_{\substack{\mathfrak{b}\in C^+\\ \gamma\mathfrak{b}\in \mathrm{Im}(Sq)}}l(\mathfrak{b})\le 2^{t-1}l(D)$$

The advantage of working with $l(D)$ rather than $l_0^-(D)$ is that it can be evaluated by a formula analogous to formula (3) for $\zeta_K(-1)$. Indeed, $l(D)$ is the sum of the lengths of all cycles occurring as the primitive period of the continued fraction of some quadratic irrationality w of discriminant D (the discriminant of w is defined as b^2-4ac, where $aw^2+bw+c=0$, $(a, b, c)=1$). This is simply the number of *reduced* quadratic irrationalities w of discriminant D (i.e. w satisfying $w>1>w'>0$), since, as discussed in 2.3, such w have purely periodic continued fractions, and a cycle $((b_0,\cdots,b_{r-1}))$ of length r gives rise to precisely r reduced numbers

$$b_i-\cfrac{1}{b_{i+1}-\ddots} \qquad (i=0, 1,\cdots, r-1).$$

If $aw^2+bw+c=0$, $b^2-4ac=D$, then the condition $(a, b, c)=1$ is automatically satisfied since D is the discriminant of a quadratic field. Therefore

$$l(D)=\#\left\{(a, b, c)\in \boldsymbol{Z}^3 \mid b^2-4ac=D,\ \frac{-b+\sqrt{D}}{2a}>1>\frac{-b-\sqrt{D}}{2a}>0\right\}.$$

The inequalities are equivalent to

$$a>0, \qquad |-b-2a|<\sqrt{D}, \qquad -b>\sqrt{D}\,;$$

therefore replacing b by $k=-b-2a$ gives

$$l(D)=\#\{(a, k)\in \boldsymbol{Z}^2 \mid a>0,\ k^2<D,\ k^2\equiv D \ (\mathrm{mod}\ 4a),\ k+2a>\sqrt{D}\}.$$

We claim that this is precisely half of

$$\#\{(a, k)\in \boldsymbol{Z}^2 \mid a>0,\ k^2<D,\ k^2\equiv D \ (\mathrm{mod}\ 4a)\}.$$

Indeed, $(a, k)\mapsto (a', k')=((D-k^2)/4a,\ -k)$ is an involution on this latter set with

$$\frac{2a'+k'-\sqrt{D}}{2a+k-\sqrt{D}}=-\frac{k+\sqrt{D}}{2a}<0,$$

so precisely half of the elements (a, k) satisfy $2a+k>\sqrt{D}$. Therefore

$$l(D)=\frac{1}{2}\sum_{\substack{k^2<D\\ k^2\equiv D (\mathrm{mod}\ 4)}}\ \sum_{\substack{a>0\\ a\mid \frac{D-k^2}{4}}} 1=\frac{1}{2}\sum_{\substack{k^2<D\\ k^2\equiv D (\mathrm{mod}\ 4)}}\sigma_0\left(\frac{D-k^2}{4}\right). \tag{10}$$

This formula will be the basis for our estimates of c_1^2.

4.3 In this section we prove the estimate (6); this case is easier than estimate (5) for composite D because of (8). We will prove (for all D, prime or composite) that

$$2\zeta_K(-1)-\frac{1}{2}l(D)\geq \frac{D+15}{180}\sqrt{D-200}-3.6 \qquad (D\geq 730)\,; \tag{11}$$

since the right-hand side is $>\frac{5D}{24}-\frac{\sqrt{D}}{2}$ for $D>1500$, this implies (6).

By (3) and (10), the left-hand side of (11) equals

$$\sum_{\substack{k^2<D\\ k^2\equiv D (\mathrm{mod}\ 4)}}\phi\left(\frac{D-k^2}{4}\right) \tag{12}$$

with

$$\phi(n)=\frac{1}{30}\sigma_1(n)-\frac{1}{4}\sigma_0(n).$$

We have

$$(13)\qquad \begin{aligned} \psi(n) &\geq -0.6 && \text{for}\quad n<50,\\ \psi(n) &\geq \frac{n-14}{30} && \text{for}\quad n\geq 50. \end{aligned}$$

Indeed, for $n\leq 56$ we can check this by hand, while, for $n>56\frac{1}{4}=\left(7\frac{1}{2}\right)^2$,

$$\begin{aligned} \psi(n) &= \frac{1}{30}\sum_{d|n}\left(d-7\frac{1}{2}\right)\\ &= \frac{1}{30}\left[\left(n-7\frac{1}{2}\right)+\left(1-7\frac{1}{2}\right)+\frac{1}{2}\sum_{\substack{d|n\\ 1<d<n}}\left(d+\frac{n}{d}-15\right)\right]\\ &\geq \frac{n-14}{30} \end{aligned}$$

