ON ENUMERATION OF b-ANGULATIONS OF SURFACES FROM AN INTEGRABILITY PERSPECTIVE

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Abstract. In this paper, we study generating series enumerating polygonal angulations of closed oriented surfaces of fixed genus, focusing on b-angulations with $b = 3$ or $b = 2\nu$, $\nu \geq 2$. Based on Toda integrability, we establish new structural results in the cases $b = 3$ and $b = 4$. Furthermore, via the Hodge–GUE correspondence, we derive a fine structure in the $b = 2\nu$ case, which implies a conjectural statement of Gharakhloo–Latimer.

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1. Introduction and statements of the main results

Enumerating ribbon graphs, also known as (combinatorial) maps, is a fundamental problem at the interface of mathematics and mathematical physics, attracting interest from combinatorics (cf. [4, 10, 26, 34, 44]), and revealing deep relations to quantum field theory [4, 7, 8, 13, 33, 45] and geometry [3, 14, 15, 16, 34, 47]. Let $\mathcal{R}_g^{\mathrm{conn}}(b_1, \dots, b_k)$ be the set of connected oriented labelled[1] ribbon graphs of genus g with k vertices of valencies $b_1, \dots, b_k$, and let $n_g(b_1, \dots, b_k) \coloneqq |\mathcal{R}_g^{\mathrm{conn}}(b_1, \dots, b_k)|$. Here, $g \geq 0$ and $b_1, \dots, b_k \geq 1$. By the Euler formula, the number $n_g(b_1, \dots, b_k)$ vanishes unless $2-2g-k+\frac{|b|}{2}$ is a positive integer. By looking at the dual graphs, one can also understand $n_g(b_1, \dots, b_k)$ as the number of polygon-angulations with k polygons of sizes $b_1, \dots, b_k$ on a genus g closed oriented surface.

Following [15, 17], define a power series of infinitely many variables $\mathbf{s} = (s_1, s_2, \dots)$ by

$$\mathcal{F}(x, \mathbf{s}; \epsilon) = \frac{x^2}{2\epsilon^2}\Big(\log x - \frac{3}{2}\Big) - \frac{\log x}{12} + \zeta'(-1) + \sum_{g\geq 2} \frac{\epsilon^{2g-2} B_{2g}}{4g(g-1)x^{2g-2}} + \sum_{g\geq 0} \epsilon^{2g-2} \sum_{k\geq 1} \frac{1}{k!} \sum_{b_1,\dots,b_k \geq 1} n_g(b_1, \dots, b_k)\, s_{b_1} \cdots s_{b_k} x^{2-2g-k+\frac{|\mathbf{b}|}{2}}, \tag{1}$$

called the *free energy*, and define $Z(x, \mathbf{s}; \epsilon) \coloneqq e^{\mathcal{F}(x,\mathbf{s};\epsilon)}$, called the *partition function*. Here, x is a formal variable, $\zeta(s)$ denotes the Riemann zeta function, and B_m denotes the mth

[1] A labelled ribbon graph refers to a ribbon graph whose half-edges are labelled.

Bernoulli number. It is known from e.g. [1, 4, 34, 17, 47] that $Z(x,\mathbf{s};\epsilon)$ can be understood as the following integral

$$2^{-n}\pi^{-\frac{n(n+1)}{2}}\epsilon^{-\frac{1}{12}}G(n+1)\int_{\mathcal{H}(n)}\exp\Big(-\frac{1}{\epsilon}\mathrm{tr}\Big(\frac{1}{2}M^2-\sum_{b\geq 1}s_bM^b\Big)\Big)dM\,, \tag{2}$$

where $x=n\epsilon$, G denotes Barnes's G-function, and

$$dM=\prod_{1\leq i\leq n}dM_{ii}\prod_{1\leq i<j\leq n}d\mathrm{Re}M_{ij}\,d\mathrm{Im}M_{ij}\,. \tag{3}$$

Define $\mathcal{F}_g(x,\mathbf{s}):=[\epsilon^{2g-2}]\mathcal{F}(x,\mathbf{s};\epsilon)$, $g\geq 0$, the *genus g free energy*. Structures for $\mathcal{F}_g(x,\mathbf{s})$ have been studied in [2, 15, 27, 16, 17, 18, 47]. By definition, $\mathcal{F}_g(x,\mathbf{s})$ encodes the enumeration of arbitrary tilings on a genus g surface (throughout this paper, all surfaces are assumed to be closed and oriented). The special case of enumeration of b-angulations, with a fixed value of b, has attracted a lot of interest [17, 21, 22, 24, 31] and will be the main focus of this paper. This corresponds to restricting $\mathcal{F}(x,\mathbf{s};\epsilon)$ to $\mathbf{s}=(0,\dots,0,s_b=s,0,\dots)$, and by rescaling we can assume $s=1$, so we will consider

