ON UNIFORM LARGE GENUS ASYMPTOTICS OF WITTEN'S INTERSECTION NUMBERS

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Abstract. Following ideas from [14], we give a uniform large genus asymptotics for primitive psi-class intersection numbers on the moduli space of stable algebraic curves, and extend this result including insertions of zeros in a certain uniform way. Application to a particular formal solution of the Painlevé I equation is given. We also use a method from [14] to give a new proof of the polynomiality conjecture on large genus asymptotic expansions of psi-class intersection numbers.

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1. Introduction and statements of the main results

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne–Mumford moduli space of stable algebraic curves of genus g with n distinct marked points. In [28] Witten proposed a striking conjecture on the relationship between topology of $\overline{\mathcal{M}}_{g,n}$ and the celebrated Korteweg–de Vries (KdV) integrable hierarchy. To be precise, let

$$Z(\mathbf{t};\epsilon) \;:=\; \exp\biggl(\sum_{g,n}\frac{\epsilon^{2g-2}}{n!}\sum_{d_1,\dots,d_n} t_{d_1}\cdots t_{d_n}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\biggr)\,, \tag{1}$$

be the *partition function of psi-class intersection numbers*, where $\mathbf{t}=(t_0,t_1,t_2,\dots)$ is an infinite vector of indeterminates, ϵ is an indeterminate often called the string coupling constant, and ψ_j denotes the first Chern class of the jth tautological line bundle on $\overline{\mathcal{M}}_{g,n}$. Here and below, $\sum_i := \sum_{i\ge 0}$. Define

$$u \;=\; u(\mathbf{t};\epsilon) \;:=\; \epsilon^2\,\frac{\partial^2\log Z(\mathbf{t};\epsilon)}{\partial t_0^2}\,. \tag{2}$$

Then Witten's conjecture claims that u satisfies the KdV hierarchy:

$$\frac{\partial u}{\partial t_k} \;=\; \frac{1}{(2k+1)!!}\Bigl[\bigl(L^{\frac{2k+1}{2}}\bigr)_+,\,L\Bigr]\,,\quad k\ge 0\,, \tag{3}$$

where $L := \epsilon^2\partial_{t_0}^2 + 2u$ is the Lax operator (cf. e.g. [7]). The $k = 1$ equation in (3) reads

$$\frac{\partial u}{\partial t_1} \;=\; u\frac{\partial u}{\partial t_0} + \frac{\epsilon^2}{12}\frac{\partial^3 u}{\partial t_0^3}\,, \tag{4}$$

which is the celebrated KdV equation. Another way of stating Witten's conjecture is that $Z(\mathbf{t};\epsilon)$ is a tau-function for the KdV hierarchy; see e.g. [2, 3, 7, 10, 11] for the notion of a KdV tau-function.

We will refer to the psi-class intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\,, \qquad g,n,d_1,\dots,d_n \ge 0\,, \tag{5}$$

as *Witten's intersection numbers*. They play important roles in studying Weil–Petersson volumes of $\overline{\mathcal{M}}_{g,n}$ [19, 22, 25, 26, 27] and Masur–Veech volumes of moduli space of quadratic differentials [1, 6]. They are called *primitive* if $d_1 \ge 2$, …, $d_n \ge 2$. By the degree-dimension matching we know that the numbers (5) vanish unless

$$d_1 + \cdots + d_n \;=\; 3g - 3 + n\,. \tag{6}$$

Witten's conjecture was first proved by Kontsevich [20]. This conjecture also can [8] be equivalently stated that the partition function $Z = Z(\mathbf{t};\epsilon)$ satisfies the Virasoro constraints:

$$L_m(Z) \;=\; 0\,, \quad \forall\, m \ge -1\,, \tag{7}$$

where L_m are linear operators defined by

$$\begin{aligned} L_m \;=\; & -\frac{(2m+3)!!}{2^{m+1}}\frac{\partial}{\partial t_{m+1}} + \sum_d \frac{(2d+2m+1)!!}{(2d-1)!!\,2^{m+1}}\,t_d\,\frac{\partial}{\partial t_{d+m}} \\ & + \frac{\epsilon^2}{2}\sum_{d=0}^{m-1}\frac{(2d+1)!!(2m-2d-1)!!}{2^{m+1}}\frac{\partial^2}{\partial t_d\partial t_{m-1-d}} + \frac{t_0^2}{2\epsilon^2}\,\delta_{m,-1} + \frac{1}{16}\,\delta_{m,0}\,, \end{aligned} \tag{8}$$

which satisfy the Virasoro commutation relations:

$$[L_{m_1}, L_{m_2}] \;=\; (m_1 - m_2)\,L_{m_1+m_2}\,, \quad m_1, m_2 \ge -1\,. \tag{9}$$

It was shown in [8] that the Virasoro constraints (7) are equivalent to the following recursive relation:

$$\begin{aligned} U(\mathbf{d}) \;=\; & \sum_{j=2}^{n}(2d_j+1)\,U(d_2,\dots,d_j+d_1-1,\dots,d_n) \\ & + \frac{1}{2}\sum_{\substack{a,b\\ a+b=d_1-2}}\left(U(a,b,d_2,\dots,d_n) + \sum_{I\sqcup J=\{2,\dots,n\}} U(a,\mathbf{d}_I)\,U(b,\mathbf{d}_J)\right), \end{aligned} \tag{10}$$

where $\mathbf{d} := (d_1,\dots,d_n) \in (\mathbb{Z}_{\ge 0})^n$ satisfying $g(\mathbf{d}) := 1 + \frac{1}{3}\sum_{j=1}^n (d_j - 1) \in \mathbb{Z}_{\ge 0}$ and $2g(\mathbf{d}) - 2 + n \ge 2$, and $U(\mathbf{d})$ is defined by

$$U(\mathbf{d}) \;:=\; \prod_{j=1}^{n}(2d_j+1)!! \int_{\overline{\mathcal{M}}_{g(\mathbf{d}),n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\,. \tag{11}$$

Note that the integral on the right-hand side of (11) is understood as 0 if $g(\mathbf{d})$ is not a non-negative integer. The relation (10) is now widely known as the *Dijkgraaf–Verlinde–Verlinde (DVV) relation*.

In [2] Bertola, Dubrovin and the second author of the present paper derived the following explicit formula of generating series of n-point Witten's intersection numbers:

$$\sum_{d_1,\dots,d_n} \frac{U(\mathbf{d})}{\prod_{i=1}^n \lambda_i^{d_i+1}} = \begin{cases} \sum_{g\geq 1} \frac{(6g-3)!!}{24^g\, g!\, \lambda^{3g-1}}\,, & n=1\,, \\ -\frac{1}{n}\sum_{\sigma\in S_n} \frac{\operatorname{tr}\prod_{i=1}^n M(\lambda_{\sigma(i)})}{\prod_{i=1}^n(\lambda_{\sigma(i)}-\lambda_{\sigma(i+1)})} - \frac{\delta_{n,2}(\lambda_1+\lambda_2)}{(\lambda_1-\lambda_2)^2}\,, & n\geq 2\,, \end{cases} \tag{12}$$

where S_n denotes the symmetric group, for an element $\sigma\in S_n$ the notation $\sigma(n+1)$ is understood as $\sigma(1)$, and

$$M(\lambda) \;:=\; \frac{1}{2}\begin{pmatrix} -\sum_{g=1}^\infty \frac{(6g-5)!!}{24^{g-1}(g-1)!}\lambda^{-3g+2} & -2\sum_{g=0}^\infty \frac{(6g-1)!!}{24^g g!}\lambda^{-3g} \\ 2\sum_{g=0}^\infty \frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^g g!}\lambda^{-3g+1} & \sum_{g=1}^\infty \frac{(6g-5)!!}{24^{g-1}(g-1)!}\lambda^{-3g+2} \end{pmatrix}. \tag{13}$$

The $n=1$ case of (12) is well known (see e.g. [28]). Following [14], an explicit formula for Witten's intersection numbers will be deduced from (12) in Section 2.

In [23] Liu–Xu obtained the large genus asymptotics for Witten's intersection numbers with fixed n and fixed $d_1,\dots,d_{n-1}$; the requirement that n should be fixed is essential in [23], meaning that it cannot be improved if one uses the normalizations considered in [23]. In [4, 6] Delecroix–Goujard–Zograf–Zorich (DGZZ) introduced a remarkable normalization of Witten's intersection numbers

$$G(\mathbf{d}) \;:=\; \frac{24^{g(\mathbf{d})}\, g(\mathbf{d})!\, \prod_{j=1}^n (2d_j+1)!!}{(6g(\mathbf{d})+2n-5)!!} \int_{\overline{\mathcal{M}}_{g(\mathbf{d}),\,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\,, \quad \mathbf{d}\in(\mathbb{Z}_{\geq 0})^n\,, \tag{14}$$

and proposed the following important conjecture: for any $C>0$,

$$\lim_{g\to+\infty} \max_{1\leq n\leq C\log g} \max_{\substack{\mathbf{d}\in(\mathbb{Z}_{\geq 0})^n \\ |\mathbf{d}|=3g-3+n}} \big|G(\mathbf{d})-1\big| = 0\,, \tag{15}$$

or stated differently, the normalized intersection numbers $G(\mathbf{d})$ tend to 1 uniformly if $n=O(\log g)$. Here $|\mathbf{d}|:=d_1+\cdots+d_n$.

The DGZZ conjecture was first proved by Aggarwal [1], and later independently by two of the authors of the present paper [15] by a different method. Actually, Aggarwal [1, Theorem 1.5] proved the following stronger version of the DGZZ conjecture:

$$\lim_{\epsilon\to 0}\left(\lim_{g\to\infty} \max_{n<\epsilon\sqrt{g}} \max_{\substack{\mathbf{d}\in(\mathbb{Z}_{\geq 0})^n \\ |\mathbf{d}|=3g-3+n}} \Big|G(\mathbf{d})-1\Big|\right) \;=\; 0\,. \tag{16}$$

In other words, $G(\mathbf{d})$ tends to 1 uniformly for $n=o(\sqrt{g})$. According to [1, 4], $o(\sqrt{g})$ cannot be replaced by $O(\sqrt{g})$ in this statement.

Recently, Norbury and the three authors of the present paper [14] discovered and proved a completely uniform large genus asymptotic formula for the Brézin–Gross–Witten (BGW) numbers (for the meaning of BGW numbers see e.g. [14]). In this paper, we aim to find and prove the corresponding result for Witten's intersection numbers.