(each term $d+n/d-15$ is ≥ 0). For $D>729$ there are at most 4 values of k (two positive and two negative) for which $k\equiv D \pmod 2$ and $0<(D-k^2)/4<50$ (because the interval $(\sqrt{D-200}, \sqrt{D})$ has length<4), so the first line of (13) is used at most four times in (12) ; the second estimate in (13) now gives

$$\begin{aligned} \sum_{\substack{k^2<D\\ k^2\equiv D(\mathrm{mod}\,4)}} \psi\left(\frac{D-k^2}{4}\right) &\geq -2.4+\sum_{\substack{k^2<D-200\\ k^2\equiv D(\mathrm{mod}\,4)}}\left(\frac{D-k^2}{120}-\frac{14}{30}\right)\\ &\geq -3.6+\sqrt{D-200}\left(\frac{D-201}{180}+\frac{200}{120}-\frac{14}{30}\right), \end{aligned}$$

where we have used the easy estimates (valid for any positive A and integer D)

$$\sum_{\substack{k^2<A\\ k\equiv D(\mathrm{mod}\,2)}} 1 \geq \sqrt{A}-1,\quad \sum_{\substack{k^2<A\\ k\equiv D(\mathrm{mod}\,2)}} (A-k^2) \geq \frac{2\sqrt{A}}{3}(A-1)$$

with $A=D-200$. This proves the inequality (11).

4. 4 We now want to prove the estimate (5). The number $l_0^-(D)$ in that equation will be estimated using (9) and (10) ; for the number $a_3^+(G)$, given exactly by 2. 1 (7), we use the estimate

$$h(-N) < \frac{\sqrt{N}}{2\pi}(2+\log N) \qquad (N>4)$$

(cf. [14]) to obtain

$$\frac{1}{3}a_3^+(G) < 0.13\, D^{1/2}(\log D+1).$$

The formula to be proved then becomes

$$(14)\quad 4\zeta_K(-1)-0.13\sqrt{D}(\log D+1) > 2^{t-2}\sum_{\substack{k^2<D\\ k^2\equiv D(\mathrm{mod}\,4)}} \sigma_0\left(\frac{D-k^2}{4}\right) \qquad (D>1500).$$

Because of the factor 2^{t-2}, the method of 4. 3 does not work here and we must have recourse to far cruder estimates. We would like to thank Henri Cohen, who suggested the method for estimating the right-hand side of (14) and carried out the necessary computer calculations.

Lemma. *Set* $\varepsilon=\log 2/\log 11=0.289064826\cdots$. *Then for all* n

$$\sigma_0(n) < 5.1039782\, n^{\varepsilon}. \tag{15}$$

Proof. The function $\sigma_0(n)/n^{\varepsilon}$ is multiplicative and $(a+1)/p^{a\varepsilon}\leq 1$ for $p\geq 11$, $a\geq 1$ by the choice of ε. Hence

$$\frac{\sigma_0(n)}{n^{\varepsilon}} \leq \prod_{p} \max_{a\geq 0}\left(\frac{a+1}{p^{a\varepsilon}}\right) = \frac{5}{2^{4\varepsilon}}\cdot\frac{3}{3^{2\varepsilon}}\cdot\frac{2}{5^{\varepsilon}}\cdot\frac{2}{7^{\varepsilon}} = 5.103978196\cdots$$

If we now estimate $\zeta_K(-1)$ by (4), and the right-hand side of (14) by the product of the number of terms in the sum with the estimate of the individual terms given by equation (15), we find as a sufficient condition for (14) the inequality

$$\frac{D^{3/2}}{90} - 0.13\, D^{1/2}(\log D+1) > 2^{t-2}(\sqrt{D}+1)\cdot 5.1039782\cdot (D/4)^{\varepsilon} \tag{16}$$

with $\varepsilon=0.289064826\cdots$ as before. A desk calculator computation now shows that (16) holds if

$$\begin{array}{lllll} & t \leq 3 & \textit{and} & D > & 9{,}000 \\ \textit{or} & t = 4 & \textit{and} & D > & 23{,}000 \\ \textit{or} & t = 5 & \textit{and} & D > & 60{,}000 \\ \textit{or} & t = 6 & \textit{and} & D > & 157{,}000 \\ \textit{or} & t = 7 & \textit{and} & D > & 420{,}000. \end{array} \tag{17}$$

But the smallest discriminant with $t=7$ is $4.\ 3.\ 5.\ 7.\ 11.\ 13.\ 17=1{,}021{,}020>420{,}000$, so (17) implies that (16) holds for all D with $t=7$. A similar argument holds for any $t\geq 7$, since a D with $t\geq 7$ distinct prime factors is greater than

$$4.\ 3.\ 5.\ 7.\ 11.\ 13.\ 16^{t-6} = \frac{60060}{65536}(2^{t-2})^4,$$

so $2^{t-2}<\sqrt[4]{1.1D}$, more than sufficient to prove (16) for $D>420{,}000$. Therefore (17) implies that (16) (and hence (14)) holds for all $D>157{,}000$, and a computer calculation showed that (14) holds for all D up to this point.