$$\begin{aligned}\mathcal{F}_g^{\{b\}}(x):=\mathcal{F}_g(x,\mathbf{s}=\mathbf{1}_b)&=\delta_{g,0}\frac{x^2}{2}\Big(\log x-\frac{3}{2}\Big)+\delta_{g,1}\Big(-\frac{\log x}{12}+\zeta'(-1)\Big)\\&+\ \delta_{g\geq 2}\frac{B_{2g}}{4g(g-1)x^{2g-2}}+\sum_{k\geq 1}\frac{n_g(b^k)}{k!}x^{2-2g+(\frac{b}{2}-1)k}\,,\end{aligned} \tag{4}$$

where $\mathbf{1}_b$ denotes the infinite vector with the bth component equal to 1 and all other components equal to 0.

According to e.g. [1, 17, 30, 47], $Z(x,\mathbf{s};\epsilon)$ is a tau-function of the Toda lattice hierarchy. The latter will be one of the main tools for this paper. Following [15, 17, 20, 47], define

$$u=\frac{\partial^2\mathcal{F}_0^{\{b\}}(x)}{\partial x^2}\,,\quad w=e^u\,. \tag{5}$$

The case $b=3$. Following [5, 17], introduce a power series $v=v(x)=6x+324x^2+31104x^3+\cdots$ as the unique solution to the following cubic equation

$$6\,x=(1-9v+18v^2)\,v\,. \tag{6}$$

According to [17] we have the identity

$$w=\frac{x}{1-6v}\,. \tag{7}$$

(We will give a new proof of (7) in Section 3.) It follows from (6), (7) that $w=x+36x^2+3240x^3+\cdots$ is the unique solution to the cubic equation

$$x^2=w^2-72w^3\,. \tag{8}$$

Equation (8) was also obtained in [5]. Building on, for example,[1, 17, 30] (see also [5]) and using a method from [19], we will prove in Section 3 the following theorem.

Theorem 1. *For $g=0$, we have $\frac{\partial^2\mathcal{F}_0^{\{3\}}(x)}{\partial x^2}=\log w$. For $g=1$,*

$$\mathcal{F}_1^{\{3\}}(x)=-\frac{1}{12}\log w-\frac{1}{24}\log(1-108w)+\zeta'(-1). \tag{9}$$

For $g \geq 2$, $\mathcal{F}_g^{\{3\}}(x)$ has the expression:

$$\mathcal{F}_g^{\{3\}}(x) = \frac{1-2g}{(2g)!}\, B_{2g}\, \partial_x^{2g-2}(\log w) + \sum_{\ell=3g-3}^{5g-5} \frac{a_{g,\ell}}{(1-108w)^\ell}\,, \tag{10}$$

where $a_{g,3g-3}, \ldots, a_{g,5g-5}$ are rational numbers.

For $g \geq 2$, the coefficient $a_{g,5g-5}$ has the expression

$$a_{g,5g-5} = \frac{162^g}{3888} \frac{C_g}{(5g-3)(5g-5)}, \tag{11}$$

which can be straightforwardly deduced using (10) and the well-known result (see e.g. [5])

$$n_g(3^{2j}) \sim \frac{16}{\sqrt{3}} \frac{\left(108\sqrt{3}\right)^j}{\left(256\sqrt{3}\right)^g} \frac{(2j)!\,(2j)^{\frac{5g-7}{2}}}{\Gamma(\frac{5g-1}{2})}\, C_g \quad (j \to \infty)\,. \tag{12}$$

Here C_g, with $C_0 = -1, \ldots$, are constants determined (cf. e.g. [36, 37, 5, 19, 26, 35, 48]) by requiring that the formal series $U = \sum_{g\geq 0} C_g X^{\frac{1-5g}{2}}$ satisfies the Painlevé I equation

$$\frac{d^2U}{dX^2} + \frac{1}{16}U^2 - \frac{1}{16}X = 0\,. \tag{13}$$

Table 1 consists of $n_g(3^{4g-4+2d})$ for $g = 0, \ldots, 4$ and $d = 1, \ldots, 5$. For the reader's convenience, we also provide

$$\mathcal{F}_2^{\{3\}} = \frac{\partial_x^2(\log w)}{240} - \frac{351}{8(1-108w)^3} + \frac{27}{8(1-108w)^4} + \frac{189}{10(1-108w)^5}\,, \tag{14}$$