Following the idea of [14], introduce a new normalization by

$$C(\mathbf{d}) \;:=\; \frac{2^{2g(\mathbf{d})}\prod_{j=1}^n (2d_j+1)!!}{3^{2g(\mathbf{d})-2+n}\,(2g(\mathbf{d})-3+n)!}\int_{\overline{\mathcal{M}}_{g(\mathbf{d}),\,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\,, \quad \mathbf{d}\in(\mathbb{Z}_{\ge 0})^n\,. \tag{17}$$

The numbers $G(\mathbf{d})$ and $C(\mathbf{d})$ are related by

$$C(0^{n-1}, 3g(\mathbf{d})-3+n)\,G(\mathbf{d}) \;=\; C(\mathbf{d})\,. \tag{18}$$

From the $n=1$ case of (12) we know that $C(3g-2)$ has the explicit expression:

$$C(3g-2) \;=\; \frac{3\,(6g-3)!!}{54^g\,g!\,(2g-2)!}\,, \quad g\ge 1\,. \tag{19}$$

Explicit values of the numbers $C(\mathbf{d})$ with $g(\mathbf{d})=2,3$ are given in Table 1.

$g=2, \quad D=3888$			
(4)	$\frac{35}{144}$	0.24306	945
$(2,3)$	$\frac{1015}{3888}$	0.26106	1015
$(2,2,2)$	$\frac{175}{648}$	0.27006	1050
$g=3, \quad D=7558272$			
(7)	$\frac{25025}{93312}$	0.26819	2027025
$(2,6)$	$\frac{77077}{279936}$	0.27534	2081079
$(3,5)$	$\frac{38731}{139968}$	0.27671	2091474
$(4,4)$	$\frac{4249}{15552}$	0.27321	2065014
$(2,2,5)$	$\frac{6545}{23328}$	0.28056	2120580
$(2,3,4)$	$\frac{39235}{139968}$	0.28031	2118690
$(3,3,3)$	$\frac{714175}{2519424}$	0.28347	2142525
$(2,2,2,4)$	$\frac{6625}{23328}$	0.28399	2146500
$(2,2,3,3)$	$\frac{179375}{629856}$	0.28479	2152500
$(2,2,2,2,3)$	$\frac{120625}{419904}$	0.28727	2171250
$(2,2,2,2,2,2)$	$\frac{546875}{1889568}$	0.28942	2187500

TABLE 1. Some normalized Witten's intersection numbers $C(\mathbf{d})$

Similar to the conjectural nesting property for BGW numbers [14, Conjecture 1], we observe from Table 1 and further numerical data that the primitive normalized Witten's intersection numbers $C(\mathbf{d})$ with a fixed genus $g\le 13$ lie between the values $C(3g-2)$ and $C(2^{3g-3})$. For example, for $g=13$ the numbers $C(\mathbf{d})$ with $d_1,\dots,d_n\ge 2$ lie between $C(37)=0.30674776\cdots$ and $C(2^{36})=0.31263595\cdots$. We formulate the following

Conjecture 1. *For $g\ge 2$, $n\ge 1$ and for $d_1,\dots,d_n\ge 1$ satisfying $|\mathbf{d}|=3g-3+n$,*

$$C(3g-2) \;\le\; C(\mathbf{d}) \;\le\; C(2^{3g-3})\,. \tag{20}$$

Remark 1. The inequality (20) does not hold if d_j is allowed to be 0. E.g., for $g = 2$,

$$C(0^6, 10) \;=\; \frac{1616615}{6718464} \;\approx\; 0.24062 \;<\; 0.24306 \;\approx\; \frac{35}{144} \;=\; C(4)\,, \tag{21}$$

$$C(0^2, 6) \;=\; \frac{5005}{15552} \;\approx\; 0.32182 \;>\; 0.27006 \;\approx\; \frac{175}{648} \;=\; C(2^3)\,. \tag{22}$$

Remark 2. A main conjecture for BGW numbers in [14] was the monotonicity of these numbers for fixed g in a suitable lexicographic ordering. However, this does not hold for the normalized numbers $C(\mathbf{d})$, e.g., $C(4,4) < C(3,5)$.

Remark 3. There is also a weak monotonicity conjecture [14] for BGW numbers, saying that the interval $I_{g,n}$ (the convex hull of normalized BGW numbers of genus g and length n) lies strictly to the left of $I_{g,n+1}$. Even this weak monotonicity fails for normalized primitive Witten's intersection numbers $C(\mathbf{d})$, a counterexample being

$$C(2^5, 8) = \frac{727759375}{2448880128} \approx 0.29718 < 0.29746 \approx \frac{419588015525}{1410554953728} = C(3^4, 5). \tag{23}$$

Following [14], we also observe from Table 1 and further numerical data that the normalized primitive Witten's intersection numbers $C(\mathbf{d})$ for each fixed g are close to each other, e.g. the minimum and maximum values $C(37)$ and $C(2^{36})$ for $g = 13$ differ by less than 2 percent. In view of the nesting property, let us focus on the two values $C(3g-2)$ and $C(2^{3g-3})$. On one hand, the value of $C(3g-2)$ is given by (19), which by Stirling's formula satisfies

$$C(3g-2) \;\sim\; \frac{1}{\pi}\Big(1 - \frac{17}{36g} + \frac{1}{2592g^2} - \frac{557}{279936g^3} + \cdots\Big)\,, \quad g \to \infty\,. \tag{24}$$

On the other hand, with the help of a deep result [17, 18] on a certain formal solution to the Painlevé I equation (cf. Section 3 for detail), one can obtain that

$$C(2^{3g-3}) \;\sim\; \frac{1}{\pi}\Big(1 - \frac{2}{9\,g} - \frac{238}{2025\,g^2} - \frac{198149}{2733750\,g^3} + \cdots\Big)\,, \quad g \to \infty\,. \tag{25}$$

Conjecture 1 together with formulas (24), (25) and (32) implies that for $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ with $g(\mathbf{d}) \in \mathbb{Z}_{\geq 1}$,

$$\frac{1}{\pi} - \frac{17}{36\pi\, g(\mathbf{d})} + O\Big(\frac{1}{g(\mathbf{d})^2}\Big) \;\leq\; C(\mathbf{d}) \;\leq\; \frac{1}{\pi} - \frac{2}{9\pi\, g(\mathbf{d})} + O\Big(\frac{1}{g(\mathbf{d})^2}\Big) \tag{26}$$

as $g(\mathbf{d}) \to \infty$. This motivates the following

Theorem 1. *For $n \geq 1$ and $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ satisfying $g(\mathbf{d}) \in \mathbb{Z}_{\geq 1}$, we have*

$$C(\mathbf{d}) \;=\; \frac{1}{\pi} + O\Big(\frac{1}{g(\mathbf{d})}\Big) \tag{27}$$

uniformly as $g(\mathbf{d}) \to \infty$.

Theorem 1 says that there exists an absolute constant K_1 such that

$$\Big|C(\mathbf{d}) - \frac{1}{\pi}\Big| \;\leq\; \frac{K_1}{g(\mathbf{d})} \tag{28}$$

holds for all $n \geq 1$ and $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ with sufficiently large $g(\mathbf{d}) \in \mathbb{Z}$.

Our proof, which does not use (25), is along the lines of [1, 14] and will be given in Section 3. It mainly uses the DVV relation (10), which written in terms of $C(\mathbf{d})$ reads

$$\begin{aligned} C(\mathbf{d}) \;=\; & \sum_{j=2}^{n} \frac{2d_j+1}{3\,(X(\mathbf{d})-1)}\, C(d_2,\dots,d_j+d_1-1,\dots,d_n) \\ & + \sum_{\substack{a,b \\ a+b=d_1-2}} \Bigg[\frac{2}{3\,(X(\mathbf{d})-1)}\, C(a,b,d_2,\dots,d_n) \\ & + \sum_{I\sqcup J=\{2,\dots,n\}} \frac{(X(a,\mathbf{d}_I)-1)!\,(X(b,\mathbf{d}_J)-1)!}{6\,(X(\mathbf{d})-1)!}\, C(a,\mathbf{d}_I)\, C(b,\mathbf{d}_J) \Bigg], \end{aligned} \tag{29}$$

where

$$X(\mathbf{d}) \;:=\; \frac{1}{3}\sum_{j=1}^{n}(2d_j+1) \;=\; 2\,g(\mathbf{d})-2+n\,. \tag{30}$$

In particular,

$$C(0,\mathbf{d}) \;=\; \sum_{j=1}^{n} \frac{2d_j+1}{3(X(0,\mathbf{d})-1)}\, C(d_1,\dots,d_j-1,\dots,d_n)\,, \tag{31}$$

$$C(1,\mathbf{d}) \;=\; C(\mathbf{d})\,. \tag{32}$$

An application of Theorem 1 will be given in Section 3.

With the help of (31) we can improve Theorem 1 as follows:

Theorem 2. *For $n\ge 1$ and $\mathbf{d}\in(\mathbb{Z}_{\ge 0})^n$ satisfying $g(\mathbf{d})\in\mathbb{Z}_{\ge 1}$, we have*

$$C(\mathbf{d}) \;=\; \frac{1}{\pi}\prod_{j=1}^{p_0(\mathbf{d})}\Big(1+\frac{2+j-p_0(\mathbf{d})}{3X(\mathbf{d})-3p_1(\mathbf{d})-3j}\Big) \;+\; O\Big(\frac{1}{g(\mathbf{d})}\Big)\,, \tag{33}$$

uniformly as $g(\mathbf{d})\to\infty$, where $p_i(\mathbf{d})$, $i\ge 0$, denotes the multiplicity of i in $\mathbf{d}$.

The proof is given in Section 3. Theorem 2 says that there exists an absolute constant $K_2>0$ such that

$$\left| C(\mathbf{d}) \;-\; \frac{1}{\pi}\prod_{j=1}^{p_0(\mathbf{d})}\Big(1+\frac{2+j-p_0(\mathbf{d})}{3X(\mathbf{d})-3p_1(\mathbf{d})-3j}\Big)\right| \;\le\; \frac{K_2}{g(\mathbf{d})}\,, \tag{34}$$

for all $n\ge 1$ and $\mathbf{d}\in(\mathbb{Z}_{\ge 0})^n$ satisfying $g(\mathbf{d})\in\mathbb{Z}_{\ge 1}$. Alternatively, we can write (33) as

$$C(\mathbf{d}) \;=\; \frac{1}{\pi}\,\frac{\left(\frac{2}{3}\right)^{p_0(\mathbf{d})}\left(\frac{3X-3p_0(\mathbf{d})-3p_1(\mathbf{d})+2}{2}\right)_{p_0(\mathbf{d})}}{(X-p_1(\mathbf{d})-p_0(\mathbf{d}))_{p_0(\mathbf{d})}} \;+\; \mathrm{O}\Big(\frac{1}{g(\mathbf{d})}\Big)\,, \tag{35}$$

as $g(\mathbf{d})\to\infty$. Here, $(a)_b := a(a+1)\cdots(a+b-1)$ denotes the Pochhammer symbol.