4. 5 As already stated, the calculation of the various invariants for $D<1500$ showed that $c_1^2(Y(D))\leq 0$ for just 50 discriminants, the largest being $D=285$. We have tabulated all numerical invariants of $Y(D)$ and $Y_-(D)$ for these discriminants. The following notation is used :

Topological Invariants :

$Z = 6\zeta_K(-1)$ (this is an integer for $D > 8$)

$l_0, l_0^- = l_0(D), l_0^-(D)$ (§§ 2. 3, 2. 4)

$a_2, a_3^+, a_3^- = a_2(G), a_3^+(G), a_3^-(G)$ (§ 2. 1 ; for $D = 5, 8$ and 12 there are also fixed points of order 5, 4 and 6 respectively)

$e = e(\mathfrak{H}^2/G) = e(\mathfrak{H}\times\mathfrak{H}^-/G)$ (2. 1 (9), 2. 2 (13))

$\tau = -\text{sign}\,(\mathfrak{H}^2/G) = \text{sign}\,(\mathfrak{H}\times\mathfrak{H}^-/G)$ (2. 1 (10), 2. 2 (13))

Invariants of $Y(D)$:

$\chi = \frac{1}{4}(e-\tau) = \chi(Y(D))$ (2. 1 (12))

$c = c_1^2(Y(D))$ (2. 4 (31))

$c^0 = c_1^2(Y^0(D)) = c+\beta(D)$ (3. 7 (22) ; c^0 is not listed if $Y(D)$ is rational, since $Y^0(D)$ was not defined in this case)

Invariants of $Y_-(D)$ (not given if D is a sum of two squares since then $Y_-(D)$ is isomorphic to $Y(D)$) :

$\chi_- = \frac{1}{4}(e+\tau) = \chi(Y_-(D))$ (2. 2 (14))

$c_- = c_1^2(Y_-(D))$ (2. 4 (32))

$c_-^0 = c_1^2(Y_-^0(D)) = c_-+\beta_-(D)$ (§ 3. 8).

D	Z	l_0	l_0^-	a_2	a_3^+	a_3^-	e	τ	χ	c	c^0	χ_-	c_-	c_-^0
5	1/5	1	1	2	1	1	4	0	1	−2	•			
8	1/2	2	2	2	1	1	4	0	1	−3	•			
12	1	2	4	3	2	0	4	0	1	−4	•	1	−4	•
13	1	3	3	2	2	2	4	0	1	−3	•			
17	2	5	5	4	1	1	4	0	1	−4	•			
21	2	2	6	4	4	1	6	2	1	−6	•	2	−1	0
24	3	4	8	6	3	0	6	2	1	−7	•	2	−2	0
28	4	4	10	4	2	2	6	2	1	−8	•	2	−2	0
29	3	5	5	6	3	3	8	0	2	−4	0			
33	6	8	12	4	3	0	6	2	1	−9	•	2	−4	0
37	5	7	7	2	4	4	8	0	2	−5	0			
40	7	12	12	6	2	2	8	0	2	−8	0			
41	8	11	11	8	1	1	8	0	2	−6	0			
44	7	6	12	10	2	2	10	2	2	−8	0	3	−2	0
53	7	7	7	6	5	5	12	0	3	−4	0			
56	10	4	16	12	2	2	12	4	2	−10	0	4	2	2
57	14	14	18	4	4	1	10	2	2	−10	0	3	−5	0
60	12	4	24	8	6	0	12	8	1	−18	•	5	4	4
61	11	11	11	6	4	4	12	0	3	−5	0			
65	16	18	18	8	2	2	12	0	3	−8	0			
69	12	4	16	8	9	0	14	6	2	−11	0	5	4	4
73	22	21	21	4	2	2	12	0	3	−7	0			
76	19	14	20	10	2	2	14	2	3	−8	0	4	−2	2
77	12	2	14	8	6	6	16	4	3	−8	0	5	4	4
85	18	18	18	4	6	6	16	0	4	−8	0			
88	23	12	24	6	4	4	16	4	3	−10	0	5	2	4
89	26	21	21	12	1	1	16	0	4	−4	2			
92	20	4	22	12	4	4	18	6	3	−10	0	6	8	8
93	18	6	18	4	12	3	18	6	3	−10	0	6	5	6
97	34	27	27	4	2	2	16	0	4	−5	2			
101	19	11	11	14	5	5	20	0	5	0	4			
104	25	20	20	18	2	2	20	0	5	−4	4			
105	36	12	44	8	6	0	20	12	2	−22	0	8	12	12
109	27	17	17	6	6	6	20	0	5	−1	4			
113	36	23	23	8	3	3	20	0	5	0	6			
120	34	8	40	12	8	2	24	12	3	−20	0	9	14	14
124	40	16	34	12	2	2	22	6	4	−8	2	7	10	12