$$\mathcal{F}_3^{\{3\}} = -\frac{\partial_x^4(\log w)}{6048} + \frac{589761}{4(1-108w)^6} - \frac{8203437}{28(1-108w)^7} - \frac{448335}{2(1-108w)^8} + \frac{324405}{(1-108w)^9} + \frac{178605}{(1-108w)^{10}}\,. \tag{15}$$

d	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$
1	0	3	3061800	357485480352000	561734730904309522560000
2	0	4536	89414357760	475379823378087 93600	208281465835272806 019563520000
3	12	19362240	2834113460935 680	514591710352541 8098278400	547188956214674466 69373094461440000
4	5184	164367221760	1107578328829 37856000	565109847632479 817270034432000	131217479838294406 811733692434863882 24000
5	9797760	233201956829 1840	5405486118155 731877068800	672926687318093 573568845764362 24000	313985184119369209 780057086172883719 8151680000

TABLE 1. The numbers $n_g(3^{4g-4+2d})$ for $g = 0, \ldots, 4$ and $d = 1, \ldots, 5$.

The case $b = 4$. According to [4, 18, 24], $w = x + 12x^2 + 288x^3 + \cdots$ is the unique solution to

$$x = w - 12\,w^2\,. \tag{16}$$

Explicitly, $w = \frac{1-\sqrt{1-48x}}{24}$.

Theorem 2. *For $g = 0$, we have $\frac{\partial^2 \mathcal{F}_0^{\{4\}}(x)}{\partial x^2} = \log w$. For $g = 1$,*

$$\mathcal{F}_1^{\{4\}}(x) = -\frac{1}{12}\log w - \frac{1}{12}\log(1-24w) + \zeta'(-1). \tag{17}$$

For $g \geq 2$, $\mathcal{F}_g^{\{4\}}(x)$ has the expression:

$$\mathcal{F}_g^{\{4\}}(x) = \frac{1-2g}{(2g)!}\,B_{2g}\,\partial_x^{2g-2}(\log w) + \sum_{\ell=4g-4}^{5g-5} \frac{\tilde{a}_{g,\ell}}{(1-24w)^\ell}\,, \tag{18}$$

where $\tilde{a}_{g,4g-4}, \ldots, \tilde{a}_{g,5g-5}$ are rational numbers.

The proof, which is based on [1, 17, 30] and uses a method from [19], is given in Section 4.

It is easy to deduce using (18) and a result of [22] that the number $\tilde{a}_{g,5g-5}$ equals to $\frac{48^g}{576}\frac{C_g}{(5g-3)(5g-5)}$ (this statement can also be proved using just the Toda lattice theory), where C_g are the universal constants introduced in (12).

The case $b = 2\nu$. In this case, the following theorem (using our notations) was originally conjectured in [24, Conjecture 5.2] and confirmed recently in the third arXiv version of [24].

Theorem A ([24]) *For $\nu \geq 2$, define $q = q(x)$ as the power-series-in-x solution to the equation*

$$1 - q + \frac{(2\nu)!}{\nu!(\nu-1)!}\,x^{\nu-1}q^\nu = 0\,. \tag{19}$$

Then for $g \geq 2$,

$$x^{2g-2}\mathcal{F}_g^{\{2\nu\}}(x) = \sum_{\ell=2g-2}^{5g-5} \frac{r_{g,\ell}(\nu)}{(\nu-(\nu-1)q)^\ell}\,, \tag{20}$$

where for each $\nu \geq 2$, $r_{g,2g-2}(\nu), \ldots, r_{g,5g-5}(\nu)$ are rational numbers.

We mention here that for the case when $b = 4$ one can use g parameters to express the genus g free energy with $g \geq 2$ (see (18)) instead of $3g-2$ parameters (see (20)). For the case when $b = 6$ we have similar observations in which $2g-1$ parameters are sufficient, whose proof will be given elsewhere.

We will also give several proofs of Theorem A, which were completed before noticing the updated version of [24].

In Section 4, a proof of Theorem A will be achieved based on Theorem 2. By using a formula of [22], we will give a second proof, which also corresponds to the proof in [24].

In [16, 18] a relationship between $n_g(2\nu_1, \ldots, 2\nu_k)$ and certain cubic Hodge integrals on the Deligne–Mumford moduli spaces, called the *Hodge–GUE correspondence*, was established. As an application of the Hodge–GUE correspondence, we will give a third proof of Theorem A.

Motivated by a study of Gharakhloo–Latimer [31], we give in the following theorem a fine structure for $\mathcal{F}_g^{\{2\nu\}}(x)$ as another application of the Hodge–GUE correspondence.