As a particular example of Theorem 2, we will prove the following

Corollary 1. *As $g\to\infty$, for $k=O(\sqrt{g})$ we have the uniform leading asymptotics*

$$C(0^k,2^{3g-3+k}) \;\sim\; \frac{1}{\pi}\,e^{-\frac{k^2}{30g}}\,, \tag{36}$$

and for $k/\sqrt{g}\to\infty$ we have $C(0^k,2^{3g-3+k})\to 0$.

In terms of the original Witten's intersection numbers, (36) reads

$$\frac{2^{2g}\,5^{3g-3+2k}}{3^{2g-2}\,(5g-6+2k)!}\int_{\overline{\mathcal{M}}_{g,\,3g-3+2k}}\psi_1^2\cdots\psi_{3g-3+k}^2\;\sim\;\frac{1}{\pi}\,e^{-\frac{k^2}{30g}}\,. \tag{37}$$

Remark 4. Theorem 2 implies Aggarwal's result (16). Actually, Corollary 1 cannot be deduced from (16).

Let us proceed to consider the full asymptotics of the normalized Witten's intersection numbers. It was shown by Liu and Xu [23] (cf. also [15]) that for fixed $n\geq 1$ and fixed $\mathbf{d}'=(d_1,\ldots,d_{n-1})\in(\mathbb{Z}_{\geq 0})^{n-1}$, the quotient $C(\mathbf{d}',3g-3+n-|\mathbf{d}'|)/C(3g-2)$ is a rational function of g, and therefore has a full asymptotic expansion which is a power series of g^{-1}. Then by using Stirling's formula and the 1-point formula (19) we know that $C(\mathbf{d}',3g-3+n-|\mathbf{d}'|)$ also has a full asymptotic expansion which is a power series of g^{-1}. The polynomiality conjecture from [15] predicts that for each k the coefficient of g^{-k} in the large genus expansion of $C(\mathbf{d}',3g-3+n-|\mathbf{d}'|)$ is a polynomial of n and the multiplicities of the arguments, and also that only the multiplicities of $0,1,\ldots,[3k/2]-1$ are involved. This conjecture was proved by Eynard *et al* [12] based on a formula proved in [29, 10] (cf. [2]).

Following [14], we will present and prove an improved version (see Theorem 3 below) of the polynomiality conjecture. For $n\geq 1$, $\mathbf{d}=(d_1,\ldots,d_n)\in(\mathbb{Z}_{\geq 0})^n$ satisfying $g(\mathbf{d})\in\mathbb{Z}$, we introduce a new normalization $\widehat{C}(\mathbf{d})$ of Witten's intersection numbers by

$$\widehat{C}(\mathbf{d})\;:=\;\frac{C(\mathbf{d})}{\gamma(X(\mathbf{d}))}\,, \tag{38}$$

where

$$\gamma(X)\;:=\;\frac{2^X}{\sqrt{\pi}\,3^{\frac{3X+1}{2}}}\,\frac{\Gamma\big(\frac{3X}{2}+1\big)}{\Gamma\big(\frac{X+3}{2}\big)\,\Gamma(X)}\,. \tag{39}$$

According to (19), we know that $\widehat{C}(3g-2)\equiv 1$ for any $g\geq 1$.

Theorem 3. *For any fixed n and fixed $\mathbf{d}'\in(\mathbb{Z}_{\geq 2})^{n-1}$, the numbers $\widehat{C}(\mathbf{d})$ satisfy*

$$\widehat{C}(\mathbf{d})\;\sim\;\sum_k\frac{\widehat{c}_k(p_2(\mathbf{d}'),p_3(\mathbf{d}'),\ldots)}{X(\mathbf{d})^k}\,,\qquad d_n\to\infty\,, \tag{40}$$

where $\mathbf{d}=(\mathbf{d}',d_n)$, $\widehat{c}_k$ are universal polynomials of $p_2,p_3,\ldots$ having rational coefficients, with $\widehat{c}_0\equiv 1$ and $\widehat{c}_k|_{p_b\equiv 0}=0$ $(k\geq 1)$. Moreover, under the degree assignments

$$\deg\,p_d\;=\;2d+1\quad(d\geq 1)\,, \tag{41}$$

the polynomials $\widehat{c}_k$, $k\geq 1$, satisfy the degree estimates

$$\deg\,\widehat{c}_k\;\leq\;3k-1\,. \tag{42}$$

Several explicit expressions for $\tilde{c}_k$ and $\widehat{c}_k$ are given in the following table.

Organization of the paper In Section 2 we give a closed formula for Witten's intersection numbers. In Section 3 we prove Theorems 1 and 2. In Section 4 we prove Theorem 3.

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k	$\tilde{c}_k(p_1,p_2,\dots)$	$\widehat{c}_k(p_1,p_2,\dots)$
0	1	1
1	$-\frac{17}{18}$	0
2	$\frac{613}{648}-\frac{5}{72}\,p_2$	$-\frac{5}{72}\,p_2$
3	$\frac{65}{648}\,p_2+\frac{35}{216}\,p_3-\frac{33713}{34992}$	$\frac{5}{144}\,p_2+\frac{35}{216}\,p_3$
4	$\frac{1225}{10368}\,p_2^2+\frac{130}{729}\,p_2+\frac{9415}{31104}\,p_3$ $-\frac{1225}{3456}\,p_4-\frac{385}{3456}\,p_5+\frac{2424889}{2519424}$	$\frac{1225}{10368}\,p_2^2+\frac{1435}{5184}\,p_2+\frac{175}{384}\,p_3$ $-\frac{1225}{3456}\,p_4-\frac{385}{3456}\,p_5$

TABLE 2. Explicit expressions for $\tilde{c}_k(p_2,p_3,\dots)$, $\widehat{c}_k(p_2,p_3,\dots)$ with $k=0,\dots,4$

2. An explicit formula for Witten's intersection numbers

In this section, based on (12) we derive an explicit formula for Witten's intersection numbers.

By taking the Laurent expansion on the right-hand side of (12) with $n=2$, the following formula was obtained in [2]:

$$\int_{\overline{\mathcal{M}}_{g,2}}\psi_1^{d_1}\psi_2^{d_2} \;=\; \frac{\sum_{l=0}^{d_1}(d_1+1-l)\,\xi_{l-1,3g-l}}{(2d_1+1)!!(2d_2+1)!!}\,, \tag{43}$$

where $g,d_1,d_2\ge 0$ satisfying $d_1+d_2=3g-1$ and

$$\xi_{k_1,k_2} \;=\; \begin{cases} \frac{(6g_1-5)!!\,(6g_2-5)!!}{2\cdot 24^{g_1+g_2-2}\,(g_1-1)!\,(g_2-1)!}\,, & k_1=3g_1-2, k_2=3g_2-2\,,\\ -\frac{(6g_1-1)!!\,(6g_2-1)!!}{24^{g_1+g_2}\,g_1!\,g_2!}\,\frac{6g_2+1}{6g_2-1}\,, & k_1=3g_1, k_2=3g_2-1\,,\\ \frac{(6g_1-1)!!\,(6g_2-1)!!}{2\cdot 24^{g_1+g_2}\,g_1!\,g_2!}\,\frac{6g_1+1}{6g_1-1}\,, & k_1=3g_1-1, k_2=3g_2\,,\\ 0\,, & \text{otherwise}\,. \end{cases}$$

Zograf [30] derived another formula for two-point Witten's intersection numbers:

$$C(d_1,3g-1-d_1) \;=\; \frac{1}{54^g\,(2g-1)!\,g}\sum_{d=-1}^{d_1-1}\eta_{g,d}\,, \tag{44}$$

where $\eta_{g,d}$, $d\ge -1$, are defined by

$$\eta_{g,d} \;=\; (6g-3-2d)!!\,(2d+1)!!\cdot\begin{cases} \frac{g-2j}{j!\,(g-j)!}\,, & d=3j-1\,,\\ -\frac{2}{j!\,(g-1-j)!}\,, & d=3j\,,\\ \frac{2}{j!\,(g-1-j)!}\,, & d=3j+1\,. \end{cases} \tag{45}$$

The equivalence of (43) and (44) was shown in [13].

By taking the Laurent expansion on the right-hand side of (12) with $n\ge 2$, a formula for Witten's intersection numbers was derived in [15], whose expression written in terms of $C(\mathbf{d})$ is given in the following proposition. Before stating it, we introduce a notation:

$$a_{k_1,\dots,k_n} \;:=\; \frac{2^{2g(\mathbf{k})}\,\operatorname{tr}A_{k_1}\cdots A_{k_n}}{3^{2g(\mathbf{k})+n-2}\,(2g(\mathbf{k})+n-3)!}\,, \tag{46}$$

for $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq -1}$, and $a_{k_1,\dots,k_n}$ are defined as 0 if some of k_j is less than or equal to -2. Here

$$M(\lambda) \;=:\; \sum_{k \geq -1} A_k \, \lambda^{-k} \,.$$

Proposition 1 ([15])**.** *For $n \geq 2$ and $\mathbf{d} = (d_1, \dots, d_n) \in (\mathbb{Z}_{\geq 0})^n$, we have*

$$C(\mathbf{d}) \;=\; \sum_{\substack{\sigma \in S_n \\ \sigma(n) = n}} (-1)^{|S^-_\sigma|+1} \sum_{\substack{\underline{J} \in (\mathbb{Z}+\frac{1}{2})^n \\ \{1 \leq q \leq n \,|\, J_q > 0\} = S^+_\sigma}} a_{d_{\sigma(1)}+J_1-J_n,\dots,d_{\sigma(n)}+J_n-J_{n-1}} \,. \tag{47}$$

The following proposition is an analogue of [14, Proposition 3].

Proposition 2. *For $n \geq 2$ and $\mathbf{d} = (d_1, \dots, d_n) \in (\mathbb{Z}_{\geq 0})^n$,*

$$C(\mathbf{d}) \;=\; \sum_{\substack{\sigma \in S_n \\ \sigma(n) = n}} (-1)^{|S^-_\sigma|+1} \sum_{\substack{k_1,\dots,k_n \geq -1 \\ k_1+\dots+k_n = d_1+\dots+d_n}} a_{k_1,\dots,k_n} \, \omega_{\mathbf{d},\sigma,\mathbf{k}} \,, \tag{48}$$

where the sets S^+_σ, $S^-_\sigma \subseteq \{1, \dots, n\}$ are denoted as

$$S^+_\sigma \;=\; \big\{ 1 \leq r \leq n \,\big|\, \sigma(r+1) > \sigma(r) \big\} \,, \quad S^-_\sigma \;=\; \big\{ 1 \leq r \leq n \,\big|\, \sigma(r+1) < \sigma(r) \big\} \,,$$

and the numbers $\omega_{\mathbf{d},\sigma,\mathbf{k}}$ have the following explicit expression

$$\omega_{\mathbf{d},\sigma,\mathbf{k}} \;=\; \max\Bigg\{ 0 \,,\, \min_{r \in S^+_\sigma} \Bigg\{ \sum_{q=1}^{r} (d_{\sigma(q)} - k_q) \Bigg\} + \min_{r \in S^-_\sigma} \Bigg\{ \sum_{q=1}^{r} (k_q - d_{\sigma(q)}) \Bigg\} \Bigg\} \,. \tag{49}$$

The proof is almost identical to that of [14, Proposition 3] and is therefore omitted.