D	Z	l_0	l_0^-	a_2	a_3^+	a_3^-	e	τ	χ	c	c^0	χ_-	c_-	c_-^0
129	50	30	34	12	4	1	26	2	6	−2	8	7	3	8
133	34	12	24	4	8	8	24	4	5	−4	4	7	8	10
136	46	32	32	12	2	2	24	0	6	−2	8			
140	38	4	40	20	4	4	28	12	4	−16	0	10	20	20
141	36	8	28	8	15	0	26	10	4	−9	2	9	16	16
156	52	16	48	16	8	2	32	12	5	−16	4	11	18	20
161	64	14	50	16	2	2	32	12	5	−8	4	11	28	28
165	44	4	44	8	16	4	32	16	4	−20	0	12	24	24
168	54	8	48	12	12	0	32	16	4	−16	4	12	28	28
184	74	16	52	12	4	4	36	12	6	−4	8	12	32	32
204	78	28	56	20	12	0	44	12	8	−8	12	14	24	28
220	92	16	64	16	4	4	44	16	7	−4	12	15	44	44
285	96	4	60	16	24	0	56	24	8	−4	16	20	60	60

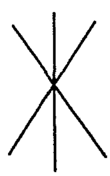
V. The Rough Classification of Hilbert Modular Surfaces

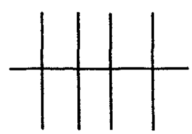
5. 1 In this chapter we prove Theorems 2 and 3 of the introduction (Chap. I). Our basic reference for the classification is the joint paper [9] with Van de Ven. In the proposition below we state the main classification principle. A (-2)-curve is a non-singular rational curve with self-intersection number -2. An elliptic configuration on an algebraic surface X is a finite set of irreducible curves on X having the same genera and intersection numbers as the configurations occurring as fibres (without exceptional curves) in an elliptic fibration of some surface ([11] Part II). We give a complete list of the elliptic configurations:

A non-singular curve E of genus 1 with $EE=0$; a rational curve E having exactly one singular point (a cusp or a double point) with $EE=0$; a configuration of (-2)-curves with one of the following intersection diagrams

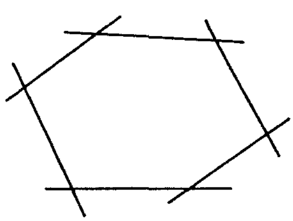
A_1'

intersection number 2



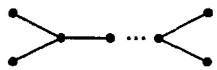


D_4'

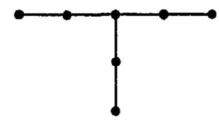


A_{k-1}' (cycle of any length $k\geq 3$)

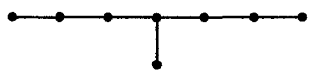
or of the diagrams, better indicated by their dual graphs (a dot indicates a (-2)-curve and a line a transversal intersection):



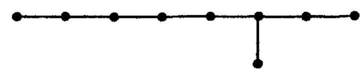
D'_{k-1} (k curves, $k-1 \geqq 5$)



E'_6



E'_7



E'_8

Proposition. *Let X be a simply-connected non-rational algebraic surface. If X contains an elliptic configuration, then X is a blown-up K3-surface or a blown-up honestly elliptic surface. If X contains an elliptic configuration which intersects a (-2)-curve on X not belonging to the configuration, then X is a blown-up K3-surface. If X is simply-connected, not rational, and not a blown-up K3-surface and E is an irreducible curve on X such that $c_1 \cdot E \geqq 0$ (where $c_1 \in H^2(X, \boldsymbol{Z})$ is the first Chern class), then either $c_1 \cdot E = 1$ and E is an exceptional curve or $c_1 \cdot E = 0$ and E is either a (-2)-curve or a curve of genus 1 or 0 with $EE = 0$.*

The proof is obtained as in [9] (compare Proposition I. 9). For the second part of the proposition we use [9] (Propositions I. 1 and I. 5) and in particular the fact that $c_1 \cdot E \geqq 2$ implies the rationality of X. When passing to the minimal model X' of X, a certain configuration L of rational curves on X is blown down. If $c_1 \cdot E \geq 0$, then either E belongs to L and is an exceptional curve or a (-2)-curve on X, or E and L are disjoint, $c_1 \cdot E = 0$, and E is a component of the unique elliptic fibration of X' or a (-2)-curve on the surface X' of general type.

5. 2 The results and the tables of the preceding chapter have shown that $c_1^2(Y(D)) > 0$ except for 50 discriminants. The arithmetic genus equals 1 for 10 discriminants (5, 8, 12, 13, 17, 21, 24, 28, 33, 60); they are among those 50. The corresponding 10 surfaces $Y(D)$ are known to be rational ([6] 4. 5 Theorem). For the remaining 40 discriminants we calculated $c_1^2(Y^0(D))$ (table in Chap. IV) using 3. 7 (21), (22) and obtained $c_1^2(Y^0(D)) > 0$ (which implies general type !) except for the 22 discriminants

(1) 29, 37, 40, 41, 44, 53, 56, 57, 61, 65, 69
73, 76, 77, 85, 88, 92, 93, 105, 120, 140, 165,

for which we get $c_1^2(Y^0(D)) = 0$. These 22 have to be investigated by hand.

