The Rankin-Selberg method for automorphic functions which are not of rapid decay

By Don ZAGIER

To the memory of Takuro Shintani

One of the most fruitful ideas in the theory of automorphic forms is the observation, made independently by Rankin [4] and Selberg [5] around 1939, that the Mellin transform of the constant term in the Fourier development of an automorphic function can be represented as the scalar product of the automorphic function with an Eisenstein series and hence inherits the analytic properties of the Eisenstein series. More precisely, let $F(z)$ be a continuous function on the complex upper half-plane $\mathfrak{H}$ which is invariant under the action of the modular group $\Gamma=PSL_2(\boldsymbol{Z})$, let

$$F(z)=\sum_{n=-\infty}^{\infty} a_n(y)e^{2\pi inx} \qquad (z=x+iy\in\mathfrak{H}) \tag{1}$$

be the Fourier expansion of F, and let $E(z, s)$ $(z\in\mathfrak{H},\ s\in\boldsymbol{C})$ be the standard non-holomorphic Eisenstein series, defined for $\mathrm{Re}(s)>1$ by

$$E(z, s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\gamma z)^s=\frac{1}{2}y^s\sum_{\substack{c,d\in\boldsymbol{Z}\\(c,d)=1}}|cz+d|^{-2s} \qquad (\mathrm{Re}(s)>1) \tag{2}$$

$\left(\Gamma_\infty=\left\{\pm\begin{pmatrix}1 & b\\0 & 1\end{pmatrix}\right\}\right)$; then denoting by $d\mu$ the invariant volume element $y^{-2}dxdy$ on $\mathfrak{H}$ we find by a standard "unfolding" trick

$$\begin{aligned}\int_{\Gamma\backslash\mathfrak{H}}F(z)E(z, s)d\mu&=\int_{\Gamma\backslash\mathfrak{H}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}F(\gamma z)\,\mathrm{Im}(\gamma z)^s d\mu=\int_{\Gamma_\infty\backslash\mathfrak{H}}F(z)\,\mathrm{Im}(z)^s d\mu\\&=\int_0^\infty y^s\int_{\boldsymbol{R}/\boldsymbol{Z}}F(x+iy)dx\,y^{-2}dy\\&=\int_0^\infty a_0(y)y^{s-2}dy \qquad (\mathrm{Re}(s)>1),\end{aligned} \tag{3}$$

the steps being justified by the absolute convergence for $\mathrm{Re}(s)>1$ if $F(z)$ is sufficiently small as $z\to\infty$. On the other hand, $E(z, s)$ is known to have a meromorphic continuation (in s) with the following properties:

i) $E(z, s)$ is holomorphic in $\mathrm{Re}(s)>1/2$ except for a simple pole of residue

$3/\pi$ (independent of z!) at $s=1$;

ii) the function

$$E^*(z, s)=\zeta^*(2s)E(z, s), \tag{4}$$

where

$$\zeta^*(s)=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\zeta^*(1-s) \tag{5}$$

is the Riemann zeta-function together with its gamma factor, is holomorphic for $s\neq 0, 1$;

iii) $E(z, s)$ satisfies the functional equation

$$E^*(z, s)=E^*(z, 1-s). \tag{6}$$

Equation (3) then gives the corresponding properties for the Mellin transform of the constant term of F: if we define transforms $R(F; s)$ and $R^*(F; s)$ by

$$R(F; s)=\int_0^\infty a_0(y)y^{s-2}dy, \quad R^*(F; s)=\zeta^*(2s)R(F, s) \qquad (\mathrm{Re}(s)>1), \tag{7}$$

then $R(F; s)$ can be meromorphically continued to all s, the only possible poles being at $s=1$ and $s=\rho/2$ (ρ a non-trivial zero of the Riemann zeta-function), the function $s(s-1)R^*(F; s)$ is entire and invariant under $s\to 1-s$, and we have the formula

$$\mathrm{Res}_{s=1}R(F; s)=\frac{3}{\pi}\int_{\Gamma\backslash\mathfrak{H}}F(z)d\mu \tag{8}$$

(note that $\int_{\Gamma\backslash\mathfrak{H}}d\mu=\pi/3$, so this formula says that the residue of $R(F; s)$ at $s=1$ equals the average value of F in $\mathfrak{H}$).

These facts have a threefold importance in the theory of automorphic forms:

I. For many functions F, the Mellin transform $R(F; s)$ can be computed as a Dirichlet series with Euler product, and we obtain the analytic properties of this L-series. As an example (the one originally studied by Rankin and Selberg), take

$$F(z)=y^{12}|\Delta(z)|^2, \tag{9}$$

where

$$\Delta(z)=e^{2\pi iz}\prod_{n=1}^\infty(1-e^{2\pi inz})^{24}=\sum_{n=1}^\infty\tau(n)e^{2\pi inz}$$

is the discriminant function from the theory of elliptic functions; then we find

$$a_0(y)=y^{12}\sum_{n=1}^\infty\tau(n)^2e^{-4\pi ny}, \tag{10}$$

$$\frac{(4\pi)^s}{\Gamma(s)}R(F,\ s-11)=\sum_{n=1}^{\infty}\frac{\tau(n)^2}{n^s}\qquad(\mathrm{Re}\,(s)>12)\,,$$

from which the meromorphic continuation and functional equation of the Dirichlet series on the right follow.

II. In some cases it may be easier to compute $R(F;\,s)$ than $\int_{\Gamma\backslash\mathfrak{H}}F(z)\,d\mu$ (since its definition involves only the constant term of F and a one-dimensional integration rather than the whole function F and an integration over a complicated fundamental domain), and in such a situation it can be advantageous to view (8) as a means of evaluating the average value of F rather than as a formula for the residue. An example is the calculation in [6] (or, in the non-holomorphic and adelic cases, in [7] or [2], respectively), where the Selberg trace formula was derived by computing $R(F;\,s)$ for $F(z)=K_0(z,\,z)$ ($K_0(z_1,\,z_2)=$ Selberg kernel function) and applying formula (8).

III. If a function $\alpha(y)$ ($y>0$) is of rapid decay at ∞ and has an asymptotic expansion $\sum c_i y^{\lambda_i}$ (resp. $\sum c_i y^{\lambda_i}(\log y)^{n_i}$) as $y\to 0$, then its Mellin transform $\int_0^\infty \alpha(y)y^{s-2}dy$ has a meromorphic continuation with simple poles of residue c_i at $s=1-\lambda_i$ (resp. poles with principal parts $(-1)^{n_i}n_i!\,c_i(s-1+\lambda_i)^{-n_i-1}$) and no other poles. Therefore the fact that the Mellin transform of $a_0(y)$ is holomorphic except for poles at $s=1$ and $s=\rho/2$ suggests (though it unfortunately does not seem to imply) that the constant term $a_0(y)$ has the asymptotic behavior

$$a_0(y)\sim C+\sum_{\zeta^*(\rho)=0}A_\rho y^{1-\rho/2}\tag{11}$$

as $y\to 0$, where C is the average value of F and the A_ρ are certain complex numbers (if ρ is an n-fold zero, A_ρ must be replaced by a polynomial of degree $n-1$ in $\log y$). This behavior, which if valid is certainly very intriguing, seems to be substantiated by the graph of the constant term (10) of the function (9), where the oscillatory behavior is clearly evident (Fig. 1). Assuming the Riemann hypothesis (that all ρ have the form $1/2\pm i\gamma_m$ with γ_m real), we can rewrite (11) as

$$a_0(y)\sim C+y^{3/4}\sum_{m=1}^{\infty}a_m\cos\Big(\frac{1}{2}\gamma_m\log y+\phi_m\Big),\tag{12}$$

the a_m and ϕ_m now being real constants. The graph of $(a_0(y)-C)/y^{3/4}$ against $\log y$ for the function (10) (with $C=3\pi^{-1}(\Delta,\,\Delta)\cong 9.88698\times 10^{-7}$) is shown in Fig. 2. Doing a rough numerical Fourier analysis on the data plotted there we obtained the values 14.138 and 20.8 for 4π times the first two frequencies; this is to be compared to $\gamma_1\approx 14.135$, $\gamma_2\approx 21.0$. Thus the formula (12) is not only vindicated

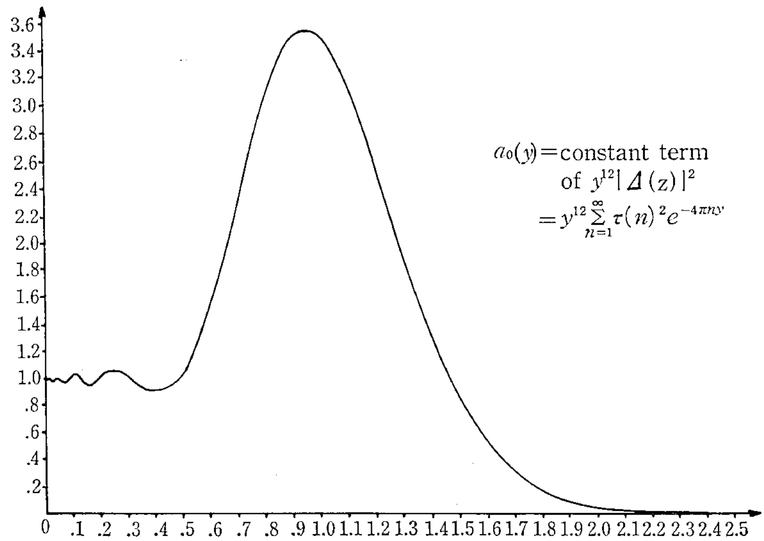

Fig. 1.