Theorem 3. *For $g \geq 2$, the coefficients $r_{g,\ell}(\nu)$ in (20) are polynomials in ν of degree $3g-3$.*

The proof is in Section 5.

Following [31], define $S_{g,k}(\nu)$ by

$$n_g((2\nu)^k) =: \left(\frac{(2\nu)!}{(\nu+1)!\nu!}\right)^k S_{g,k}(\nu)\,. \tag{21}$$

By (20), (19) we have

$$S_{g,k}(\nu) = k!(\nu(\nu+1))^k \sum_{m=0}^{k} \binom{\nu k-1}{k-m} \frac{(\nu-1)^m}{m!} \sum_{\ell=2g-2}^{5g-5} r_{g,\ell}(\nu)(\ell-1)_m\,. \tag{22}$$

Theorem 3 then implies a conjectural statement given by Gharakhloo–Latimer in [31, Remark 2.9]: *for $g \geq 0$ and $k \geq 1$, $S_{g,k}(\nu)$ is a polynomial in ν of degree $3g-3+3k$.* A further conjectural statement on distributions of zeros of $S_{g,k}(\nu)$ was also proposed in [31]. We recall that, when $g=0$, the expressions of $n_0((2\nu)^k)$, $\nu \geq 2$, were given in [25] (the $\nu=2$ case also given in [9]); when $g=1$, the expressions of $n_1((2\nu)^k)$ were obtained in [9] for $\nu=2$ and in [23] for $\nu \geq 2$; when $g=2$, the expressions of $n_2((2\nu)^k)$, $\nu \geq 2$, $k=1,2,3$, were given in [31]; the expressions of $n_g(4^k)$, $g=2,\ldots,7$, $k \geq 1$, were given in [23], and the expressions of $n_g(6^k)$, $g=2,\ldots,5$, $k \geq 1$, were given in [31]. We also note that, the above-proved polynomiality of $S_{g,k}(\nu)$ can alternatively be deduced from the celebrated quasi-polynomiality of [12, 38] (cf. also [32, 43]), which is deeply related to topological recursion.

Organization of the paper In Section 2 we review earlier works on the free energy $\mathcal{F}(x,\mathbf{s};\epsilon)$. In Section 3 we prove Theorem 1. In Section 4 we prove Theorem 2. In Section 5 we prove Theorem 3.

Acknowledgements We thank Paul Norbury for pointing out the reference [27]. The work of E.G-F. is supported by the Ramón y Cajal Fellowship RYC2023-045188-I, funded by MCIN/AEI/10.13039/501100011033 and by the FSE+. She also acknowledges support by the project PID2024-155686NB-I00 of the Spanish Ministry of Science and Innovation, the ANR CarteEtPlus ANR-23-CE48-0018, a Tremplin grant from Sorbonne Université, a PEPS grant from the CNRS and the ERC Synergy Grant ReNewQuantum. The work of D.Y. and J.X. is supported by NSFC 12371254 and CAS YSBR-032.

2. Review of the GUE free energy $\mathcal{F}(x,\mathbf{s};\epsilon)$

In this section we review several properties of the GUE free energy $\mathcal{F}(x,\mathbf{s};\epsilon)$.

It is known (cf. e.g. [30, 34, 39, 41, 42]) that the partition function $Z = Z(x,\mathbf{s};\epsilon)$ satisfies the following *Virasoro constraints*:

$$L_k(Z(x,\mathbf{s};\epsilon)) = 0\,, \quad k \geq -1\,, \tag{23}$$

where L_k are linear operators given by

$$L_{-1} = \sum_{j\geq 2} j s_j \frac{\partial}{\partial s_{j-1}} - \frac{\partial}{\partial s_1} + \frac{x s_1}{\epsilon^2}\,, \tag{24}$$

$$L_0 = \sum_{j\geq 1} j s_j \frac{\partial}{\partial s_j} - \frac{\partial}{\partial s_2} + \frac{x^2}{\epsilon^2}\,, \tag{25}$$

$$L_k = \epsilon^2 \sum_{j=1}^{k-1} \frac{\partial^2}{\partial s_j \partial s_{k-j}} + 2x \frac{\partial}{\partial s_k} + \sum_{j\geq 1} j s_j \frac{\partial}{\partial s_{j+k}} - \frac{\partial}{\partial s_{k+2}}\,, \quad k \geq 1\,, \tag{26}$$

which satisfy the Virasoro commutation relations:

$$[L_k, L_\ell] = (k-\ell) L_{k+\ell}\,, \quad k, \ell \geq -1\,. \tag{27}$$

The $k = -1$ equation in (23) is also known as the *string equation*. It also follows from (2) that Z satisfies the following *dilaton equation*:

$$\sum_{j\geq 1} s_j \frac{\partial Z}{\partial s_j} + \epsilon \frac{\partial Z}{\partial \epsilon} + x \frac{\partial Z}{\partial x} + \frac{1}{12} Z = \frac{1}{2} \frac{\partial Z}{\partial s_2}\,. \tag{28}$$