The surface $Y_-(D)$ has arithmetic genus 1 for 5 discriminants (5, 8, 12, 13, 17). These surfaces are rational. Namely, except for $D=12$ they are isomorphic to $Y(D)$, and for $D=12$ it was shown in [6] 4. 5 that $Y_-(D)$ is rational. For $D \neq 5$, 8, 12, 13, 17 (i.e. $D > 17$) there are 23 discriminants for which $c_1^2(Y_-(D)) \leqq 0$. For these we consider $Y_-^0(D)$ (see 3. 8) and obtain $c_1^2(Y_-^0(D)) > 0$ except for 15 discriminants (see table in Chap. IV)

(2) 21, 24, 28, 29, 33, 37, 40, 41
44, 53, 57, 61, 65, 73, 85

for which we get $c_1^2(Y_-^0(D))=0$. These 15 surfaces have to be investigated by hand. All other $Y_-(D)$ are rational (5 cases) or of general type.

5. 3 The components of the curves F_N in $Y(D)$ or $Y_-(D)$ all have the same non-singular model if N is admissible (3. 3 and 3. 8). This model is $\overline{\mathfrak{H}/\Gamma_0(N)}$ if N is not divisible by the square free part d of D.

The values of $c_1(\Gamma_0(N)/\{1,-1\})$ (see 3. 7 (17)) are denoted by $c_1(N)$ and were listed in [6] 4. 3 for the case that the genus of $\overline{\mathfrak{H}/\Gamma_0(N)}$ is 0 or 1.

Let c_1 be the first Chern class of $Y(D)$ or $Y_-(D)$ respectively. Then

$$c_1\cdot E \geqq c_1(N)+\sum\left(\frac{p+q}{(p,q)}-1\right) \tag{3}$$

for any component E of F_N (N admissible, $N\not\equiv 0 \bmod d$) where the sum is over all the branches of F_N belonging to E near the cusps (see 3. 3 (6)). For (3) compare [6] 4. 5 (34). The sum in (3) equals the intersection number of E with the Chern divisors of the cusps minus $\sigma(\Gamma_0(N))$.

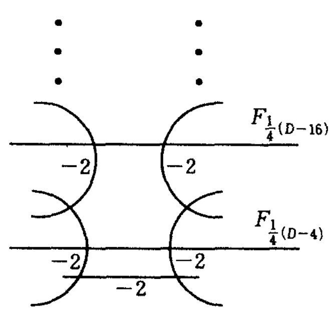
For $N=5, 6, 7, 8, 9$ the curve $\overline{\mathfrak{H}/\Gamma_0(N)}$ is rational and $c_1(N)=0$. In these cases $c_1\cdot E\geqq 0$ for all components E of F_N. If $c_1\cdot E=0$ and E is non-singular, then E is a (-2)-curve.

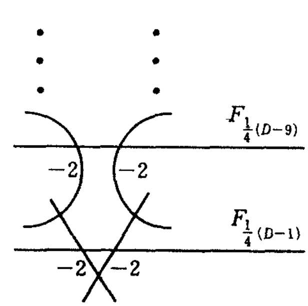
5. 4 In this section we settle the rough classification of the surfaces $Y_-(D)$ using the proposition in 5. 1.

The principal cusp of $Y_-(D)$ has the resolution cycle belonging to the strict ideal class of $(\sqrt{D})$. We consider the reduced quadratic irrationalities

$$w_{(b)}=\frac{b+2+\sqrt{D}}{b+\sqrt{D}},\qquad b\in\boldsymbol{Z},\ b\equiv D \bmod 2,\ -\sqrt{D}<b<\sqrt{D}-2. \tag{4}$$

The module $\boldsymbol{Z}w_{(b)}+\boldsymbol{Z}$ is strictly equivalent to the ideal $(\sqrt{D})$. We have $w_{(b)}=2-\dfrac{1}{w_{(b-2)}}$ for $|b|<\sqrt{D}-2$. Furthermore $w_{(b)}$ is of the form $\dfrac{1}{2N}(M+\sqrt{D})$ with $N=\dfrac{1}{4}(D-b^2)$. Hence we have on $Y_-(D)$ a configuration

(5) 

or 

$(\pm b=2,4,\cdots;|b|<\sqrt{D}-2)$ $(\pm b=1,3,\cdots;|b|<\sqrt{D}-2)$

depending on whether D is even or odd. The (-2)-curves belong to the resolution of the cusp. The symmetry in the configutation comes from the canonical involution on $Y_-(D)$ induced by $(z_1, z_2) \mapsto (-z_2, -z_1)$. Two b's differing only up to sign give the same component of $F_{(D-b^2)/4}$. It is carried to itself by the involution ([6] § 4. 5).