Let us give some examples for Proposition 2. The $n = 2$ case of Proposition 2 is equivalent to (43). For $n \geq 1$, $\mathbf{e} \in \mathbb{Z}^n$, introduce the notation

$$M(\mathbf{e}) \;:=\; \max\big\{ 0, \min_{1 \leq i \leq n} \{ e_i \} \big\} \,, \tag{50}$$

which has the generating function

$$\sum_{e_1,\dots,e_n \geq 0} x_1^{e_1-1} \cdots x_n^{e_n-1} \, M(e_1, \dots, e_n) \;=\; \frac{1}{(1 - x_1 \cdots x_n) \prod_{i=1}^n (1 - x_i)} \,. \tag{51}$$

Using (50), the $n = 3$ case of Proposition 2 can be written as

$$C(d_1, d_2, d_3) \;=\; 2 \sum_{k_1+k_2+k_3 = d_1+d_2+d_3} a_{k_1,k_2,k_3} \, M(d_1 - k_1, d_1 + d_2 - k_1 - k_2) \,, \tag{52}$$

and the $n = 4$ case of Proposition 2 can be written as

$$\begin{aligned} C(d_1, d_2, d_3, d_4) \;=\; & 2 \sum_{k_1+k_2+k_3+k_4 = d_1+d_2+d_3+d_4} a_{k_1,k_2,k_3,k_4} \\ & \times \Big(M\big(d_1 - k_1, d_1 + d_2 - k_1 - k_2, k_4 - d_4\big) \\ & \quad - M\big(d_1 - k_2, d_1 + d_2 - k_2 - k_3, d_1 + d_3 - k_1 - k_2, k_4 - d_4\big) \\ & \quad - M\big(d_1 - k_1, d_2 - k_3, k_2 - d_3, k_4 - d_4\big) \Big) \,, \end{aligned} \tag{53}$$

where we have used the fact that $a_{k_1,k_2,k_3,k_4} = a_{k_2,k_3,k_4,k_1} = a_{k_1,k_4,k_3,k_2}$.

3. Uniform large genus asymptotics of Witten's intersection numbers

In this section, we prove Theorem 1. Our proof will mainly use the DVV relation (10), the techniques introduced by Aggarwal [1] (see also [14]), and the improvement in [14].

3.1. **Lower Bound.**

In this subsection we give lower bounds for the normalized Witten's intersection numbers $C(\mathbf{d})$.

Lemma 1. *For $n \geq 1$ and for $\mathbf{d} = (d_1, \dots, d_n) \in (\mathbb{Z}_{\geq 0})^n$ satisfying $g(\mathbf{d}) \in \mathbb{Z}_{\geq 0}$ and $2g(\mathbf{d}) - 2 + n > 0$, we have*

$$C(\mathbf{d}) \;>\; 0\,. \tag{54}$$

Proof. By using (29) and by recalling that $C(0^3) = 1/3$, $C(1) = 1/6$. □

The following lemma proves one side of Conjecture 1.

Lemma 2. *For $g, n \geq 1$ and $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ satisfying $|\mathbf{d}| = 3g - 3 + n$, we have*

$$C(\mathbf{d}) \;\geq\; C(3g-2)\,. \tag{55}$$

Proof. For $n = 1$, inequality (55) is trivial. For $n = 2$ inequality (55) was obtained in [4, 5]. Now assume $n \geq 3$, and let us prove (55) by induction on $X(\mathbf{d})$. For $X(\mathbf{d}) \leq 5$, inequality (55) can be checked directly from Table 1. For $\mathbf{d} = (d_1, \dots, d_n) \in (\mathbb{Z}_{\geq 1})^n$ with $X(\mathbf{d}) \geq 6$, because of the symmetry of C-values and formula (32), we can assume that $2 \leq d_1 \leq \cdots \leq d_n$. For $d_1 \geq 3$, by using (19), (29) and the inductive assumption we have

$$\begin{aligned} C(\mathbf{d}) \;&\geq\; \sum_{j=2}^{n} \frac{2d_j+1}{3(X(\mathbf{d})-1)}\, C(3g-2) + \frac{2(d_1 - 1 + \frac{4}{3\,(X(\mathbf{d})-2)})}{3\,(X(\mathbf{d})-1)}\, C(3g-5) \\ &=\; C(3g-2) \;-\; \frac{2(d_1-1)}{3(X(\mathbf{d})-1)}\Big(C(3g-2) - C(3g-5)\Big) + \frac{8\,C(3g-5)}{9\,(X(\mathbf{d})-1)\,(X(\mathbf{d})-2)} \\ &=\; C(3g-2) \;+\; \frac{2\,C(3g-5)}{3(X(\mathbf{d})-1)}\left(-\frac{(d_1-1)(34g-35)}{36g(2g-3)(g-1)} + \frac{4}{3(X(\mathbf{d})-2)}\right), \end{aligned} \tag{56}$$

where in the inequality we have omitted the quadratic-in-C term since they are non-negative and have used the following estimate which is implied by equation (31)

$$C(0, d_1 - 2, d_2, \dots, d_n) \;\geq\; \Big(1 + \frac{2}{3(X(\mathbf{d})-2)}\Big)\, C(3g-5)\,. \tag{57}$$

The expression in the parenthesis of the right-hand side of (56) is nonnegative since

$$\begin{aligned} -\frac{(d_1-1)(34g-35)}{36g(2g-3)(g-1)} + \frac{4}{3(X(\mathbf{d})-2)} \;&\geq\; -\frac{(34g-35)}{12ng(2g-3)} + \frac{4}{3(2g-4+n)} \\ &\geq\; \frac{1}{3n}\Big(-\frac{(34g-35)}{4g(2g-3)} + \frac{12}{(2g-4+3)}\Big) \;\geq\; 0\,, \end{aligned} \tag{58}$$

where we have used the conditions $d_1 - 1 \leq \frac{3g-3}{n}$, $3 \leq n \leq 3g-3$ and $g \geq 3$ (since $X(\mathbf{d}) = 2g - 2 + n \geq 6$). Inequalities (56) and (58) imply that $C(\mathbf{d}) \geq C(3g-2)$ when $d_1 \geq 3$.

For $d_1 = 2$, we have the estimate

$$\begin{aligned} C(0,0,d_2,\ldots,d_n) \;&=\; \sum_{j=2}^{n} \frac{2d_j+1}{3(X(\mathbf{d})-2)} C(0,d_2,\ldots,d_j-1,\ldots,d_n) \\ &\geq\; \Big(1+\frac{1}{3(X(\mathbf{d})-2)}\Big)\Big(1+\frac{2}{3(X(\mathbf{d})-3)}\Big) C(3g-5) \;\geq\; C(3g-2)\,. \end{aligned} \tag{59}$$

where for the equality we have used (31), for the first inequality we have used the inductive assumption and an analogue of (57), and for the second inequality we have used the one point forumla (19) and the condition $n \leq 3g-3$ and $g \geq 3$. By using (29), (59), we obtain that

$$\begin{aligned} C(d_1=2,\ldots,d_n) \;&\geq\; \sum_{j=2}^{n} \frac{2d_j+1}{3(X(\mathbf{d})-1)} C(d_2,\ldots,d_j+1,\ldots,d_n) \\ &\quad+\; \frac{2}{3(X(\mathbf{d})-1)}\, C(0,0,d_2,\ldots,d_n) \;\geq\; C(3g-2)\,. \end{aligned} \tag{60}$$

This completes the proof. □

Corollary 2. *For $g,n \geq 1$ and $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^n$ satisfying $|\mathbf{d}| = 3g-3+n$, we have*

$$C(\mathbf{d}) \;\geq\; C(3g-2) \prod_{j=1}^{p_0(\mathbf{d})} \Big(1+\frac{j+2-p_0(\mathbf{d})}{6g-6+3n-3j}\Big)\,, \tag{61}$$

where $p_0(\mathbf{d})$ denotes the multiplicity of 0 in $\mathbf{d}$.

Proof. By (31) and Lemma 2 (some details can be found in the proof of Theorem 2 below). □

We mention that several other lower bounds for Witten's intersection numbers are obtained in [1, Proposition 4.1], [4, Theorem 5] and [21, Corollary 5.4].

3.2. **Upper Bound.** Analogous to [1, Lemma 3.1], we first prove the following lemma.

Lemma 3. *For $n \geq 1$, $g \geq 2$ and for $\mathbf{d} \in (\mathbb{Z}_{\geq 2})^n$ satisfying $|\mathbf{d}| = 3g-3+n$, we have*

$$\sum_{\substack{a,b\geq 0 \\ a+b=d_1-2}} \sum_{\substack{I\sqcup J=\{2,\ldots,n\} \\ X(a,\mathbf{d}_I),X(b,\mathbf{d}_J)\in\mathbb{Z}}} \frac{\big(X(a,\mathbf{d}_I)-1\big)!\,\big(X(b,\mathbf{d}_J)-1\big)!}{6\,\big(X(\mathbf{d})-1\big)!} \;\leq\; \frac{2}{(X(\mathbf{d})-1)(X(\mathbf{d})-2)}\,. \tag{62}$$

Proof. For $n=1$, the inequality (62) is trivial. Assume that $n \geq 2$. For each a,b,I,J satisfying $a+b=d_1-2$, $I \sqcup J = \{2,\ldots,n\}$ and $X(a,\mathbf{d}_I), X(b,\mathbf{d}_J) \in \mathbb{Z}$, we set $n_1 = |I|$, $g_1 = g(a,\mathbf{d}_I) = (a+|\mathbf{d}_I|+2-n_1)/3 \in \mathbb{Z}$, $n_2 = |J|$, $g_2 = g(a,\mathbf{d}_J) = (b+|\mathbf{d}_J|+2-n_2)/3 \in$