Denote by $\Lambda = e^{\epsilon \partial_x}$ the shift operator. It is known (cf. [1, 17, 30, 41]) that ($V^{\mathrm{GUE}} = V^{\mathrm{GUE}}(x, \mathbf{s}; \epsilon), W^{\mathrm{GUE}} = W^{\mathrm{GUE}}(x, \mathbf{s}; \epsilon)$) defined by

$$V^{\mathrm{GUE}}(x, \mathbf{s}; \epsilon) = \epsilon(\Lambda - 1) \frac{\partial \mathcal{F}(x, \mathbf{s}; \epsilon)}{\partial s_1}\,, \tag{29}$$

$$W^{\mathrm{GUE}}(x, \mathbf{s}; \epsilon) = \epsilon^2 \frac{\partial^2 \mathcal{F}(x, \mathbf{s}; \epsilon)}{\partial s_1 \partial s_1} \tag{30}$$

is a solution to the *Toda lattice hierarchy* [28, 40]:

$$\epsilon \frac{\partial L}{\partial s_j} = [A_j, L]\,, \quad j \geq 1\,, \tag{31}$$

where

$$L = \Lambda + V + W\Lambda^{-1}\,, \quad A_j \coloneqq (L^j)_+\,. \tag{32}$$

Here and below, for a difference operator $P = \sum_{k\in\mathbb{Z}} P_k \Lambda^k$, $P_+ \coloneqq \sum_{k\geq 0} P_k \Lambda^k$ and $\operatorname{res} P \coloneqq P_0$. Denote

$$V^{\{b\}}(x, \epsilon) \coloneqq V^{\mathrm{GUE}}(x, \mathbf{1}_b; \epsilon)\,, \quad W^{\{b\}}(x, \epsilon) \coloneqq W^{\mathrm{GUE}}(x, \mathbf{1}_b; \epsilon)\,. \tag{33}$$

By definition, we know that $W^{\{b\}}(x, \epsilon)$ has the following genus expansion:

$$W^{\{b\}}(x, \epsilon) = \sum_{g\geq 0} \epsilon^{2g} W_g^{\{b\}}(x)\,. \tag{34}$$

Dividing the $k = -1$ equation in (23) by $Z(x, \mathbf{s}; \epsilon)$, we have

$$\sum_{j\geq 2} j s_j \frac{\partial \mathcal{F}(x, \mathbf{s}; \epsilon)}{\partial s_{j-1}} + \frac{x s_1}{\epsilon^2} = \frac{\partial \mathcal{F}(x, \mathbf{s}; \epsilon)}{\partial s_1}\,. \tag{35}$$

Applying $\epsilon(\Lambda - 1)$ on both sides of (35) yields

$$\epsilon \sum_{j\geq 2} j s_j (\Lambda - 1) \frac{\partial \mathcal{F}(x, \mathbf{s}; \epsilon)}{\partial s_{j-1}} + s_1 = \epsilon(\Lambda - 1) \frac{\partial \mathcal{F}(x, \mathbf{s}; \epsilon)}{\partial s_1}\,. \tag{36}$$

Following [17, (2.1.4)], define $h_j := \frac{1}{j+2}\mathrm{res}L^{j+2} \in \mathbb{Q}[V, W, \Lambda^{\pm 1}V, \Lambda^{\pm 1}W, \dots]$. In particular, $h_{-1} = V$. From [17, (1.2.9) and Corollary 2.2.2] we know

$$\epsilon(\Lambda - 1)\frac{\partial \mathcal{F}(x, \mathbf{s}; \epsilon)}{\partial s_j} = j h_{j-2}\big|_{\Lambda^i V = \Lambda^i V^{\mathrm{GUE}},\, \Lambda^i W = \Lambda^i W^{\mathrm{GUE}},\, i \in \mathbb{Z}}\,, \tag{37}$$

then (36) becomes

$$\sum_{j \geq 2} j(j-1) s_j\, h_{j-3}\big|_{\Lambda^i V = \Lambda^i V^{\mathrm{GUE}},\, \Lambda^i W = \Lambda^i W^{\mathrm{GUE}},\, i \in \mathbb{Z}} + s_1 = V^{\mathrm{GUE}}\,. \tag{38}$$