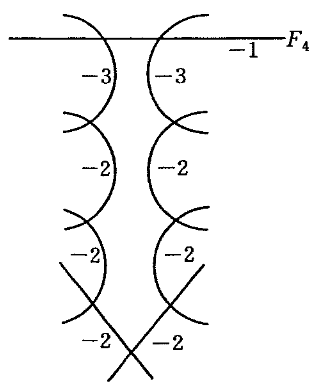
For the rest of this section 5. 4 we suppose that $D>17$ so that $Y_-(D)$ is *not rational*. Then we can use (5) for the rough classification as follows.

If in (5) *any of the* $F_{(D-b^2)/4}$ (*with* $|b|<\sqrt{D}-2$) *is* F_5, F_6, F_7, F_8 *or* F_9, *then* $Y_-(D)$ *is a blown-up* $K3$ *or an honestly elliptic surface.*

Namely, let E be the component drawn in (5) of such an $F_{(D-b^2)/4}$. Then $c_1 \cdot E \geqq 0$, but E is not an exceptional curve, because blowing it down would give by (5) two intersecting exceptional curves which is not possible on a non-rational regular surface.. The proposition in 5. 1 and diagram (5) now show: If $Y_-(D)$ is not a blown-up $K3$-surface, then E is a (-2)-curve (in fact it has to belong to the largest $|b|$ in (5)) and E and the (-2)-curves of the resolved cusp indicated in (5) give a cyclic elliptic configuration proving that the surface is blown-up honestly elliptic.

Let us first consider the values D in (2) for which $\chi(Y_-(D)) \geqq 3$. These are certainly not blown-up $K3$-surfaces. For $D=44$, 53, 57, 61, 85 we get on $Y_-(D)$ a cycle of non-singular rational curves of self-intersection number -2 using F_7, F_7, F_8, F_9, F_9 respectively. For $D=65$ we have a configuration

(6)

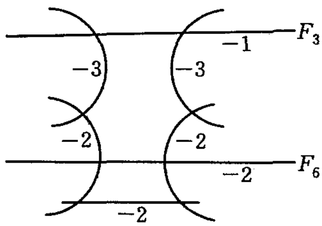


where the F_4 belongs to $w_{(-7)}=(-5+\sqrt{65})/(-7+\sqrt{65})$ and $[w_{(-7)}+1]=3$. Blowing down the component of F_4 drawn in (6) we get again a cyclic elliptic configuration. For $D=73$ see [9] (for prime discriminants D the surfaces $Y(D)$ and $Y_-(D)$ are isomorphic). In all cases we have a cyclic elliptic configuration. The surfaces are blown-up honestly elliptic.

Now we study the eight values in (2) for which $\chi(Y_-(D))=2$. These are 21, 24, 28, 29, 33, 37, 40, 41.

We wish to prove that the corresponding surfaces are blown-up $K3$-surfaces. We may assume that the components of F_5, F_6, F_7, F_8, F_9 occurring in (5) are (-2)-curves, since if they are not, the surfaces are certainly blown-up $K3$-surfaces (Proposition 5. 1). For $D=21$, 24, 29 the curve F_5 occurs in configuration (5). Every component of F_5 passes through a curve A of self-intersection number -2 coming from the resolution of a quotient singularity of order 2, because $\Gamma_0(5)$ has fixed points of order 2 in $\mathfrak{H}$. Thus (5) leads to an elliptic configuration which intersects A, so the surface is a blown-up $K3$-surface. For $D=33$, 37, 40 we have in (5) two different values of b for which $(D-b^2)/4=6$, 7, 8, 9. Hence the surfaces are blown-up $K3$-surfaces.

For $D=28$ we have a configuration



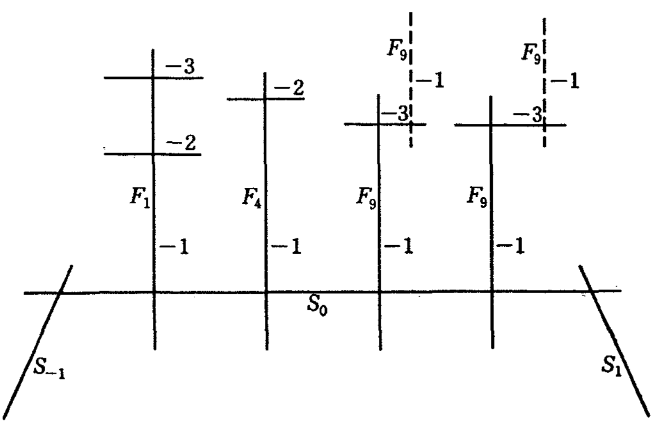
which proves by blowing down F_3 that $Y_-(28)$ is blown-up $K3$. For $D=41$ the same argument works: The curve F_8 occurs and one has to blow down F_4.