$\mathbb{Z}$. By counting the number of 4-tuples (a,b,I,J) with given values of n_i and g_i, we find

$$\begin{aligned}
&\sum_{\substack{a,b\geq 0\\ a+b=d_1-2}} \sum_{\substack{I\sqcup J=\{2,\dots,n\}\\ X(a,\mathbf{d}_I),X(b,\mathbf{d}_J)\in\mathbb{Z}}} \frac{\big(X(a,\mathbf{d}_I)-1\big)!\,\big(X(b,\mathbf{d}_J)-1\big)!}{6\,\big(X(\mathbf{d})-1\big)!}\\
&\leq \sum_{n_1+n_2=n-1} \sum_{\substack{g_1,g_2\geq 1\\ g_1+g_2=g}} \binom{n-1}{n_1} \frac{(2g_1+n_1-2)!\,(2g_1+n_2-2)!}{6\,(X(\mathbf{d})-1)!}\\
&= \sum_{n_1+n_2=n-1} \binom{n-1}{n_1} \left(\frac{n_1!\,(X(\mathbf{d})-n_1-3)!}{3\,(X(\mathbf{d})-1)!} + \sum_{\substack{g_1,g_2\geq 2\\ g_1+g_2=g}} \frac{(2g_1+n_1-2)!\,(2g_1+n_2-2)!}{6\,(X(\mathbf{d})-1)!} \right).
\end{aligned} \tag{63}$$

We estimate the two terms on the right-hand side of (63) separately. For the first term we have

$$\begin{aligned}
\sum_{n_1+n_2=n-1} \binom{n-1}{n_1} \frac{n_1!\,(X(\mathbf{d})-n_1-3)!}{3\,(X(\mathbf{d})-1)!} &= \frac{1}{3\,(X(\mathbf{d})-1)\,(X(\mathbf{d})-2)} \sum_{n_1=0}^{n-1} \prod_{j=1}^{n_1} \frac{n-j}{X(\mathbf{d})-2-j}\\
&\leq \frac{1}{3\,(X(\mathbf{d})-1)\,(X(\mathbf{d})-2)} \left(2 + \sum_{n_1=2}^{\infty} \Big(\frac{3}{5}\Big)^{n_1-2} \right) = \frac{3}{2\,(X(\mathbf{d})-1)\,(X(\mathbf{d})-2)},
\end{aligned} \tag{64}$$

where in the inequality we have used the fact that $n \leq \frac{3X(\mathbf{d})}{5}$ (implied by $\mathbf{d} \in (\mathbb{Z}_{\geq 2})^n$) and $n \leq X(\mathbf{d})-2$ (implied by $g \geq 2$). For the second term, we note that it vanishes if $g \leq 3$, otherwise it has the upper bound

$$\begin{aligned}
&\sum_{n_1+n_2=n-1} \sum_{\substack{g_1,g_2\geq 2\\ g_1+g_2=g}} \binom{n-1}{n_1} \frac{(2g_1+n_1-2)!\,(2g_2+n_2-2)!}{6\,(X(\mathbf{d})-1)!}\\
&\leq \sum_{\substack{g_1,g_2\geq 2\\ g_1+g_2=g\\ n_1+n_2=n-1}} \frac{\binom{2g-4}{2g_1-2}^{-1}}{6\,(X(\mathbf{d})-1)(X(\mathbf{d})-2)} \leq \frac{n\,(g-3)\,\binom{2g-4}{2}^{-1}}{6\,(X(\mathbf{d})-1)(X(\mathbf{d})-2)}\\
&\leq \frac{3}{8\,(X(\mathbf{d})-1)\,(X(\mathbf{d})-2)},
\end{aligned} \tag{65}$$

where for the first inequality we used the fact that $X(\mathbf{d})-3 = (2g_1+n_1-2)+(2g_2+n_2-2)$ and that

$$\binom{a_1}{b_1}\binom{a_2}{b_2} \leq \binom{a_1+a_2}{b_1+b_2}, \tag{66}$$

and for the last inequality we used $g \geq 4$ and $n \leq 3g-3$. Combining (63), (64), (65), we obtain the lemma. □

Following again the idea of Aggarwal [1] (cf. [14]), we now define

$$\theta_{\mathcal{X},n} = \max_{\substack{\mathbf{d}\in(\mathbb{Z}_{\geq 1})^n\\ X(\mathbf{d})=\mathcal{X}}} C(\mathbf{d}), \quad \mathcal{X}, n \geq 1. \tag{67}$$

According to (32) we have

$$\theta_{\mathcal{X},n} \geq \theta_{\mathcal{X}-1,n-1}\,, \qquad \forall\, \mathcal{X}, n \geq 2\,. \tag{68}$$

We have the following

Lemma 4. *For $n \geq 1$, $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ we have*

$$C(0,\mathbf{d}) \;\leq\; \frac{3X(0,\mathbf{d})-3p_1(\mathbf{d})-1}{3X(0,\mathbf{d})-3p_1(\mathbf{d})-3}\,\theta_{X(0,\mathbf{d})-p_1(\mathbf{d})-1,n-p_1(\mathbf{d})}\,, \tag{69}$$

$$C(0,0,\mathbf{d}) \;\leq\; \frac{(3X(0^2,\mathbf{d})-3p_1(\mathbf{d})-2)(3X(0^2,\mathbf{d})-3p_1(\mathbf{d})-7)}{(3X(0^2,\mathbf{d})-3p_1(\mathbf{d})-3)(3X(0^2,\mathbf{d})-3p_1(\mathbf{d})-9)}\,\theta_{X(0^2,\mathbf{d})-p_1(\mathbf{d})-2,n-p_1(\mathbf{d})}\,. \tag{70}$$

Proof. Let us first prove (69). Since both sides of (69) are invariant if any argument 1 in $\mathbf{d}$ is removed, it is sufficient to prove (69) for the case when $\mathbf{d} \in (\mathbb{Z}_{\geq 2})^n$. By using (31), (32) and (68) we have

$$C(0,\mathbf{d}) \;=\; \sum_{j=1}^{n} \frac{2d_j+1}{3X(0,\mathbf{d})-3} C(d_1,\dots,d_j-1,\dots,d_n) \;\leq\; \frac{3X(0,\mathbf{d})-1}{3X(0,\mathbf{d})-3}\,\theta_{X(0,\mathbf{d})-1,n}\,, \tag{71}$$

which proves inequality (69). To prove (70), for the same reason we can assume $\mathbf{d} \in (\mathbb{Z}_{\geq 2})^n$. By using (31), (68) and (69) we have

$$\begin{aligned} C(0^2,\mathbf{d}) \;&=\; \sum_{j=1}^{n} \frac{2d_j+1}{3X(0^2,\mathbf{d})-3} C(0,d_1,\dots,d_j-1,\dots,d_n) \\ &\leq\; \frac{3X(0^2,\mathbf{d})-2}{3X(0^2,\mathbf{d})-3}\,\max\left\{\frac{3X(0^2,\mathbf{d})-4}{3X(0^2,\mathbf{d})-6}\,\theta_{X(0^2,\mathbf{d})-2,n}, \frac{3X(0^2,\mathbf{d})-7}{3X(0^2,\mathbf{d})-9}\,\theta_{X(0^2,\mathbf{d})-3,n-1}\right\} \\ &\leq\; \frac{3X(0^2,\mathbf{d})-2}{3X(0^2,\mathbf{d})-3}\,\frac{3X(0^2,\mathbf{d})-7}{3X(0^2,\mathbf{d})-9}\,\theta_{X(0^2,\mathbf{d})-2,n}\,, \end{aligned} \tag{72}$$

which finishes the proof of (70). □

Following [1] (see also [14]), introduce a number-theoretic function $f(\mathcal{X},n)$, defined through the recursion

$$f(\mathcal{X},n) \;=\; \frac{2}{3}\,f(\mathcal{X}-1,n-1) \;+\; \frac{1}{3}\,f(\mathcal{X}-1,n+1) \;+\; \frac{4}{(\mathcal{X}-1)(\mathcal{X}-2)}\,, \tag{73}$$

for any $n \geq 3$, $\mathcal{X} \geq 8$ together with the initial data $f(\mathcal{X},n) = 1/\pi$ for $1 \leq \mathcal{X} \leq 7$ or $n = 1,2$. We notice that the function $f(\mathcal{X},n)$ is the same as the one introduced in [14]. The following proposition is proved in [14].

Lemma 5 ([14])**.** *The function $f(\mathcal{X},n)$ satisfies the following properties.*

1. *$1/\pi \leq f(\mathcal{X},n) \leq 1$ for any $n \geq 1$, $X \geq 1$.*
2. *$f(\mathcal{X},n)$ is monotone increasing with respect to n.*
3. *For $1 \leq n \leq \mathcal{X}/5$, as $\mathcal{X} \to \infty$, $f(\mathcal{X},n) = 1/\pi + O(1/\mathcal{X})$ where the O-constant is uniform in n.*

The significance of the function $f(\mathcal{X},n)$ is given by the following important lemma.

Lemma 6. *For $n \geq 1$, $\mathcal{X} \geq 1$, the numbers $\theta_{\mathcal{X},n}$ have the upper bound*

$$\theta_{\mathcal{X},n} \;\leq\; f(\mathcal{X},n)\,. \tag{74}$$

Proof. For $1 \leq \mathcal{X} \leq 7$, we check individually that inequality (74) holds. For $n = 1$, by (19) inequality (74) is equivalent to the inequality

$$\frac{3\,(6g-3)!!}{54^g\,g!\,(2g-2)!} \;\leq\; \frac{1}{\pi}\,, \quad \forall\, g \geq 1\,, \tag{75}$$

which is true because the left-hand side is monotone increasing and asymptotic to $\frac{1}{\pi}$ by Stirling's formula. For $n = 2$, according to [4] we have for $g \geq 1$, $d_1, d_2 \geq 0$ satisfying $d_1 + d_2 = 3g - 1$,

$$C(d_1,d_2) \;\leq\; C(0,3g-1)\,. \tag{76}$$

Applying (31) and (19), we have

$$C(d_1,d_2) \;\leq\; \frac{6g-1}{6g-3}\,\frac{3\,(6g-3)!!}{54^g\,g!\,(2g-2)!} \;\leq\; \frac{1}{\pi}\,, \tag{77}$$

where the second inequality holds for the similar reason as (75).