Applying $\epsilon^2 \frac{\partial}{\partial s_1}$ on both sides of (35), we get

$$\epsilon^2 \sum_{j \geq 2} j s_j \frac{\partial^2 \mathcal{F}(x, \mathbf{s}; \epsilon)}{\partial s_1 \partial s_{j-1}} + x = \epsilon^2 \frac{\partial^2 \mathcal{F}(x, \mathbf{s}; \epsilon)}{\partial s_1^2}\,. \tag{39}$$

Then according to [17, (1.2.8), (2.1.10)] and [46, (56)], we obtain

$$\sum_{j \geq 2} j s_j (\Lambda + 1)^{-1} (j h_{j-2} - (j-1) V h_{j-3})|_{\Lambda^i V = \Lambda^i V^{\mathrm{GUE}},\, \Lambda^i W = \Lambda^i W^{\mathrm{GUE}},\, i \in \mathbb{Z}} + x = W^{\mathrm{GUE}}\,. \tag{40}$$

We note that special cases of (38) and (40) were given in e.g. [5, 8, 9, 29].

Define $\mathbf{v}^{\mathrm{GUE}}(x, \mathbf{s}) = (v^{\mathrm{GUE}}(x, \mathbf{s}), u^{\mathrm{GUE}}(x, \mathbf{s})) := (V^{\mathrm{GUE}}(x, \mathbf{s}; 0), \log W^{\mathrm{GUE}}(x, \mathbf{s}; 0))$. Based on the Dubrovin–Zhang theory, the following formulas were obtained in [15] (cf. also [2, 11, 26, 47]):

$$\begin{aligned}
\mathcal{F}_0(x, \mathbf{s}) =& \frac{1}{2} \sum_{p,q \geq 0} (p+1)!(q+1)! \big(s_{p+1} - \frac{1}{2}\delta_{p,1}\big)\big(s_{q+1} - \frac{1}{2}\delta_{q,1}\big) \Omega^{\mathbb{P}^1,[0]}_{2,p;2,q}(v^{\mathrm{GUE}}(x, \mathbf{s}), u^{\mathrm{GUE}}(x, \mathbf{s})) \\
&+ x \sum_{p \geq 0} (p+1)! \big(s_{p+1} - \frac{1}{2}\delta_{p,1}\big) \theta^{\mathbb{P}^1}_{2,p}(v^{\mathrm{GUE}}(x, \mathbf{s}), u^{\mathrm{GUE}}(x, \mathbf{s})) + \frac{1}{2} x^2 u^{\mathrm{GUE}}(x, \mathbf{s})\,, \qquad (41) \\
\mathcal{F}_g(x, \mathbf{s}) =& F^{\mathbb{P}^1}_g \left(v^{\mathrm{GUE}}(x, \mathbf{s})\,, u^{\mathrm{GUE}}(x, \mathbf{s}), \dots, \frac{\partial^{3g-2} v^{\mathrm{GUE}}(x, \mathbf{s})}{\partial x^{3g-2}}, \frac{\partial^{3g-2} u^{\mathrm{GUE}}(x, \mathbf{s})}{\partial x^{3g-2}} \right) \\
&+ \big(\zeta'(-1) - \frac{1}{24}\log(-1)\big)\delta_{g,1}\,, \quad g \geq 1\,. \qquad (42)
\end{aligned}$$

Here, $\Omega^{\mathbb{P}^1,[0]}_{2,p;2,q}(v, u)$ and $\theta^{\mathbb{P}^1}_{2,p}(v, u)$ are certain genus 0 two-point correlation functions of the $\mathbb{P}^1$-Frobenius manifold, and $F^{\mathbb{P}^1}_g(v, u, v_1, u_1, \dots, v_{3g-2}, u_{3g-2})$ denotes the genus g free energy in jet variables. For $g = 1$,

$$F^{\mathbb{P}^1}_1(v, u, v_1, u_1) = \frac{1}{24}\log(v_1^2 - e^u u_1^2) - \frac{1}{24}u\,. \tag{43}$$

For $g \geq 2$, $F^{\mathbb{P}^1}_g(v, u, v_1, u_1, \dots, v_{3g-2}, u_{3g-2})$ depends polynomially on $v_2, u_2, \dots, v_{3g-2}, u_{3g-2}$ and rationally on v_1, u_1 with coefficients being smooth functions of v and u. Moreover,

$$\deg F^{\mathbb{P}^1}_g(v, u, v_1, u_1, \dots, v_{3g-2}, u_{3g-2}) = 2g - 2\,, \tag{44}$$

where $\deg v_k = \deg u_k := k$. The reader is referred to e.g. [14, 20, 47] for details on the $\mathbb{P}^1$-Frobenius manifold, $\theta^{\mathbb{P}^1}_{2,p}(v, u)$, $\Omega^{\mathbb{P}^1,[0]}_{2,p;2,q}(v, u)$, and $F^{\mathbb{P}^1}_g(v, u, v_1, u_1, \dots, v_{3g-2}, u_{3g-2})$.