Theorem 3 in Chapter I is now completely proved.

5. 5 In this section and in the following one we shall do the rough classification of the surfaces $Y(D)$ and prove Theorem 2 of Chap. I. Since $Y(D)$ is equivalent to $Y_-(D)$ if D is not divisible by a prime $\equiv 3 \bmod 4$, it remains to study the following 13 discriminants from the list (1)

$$\text{(7)} \quad \begin{array}{ll} D = 44, 56, 57, 69, 105 & (\chi(Y(D)) = 2) \\ D = 76, 77, 88, 92, 93, 120, 140, 165 & (\chi(Y(D)) \geqq 3). \end{array}$$

For these 13 discriminants we indicate the resolution of the cusps with the notation of 3. 3 (5). The reader should consult these diagrams, which are printed at the end of the paper, during the course of the proofs.



For $D=44, 56, 57, 69, 105$ we consider the curve S_0 in the resolution of the principal cusp (see 3. 6).

The exceptional curves F_1, F_4 and F_9 (for $3|D$) give on $Y(D)$ the configuration in the previous page. The curve of self-intersection -2 intersecting F_4 exists if and only if D is even. The exceptional curves F_9 exist if and only if $3|D$. The dotted components of F_9 intersect the (-3)-curves if and only if $D=105$ (see 3. 7). (It has to be checked for $D=105$ that the two components of F_9 intersecting S_0 do not meet in $\mathfrak{H}^2/G$.) We have $S_0\cdot S_0=-b_0=-[w_0]-1$ and $c_1\cdot S_0=-b_0+2$. On passing to $Y^0(D)$ the exceptional curves in the above diagram are blown down successively and we get on $Y^0(D)$ (whose first Chern class we denote by $\tilde{c}_1$) an image curve $\tilde{S}_0$ which has exactly one singular point (a cusp) and for which

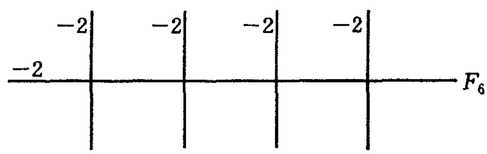
$$\tilde{c}_1\cdot\tilde{S}_0 = -b_0+7+1(\text{if } D \text{ is even})+2(\text{if } 3|D)+2(\text{if } D=105).$$

For $D=44, 56, 57, 69, 105$ the values of b_0 are 8, 8, 9, 9, 11 and we get $\tilde{c}_1\cdot\tilde{S}_0=0$ and hence $\tilde{S}_0\cdot\tilde{S}_0=0$. Thus the single curve $\tilde{S}_0$ is an elliptic configuration.

The curve S_1 has for $D=44, 57$ the self-intersection number -2. For $D=69, 105$ the curve S_1 has self-intersection number -3 and intersects F_3, F_4 respectively; therefore in $Y^0(D)$ the image curve $\tilde{S}_1$ has self-intersection number -2. For $D=56$ we have $S_1\cdot S_1=-4$, but the curve S_1 meets the exceptional curve F_2 which by 3. 7 (19) leads to two blow-downs, so the image curve $\tilde{S}_1$ on $Y^0(56)$ has the self-intersection number -2. Thus by the proposition in 5. 1 (and because $c_1^2(Y^0(D))=0$), the surfaces $Y^0(D)$ are $K3$-surfaces for $D=44, 56, 57, 69, 105$.

5. 6 We now study the discriminants in the second line of (7). In all cases we shall find an elliptic configuration on $Y(D)$ which proves that $Y(D)$ is blown-up honestly elliptic and finishes the proof of Theorem 2 in Chapter I.

For $D=76$ consider the curve F_6. It has one component which meets four curves of the resolution of the cusp. This gives rise to an elliptic configuration:

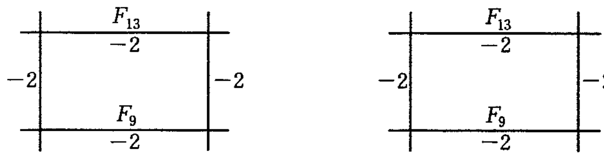


(The proposition in 5. 1 implies that F_6 is a (-2)-curve.)

For $D=77$ the irreducible curve F_{11} passes through the two corners of the resolution of the cusp. The genus of F_{11} is 1. We have $c_1(11)=-2$ and $c_1\cdot F_{11}\geqq 0$ by (3). By the proposition in 5. 1 the curve F_{11} is an elliptic configuration ($c_1\cdot F_{11}=F_{11}\cdot F_{11}=0$).

For $D=88$ the two components of F_9 together with 6 curves of the resolved cusp give a cyclic elliptic configuration of length 8.