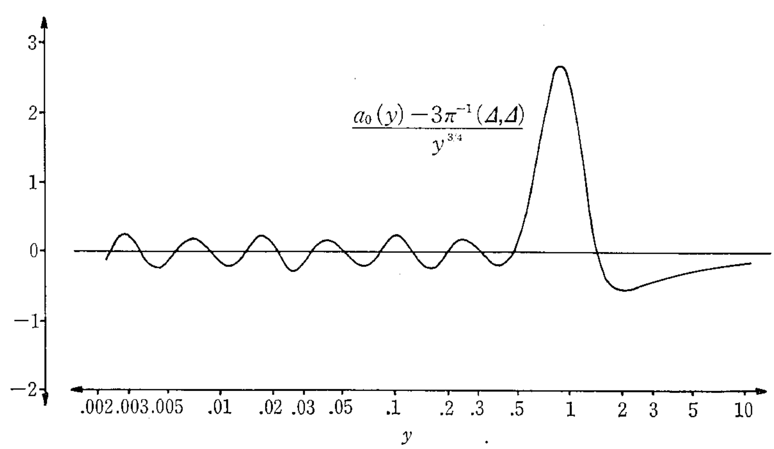

Fig. 2.

numerically but can actually be used to evaluate the zeros of the Riemann zeta-function using only the values of the Ramanujan τ-function!

After this very long preamble extolling the merits of the Rankin-Selberg method in its classical form, we come to the object of this paper, namely to extend its range of applicability to functions $F(z)$ which do not fall off rapidly enough in a fundamental domain to permit the calculation (3) for any s. This extension is useful because many of the automorphic functions encountered in real life are of slow growth (i.e. $O(y^N)$ for some N as $y\to\infty$) rather than rapid decay (i.e. $O(y^{-N})$ for all N). Also, it may happen that a function $F(z)$ is itself of rapid decay but has a natural decomposition as a sum of $SL_2(\boldsymbol{Z})$-invariant functions which are only of slow growth (this is the case, for instance, for the function $K_0(z, z)$ mentioned in II. above, and is the primary reason for the difficulty of the Selberg trace formula), and we would like to have a version of the Rankin-Selberg method which can be applied to each summand separately. Such a version is provided by the following theorem, which is the main result of this paper.

THEOREM. *Let $F(z)$ be a continuous $SL_2(\boldsymbol{Z})$-invariant function on $\mathfrak{H}$ with the Fourier development* (1) *and suppose that*

$$F(z)=\varphi(y)+O(y^{-N}) \qquad (\forall N) \text{ as } y=\operatorname{Im}(z)\to\infty, \tag{13}$$

where $\varphi(y)$ is a function of the form

$$\varphi(y)=\sum_{i=1}^{l}\frac{c_i}{n_i!}y^{\alpha_i}\log^{n_i}y \qquad (c_i,\ \alpha_i\in\boldsymbol{C},\ n_i\in\boldsymbol{N}_{\geq 0}). \tag{14}$$

Define the Rankin-Selberg transform of F by

$$R(F;s)=\int_0^\infty(a_0(y)-\varphi(y))y^{s-2}dy \qquad (\operatorname{Re}(s)\gg 0) \tag{15}$$

(*the integral converges for* $\operatorname{Re}(s)$ *sufficiently large*). *Then $R(F;s)$ can be meromorphically continued to all s, the only possible poles being at $s=0,\ 1,\ \alpha_i,\ 1-\alpha_i$ and $\rho/2$ (ρ=non-trivial zero of the Riemann zeta-function) and satisfies a functional equation under $s\to 1-s$. More precisely, the function*

$$R^*(F;s)=\zeta^*(2s)R(F;s) \tag{16}$$

satisfies

$$R^*(F;s)=\sum_{i=1}^{l}c_i\left(\frac{\zeta^*(2s)}{(1-\alpha_i-s)^{n_i+1}}+\frac{\zeta^*(2s-1)}{(s-\alpha_i)^{n_i+1}}\right)+\frac{\text{entire function of } s}{s(s-1)} \tag{17}$$

and

$$R^*(F;s)=R^*(F;1-s). \tag{18}$$

In particular, if no α_i equals 0 or 1 then $R(F;s)$ has a simple pole at $s=1$; its residue is given by (8) *if $\int_{\Gamma\backslash\mathfrak{H}}F(z)d\mu$ converges (i.e. if $\operatorname{Re}(\alpha_i)<1$ for all i). Finally, if $\Theta:=\max\operatorname{Re}(\alpha_i)$ is less than $1/2$ then we have the Rankin-Selberg identity*

$$R(F;s)=\int_{\Gamma\backslash\mathfrak{H}}F(z)E(z,s)dz \qquad (\Theta<\operatorname{Re}(s)<1-\Theta). \tag{19}$$

REMARKS 1. The class of functions F allowed in the theorem may seem artificial at first, but is in fact quite natural and, as we shall see, contains many of the functions of slow growth encountered in practice. The seemingly odd definition (15) of the Rankin zeta-function, in which we simply throw out the part of $a_0(y)$ which is not of rapid decay, is natural because the Mellin transform of any function of the form (14) vanishes identically (in the sense that the integrals $\int_0^T \varphi(y)y^{s-2}dy$ and $\int_T^\infty \varphi(y)y^{s-2}dy$, each of which converges in a half-plane, both have meromorphic continuations to all s and the sum of these continuations is identically zero). Note that we have not really lost $\varphi(y)$ as the information about the numbers c_i, n_i, α_i is encoded in the poles of $R(F; s)$ (equation (17)). The class of functions (14), as was pointed out to me by Deligne, also has an intrinsic characterization as the space of G-finite functions on the group $G=\boldsymbol{R}_+^\times$ acting by translation on the space of continuous functions of G, i.e. as the set of functions $\varphi(y)$ such that the functions $y \mapsto \varphi(ay)$ $(a>0)$ lie in a finite-dimensional vector space.

2. As will be clear from the proof, the statement of the theorem can be generalized to the case where Γ is a congruence subgroup of $SL_2(\boldsymbol{Z})$, the only complication being that there is now a separate Eisenstein series and a separate growth condition on F at each cusp. In fact, it is not hard to give an adelic formulation of the theorem and its proof (cf. [2]).

PROOF. First let F be an arbitrary continuous function on $\Gamma\backslash\mathfrak{H}$. We denote by $\mathcal{D}$ the standard fundamental domain

$$\mathcal{D}=\left\{z\in\mathfrak{H} \,\middle|\, |z|\geqq 1,\ |x|\leqq\frac{1}{2}\right\}$$

for the action of Γ on $\mathfrak{H}$ and by $\mathcal{D}_T$ $(T\geqq 1)$ the truncated domain

$$\mathcal{D}_T=\left\{z\in\mathfrak{H} \,\middle|\, |z|\geqq 1,\ |x|\leqq\frac{1}{2},\ y\leqq T\right\}.$$

Then $\mathcal{D}_T$ is a fundamental domain for the action of Γ on

$$\mathfrak{H}_T=\bigcup_{\gamma\in\Gamma}\gamma\mathcal{D}_T=\left\{z\in\mathfrak{H} \,\middle|\, \max_{\gamma\in\Gamma}\operatorname{Im}(\gamma z)\leqq T\right\}.$$

It is easily checked that

$$\mathfrak{H}_T=\{z\in\mathfrak{H} \mid \operatorname{Im}(z)\leqq T\}-\bigcup_{\substack{c\geqq 1\\ (a,c)=1}}\bigcup_{a\in\boldsymbol{Z}} S_{a/c}, \tag{20}$$

where $S_{a/c}$ $(a, c\in\boldsymbol{Z}, (a, c)=1)$ is the disc of radius $1/2c^2T$ tangent to the real axis at a/c (see Fig. 3). We denote by χ_T the characteristic function of $\mathfrak{H}_T$. Applying the Rankin-Selberg identity (3) to the Γ-invariant function $F\cdot\chi_T$, we obtain

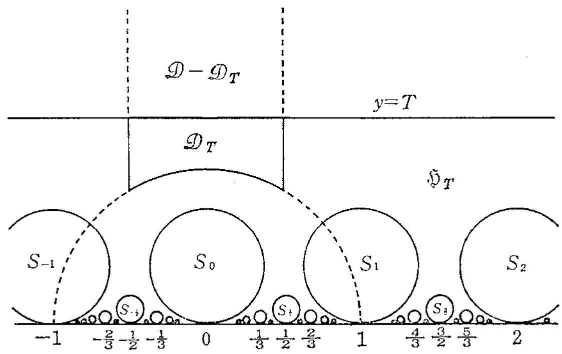

Fig. 3.