Now consider the case that $n \geq 3$ and $\mathcal{X} \geq 8$. Let us prove inequality (74) by induction on $\mathcal{X}$. For $\mathcal{X} = 8$, it is easy to check the validity of inequality (74). Consider $\mathcal{X} \geq 9$ and $\mathbf{d} = (d_1,\dots,d_n) \in \big(\mathbb{Z}_{\geq 1}\big)^n$ satisfying $X(\mathbf{d}) = \mathcal{X}$. Without loss of generality, we assume $d_1 = \min\{d_j\}$. For $d_1 \geq 3$, by using (29), Lemma 3 and (69) we can obtain

$$\begin{aligned} C(\mathbf{d}) \;\leq\; &\Big(1 - \frac{2d_1-2}{3\mathcal{X}-3}\Big)\, f(\mathcal{X}-1,n-1) \;+\; \frac{2d_1-6}{3\mathcal{X}-3}\, f(\mathcal{X}-1,n+1) \\ &+\; \frac{4}{3\mathcal{X}-3}\,\frac{3\mathcal{X}-4}{3\mathcal{X}-6}\, f(\mathcal{X}-2,n) \;+\; \frac{2}{(\mathcal{X}-1)(\mathcal{X}-2)}\,. \end{aligned}$$

So

$$\begin{aligned} C(\mathbf{d}) \;&\leq\; \Big(1 - \frac{2d_1-2}{3\mathcal{X}-3}\Big)\, f(\mathcal{X}-1,n-1) \;+\; \frac{2d_1-2}{3\mathcal{X}-3}\, f(\mathcal{X}-1,n+1) \;+\; \frac{26}{9(\mathcal{X}-1)(\mathcal{X}-2)} \\ &\leq\; \frac{2}{3}\, f(\mathcal{X}-1,n-1) \;+\; \frac{1}{3}\, f(\mathcal{X}-1,n+1) \;+\; \frac{26}{9(\mathcal{X}-1)(\mathcal{X}-2)}\,, \end{aligned} \tag{78}$$

where for the first inequality we used the facts that $f(\mathcal{X}-2,n) \leq f(\mathcal{X}-1,n+1)$ and $f(\mathcal{X}-1,n+1) \leq 1$, and for the second inequality we used the facts that $d_1 \leq \frac{3\mathcal{X}-3}{2n}$, $n \geq 3$ and $f(X-1,n+1) \geq f(\mathcal{X}-1,n-1)$. For $d_1 = 2$, by using (29), Lemma 3 and (70) we obtain

$$\begin{aligned} C(\mathbf{d}) \;\leq\; &\Big(1 - \frac{2}{3\mathcal{X}-3}\Big)\, f(\mathcal{X}-1,n-1) \;+\; \frac{2}{(\mathcal{X}-1)(\mathcal{X}-2)} \\ &+\; \frac{2}{3X-3}\,\frac{3\mathcal{X}-5}{3\mathcal{X}-6}\,\frac{3\mathcal{X}-10}{3\mathcal{X}-12} f(\mathcal{X}-3,n-1)\,. \end{aligned}$$

So

$$C(\mathbf{d}) \;\leq\; \frac{1}{3}\, f(\mathcal{X}-1,n-1) \;+\; \frac{2}{3}\, f(\mathcal{X}-1,n+1) \;+\; \frac{79}{27(\mathcal{X}-1)(\mathcal{X}-2)}\,, \tag{79}$$

where we used the facts that $f(\mathcal{X}-3,n-1)\le f(\mathcal{X}-1,n+1)$, $f(\mathcal{X}-1,n+1)\le 1$ and $f(X-1,n+1)\ge f(\mathcal{X}-1,n-1)$. For $d_1=1$, by using (32) we have

$$C(\mathbf{d}) \;\le\; f(\mathcal{X}-1,n-1) \;\le\; \frac{1}{3}\,f(\mathcal{X}-1,n-1) \,+\, \frac{2}{3}\,f(\mathcal{X}-1,n+1)\,. \tag{80}$$

From (78), (79), (80), we get the validity of (74). The lemma is proved. □

Lemma 7. *For $\mathcal{X}\ge 4, n\ge 2$, we have*

$$\theta_{\mathcal{X}-3,n-1} \;\le\; \theta_{\mathcal{X},n} \,+\, \frac{M}{\mathcal{X}-1} \tag{81}$$

for some constant $M>0$.

Proof. To prove (81), it suffices to prove

$$C(4,d_1,\ldots,d_n) \;\ge\; C(d_1,\ldots,d_n) \,-\, \frac{M}{X(4,d_1,\ldots,d_n)-1} \tag{82}$$

for some constant $M>0$. According to e.g. [2, 7, 10], for a KdV tau-function $\tau=\tau(\mathbf{t})$ the following identities hold:

$$\epsilon^2\frac{\partial^2\log\tau}{\partial t_1\partial t_1} \;=\; \frac{u_4}{144} \,+\, \frac{uu_2}{6} \,+\, \frac{u_1^2}{24} \,+\, \frac{u^3}{3}\,, \tag{83}$$

$$\begin{aligned}\epsilon^2\frac{\partial^2\log\tau}{\partial t_1\partial t_4} \;=\;& \frac{u_{10}}{2903040}+\frac{u\,u_8}{60480}+\frac{13u^2\,u_6}{40320}+\frac{7u^3\,u_4}{2160}+\frac{17u_4^2}{96768}+\frac{u\,u_3^2}{448}+\frac{5u^4u_2}{288}\\ &+\frac{41u^2u_2^2}{2880}+\frac{79u_2^3}{40320}+\frac{u^3u_1^2}{36}+\frac{u_1^4}{384}+\frac{u_7u_1}{17280}+\frac{41u_2\,u_6}{241920}+\frac{23u_3u_5}{80640}+\frac{17u\,u_1\,u_5}{10080}\\ &+\frac{13u\,u_2\,u_4}{3360}+\frac{251u_1^2\,u_4}{120960}+\frac{1}{60}u^2\,u_1\,u_3+\frac{1}{36}u\,u_1^2\,u_2+\frac{151u_1\,u_2\,u_3}{20160}+\frac{u^6}{144}\,,\end{aligned} \tag{84}$$

where $u=\partial_x^2(\log\tau)$, $u_k=\partial_{t_0}^k(u)$. Similar to [24], using (83) one can obtain the following recursion for $C(\mathbf{d})$:

$$\begin{aligned}C(\mathbf{d}) \;=\;& C(0^6,\mathbf{d}) \,+\, \frac{1}{6}\sum_{I\sqcup J=\{1,\ldots,n\}}\frac{(X(0^3,d_I)-1)!(X(0^3,d_J)-1)!}{(X(\mathbf{d})+1)!}\,C(0^3,d_I)\,C(0^3,d_J)\\ &+\,\frac{2}{3}\sum_{I\sqcup J=\{1,\ldots,n\}}\frac{(X(0^2,d_I)-1)!(X(0^4,d_J)-1)!}{(X(\mathbf{d})+1)!}\,C(0^2,d_I)\,C(0^4,d_J)\\ &+\,\frac{1}{3}\sum_{I\sqcup J\sqcup K=\{1,\ldots,n\}}\frac{(X(0^2,d_I)-1)!(X(0^2,d_J)-1)!(X(0^2,d_K)-1)!}{(X(\mathbf{d})+1)!}\,C(0^2,d_I)\,C(0^2,d_J)\,C(0^2,d_K)\,,\end{aligned} \tag{85}$$

which can imply

$$C(0^6,\mathbf{d}) \;\ge\; C(\mathbf{d}) \,-\, \frac{M_1}{X(\mathbf{d})+1} \tag{86}$$

for some constant $M_1>0$. Similarly, the identity (84) leads to the inequality

$$C(4,\mathbf{d}) \;\ge\; C(0^{12},\mathbf{d})\,. \tag{87}$$

By applying (86) and (87), we obtain (82). The lemma is proved. □

We are ready to prove Theorem 1.

Proof of Theorem 1. For $n \geq 1$, $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ satisfying $g(\mathbf{d}) \in \mathbb{Z}$, by Lemma 2 and (67) we have

$$C(3g(\mathbf{d})-2) \;\leq\; C(\mathbf{d}) \;\leq\; \theta_{X(\mathbf{d}),n}\,. \tag{88}$$

On one hand, we know from (24) that $C(3g(\mathbf{d})-2) = \frac{1}{\pi} + O(\frac{1}{g(\mathbf{d})})$ with an absolute O-constant. On the other hand, according to Lemma 6, $\theta_{X(\mathbf{d}),n} \leq f(X(\mathbf{d}),n)$, then by Lemma 5 we know that when $n \leq \frac{\mathcal{X}}{5}$, $\theta_{\mathcal{X},n} = \frac{1}{\pi} + O(\frac{1}{\mathcal{X}})$ with an absolute O-constant.

Consider $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ such that $\frac{X(\mathbf{d})}{5} < n < \frac{3X(\mathbf{d})}{5}$. Without loss of generality, assume $d_1 = \min\{d_j\}$. We then conclude that $d_1 \in \{1,\dots,6\}$. Using the recursion (29) and using an estimate similar to (78), (79), we have

$$\begin{aligned} C(\mathbf{d}) \;\leq\; \Big(1-\frac{10}{3X(\mathbf{d})-3}\Big)\,\theta_{X(\mathbf{d})-1,n-1} \;+\; \frac{10}{3X(\mathbf{d})-3}\,\max\{\theta_{X(\mathbf{d})-1,n-1},\theta_{X(\mathbf{d})-3,n-1}\} \\ +\;\frac{M_2}{(X(\mathbf{d})-1)(X(\mathbf{d})-2)} \end{aligned} \tag{89}$$

for some $M_2 > 0$. By applying Lemma 7, we obtain

$$\theta_{\mathcal{X},n} \;\leq\; \theta_{\mathcal{X}-1,n-1} \;+\; \frac{M_3}{(\mathcal{X}-1)(\mathcal{X}-2)} \tag{90}$$

for some constant $M_3 > 0$. Iterating (90) $t = [\frac{5n-\mathcal{X}+1}{4}]$ times and using Lemma 5 and Lemma 7 we obtain that $\theta_{\mathcal{X},n}$ is bounded by $\frac{1}{\pi} + O(\frac{1}{\mathcal{X}})$ uniformly when $\frac{\mathcal{X}}{5} < n \leq \frac{3\mathcal{X}}{5}$. This together with (88) implies that formula (27) holds for $\frac{X(\mathbf{d})}{5} \leq n \leq \frac{3X(\mathbf{d})}{5}$. For $n > \frac{3X(\mathbf{d})}{5}$, we have $\theta_{X(\mathbf{d}),n} = \theta_{X(\mathbf{d})-1,n-1}$ (indeed, since $C(\mathbf{d})$ is unchanged by removing any 1 argument, we have by definition $\theta_{\mathcal{X},n} = \theta_{\mathcal{X}-1,n-1}$ for $n > \frac{3\mathcal{X}}{5}$), which implies that formula (27) still holds. Combining all three cases, we obtain the statement of Theorem 1. □

Before proceeding, let us give an application of Theorem 1. Consider the *Painlevé I equation*:

$$\frac{d^2U}{dX^2} + \frac{1}{16}U^2 - \frac{1}{16}X \;=\; 0\,. \tag{91}$$

This equation has (cf. e.g. [17, 18]) a unique formal solution $U(X)$ of the form

$$U(X) \;=\; \sum_{g=0}^{\infty} c_g\, X^{\frac{1-5g}{2}}\,, \qquad c_0 = -1\,,\; c_g \in \mathbb{C}\,, \tag{92}$$

where c_g, $g \geq 0$, are determined by the recursion

$$c_g \;=\; 50\,(g-1)^2\,c_{g-1} \;+\; \frac{1}{2}\sum_{h=2}^{g-2} c_h\,c_{g-h}\,, \quad g \geq 3\,, \tag{93}$$

with the initial data $c_0 = -1$, $c_1 = 2$, $c_2 = 98$. Here we use normalizations as in [9]. From (93) one can deduce the following asymptotics for c_g:

$$c_g \;\sim\; A\cdot 50^g\,(g-1)!^2\,\Big(1-\frac{49}{3750g^3}-\frac{49}{1250g^4}+\cdots\Big)\,, \quad g\to\infty\,, \tag{94}$$

where A is some constant whose determination requires deep analysis, which was obtained in [17, 18] and we will give a new proof.