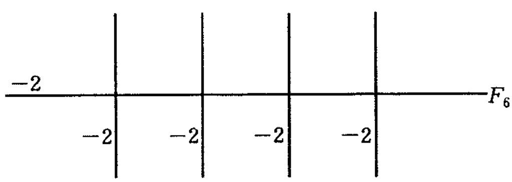
For $D=92$ the two components of F_{13} pass through the four corners of the resolution of the cusp. We have $c_1(13)=-2$, but (3) implies that $c_1\cdot E\geqq 0$ for each component E of F_{13}. Since E is a rational curve and meets two (-2)-curves (coming from two quotient singularities of order 2), it follows from the proposition in 5. 1 that E is a (-2)-curve. The curve F_9 was considered in 3. 6. Here 9 is not admissible ; the group Γ in 3. 6 (15) is $\Gamma'(3)/\{1, -1\}$ for which $a_2(\Gamma)=2$, $a_3(\Gamma)=0$, $\sigma(\Gamma)=2$, $e(\overline{\mathfrak{H}/\Gamma})=2$ and $c_1(\Gamma)=0$. The curve F_9 has two components. It follows as before that each component of F_9 is a (-2)-curve. As can be checked, F_9 and F_{13} intersect in $\mathfrak{H}^2/G$ in quotient singularities of order 2. (Condition 3. 3 (10) is satisfied : $(4\cdot 9\cdot 13-10^2)/92=4$.) We have on $Y(D)$ the following elliptic configurations :



where the "vertical" curves come from the resolution of quotient singularities of order 2.

For $D=93$ the two components of F_7 together with 4 curves of the resolved cusp give a cyclic elliptic configuration of length 6.

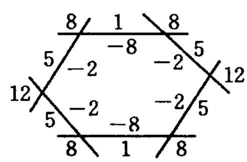
For $D=120$ the curve F_6 has one component. It passes through both cusps and gives rise to the elliptic configuration:



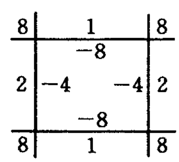
For $D=140$ the curve F_{14} passes through the four corners of the 2 resolved cusps. It has one component. The genus of F_{14} is 1. We have $c_1(14)=-4$, and $c_1\cdot F_{14}=0$ by (3). By proposition 5. 1 the curve F_{14} is an elliptic configuration.

For $D=165$ the same argument works with F_{15}.

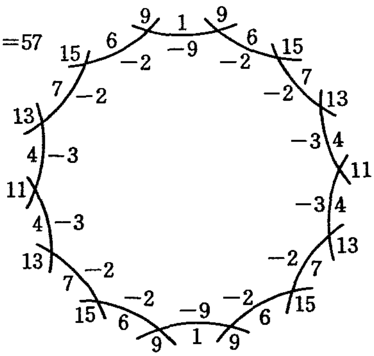
$D=44$



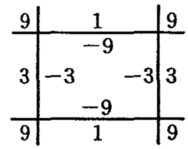
$D=56$



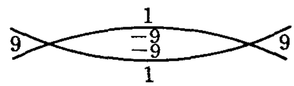
$D=57$



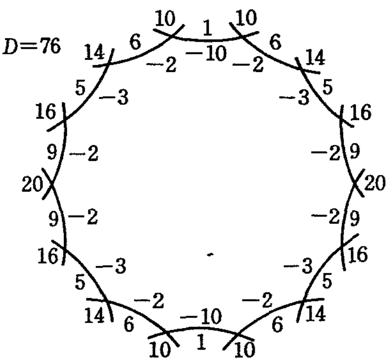
$D=69$



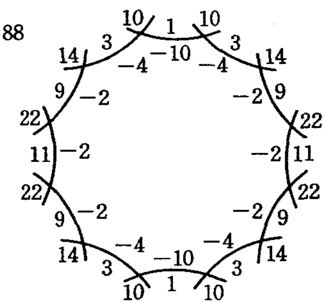
$D=77$



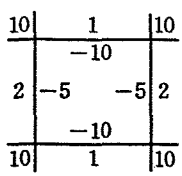
$D=76$



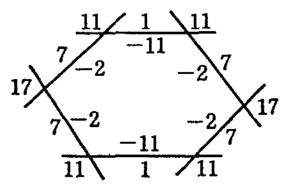
$D=88$



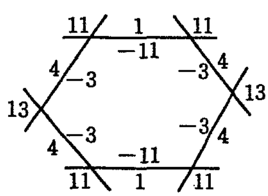
$D=92$



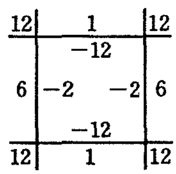
$D=93$



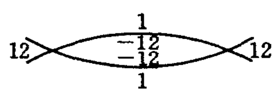
$D=105$
(two cusps)



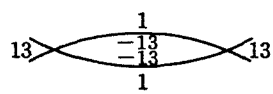
$D=120$
(two cusps)



$D=140$
(two cusps)



$D=165$
(two cusps)



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