(21) $$\int_{\mathcal{D}_T} F(z)E(z,\, s)d\mu = \int_{\Gamma_\infty\backslash\mathfrak{H}_T} F(z)y^s\, d\mu$$

for $\mathrm{Re}(s)>1$. This is valid without restrictions on F since $F\cdot\chi_T$ has compact support modulo Γ.

Now assume that F has slow growth (i. e. $F(x+iy)=O(y^C)$ as $y\to\infty$, for some C; under the assumption of the theorem this holds for any $C>\Theta$). For $z\in\mathfrak{H}$ we have

$$\max_{\gamma\in\Gamma} \mathrm{Im}(\gamma z) = \max_{(c,d)=1} \frac{y}{|cz+d|^2} \leqq \max\left(y,\, \frac{1}{y}\right)$$

(since $c\neq 0 \Rightarrow |cz+d|^2 \geqq c^2y^2 \geqq y^2$), so F then satisfies

$$F(x+iy)=O(y^{-\max(0,C)}) \qquad (y\to 0)$$

and therefore the integral $\int_0^T\int_0^1 F(x+iy)y^{s-2}dxdy$ converges absolutely and uniformly for $\mathrm{Re}(s)>1+\max(0,\, C)$. On the other hand, from (20) we have

$$\Gamma_\infty\backslash\mathfrak{H}_T = \{x+iy \mid x (\mathrm{mod}\ 1),\ 0\leqq y\leqq T\} - \bigcup_{c\geqq 1} \bigcup_{\substack{a\ (\mathrm{mod}\ c)\\ (a,c)=1}} S_{a/c},$$

so (21) gives

(22) $$\int_{\mathcal{D}_T} F(z)E(z,\, s)d\mu = \int_0^T\int_0^1 F(x+iy)y^s d\mu - \sum_{c=1}^{\infty} \sum_{\substack{a\ (\mathrm{mod}\ c)\\ (a,c)=1}} \iint_{S_{a/c}} F(z)y^s d\mu$$

for $\mathrm{Re}(s)>1+\max(0,\, C)$. The first integral is simply $\int_0^T a_0(y)y^{s-2}dy$ with a_0 as in (1). The integral over $S_{a/c}$ can be transformed by choosing an element $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in\Gamma$ with first column $\begin{pmatrix} a \\ c \end{pmatrix}$; then $\gamma_0 S_{a/c} = \{z\in\mathfrak{H} \mid y\geqq T\}$ and hence (since $F(z)$ and $d\mu$ are Γ-invariant)

$$\iint_{S_{a/c}} F(z)y^s d\mu=\int_T^\infty\int_{-\infty}^\infty F(z)\,\mathrm{Im}\,(\gamma_0 z)^s\,d\mu$$

$$=\int_T^\infty\int_0^1 F(z)\sum_{n=-\infty}^{\infty}\mathrm{Im}\,(\gamma_0(z+n))^s d\mu$$

$$=\iint_{\mathcal{D}-\mathcal{D}_T} F(z)\sum_{\gamma=\binom{a\ \cdot}{c\ \cdot}}\mathrm{Im}\,(\gamma z)^s d\mu\,,$$

where the sum is over all $\gamma\in\Gamma$ with first column $\binom{a}{c}$ (all such γ have the form $\gamma_0\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ for some $n\in\boldsymbol{Z}$). When we substitute this identity into (22), we find that the sum over a and c produces (all but one term of) the Eisenstein series $E(z, s)$, so

$$\iint_{\mathcal{D}_T} F(z)E(z, s)d\mu=\int_0^T a_0(y)y^{s-2}\,dy-\iint_{\mathcal{D}-\mathcal{D}_T} F(z)\sum_{\substack{\gamma=\binom{a\ b}{c\ d}\in\Gamma_\infty\backslash\Gamma\\ c>0}}\mathrm{Im}\,(\gamma z)^s d\mu$$

$$=\int_0^T a_0(y)y^{s-2}dy-\iint_{\mathcal{D}-\mathcal{D}_T} F(z)(E(z, s)-y^s)d\mu\,.$$

Note that all steps are justified for $\mathrm{Re}\,(s)>1+\max\,(0, C)$ because of the absolute convergence.

We now use some well-known facts about the Eisenstein series. The constant terms in the Fourier expansions of $E(z, s)$ and $E^*(z, s)$ are given by

$$\int_0^1 E(x+iy,\ s)dx=y^s+\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\,y^{1-s}\qquad\left(s\neq 0,\ \frac{1}{2},\ 1\right)\tag{23}$$

and

$$e(y,\ s):=\int_0^1 E^*(x+iy,\ s)dx$$

$$=\begin{cases}\zeta^*(2s)y^s+\zeta^*(2s-1)y^{1-s} & \left(s\neq 0,\ \frac{1}{2},\ 1\right)\\[1ex] y^{1/2}\log y+(\gamma-\log 4\pi)y^{1/2} & \left(s=\frac{1}{2},\ \gamma=\text{Euler's constant}\right)\end{cases}\tag{24}$$

$$=e(y,\ 1-s)\,;$$

the difference $E^*(z, s)-e(y, s)$ is an entire function of s and is of rapid decay with respect to y. Hence, multiplying both sides of the above equation by $\zeta^*(2s)$ we find

$$\iint_{\mathcal{D}_T} F(z)E^*(z, s)d\mu = \zeta^*(2s)\int_0^T a_0(y)y^{s-2}dy$$

$$-\iint_{\mathcal{D}-\mathcal{D}_T} F(z)(E^*(z, s)-e(y, s))d\mu - \zeta^*(2s-1)\iint_{\mathcal{D}-\mathcal{D}_T} F(z)y^{1-s}d\mu$$

or, rearranging,

$$\zeta^*(2s)\int_0^T a_0(y)y^{s-2}dy - \zeta^*(2s-1)\int_T^\infty a_0(y)y^{-1-s}dy \tag{25}$$

$$= \iint_{\mathcal{D}_T} F(z)E^*(z, s)d\mu + \iint_{\mathcal{D}-\mathcal{D}_T} F(z)(E^*(z, s)-e(y, s))d\mu\,.$$

This has been proved only for $\mathrm{Re}(s)$ sufficiently large, but the right-hand side makes sense for all s ($\neq 0, 1$), since the first integral is over a compact set and the integrand in the second is of rapid decay.

To derive (25) we used only that F is of slow growth. Assume now that F is as in the theorem, i.e. F satisfies (13) for some φ as in (14). Then for $\mathrm{Re}(s)$ sufficiently large we have

$$\int_0^T a_0(y)y^{s-2}dy = \int_0^T (a_0(y)-\varphi(y))y^{s-2}dy + \sum_{i=1}^{l} \frac{c_i}{n_i!}\int_0^T y^{s+\alpha_i-2}\log^{n_i} y\, dy$$

$$= R(F;\, s) - \int_T^\infty (a_0(y)-\varphi(y))y^{s-2}dy + h_T(s)\,,$$

where $R(F;\, s)$ is defined by (15) and

$$h_T(s) = \sum_{i=1}^{l} \frac{c_i}{n_i!}\frac{\partial^{n_i}}{\partial s^{n_i}}\left(\frac{T^{s+\alpha_i-1}}{s+\alpha_i-1}\right) = \sum_{i=1}^{l} c_i \sum_{m=0}^{n_i} \frac{(-1)^{n_i-m}}{m!}\frac{T^{s+\alpha_i-1}\log^m T}{(s+\alpha_i-1)^{n_i-m+1}}\,. \tag{26}$$

Similarly,

$$\int_T^\infty a_0(y)y^{-s-1}dy = \int_T^\infty (a_0(y)-\varphi(y))y^{-s-1} - h_T(1-s)\,.$$

Substituting these two formulas into (25) we obtain (still for $\mathrm{Re}(s)$ sufficiently large)

$$R^*(F;\, s) + \zeta^*(2s)h_T(s) + \zeta^*(2s-1)h_T(1-s)$$

$$= \iint_{\mathcal{D}_T} F(z)E^*(z, s)d\mu + \iint_{\mathcal{D}-\mathcal{D}_T} F(z)(E^*(z, s)-e(y, s))\,d\mu$$

$$+ \zeta^*(2s)\int_T^\infty (a_0(y)-\varphi(y))y^{s-2}dy + \zeta^*(2s-1)\int_0^\infty (a_0(y)-\varphi(y))y^{-s-1}dy\,.$$