Theorem A ([17, 18]). *There holds that*

$$A \;=\; \frac{1}{2\pi^2}\sqrt{\frac{3}{5}}\,.$$

Proof. According to [16] (cf. also e.g. [9]), we know that c_g can be expressed in terms of Witten's intersection numbers as follows:

$$c_g \;=\; \frac{2^g\,3^{3g-2}}{5^{3g-3}}\,\frac{(5g-5)!\,(5g-3)}{(3g-3)!}\,C(2^{3g-3})\,,\quad g\geq 2\,. \tag{95}$$

The theorem is then proved by using Theorem 1 and Stirling's formula. □

Note that Theorem A cannot be deduced from Aggarwal's result (16), as $n=3g-3$ which is certainly beyond $o(\sqrt{g})$.

We now proceed and prove Theorem 2.

Proof of Theorem 2. For $Y\geq 1$ and for $\alpha\geq 0$, define

$$\theta_Y^{(\alpha)} \;:=\; \sup_{n\geq 1}\max_{\substack{\mathbf{d}\in(\mathbb{Z}_{\geq 1})^n\\ X(0^\alpha,\mathbf{d})-p_1(\mathbf{d})=Y}} C(0^\alpha,\mathbf{d})\,. \tag{96}$$

From Theorem 1 and Lemma 4 we know that as $Y\to\infty$, $\theta_Y^{(\alpha)} \;=\; 1/\pi+O(Y^{-1})$, $\alpha=0,1,2$, with an absolute O-constant. Consider $\alpha\geq 3$. For $n,Y\geq 1$, and for any $\mathbf{d}\in(\mathbb{Z}_{\geq 1})^n$ satisfying $X(0^\alpha,\mathbf{d})=p_1(\mathbf{d})+Y$, write $\mathbf{d}=(1^{p_1(\mathbf{d})},d_{p_1(\mathbf{d})+1},\ldots,d_n)$. By using (31) we have

$$\begin{aligned} C(0^\alpha,\mathbf{d}) \;&=\; C(0^\alpha,d_{p_1(\mathbf{d})+1},\ldots,d_n)\\ &=\; \sum_{j=p_1(\mathbf{d})+1}^{n}\frac{2d_j+1}{3(Y-1)}C(0^{\alpha-1},d_{p_1(\mathbf{d})+1},\ldots,d_j-1,\ldots,d_n)\\ &\leq\; \Big(1+\frac{3-\alpha}{3(Y-1)}\Big)\,\max\{\theta_{Y-1}^{(\alpha-1)},\theta_{Y-2}^{(\alpha-1)}\}\,. \end{aligned} \tag{97}$$

By taking maximum we get

$$\begin{aligned} \theta_Y^{(\alpha)} \;&\leq\; \Big(1+\frac{3-\alpha}{3(Y-1)}\Big)\,\max\{\theta_{Y-1}^{(\alpha-1)},\theta_{Y-2}^{(\alpha-1)}\}\\ &\leq\;\cdots\;\leq\; \max_{j=0,\ldots,\alpha-2}\{\theta_{Y+2-\alpha-j}^{(2)}\}\,\prod_{j=1}^{\alpha-2}\Big(1+\frac{2+j-\alpha}{3(Y-j)}\Big)\,. \end{aligned} \tag{98}$$

Given sufficiently small $\epsilon>0$, when $\alpha\leq\epsilon Y$ we know from $\theta_Y^{(2)} \;=\; 1/\pi+O(Y^{-1})$ that the maximum in the right-hand side of (98) equals $1/\pi+O(Y^{-1})$ with an absolute O-constant, and when $\alpha>\epsilon Y$, the maximum is still bounded by some constant but the product in the right-hand side of (98) has the estimate

$$\prod_{j=1}^{\alpha-2}\Big(1+\frac{2+j-\alpha}{3(Y-j)}\Big)\;\leq\;\exp\Big(\sum_{j=1}^{\alpha-2}\frac{2+j-\alpha}{3(Y-j)}\Big)\;\leq\;\exp\Big(-\frac{(\alpha-2)(\alpha-3)}{6Y}\Big)\,, \tag{99}$$

and hence is $O(Y^{-1})$ with an absolute O-constant. Therefore, we conclude that

$$\theta_Y^{(\alpha)} \;=\; \frac{1}{\pi}\prod_{j=1}^{\alpha-2}\Big(1+\frac{2+j-\alpha}{3(Y-j)}\Big)+O(Y^{-1})\,, \tag{100}$$

with an absolute O-constant (independent of α). Let us now consider the lower bound. For $Y\ge 1$ and for $\alpha\ge 0$, define

$$\phi_Y^{(\alpha)} \;:=\; \inf_{n\ge 1}\min_{\substack{\mathbf{d}\in(\mathbb{Z}_{\ge 1})^n\\ X(0^\alpha,\mathbf{d})-p_1(\mathbf{d})=Y}} C(0^\alpha,\mathbf{d})\,. \tag{101}$$

From Theorem 1, inequalities (57) and (59) we find that as $Y\to\infty$, $\theta_Y^{(\alpha)} = 1/\pi + O(Y^{-1})$, $\alpha=0,1,2$, with an absolute O-constant. By using a similar estimate of (97) and by taking minimum we get

$$\begin{aligned}\phi_Y^{(\alpha)} \;&\ge\; \Big(1+\frac{3-\alpha}{3(Y-1)}\Big)\,\min\{\theta_{Y-1}^{(\phi-1)},\phi_{Y-2}^{(\alpha-1)}\}\\ &\ge\;\cdots\;\ge\;\min_{j=0,\dots,\alpha-2}\{\phi_{Y+2-\alpha-j}^{(2)}\}\prod_{j=1}^{\alpha-2}\Big(1+\frac{2+j-\alpha}{3(Y+1-2j)}\Big)\,.\end{aligned} \tag{102}$$

Using similar arguments that proves (100) we can obtain that

$$\phi_Y^{(\alpha)} \;=\; \frac{1}{\pi}\prod_{j=1}^{\alpha-2}\Big(1+\frac{2+j-\alpha}{3(Y+1-2j)}\Big)+O(Y^{-1})\,, \tag{103}$$

with an absolute O-constant (independent of α). Noticing that the upper bound (100) and the lower bound (103) differ by $O(Y^{-1})$ with an absolute O-constant independent of α and that $X(\mathbf{d})-p_1(\mathbf{d})\ge 2g(\mathbf{d})-2$, we finish the proof of Theorem 2. □

Proof of Corollary 1. For $k=k(g)=O(\sqrt{g})$, we know from Theorem 2 and Stirling's formula that

$$\begin{aligned}C\big(0^k,2^{3g-3+k}\big) \;&=\; \frac{1}{\pi}\,\frac{\big(\frac{2}{3}\big)^k\,\big(\frac{15g+3k-13}{2}\big)_k}{(5g-5+k)_k}\;+\;O\Big(\frac{1}{g}\Big)\\ &=\; \frac{1}{\pi}\,\frac{\Big(\frac{15g+5k-13}{3}\Big)^k\,e^{-\frac{k^2}{15g+3k-13}}}{(5g-5+2k)^k\,e^{-\frac{k^2}{2(5g-5+k)}}}\,(1+o(1))\;+\;O\Big(\frac{1}{g}\Big)\\ &=\; \frac{1}{\pi}\,e^{-\frac{k^2}{30g}}\,(1+o(1))\;+\;O\Big(\frac{1}{g}\Big)\,.\end{aligned} \tag{104}$$

For $k/\sqrt{g}\to\infty$, we have

$$\prod_{j=1}^{k}\Big(1+\frac{2+j-k}{3(5g-5+2k-j)}\Big)\;\le\;\exp\Big(\frac{k(5-k)}{6(5g-5+k)}\Big)\;\to\;0\,,\quad g\to\infty\,. \tag{105}$$

By using Theorem 2 we know that $C(0^k,2^{3g-3+k})\to 0$ as $g\to\infty$. This proves Corollary 1. □

4. Polynomiality in large genus

In this section, we prove Theorem 3 by using the recursion (29).

Proof of Theorem 3. Fix $\mathbf{d}' = (d_1, \dots, d_{n-1}) \in (\mathbb{Z}_{\geq 0})^{n-1}$. Write $\mathbf{d} = (\mathbf{d}', d_n)$ with $d_n \geq 0$. According to [23] (cf. [15]) and Stirling's formula we know that $C(\mathbf{d})$ has the asymptotic expansion

$$C(\mathbf{d}) \;=\; \frac{1}{\pi} \sum_k \frac{C_k(\mathbf{d}')}{X(\mathbf{d})^k}\,, \qquad \text{as } d_n \to \infty\,, \tag{106}$$

where C_k are functions of $\mathbf{d}'$. By using the recursion (29) and by performing Laurent expansions, we obtain

$$\begin{aligned} C_k(d, \mathbf{d}') - C_k(\mathbf{d}') \;=\; & -\sum_{l=1}^{k-1} (-1)^{k-l} \binom{k-1}{l-1} C_l(d, \mathbf{d}') \\ & + \frac{1}{3} \sum_{j=1}^{n-1} (2d_j + 1) \left(C_{k-1}(d_1, \dots, d_j + d - 1, \dots, d_{n-1}) - C_{k-1}(\mathbf{d}') \right) \\ & + \sum_{\substack{a,b \\ a+b=d-2}} \left[\frac{2}{3} \left(C_{k-1}(a, b, \mathbf{d}') - C_{k-1}(\mathbf{d}') \right) + \frac{1}{3} \sum_{I \sqcup J = \{1,\dots,n-1\}} \sum_{l=0}^{k-2} \mathfrak{a}_{(a,\mathbf{d}'_I),l,k}\, C_l(b, \mathbf{d}'_J) \right], \end{aligned} \tag{107}$$

where $d \geq 0$ and $\mathfrak{a}_{\mathbf{w},l,k}$ are certain numbers defined via the following generating function

$$(X(\mathbf{w}) - 1)!\, C(\mathbf{w})\, \frac{(X - X(\mathbf{w}) - 1)!}{X!\, (X - X(\mathbf{w}))^l} \;=:\; \sum_k \frac{\mathfrak{a}_{\mathbf{w},l,k}}{X^k}\,. \tag{108}$$