Following for example [45], consider *the even GUE free energy* $\mathcal{F}^{\text{even}}$, which is the GUE free energy restricted to even couplings, i.e.,

$$\mathcal{F}^{\text{even}} = \mathcal{F}^{\text{even}}(x, s_2, s_4, \dots; \epsilon) \coloneqq \mathcal{F}(x, \mathbf{s}; \epsilon)|_{s_1=s_3=\cdots=0} \tag{45}$$

and plays an important role in two-dimensional quantum gravity [45]. Denote by $\mathcal{F}_g^{\text{even}} \coloneqq [\epsilon^{2g-2}]\mathcal{F}^{\text{even}}$ the genus g part of $\mathcal{F}^{\text{even}}$, and denote

$$v_{\text{even}} = v^{\text{GUE}}(x, \mathbf{s})|_{s_1=s_3=\cdots=0}\,,\ u_{\text{even}} = u^{\text{GUE}}(x, \mathbf{s})|_{s_1=s_3=\cdots=0}\,. \tag{46}$$

It was proved in [18] that $v_{\text{even}} \equiv 0$ and $w_{\text{even}} \coloneqq e^{u_{\text{even}}}$ satisifies

$$x = w_{\text{even}} - \sum_{\nu\geq 1} \frac{(2\nu)!}{\nu!(\nu-1)!} s_{2\nu} w_{\text{even}}^{\nu}\,. \tag{47}$$

The *Hodge–GUE correspondence*, which gives an explicit relationship between a certain cubic Hodge free energy and $\mathcal{F}^{\text{even}}$, was established in [16, 18]. Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne–Mumford moduli space of stable genus-g curves with n marked points. Denote by ψ_i the first Chern class of the ith tautological line bundle over $\overline{\mathcal{M}}_{g,n}$, and by λ_i the ith Chern class of the rank-g Hodge bundle on $\overline{\mathcal{M}}_{g,n}$.

We define the Chern polynomial $\Lambda_g(z) \coloneqq \sum_{i=0}^{g} \lambda_i z^i$, and set

$$\mathcal{H}(\mathbf{t}; \epsilon) \coloneqq \sum_{g\geq 0} \epsilon^{2g-2} \sum_{n\geq 0} \frac{1}{n!} \sum_{i_1,\cdots,i_n\geq 0} \prod_{m=1}^{n} t_{i_m} \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_g(-1)\Lambda_g(-1)\Lambda_g(\tfrac{1}{2})\psi_1^{i_1}\cdots\psi_n^{i_n}\,. \tag{48}$$

The Hodge–GUE correspondence says that

$$\mathcal{F}^{\text{even}}(x, s_2, s_4, \dots) = \epsilon^{-2} A(x, s_2, s_4, \dots) + \zeta'(-1) + (\Lambda^{\frac{1}{2}} + \Lambda^{-\frac{1}{2}})\mathcal{H}(\mathbf{t}(x, s_2, s_4, \dots); \epsilon)\,, \tag{49}$$

where $A = A(x, s_2, s_4, \dots)$ is defined by

$$A = \frac{1}{2} \sum_{k_1,k_2\geq 1} \frac{k_1 k_2}{k_1+k_2} \binom{2k_1}{k_1} \binom{2k_2}{k_2} s_{2k_1} s_{2k_2} - \sum_{k\geq 1} \frac{k}{1+k} \binom{2k}{k} s_{2k} + x \sum_{k\geq 1} \binom{2k}{k} s_{2k} + \frac{1}{4} - x\,, \tag{50}$$

and

$$t_i(x, s_2, s_4, \dots) \coloneqq \sum_{k\geq 1} k^{i+1} \binom{2k}{k} s_{2k} - 1 + \delta_{i,1} + x\delta_{i,0}\,, \quad i \geq 0\,. \tag{51}$$

This correspondence yields the only known ELSV-type formula for strictly monotone Hurwitz numbers [6], to the best of our knowledge. Explicit expressions for $\mathcal{F}_g^{\text{even}}$ for $g = 1, \dots, 5$ in terms of u_{even} and its x-derivatives were given in [18] and its arXiv preprint version. For example, [18] shows that

$$\mathcal{F}_1^{\text{even}} = \frac{1}{12} \log \frac{\partial u_{\text{even}}}{\partial x} + \zeta'(-1). \tag{52}$$

3. On triangulations

In this section we prove Theorem 1.