But the function $e(y, s)$ is independent of x and hence has the same integral against $F(z)$ as against $a_0(y)$ in the rectangle $\mathcal{D}-\mathcal{D}_T$, so we obtain (after some rearrangement)

$$R^*(F;s)=\iint_{\mathscr{D}_T}F(z)E^*(z,s)d\mu+\iint_{\mathscr{D}-\mathscr{D}_T}[F(z)E^*(z,s)-\varphi(y)e(y,s)]d\mu \tag{27}$$
$$-\zeta^*(2s)h_T(s)-\zeta^*(2s-1)h_T(1-s).$$

Equation (27) is the basic identity from which all of the statements of the theorem will follow. Indeed, the integrals on the right of equation (27) converge for all s and clearly define a meromorphic function of s, invariant under $s\to1-s$, whose only singularities are simple poles at $s=0$ and $s=1$. Since the function $h_T(s)$ defined by (26) is also meromorphic, equation (27) immediately gives the meromorphic continuation and functional equation of $R^*(F;s)$. Observing that

$$h_1(s)=-\sum_{i=1}^{l}\frac{c_i}{(1-\alpha_i-s)^{n_i+1}}$$

and that

$$h_T(s)-h_1(s)=\sum_{i=1}^{l}\frac{c_i}{n_i!}\frac{\partial^{n_i}}{\partial s^{n_i}}\left(\frac{T^{s+\alpha_i-1}-1}{s+\alpha_i-1}\right)$$

is an entire function of s, we also obtain the statement in the theorem about the position of the poles of $R^*(F;s)$. Finally, taking residues at $s=1$ on both sides of (27) we obtain

$$\operatorname{Res}_{s=1}R^*(F;s)=-\operatorname{Res}_{s=1}(\zeta^*(2s)h_T(s))-\operatorname{Res}_{s=1}(\zeta^*(2s-1)h_T(1-s)) \tag{28}$$
$$+\frac{1}{2}\left(\iint_{\mathscr{D}_T}F(z)d\mu+\iint_{\mathscr{D}-\mathscr{D}_T}(F(z)-\varphi(y))d\mu\right).$$

The first term on the right is independent of T (since $h_T(s)-h_1(s)$ is holomorphic) and vanishes if all α_i are different from zero. The second term is a linear combination of terms $T^{\alpha_i-1}\log^r T$ and hence is $O(1)$ as $T\to\infty$ if $\operatorname{Re}(\alpha_i)<1$ for all i, as letting $T\to\infty$ we deduce also the statement about the residue of $R(F;s)$ made in the theorem. Finally, equation (19) follows from (27) on letting $T\to\infty$, since $\lim_{T\to\infty}h_T(s)=0$ for $\operatorname{Re}(s)<1-\Theta$.

Reinterpretation. Before proceeding to applications, we digress to give a reformulation of the results obtained so far which the reader may find enlightening. We call a continuous function F on $\Gamma\backslash\mathfrak{H}$ *renormalizable* if its behavior at infinity is of the type described by equations (13) and (14) and define the *renormalized integral* of such a function by

$$R.N.\left(\int_{\Gamma\backslash\mathfrak{H}}F(z)d\mu\right):=\iint_{\mathscr{D}_T}F(z)d\mu+\iint_{\mathscr{D}-\mathscr{D}_T}(F(z)-\varphi(y))d\mu-\hat{\varphi}(T), \tag{29}$$

where $T>1$ is arbitrary and $\hat{\varphi}(y)$ is the antiderivative of φ which is a linear combination of non-constant functions of the form $y^{\beta}\log^m y$ ($\beta\in\boldsymbol{C}$, $m\in\boldsymbol{Z}$, $m\geqq 0$). Explicitly, if φ is given by (14) then

$$\hat{\varphi}(y)=\sum_{\substack{1\leqq i\leqq l\\ \alpha_i\neq 1}} c_i \sum_{m=0}^{n_i} \frac{(-1)^{n_i-m}}{m!}\frac{y^{\alpha_i-1}\log^m y}{(\alpha_i-1)^{n_i-m+1}}+\sum_{\substack{1\leqq i\leqq l\\ \alpha_i=1}} c_i\frac{\log^{n_i+1} y}{(n_i+1)!}. \tag{30}$$

Note that the right-hand side of (29) is independent of T, so the definition makes sense; letting $T\to\infty$ we can write it in the equivalent form

$$R.\,N.\Big(\int_{\Gamma\backslash\mathfrak{H}} F(z)d\mu\Big):=\lim_{T\to\infty}\Big(\iint_{\mathcal{D}_T} F(z)d\mu-\hat{\varphi}(T)\Big), \tag{31}$$

where $\hat{\varphi}$ can now be simplified by omitting the terms with $\mathrm{Re}(\alpha_i)<1$ from (30), since their contribution tends to zero as $T\to\infty$. If no α_i equals 0 or 1, then the function $h_T(s)$ defined in (26) is holomorphic at $s=0$ and $s=1$ and $h_T(0)=\hat{\varphi}(T)$, so (28) can be stated

$$\mathrm{Res}_{s=1}R^*(F;\,s)=\frac{1}{2}R.\,N.\Big(\int_{\Gamma\backslash\mathfrak{H}} F(z)d\mu\Big); \tag{32}$$

in general, we see by taking $T=1$ in (28) that this equation still holds if we replace $R^*(F;\,s)$ by

$$R^*(F;\,s)+\sum_{\alpha_i=0}\frac{(-1)^{n_i}c_i\zeta^*(2s)}{(s-1)^{n_i+1}}-\sum_{\alpha_i=1}\frac{c_i\zeta^*(2s-1)}{(s-1)^{n_i+1}},$$

which by virtue of (17) has a simple pole at $s=1$. Thus formula (8) for the integral of a function of rapid decay over a fundamental domain generalizes to arbitrary functions of the type considered in the theorem if we interpret the integral in the renormalized sense. But we can do more: Since the class of renormalized functions is closed under multiplication and Eisenstein series are renormalizable, the product $F(z)E^*(z,\,s)$ for F as in the theorem is itself renormalizable: it is the sum of $\varphi_s(y)=\varphi(y)e(y,\,s)$ and a function of rapid decay. A short calculation gives

$$\hat{\varphi}_s(y)=\zeta^*(2s)h_y(s)+\zeta^*(2s-1)h_y(1-s),$$

so the right-hand side of our basic identity (27) is simply $R.\,N.\Big(\int F(z)E^*(z,\,s)d\mu\Big)$.

To push things to their logical limit, we introduce a similar terminology in dimension one: we call a continuous function f on $[0,\,\infty)$ renormalizable if it is integrable near 0 and of the form $\varphi(y)+O(y^{-N})$ ($\forall N$) as $y\to\infty$ for some φ as in (13), and define the renormalized integral of such an f by

$$R.\,N.\Big(\int_0^{\infty} f(y)dy\Big):=\int_0^T f(y)dy+\int_T^{\infty}(f(y)-\varphi(y))dy-\hat{\varphi}(T)\quad\text{(independent of } T\text{)}$$

$$=\lim_{T\to\infty}\left(\int_0^T f(y)dy-\hat{\varphi}(T)\right).$$

Then the results we have obtained can be restated as follows:

THEOREM (second version). *Let F be a renormalizable function on $\Gamma\backslash\mathfrak{H}$ and $a_0(y)$ the constant term of F. Then the function $a_0(y)y^{s-2}$ is renormalizable for* $\mathrm{Re}(s)$ *sufficiently large and the function*

$$R(F;s):=R.N.\left(\int_0^\infty a_0(y)y^{s-2}dy\right)\qquad(\mathrm{Re}(s)\gg 0)$$

equals $R.N.\left(\int_{\Gamma\backslash\mathfrak{H}}F(z)E(z,s)d\mu\right)$. In particular, $R(F;s)$ inherits the analytic properties (meromorphic continuation, position of poles, functional equation) of the Eisenstein series. If no α_i equals 0 or 1, then the residue of $R(F;s)$ at $s=1$ equals $\frac{3}{\pi}R.N.\left(\int_{\Gamma\backslash\mathfrak{H}}F(z)d\mu\right)$.

The rest of the paper will be devoted to various examples and applications of the theorem.