Write

$$C_k(\mathbf{d}') \;=:\; \tilde{c}_k(p_0(\mathbf{d}'), p_1(\mathbf{d}'), \dots)\,, \tag{109}$$

where $p_r(\mathbf{d}')$ denotes the multiplicity of r in $\mathbf{d}'$. This defines functions $\tilde{c}_k(\mathbf{p})$, $k \geq 0$, where $\mathbf{p} = (p_0, p_1, p_2, \dots)$ is a infinite integer sequence satisfying $\sum_i p_i < \infty$. Then formula (107) becomes

$$\begin{aligned} \tilde{c}_k(\mathbf{p} + \mathbf{e}_d) - \tilde{c}_k(\mathbf{p}) \;=\; & -\sum_{l=1}^{k-1} (-1)^{k-l} \binom{k-1}{l-1} \tilde{c}_l(\mathbf{p} + \mathbf{e}_d) \\ & + \frac{1}{3} \sum_{i \geq 0} (2i+1)\, p_i \left(\tilde{c}_{k-1}(\mathbf{p} - \mathbf{e}_i + \mathbf{e}_{i+d-1}) - \tilde{c}_{k-1}(\mathbf{p}) \right) \\ & + \sum_{\substack{a,b \geq 0 \\ a+b=d-2}} \left[\frac{2}{3} \left(\tilde{c}_{k-1}(\mathbf{p} + \mathbf{e}_a + \mathbf{e}_b) - \tilde{c}_{k-1}(\mathbf{p}) \right) \right. \\ & \left. + \frac{1}{3} \sum_{\substack{\mathcal{E}(\mathbf{t}+\mathbf{e}_a) \leq k-1 \\ 0 \leq t_r \leq p_r,\, r \geq 0}} \sum_{l=0}^{k-2} \left(\tilde{c}_l(\mathbf{p} - \mathbf{t} + \mathbf{e}_b)\, \alpha_{\mathbf{t}+\mathbf{e}_a,l,k} \prod_{i \geq 0} \binom{p_i}{t_i} \right) \right], \quad d \geq 0\,, \end{aligned} \tag{110}$$

where $\mathcal{E}(\mathbf{t}) := X(0^{t_0}, 1^{t_1}, \dots)$, $\alpha_{\mathbf{t},l,k} = \mathfrak{a}_{(0^{t_0}1^{t_1}2^{t_2}\cdots),l,k}$, and $\mathbf{e}_d$ denotes $(0, \dots, 0, 1, 0, 0, \dots)$ with "1" appearing in the $(d+1)$th place.

Let us now prove by induction that $\tilde{c}_k(\mathbf{p})$, $k \geq 0$, belong to $\mathbb{Q}[p_0, p_1, \dots]$ and satisfy the degree estimates

$$\deg \tilde{c}_k(\mathbf{p}) \;\leq\; 3k-1\,, \quad k \geq 1\,, \tag{111}$$

under the degree assignments $\deg p_d = 2d+1$, $d \geq 0$. For $k = 0$, by using Theorem 1 we know that $\tilde{c}_0(\mathbf{p}) \equiv 1$. Assume that for $1 \leq l \leq k-1$, $\tilde{c}_l(\mathbf{p}) \in \mathbb{Q}[p_0, p_1, \dots]$ are polynomials satisfying $\deg \tilde{c}_l(\mathbf{p}) \leq 3l-1$. Then for k and for every $d \geq 0$, the RHS of equation (110) are polynomials in $p_0, \dots, p_{[(3k-5)/2]}$. Moreover, by the inductive assumption these polynomials are independent of d for every $d \geq 3k-1$, i.e.,

$$\tilde{c}_k(\mathbf{p}+\mathbf{e}_d) - \tilde{c}_k(\mathbf{p}) \;=\; \begin{cases} f_d(p_0, \dots, p_{[(3k-5)/2]})\,, & d \leq 3k-2\,, \\ g(p_0, \dots, p_{[(3k-5)/2]})\,, & d \geq 3k-1 \end{cases} \tag{112}$$

for some f_d ($d \leq k$) and g in $\mathbb{Q}[p_0, \dots, p_{[(3k-5)/2]}]$. The compatibility of (112) implies that $g(p_0, \dots, p_{[(k-3)/2]}) \equiv A$ is a constant. Solving (112) we obtain that $\tilde{c}_k$ have the form

$$\tilde{c}_k(\mathbf{p}) \;=\; h(p_0, \dots, p_{3k-2}) \,+\, A\, n'(\mathbf{p})\,, \tag{113}$$

where $h \in \mathbb{Q}[p_0, \dots, p_{3k-2}]$ and $n'(\mathbf{p}) := \sum_{i \geq 0} p_i$. We aim to show that $A = 0$. Consider equation (110) with k replaced by $k+1$. Using a similar analysis and using (113), we obtain that for every $d \geq 6\,k-1$,

$$\tilde{c}_{k+1}(\mathbf{p}+\mathbf{e}_d) - \tilde{c}_{k+1}(\mathbf{p}) \;=\; \frac{4}{3}\,A\,d \,+\, A'\,n'(\mathbf{p}) \,+\, s(p_0, \dots, p_{3k-2})\,, \tag{114}$$

where $A' \in \mathbb{Q}$ is a constant, and $s(p_0, \dots, p_{3k-2})$ is a polynomial in $\mathbb{Q}[p_0, \dots, p_{3k-2}]$ independent of d. Now taking $\mathbf{p} = \mathbf{0}$, we find that equation (114) contradicts with (44) unless $A = 0$. We conclude from (113) that

$$\tilde{c}_k(\mathbf{p}) \in \mathbb{Q}[p_0, \dots, p_{3k-2}]\,. \tag{115}$$

Now taking $d \geq 3k-1$ in equation (110) gives

$$\begin{aligned} 0 \;=\; & -\sum_{l=1}^{k-1} (-1)^{k-l} \binom{k-1}{l-1} \tilde{c}_l(\mathbf{p}) \,+\, \frac{1}{3} \sum_{i \geq 0} (2i+1)\, p_i \left(\tilde{c}_{k-1}(\mathbf{p}-\mathbf{e}_i) - \tilde{c}_{k-1}(\mathbf{p}) \right) \\ & + \sum_{a=0}^{[(3k-5)/2]} \left(\frac{4}{3} \left(\tilde{c}_{k-1}(\mathbf{p}+\mathbf{e}_a) - \tilde{c}_{k-1}(\mathbf{p}) \right) + \frac{1}{3} \sum_{\substack{\mathcal{E}(\mathbf{t}+\mathbf{e}_a) \leq k-1 \\ 0 \leq t_r \leq p_r,\, r \geq 0}} \sum_{l=0}^{k-2} \left(\tilde{c}_l(\mathbf{p}-\mathbf{t})\, \alpha_{\mathbf{t}+\mathbf{e}_a, l, k} \prod_{i \geq 0} \binom{p_i}{t_i} \right) \right). \end{aligned} \tag{116}$$

Using (116) and (110), we obtain

$$\begin{aligned}\Delta_{p_d}\tilde{c}_k(\mathbf{p}) \;=\; &-\sum_{l=1}^{k-1}(-1)^{k-l}\binom{k-1}{l-1}\,\Delta_{p_d}\tilde{c}_l(\mathbf{p}) \;+\; \frac{1}{3}\sum_{i\geq 0}(2i+1)\,p_i\,\Delta_{p_i+d}\tilde{c}_{k-1}(\mathbf{p}-\mathbf{e}_i)\\ &+\;\frac{2}{3}\sum_{a=0}^{d-2}\Delta_{p_a}\Delta_{p_{d-2-a}}\tilde{c}_{k-1}(\mathbf{p}) \;-\; \frac{4}{3}\sum_{a=d-1}^{[(3k-5)/2]}\Delta_{p_a}\tilde{c}_{k-1}(\mathbf{p})\\ &+\;\frac{1}{3}\sum_{a=0}^{d-2}\sum_{\substack{\mathcal{E}(\mathbf{t}+\mathbf{e}_a)\leq k-1\\ 0\leq t_r\leq p_r\,(r\geq 0)}}\sum_{l=0}^{k-2}\Bigg(\Delta_{p_{d-2-a}}\tilde{c}_l(\mathbf{p}-\mathbf{t})\,\alpha_{\mathbf{t}+\mathbf{e}_a,l,k}\prod_{i\geq 0}\binom{p_i}{t_i}\Bigg)\\ &+\;\frac{1}{3}\sum_{a=d-1}^{[(3k-5)/2]}\sum_{\substack{\mathcal{E}(\mathbf{t}+\mathbf{e}_a)\leq k-1\\ 0\leq t_r\leq p_r\,(r\geq 0)}}\sum_{l=0}^{k-2}\left(\tilde{c}_l(\mathbf{p}-\mathbf{t})\,\alpha_{\mathbf{t}+\mathbf{e}_a,l,k}\prod_{i\geq 0}\binom{p_i}{t_i}\right),\quad d\geq 0\,. \end{aligned}\tag{117}$$

We find by inductive assumption that each term of the right-hand side of (117) is of degree less than or equal to $3k-2-2d$ for every $d\geq 0$, which implies $\deg\tilde{c}_k\leq 3k-1$. In particular, $\tilde{c}_k$ is a polynomial that only depends on $p_0,\dots,p_{[3k/2]-1}$.

Now restrict to the case when $\mathbf{d}'\in(\mathbb{Z}_{\geq 2})^{n-1}$. From (106), (42) and Stirling's formula we know that

$$\widehat{C}(\mathbf{d}) \;\sim\; \sum_k \frac{\widehat{C}_k(\mathbf{d}')}{X(\mathbf{d})^k}\,,\qquad d_n\to\infty\,,\tag{118}$$

where $\widehat{C}_k$ are functions of $\mathbf{d}'$ with $\widehat{C}_0\equiv 1$. Define $\widehat{c}_k(p_2,p_3,\dots)\in\mathbb{Q}[p_2,p_3,\dots]$, $k\geq 0$, via

$$\gamma(X)\sum_k\frac{\widehat{c}_k(p_2,p_3,\dots)}{X^k} \;=\; \sum_k\frac{\tilde{c}_k(0,0,p_2,p_3,\dots)}{X^k}\,,\tag{119}$$

where the left-hand side is understood as a power series in X^{-1}. It then follows from $\deg\tilde{c}_k\leq 3k-1$ that $\deg\widehat{c}_k\leq 3k-1$. Using (118), (106), (109), we know that

$$\widehat{C}_k(\mathbf{d}') \;=\; \widehat{c}_k(p_1(\mathbf{d}'),p_2(\mathbf{d}'),\dots)\tag{120}$$

for all $\mathbf{d}'$. The statement that $\widehat{c}_k(0,0,\dots)=0$ follows from the fact that $C(3g-2)=\gamma(6g-3)$ for every $g\geq 1$. This finishes the proof of Theorem 3. □

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