1. Constant function. The simplest function satisfying the conditions of the theorem is $F(z)\equiv 1$. Here $a_0(y)=1$ and $\varphi(y)=1$, so the function $R(F;s)$ defined by (15) vanishes identically. Thus the meromorphic continuability and functional equation are trivial. Nevertheless, the formulas we have obtained are not completely trivial: Using $h_T(s)=T^{s-1}/(s-1)$ we find that equation (27) specializes to

$$\iint_{\mathcal{D}_T}E^*(z,s)d\mu+\iint_{\mathcal{D}-\mathcal{D}_T}(E^*(z,s)-e(y,s))d\mu=\frac{\zeta^*(2s)}{s-1}T^{s-1}-\frac{\zeta^*(2s-1)}{s}T^{-s}\tag{33}$$

(i.e. the renormalized integral of an Eisenstein series over $\Gamma\backslash\mathfrak{H}$ vanishes), equation (28) becomes

$$0=-\zeta^*(2)+\frac{1}{2T}+\frac{1}{2}\mathrm{vol}(\mathcal{D}_T)$$

or

$$\mathrm{vol}(\mathcal{D}_T)=\frac{\pi}{3}-\frac{1}{T},\tag{34}$$

and equation (19) states

$$\iint_{\Gamma\backslash\mathfrak{H}}E^*(z,s)d\mu=0\qquad(0<\mathrm{Re}(s)<1).\tag{35}$$

Equation (34), of course, can be checked easily by hand, and equation (35) can be proved directly by noting that $E^*(z, s)$ is an eigenfunction of the Laplace operator with eigenvalue $\neq 0$; hence, since the Laplace operator is self-adjoint, its integral over $\Gamma\backslash\mathfrak{H}$ (in the range of s for which this integral exists) must vanish. (One can also argue the same way using any Hecke operator $T(n)$, $n>1$, instead of the Laplace operator.) Equation (33) is also known and can be given a direct proof: one first checks that the difference between the left- and right-hand sides of (33) is independent of T (this is just a question of computing $\iint_{\substack{0\leq x\leq 1\\ T\leq y\leq T'}} e(y, s)d\mu$) and then shows that this difference is zero for $0<\mathrm{Re}(s)<1$ by letting T go to ∞ in (35); the result for general s then follows by analytic continuation.

As an application of (33) we can give a new definition of the functional $R.N.\left(\int_{\Gamma\backslash\mathfrak{H}}\cdot d\mu\right)$ defined above. Indeed, since the renormalized integral of an Eisenstein series is zero, we can add to F an arbitrary Eisenstein series or linear combination of derivatives of Eisenstein series without changing $R.N.\left(\int F\, d\mu\right)$, and by suitably choosing the linear combination we can make F grow sufficiently slowly at infinity that its integral converges. Explicitly, for F as in (13), (14) we define

$$F_1(z)=F(z)-\sum_{\alpha_i\geq 1/2} C_i\frac{\partial^{n_i}}{\partial\alpha_i^{n_i}}E(z, \alpha_i);$$

then F and F_1 have the same renormalized integral over $\Gamma\backslash\mathfrak{H}$ but $F_1(z)$ is $O(y^{1/2})$ and hence actually integrable, so we have

$$R.N.\left(\int_{\Gamma\backslash\mathfrak{H}}F(z)d\mu\right)=\int_{\Gamma\backslash\mathfrak{H}}F_1(z)d\mu, \tag{36}$$

and this equation could be taken as an alternative definition of the renormalization procedure.

2. Eisenstein series. Our next example is $F(z)=E^*(z, s_1)$, $s_1\in\boldsymbol{C}\backslash\{0, 1\}$. Here $a_0(y)=e(y, s_1)$, and since this is a finite linear combination of powers of y (respectively powers of y times powers of $\log y$ if $s_1=1/2$) we again have $\varphi(y)=a_0(y)$ and $R(F; s)\equiv 0$. The data l, c_i, α_i, n_i of equation (14) are given according to (24) by $l=2$ and

i	c_i	α_i	n_i		i	c_i	α_i	n_i
1	$\zeta^*(2s_1)$	s_1	0	or	1	1	$\frac{1}{2}$	1
2	$\zeta^*(2s_1-1)$	$1-s_1$	0		2	$\gamma-\log 4\pi$	$\frac{1}{2}$	0

for $s_1 \neq 1/2$ and $s_1 = 1/2$, respectively. Hence (ignoring from now on the special case $s_1 = 1/2$) we find that equation (17) reduces to the assertion that

$$\frac{\zeta^*(2s_1)\zeta^*(2s)}{s_1+s-1}+\frac{\zeta^*(2s_1)\zeta^*(2s-1)}{s_1-s}+\frac{\zeta^*(2s_1-1)\zeta^*(2s)}{-s_1+s}+\frac{\zeta^*(2s_1-1)\zeta^*(2s-1)}{1-s_1-s}$$

is holomorphic for $s,\ s_1 \neq 0,\ 1$ (this is easily checked using the functional equation of $\zeta^*(s)$); the statement about $\mathrm{Res}_{s=1} R(F, s)$ in the theorem gives $\int_{\mathcal{D}} F\, d\mu = 0$ for $0 < \mathrm{Re}(s_1) < 1$ (i.e. another proof of equation (35)), equation (28) specializes to (33) (with s_1 instead of s), and equation (19) is empty (since $\Theta = \max(\mathrm{Re}(s_1),\ 1-\mathrm{Re}(s_1))$ is always $\geqq 1/2$). Finally, equation (27) becomes

$$\begin{aligned}\iint_{\mathcal{D}_T} & E^*(z, s)E^*(z, s_1)d\mu + \iint_{\mathcal{D}-\mathcal{D}_T}(E^*(z, s)E^*(z, s_1) - e(y, s)e(y, s_1))d\mu \\ &= \frac{\zeta^*(2s)\zeta^*(2s_1)}{s+s_1-1}T^{s+s_1-1} + \frac{\zeta^*(2s)\zeta^*(2s_1-1)}{s-s_1}T^{s-s_1} \\ &\quad + \frac{\zeta^*(2s-1)\zeta^*(2s_1)}{s_1-s}T^{s_1-s} + \frac{\zeta^*(2s-1)\zeta^*(2s_1-1)}{1-s-s_1}T^{1-s-s_1},\end{aligned} \tag{37}$$

a formula known as the "Maass-Selberg relation" (see e.g. [3], Theorem 2.3.1). The reader may amuse himself by comparing the functional equations, residues, etc. on both sides of the identity (37), and also by working out the statement of the identity when $s = s_1$ or $1 - s_1$ or when s or s_1 equals $1/2$.

The Eisenstein series $E^*(z, s)$ and $E^*(z, s_1)$ are eigenfunctions of the Laplace operator with distinct eigenvalues (for $s_1 \neq s,\ 1-s$) and hence the integral $\int_{\Gamma\backslash\mathfrak{H}} E^*(z, s)E^*(z, s_1)d\mu$ should vanish identically. However, this integral is divergent for all values of s and s_1, so we have to find a suitable interpretation of this statement. Formula (37), which with the terminology introduced earlier can be written simply

$$R.N.\Big(\int_{\Gamma\backslash\mathfrak{H}} E^*(z, s)E^*(z, s_1)d\mu\Big) = 0,$$

describes one possibility; another, in view of the remarks at the end of Example 1, is to subtract from the product $E^*(z, s)E^*(z, s_1)$ a suitable combination of Eisenstein series to make the integral converge, e.g. for $\mathrm{Re}(s) > \max(\mathrm{Re}(s_1),\ \mathrm{Re}(1-s_1))$,

$$\int_{\Gamma\backslash\mathfrak{H}}\Big(E(z, s)E(z, s_1) - E(z, s+s_1) - \frac{\zeta^*(2s_1-1)}{\zeta^*(2s_1)}E(z, s+1-s_1)\Big)d\mu = 0.$$

Finally, we can use the result of this example to give a new definition of the Rankin-Selberg transform as a convergent integral in many cases. Indeed, since the Rankin-Selberg transform of an Eisenstein series vanishes, we can modify F by a linear combination of Eisenstein series and their derivatives without changing $R(F; s)$. In particular, replacing F by the function F_1 of (36) will not change $R(F; s)$ but will replace the number $\Theta=\max_i(\max(\mathrm{Re}(\alpha_i), \mathrm{Re}(1-\alpha_i)))$ by $\Theta_1=\max_i(\min(\mathrm{Re}(\alpha_i), \mathrm{Re}(1-\alpha_i)))$ which is always $\leqq 1/2$; if all α_i have real part distinct from 1/2 then $\Theta_1<1/2$ and the last statement of the theorem gives

$$R(F; s)=\int_{\Gamma\backslash\mathfrak{H}} F_1(z)E(z, s)d\mu \qquad (\Theta_1<\mathrm{Re}(s)<1-\Theta_1),$$

where now the integral is convergent in the range stated.

3. Products of Eisenstein series. Our first example with $R(F; s)$ not identically zero is $F(z)=E^*(z, s_1)E^*(z, s_2)$, where s_1 and s_2 are complex numbers different from 0, 1. To compute $R(F; s)$ in this case we need the full Fourier development of the Eisenstein series, not just the constant term; it is given by

$$E^*(z, s)=e(y, s)+2\sqrt{y}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\tau_{s-1/2}(n)K_{s-1/2}(2\pi|n|y)e^{2\pi inx}, \tag{38}$$

where $\tau_\nu(n)$ is the sum-of-divisors function

$$\tau_\nu(n)=|n|^\nu\sum_{\substack{d\mid n\\ d>0}}d^{-2\nu} \qquad (n\in\boldsymbol{Z},\ n\neq 0,\ \nu\in\boldsymbol{C}) \tag{39}$$

and $K_\nu(t)$ the Bessel function

$$K_\nu(t)=\int_0^\infty e^{-t\cosh u}\cosh\nu u\, du \qquad (\nu, t\in\boldsymbol{C},\ \mathrm{Re}(t)>0), \tag{40}$$

both of which are even functions of ν. Taking $s=s_1$ and $s=s_2$ in (38) and multiplying together the two expressions obtained, we find that the constant term of $F(z)$ is given by

$$a_0(y)=e(y, s_1)e(y, s_2)+8y\sum_{n=1}^{\infty}\tau_{s_1-1/2}(n)\tau_{s_2-1/2}(n)K_{s_1-1/2}(2\pi ny)K_{s_2-1/2}(2\pi ny). \tag{41}$$

The sum is of rapid decay, since the Bessel functions are, and the term $e(y, s_1)\cdot e(y, s_2)$ is a linear combination of powers of y (respectively powers of y times $\log y$ or $\log^2 y$ if s_1 and/or s_2 equals 1/2). Therefore $F(z)$ satisfies the conditions of the theorem with

$$\varphi(y)=e(y, s_1)e(y, s_2), \tag{42}$$

i.e. (if s_1 and s_2 are $\neq 1/2$) with $l=4$ and c_i, α_i, n_i given by

(43)

i	c_i	α_i	n_i
1	$\zeta^*(2s_1)\zeta^*(2s_2)$	s_1+s_2	0
2	$\zeta^*(2s_1-1)\zeta^*(2s_2)$	$1-s_1+s_2$	0
3	$\zeta^*(2s_1)\zeta^*(2s_2-1)$	$1+s_1-s_2$	0
4	$\zeta^*(2s_1-1)\zeta^*(2s_2-1)$	$2-s_1-s_2$	0

Substituting (41) and (42) into (15) we find

$$R(F;\, s)=8\int_0^\infty y^{s-1}\sum_{n=1}^\infty \tau_{s_1-1/2}(n)\tau_{s_2-1/2}(n)K_{s_1-1/2}(2\pi n y)K_{s_2-1/2}(2\pi n y)dy$$

$$=8(2\pi)^{-s}\Big(\sum_{n=1}^\infty \tau_{s_1-1/2}(n)\tau_{s_2-1/2}(n)n^{-s}\Big)\Big(\int_0^\infty K_{s_1-1/2}(t)K_{s_2-1/2}(t)t^{s-1}dt\Big).$$

The Dirichlet series in parentheses is equal to the product of zeta-functions

$$\zeta(s+s_1+s_2-1)\zeta(s-s_1+s_2)\zeta(s+s_1-s_2)\zeta(s-s_1-s_2+1)/\zeta(2s),$$

as one checks be expanding it in an Euler product and computing each Euler factor as a geometric series (we leave the computation to the reader), and the integral in parentheses is equal to

$$2^{s-3}\Gamma\Big(\frac{s+s_1+s_2-1}{2}\Big)\Gamma\Big(\frac{s-s_1+s_2}{2}\Big)\Gamma\Big(\frac{s+s_1-s_2}{2}\Big)\Gamma\Big(\frac{s-s_1-s_2+1}{2}\Big)/\Gamma(s)$$

([1], 6.8 (48)), i.e. to the corresponding product of gamma factors. Hence

(44) $$R^*(F;\, s)=\zeta^*(s+s_1+s_2-1)\zeta^*(s-s_1+s_2)\zeta^*(s+s_1-s_2)\zeta^*(s-s_1-s_2+1).$$

From this explicit formula we can easily check the various statements of the theorem, viz. that $R^*(F;s)$ is holomorphic except for simple poles at the ten points 0, 1, α_i, and $1-\alpha_i$ with

$$\mathrm{Res}_{s=\alpha_i}R^*(F;\, s)=c_i\zeta^*(2\alpha_i-1),\quad \mathrm{Res}_{s=1-\alpha_i}R^*(F;\, s)=-c_i\zeta^*(2\alpha_i)$$

(α_i and c_i as in (43)). Substituting (43) and (44) into equation (15), we find the identity

(45) $$\iint_{\mathcal{D}_T} E^*(z,\, s)E^*(z,\, s_1)E^*(z,\, s_2)d\mu$$
$$+\iint_{\mathcal{D}-\mathcal{D}_T}(E^*(z,\, s_1)E^*(z,\, s_2)E^*(z,\, s)-e(y,\, s_1)e(y,\, s_2)e(y,\, s))d\mu$$

$$=\zeta^*(s+s_1+s_2-1)\zeta^*(s-s_1+s_2)\zeta^*(s+s_1-s_2)\zeta^*(s-s_1-s_2+1)$$
$$+\sum \zeta^*(2x)\zeta^*(2x_1)\zeta^*(2x_2)\frac{T^{x+x_1+x_2-1}}{x+x_1+x_2-1},$$

where the sum is over the eight combinations of values

$$x=s \text{ or } 1-s, \quad x_1=s_1 \text{ or } 1-s_1, \quad x_2=s_2 \text{ or } 1-s_2.$$

Equation (45) states that $R.N.\left(\int E^*(z, s)E^*(z, s_1)E^*(z, s_2)d\mu\right)$ equals the product of zeta-functions in (44) for all values of s, s_1, s_2; as in the last two examples, we can replace the renormalized integral by convergent integrals in certain domains, e.g. for $\mathrm{Re}(s)$ sufficiently large by

$$\iint_{\Gamma\backslash\mathfrak{H}}\left(E^*(z, s)E^*(z, s_1)E^*(z, s_2)-\zeta^*(2s)\sum_{x_1, x_2}\frac{\zeta^*(2x_1)\zeta^*(2x_2)}{\zeta^*(2s+2x_1+2x_2)}E^*(z, s+x_1+x_2)\right)d\mu$$

(same conventions on x_1, x_2 as before) or by

$$\iint_{\Gamma\backslash\mathfrak{H}}\left(E^*(z, s)E^*(z, s_1)-\zeta^*(2s)\sum_{x_1}\frac{\zeta^*(2x_1)}{\zeta^*(2s_1+2x_1)}E^*(z, s+x_1)\right)E^*(z, s_2)d\mu.$$

We can think of equation (45) as a generalization of the Maass-Selberg relation (37), which can be recovered from it by computing residues at $s_2=1$.

After the last three examples it is natural to ask for the renormalized integral of a product of four or more Eisenstein series. The function $F(z) = \prod_{j=1}^{n} E^*(z, s_j)$ satisfies the conditions of our theorem (with $\varphi(y)=\prod e(y, s_j)$) for any n and any $s_j\in \boldsymbol{C}\backslash\{0, 1\}$, so $R.N.\left(\int F(z)E(z, s)d\mu\right)$ can be written as a Mellin transform. But already for $n=3$ this transform cannot be computed in closed form since this would involve evaluating sums like

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\tau_{s_1-1/2}(n_1)\tau_{s_2-1/2}(n_2)\tau_{s_3-1/2}(n_1+n_2)$$
$$\cdot\int_0^{\infty}K_{s_1-1/2}(n_1t)K_{s_2-1/2}(n_2t)K_{s_3-1/2}((n_1+n_2)t)t^{s-1/2}dt,$$

which cannot be reduced to Dirichlet series as was done for $n=2$.

4. Restrictions of Hecke-Eisenstein series (I am indebted to Carlos Moreno for suggesting this application of the theorem). If K is a totally real number field of degree n, then following Hecke one can define Eisenstein series $E_K(z_1, \cdots, z_n; s)$, $E_K^*(z_1, \cdots, z_n; s)$ ($z_j\in\mathfrak{H}$, $s\in \boldsymbol{C}$) which are invariant under the action

$$z=(z_1,\cdots,z_n)\longrightarrow\left(\frac{a^{(1)}z_1+b^{(1)}}{c^{(1)}z_1+d^{(1)}},\cdots,\frac{a^{(n)}z_n+b^{(n)}}{c^{(n)}z_n+d^{(n)}}\right)$$

$(a, b, c, d$ integers of K, $ad-bc=1$, $x^{(j)}=j^{th}$ conjugate of $x)$

of the Hilbert modular group. Specializing to $z_1=\cdots=z_n=z$ gives functions on $SL_2(\boldsymbol{Z})\backslash\mathfrak{H}$ to which our theorem applies. The Eisenstein series are defined by

$$E_K(z,s)=\sum_{A\in C}E_{K,A}(z,s),\quad E_K^*(z,s)=\sum_{A\in C}E_{K,A}^*(z,s);$$

here C is the ideal class group of K and, for an ideal class $A\in C$,

$$E_{K,A}(z,s)=N(\mathfrak{a})^{2s}\sum_{\substack{(\alpha,\beta)\in\mathfrak{a}\times\mathfrak{a}/\text{units}\\(\alpha,\beta)=\mathfrak{a}}}\prod_{j=1}^{n}\frac{\operatorname{Im}(z_j)^s}{(\alpha^{(j)}z_j+\beta^{(j)})^{2s}}\qquad(\operatorname{Re}(s)>1)$$

($\mathfrak{a}$=any ideal in A^{-1}, sum over non-associated pairs of numbers $\alpha,\beta\in\mathfrak{a}$ with greatest common divisor $\mathfrak{a}$), while $E_{K,A}^*$ is defined by the same series but without the restriction $(\alpha,\beta)=\mathfrak{a}$ and with a factor $\pi^{-ns}\Gamma(s)^nD^s$ (D=discriminant of K) in front. The series E and E^* are related by

$$E_{K,A}^*(z,s)=\sum_{B\in C}\zeta_{K,AB^{-1}}^*(2s)E_{K,B}(z,s),\qquad E_K^*(z,s)=\zeta_K^*(2s)E_K(z,s),$$

where

$$\zeta_{K,A}^*(s)=\pi^{-ns/2}\Gamma\left(\frac{s}{2}\right)^nD^{s/2}\sum_{\substack{\mathfrak{a}\in A\\\mathfrak{a}\text{ integral}}}N(\mathfrak{a})^{-s},\quad\zeta_K^*(s)=\sum_A\zeta_{K,A}^*(s)$$

denote the zeta-function of an ideal class A and the Dedekind zeta-function of K together with their gamma factors. We have the functional equations

$$\zeta_{K,A}^*(s)=\zeta_{K,\mathfrak{d}^{-1}A^{-1}}^*(1-s),\quad\zeta_K^*(s)=\zeta_K^*(1-s),\quad\zeta_{K,A}^*(z,s)=E_{K,\mathfrak{d}^{-1}A^{-1}}^*(1-s),$$
$$E_K^*(z,s)=E_K^*(z,1-s)$$

($\mathfrak{d}$=different of K) and the Fourier expansion

$$E_K^*(z,s)=\zeta_K^*(2s)y_1^s\cdots y_n^s+\zeta_K^*(2s-1)y_1^{1-s}\cdots y_n^{1-s}$$
$$+2^ny_1^{1/2}\cdots y_n^{1/2}\sum_{\substack{\xi\in\mathfrak{d}^{-1}\\\xi\neq0}}\tau_{s-1/2}^K(\xi\mathfrak{d})\prod_{j=1}^nK_{s-1/2}(2\pi|\xi^{(j)}|y_j)e^{2\pi i\xi^{(j)}x_j},$$

where $z_j=x_j+iy_j$ and $\tau_{s-1/2}^K(\mathfrak{a})$ ($\mathfrak{a}$ an integral ideal) is defined as the sum over all integral divisors $\mathfrak{b}$ of $\mathfrak{a}$ of $N(\mathfrak{a})^{s-1/2}/N(\mathfrak{b})^{2s-1}$. A similar expansion holds for $E_{K,A}^*$ if we replace ζ_K^* by $\zeta_{K,A}^*$ and $\tau_{s-1/2}^K$ by $\tau_{s-1/2}^{K,A}$ (defined by summing only over $\mathfrak{b}\in A$). It follows that the function

$$F_{K,s_1}(z)=E_K^*(\underbrace{(z,\cdots,z)}_{n},s_1)\qquad(s_1\in\boldsymbol{C},\ z\in\mathfrak{H})$$

satisfies the condition of the theorem with

$$\varphi(y)=\zeta_K^*(2s_1)y^{ns_1}+\zeta_K^*(2s-1)y^{n(1-s_1)},$$

so the theorem implies:

PROPOSITION. *Let K be a totally real number field of degree n, $s_1\in\boldsymbol{C}$. Then the sum*

$$R_K(s,\ s_1)=2^n\sum_{\substack{\xi\in\mathfrak{d}^{-1}\\ \xi\neq 0\\ \mathrm{Tr}\,\xi=0}}\tau_{s_1-1/2}^K(\xi\mathfrak{d})\int_0^\infty\prod_{j=1}^n K_{s_1-1/2}(2\pi|\xi^{(j)}|y)y^{s+n/2-2}dy\,, \tag{46}$$

converges for $s\in\boldsymbol{C}$ with $\mathrm{Re}(s)>n\max(\mathrm{Re}(s_1),\ \mathrm{Re}(1-s_1))$ *and has a meromorphic continuation in s; the function $R_K^*(s,\ s_1)=\zeta^*(2s)R_K(s,\ s_1)$ has (at most) simple poles at $s=0$, 1, ns_1, $n(1-s_1)$, $1-ns_1$ and $1-n(1-s_1)$, and satisfies the functional equation $R_K^*(s,\ s_1)=R_K^*(1-s,\ s_1)$ (as well as the obvious invariance under $s_1\to 1-s_1$). The residue of $R_K^*(\cdot,\ s_1)$ at $s=ns_1$ is $\zeta_K^*(2s_1)$; the residue at $s=1$ is $1/2$ times the renormalized integral over $\Gamma\backslash\mathfrak{H}$ of the restriction to the diagonal of the Hecke-Eisenstein series $E_K^*(z_1,\cdots,z_n;s_1)$. Similar statements hold for the function $R_{K,A}$ obtained by replacing τ^K by $\tau^{K,A}$ in* (46).

The case $n=1$ corresponds to our example 2; the case $n=2$ contains (if we take for K the algebra $\boldsymbol{Q}\times\boldsymbol{Q}$ instead of a real quadratic field) the case $s_1=s_2$ of Example 3. For $n=2$ we can explicitly compute $R_K(s,\ s_1)$: writing $\xi=m/\sqrt{D}$ (D the discriminant of the real quadratic field K) we find

$$R_K(s,\ s_1)=\sum_{m=1}^{\infty}\tau_{s_1-1/2}^K((m))m^{-s}\cdot 8(2\pi/\sqrt{D})^{-s}\int_0^\infty K_{s_1-1/2}(y)^2y^{s-1}dy\,;$$

the Dirichlet series is easily computed to be

$$\zeta(2s)^{-1}\zeta(s+2s_1-1)\zeta_K(s)\zeta(s-2s_1+1)$$

and the second factor to be

$$\pi^{-s}D^{s/2}\Gamma(s)^{-1}\Gamma\Big(\frac{s-1}{2}+s_1\Big)\Gamma\Big(\frac{s}{2}\Big)^2\Gamma\Big(\frac{s+1}{2}-s_1\Big)$$

(compare the case $s_1=s_2$ of Example 3), so

$$R_K^*(s,\ s_1)=\zeta^*(s+2s_1-1)\zeta^*(s-2s_1+1)\zeta_K^*(s)$$

and similarly for $R_{K,A}^*$ with ζ_K^* replaced by $\zeta_{K,A}^*$. For $n\geqq 3$, we cannot compute R_K explicitly for the same reasons as in the case when F was a product of n Eisenstein series.

5. Petersson scalar product for non-cusp forms. If $f(z)=\sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ and $g(z)=\sum_{n=0}^{\infty} b_n e^{2\pi i n z}$ are holomorphic modular forms of weight k on $SL_2(\boldsymbol{Z})$, with at least one of them a cusp form (i.e. $a_0 b_0=0$), then the Petersson scalar product (f, g) is defined by

$$(f, g)=\int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)} y^k d\mu ; \tag{47}$$

according to Rankin [4], this can also be computed as

$$(f, g)=\frac{\pi}{3}\cdot(k-1)!\cdot(4\pi)^{-k} \operatorname{Res}_{s=k} L_{f,g}(s), \tag{48}$$

where $L_{f,g}(s)=\sum_{n=1}^{\infty} a_n \bar{b}_n n^{-s}$. If neither f nor g is a cusp form, then the integral in (47) diverges, but the Γ-invariant function $F(z)=y^k f(z)\,\overline{g(z)}$ is renormalizable and we could define (f, g) as its renormalized integral over $\Gamma\backslash\mathfrak{H}$. Then our theorem shows that (48) still holds, so we have the various equivalent definitions

$$\begin{aligned}(f, g)&=\iint_{\mathscr{D}_T} f(z)\overline{g(z)} y^k d\mu+\iint_{\mathscr{D}-\mathscr{D}_T}(f(z)\overline{g(z)}-a_0\bar{b}_0) y^k d\mu-a_0\bar{b}_0\frac{T^{k-1}}{k-1} \quad (\text{any } T)\\ &=\lim_{T\to\infty}\left(\iint_{\mathscr{D}_T} f(z)\overline{g(z)} y^k d\mu-\frac{a_0\bar{b}_0}{k-1}T^{k-1}\right)\\ &=\iint_{\Gamma\backslash\mathfrak{H}}(f(z)\overline{g(z)}y^k-a_0\bar{b}_0 E(z, k))d\mu\\ &=\frac{\pi}{3}\frac{(k-1)!}{(4\pi)^k}\operatorname{Res}_{s=k}\left(\sum_{n=1}^{\infty}\frac{a_n\bar{b}_n}{n^s}\right).\end{aligned}$$

(Note that in the last line the residue can only be defined after continuing the series analytically.)

We compute this value for f and g both equal to the Eisenstein series

$$G_k(z)=-\frac{B_k}{2k}+\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n z} \qquad (k\geqq 4,\ k \text{ even})$$

(this is sufficient since any modular form is a multiple of G_k plus a cusp form). With $F(z)=y^k|G_k(z)|^2$ we find

$$a_0(y)=\frac{B_k^2}{4k^2}y^k+\sum_{n=1}^{\infty}\sigma_{k-1}(n)^2 y^k e^{-4\pi n y}, \qquad \varphi(y)=\frac{B_k^2}{4k^2}y^k,$$

$$R(F; s)=\sum_{n=1}^{\infty}\sigma_{k-1}(n)^2 n^{-s-k+1}\cdot\int_0^{\infty} e^{-4\pi y} y^{k+s-2} dy$$

$$=\underbrace{\frac{\zeta(s)^2\zeta(s+k-1)\zeta(s-k+1)}{\zeta(2s)}}_{=L_{G_k,G_k}(s+k-1)}\cdot(4\pi)^{-s-k+1}\Gamma(s+k-1),$$

$$R^*(F;s)=\pi^{-2s-k+1}2^{-2s-2k+1}\Gamma(s)\Gamma(s+k-1)\zeta(s)^2\zeta(s+k-1)\zeta(s-k+1)$$

$$=(-1)^{k/2}(2\pi)^{-k}\prod_{j=0}^{k/2-1}\Big(j+\frac{s}{2}\Big)\Big(j+\frac{1-s}{2}\Big)\cdot\zeta^*(s)^2\zeta^*(s+k-1)\zeta^*(s-k+1).$$

The last formula puts the functional equation into evidence and shows that $R^*(F;s)$ has simple poles at $s=0$, 1, k and $1-k$, in accordance with our theorem; computing the residue at $s=k$ we find

$$(G_k,G_k)=R.N.\Big(\int F(z)d\mu\Big)=2\operatorname{Res}_{s=1}R^*(F;s)$$

$$=\frac{(-1)^{k/2-1}}{\pi^k}\frac{(k-1)!}{2^{2k-1}}\zeta^*(k)\zeta^*(2-k)$$

$$=(-1)^{k/2-1}\frac{(k-1)!(k-2)!}{2^{3k-3}\pi^{2k-1}}\zeta(k)\zeta(k-1).$$

Thus there is a natural extension of the Petersson scalar product from the space of cusp forms to the space of all modular forms, and this scalar product is always non-degenerate, but it is positive-definite if and only if $k\equiv 2\pmod 4$.

6. Functions constructed from the Weil representation. In [8] it was shown that, for Φ any rapidly decaying function on $\boldsymbol{R}$ and D an integer which is congruent to 0 or 1 (mod 4) and not a perfect square, the function

$$L_D\Phi(z)=\sum_{\substack{(a,b,c)\in Z^3\\ b^2-4ac=D}}\Phi\Big(\frac{a|z|^2+bx+c}{y}\Big)\qquad(z=x+iy\in\mathfrak{H})$$

is a Γ-invariant function of rapid decay whose Rankin-Selberg transform is divisible by $\zeta(s)$; the fact was used to give a characterization of the zeros of the Riemann zeta-function. If D is a square, then $L_D\Phi(z)$ is the sum of Cy and a function of rapid decay as $y\to\infty$, where

$$C=\begin{cases}\dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty}\Phi(t)dt & D=0,\\[2ex] \displaystyle\int_{-\infty}^{\infty}\Phi(t)dt & D>0,\ \sqrt{D}\in\boldsymbol{Z},\end{cases}$$

and in order to make sense of the assertion we had to assume that the integral of Φ vanished. With the definition made in this paper, this assumption is not needed; we can define the Rankin-Selberg transform in a uniform way and find that $R^*(L_D\Phi, s)/\zeta^*(s)$ is entire for any D.

7. Selberg trace formula. The Selberg trace formula is the formula for the trace of a Hecke operator T on a space of cusp forms obtained by constructing a kernel function $K(z_1, z_2)$ for T and then computing $\int_{\Gamma\backslash\mathfrak{H}} K(z, z)d\mu$. In [6] and [7] it was shown how to derive this formula in the holomorphic and non-holomorphic cases, respectively, by computing the Rankin-Selberg transform of $K(z, z)$ and then taking the residue at $s=1$. The function $K(z, z)$ is of rapid decay, but has a natural decomposition as $\sum_C K_C(z)+K_\infty(z)$, where the sum runs over conjugacy classes in Γ and K_∞ describes the contribution from the cusp. The functions K_C (except for C elliptic) and K_∞ are not of rapid decay but are renormalizable (their growth at ∞ is typically of the type $Ay\log y+By+O(\log y)$), so we can compute $\operatorname{tr}(T)$ as the sum of the renormalized integrals of the K_C and K_∞, and then choose for each case whether it is advantageous to calculate this integral directly (e.g. for C elliptic) or by computing the Rankin-Selberg transform and its residue. Each of the functions K_C is of the type discussed in Example 6 above (with $D=t^2-4$, where $\pm t$ is the trace of the elements of C), so the Rankin-Selberg transforms $R^*(K_C, s)$ can be computed explicitly and are divisible by $\zeta^*(s)$. The function K_∞ is zero in the holomorphic case; in the case treated in [7], namely non-holomorphic modular forms of weight 0 (Maass wave forms), it is given by

$$K_\infty(z)=-\frac{3}{\pi}h\Big(\frac{i}{2}\Big)-\frac{1}{4\pi}\int_{-\infty}^{\infty}E\Big(z, \frac{1}{2}+ir\Big)E\Big(z, \frac{1}{2}-ir\Big)h(r)dr\,,$$

where $h(r)$ (Selberg transform) is a holomorphic function of rapid decay in a strip around the real axis, so the results of Examples 1 and 3 give

$$R^*(K_\infty, s)=-\frac{1}{4\pi}\zeta^*(s)^2\int_{-\infty}^{\infty}\frac{\zeta^*(s+2ir)\zeta^*(s-2ir)}{\zeta^*(1+2ir)\zeta^*(1-2ir)}h(r)dr\,,$$

again divisible by $\zeta^*(s)$. (Here the function $K(z_1, z_2)$ equals $\sum_j h(r_j)f_j(z_1)\overline{f_j(z_2)}$, where $\{f_j\}$ is an orthonormal basis for the space of Maass cusp forms of weight 0 and $r_j^2+1/4$ the corresponding eigenvalues of the Laplacian, so we get the identity $\sum_j h(r_j)R^*(|f_j|^2, s)=\sum_C R^*(K_C, s)+R^*(K_\infty, s)$, from which the divisibility of each function $R^*(|f_j|^2, s)$ by $\zeta^*(s)$ can be deduced.) The point is that, using the method of this paper, we can calculate the various Rankin-Selberg transforms $R^*(K_C, s)$, $R^*(K_\infty, s)$ directly, whereas in [6] and [7] it was necessary to first calculate the constant terms of K_C and K_∞ and then split them up into pieces and recombine the pieces into functions of rapid decay. In this way some of the analytic difficulties in the proof of the Selberg trace formula (or, more generally, in the calculation of $\int K(z, z)E(z, s)d\mu$) are avoided.

Bibliography

[1] Erdelyi, A. et al., Tables of Integral Transforms, Vol. I., McGraw-Hill, New York, 1954.

[2] Jacquet, H. and D. Zagier, Eisenstein series and the Selberg trace formula, II, in preparation.

[3] Kubota, T., Elementary Theory of Eisenstein Series, Kodansha and John Wiley, Tokyo-New York, 1973.

[4] Rankin, R., Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions, I, Proc. Cambridge Philos. Soc. **35** (1939), 351-372.

[5] Selberg, A., Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist, Arch. Math. Naturvid. **43** (1940), 47-50.

[6] Zagier, D., Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields, in Modular Functions of One Variable VI, Lecture Notes in Math. No. 627, Springer, Berlin-Heidelberg-New York, 1977, pp. 107-169.

[7] Zagier, D., Eisenstein series and the Selberg trace formula I, in Automorphic Forms, Representation Theory and Arithmetic, Bombay 1979, Springer-Verlag, Berlin-Heidelberg-New York, 1981, pp. 303-355.

[8] Zagier, D., Eisenstein series and the Riemann zeta-function, in Automorphic Forms, Representation Theory and Arithmetic, Bombay 1979, Springer-Verlag, Berlin-Heidelberg-New York, 1981, pp. 275-301.

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