Taking $\mathbf{s} = \mathbf{1}_3$ in (38) and (40), we have

$$3\left(V^{\{3\}}(x,\epsilon)^2 + W^{\{3\}}(x,\epsilon) + W^{\{3\}}(x+\epsilon,\epsilon)\right) = V^{\{3\}}(x,\epsilon)\,, \tag{53}$$

$$x + 3\,W^{\{3\}}(x,\epsilon)\left(V^{\{3\}}(x,\epsilon) + V^{\{3\}}(x-\epsilon,\epsilon)\right) = W^{\{3\}}(x,\epsilon) \tag{54}$$

(cf. also e.g. [5, 9]). Introduce

$$\widetilde{V}^{\{3\}}(x,\epsilon) \coloneqq V^{\{3\}}\big(x-\frac{\epsilon}{2},\epsilon\big)\,, \tag{55}$$

then equations (53)–(54) become

$$3\left(\widetilde{V}^{\{3\}}(x,\epsilon)^2+W^{\{3\}}\big(x-\frac{\epsilon}{2},\epsilon\big)\;+\;W^{\{3\}}\big(x+\frac{\epsilon}{2},\epsilon\big)\right)=\widetilde{V}^{\{3\}}(x,\epsilon)\,, \tag{56}$$

$$x+3W^{\{3\}}(x,\epsilon)\Big(\widetilde{V}^{\{3\}}\big(x-\frac{\epsilon}{2},\epsilon\big)+\widetilde{V}^{\{3\}}\big(x+\frac{\epsilon}{2},\epsilon\big)\Big)=W^{\{3\}}(x,\epsilon)\,. \tag{57}$$

From the definition of $V^{\mathrm{GUE}}(x,\mathbf{s};\epsilon)$, we know that $\widetilde{V}^{\{3\}}(x,\epsilon)$ has the following genus expansion:

$$\widetilde{V}^{\{3\}}(x,\epsilon)=\sum_{g\geq 0}\epsilon^{2g}\widetilde{V}^{\{3\}}_g(x). \tag{58}$$

Substituting (34) and (58) in equations (56)–(57), we obtain

$$3\sum_{g_1,g_2\geq 0}\epsilon^{2(g_1+g_2)}\,\widetilde{V}^{\{3\}}_{g_1}\,\widetilde{V}^{\{3\}}_{g_2}+6\sum_{g,k}\frac{\epsilon^{2g+2k}}{2^{2k}(2k)!}\frac{\partial^{2k}W^{\{3\}}_g}{\partial x^{2k}}=\sum_{g\geq 0}\epsilon^{2g}\,\widetilde{V}^{\{3\}}_g(x)\,, \tag{59}$$

$$x+6\sum_{g_1,g_2,k\geq 0}\frac{\epsilon^{2g_1+2g_2+2k}}{2^{2k}(2k)!}W^{\{3\}}_{g_1}(x)\,\frac{\partial^{2k}\widetilde{V}^{\{3\}}_{g_2}}{\partial x^{2k}}=\sum_{g\geq 0}\epsilon^{2g}\,W^{\{3\}}_g(x)\,. \tag{60}$$

Comparing coefficients of ϵ^{2g} $(g\geq 0)$ on both sides of (59)–(60), we find

$$3\,\widetilde{V}^{\{3\}}_0(x)^2+6\,W^{\{3\}}_0(x)=\widetilde{V}^{\{3\}}_0(x)\,, \tag{61}$$

$$x+6\,W^{\{3\}}_0(x)\,\widetilde{V}^{\{3\}}_0(x)=W^{\{3\}}_0(x)\,, \tag{62}$$

and for $g\geq 1$,

$$\big(1-6\widetilde{V}^{\{3\}}_0\big)\,\widetilde{V}^{\{3\}}_g\;-\;6\,W^{\{3\}}_g=3\sum_{g_1=1}^{g-1}\widetilde{V}^{\{3\}}_{g_1}\,\widetilde{V}^{\{3\}}_{g-g_1}+6\sum_{k=1}^{g}\frac{1}{2^{2k}(2k)!}\frac{\partial^{2k}W^{\{3\}}_{g-k}}{\partial x^{2k}}\,, \tag{63}$$

$$-\;6\,W^{\{3\}}_0\,\widetilde{V}^{\{3\}}_g+(1-6\,\widetilde{V}^{\{3\}}_0)\,W^{\{3\}}_g=6\sum_{\substack{g_1,g_2\leq g-1\\ g_1+g_2+k=g}}\frac{1}{2^{2k}(2k)!}W^{\{3\}}_{g_1}\,\frac{\partial^{2k}\widetilde{V}^{\{3\}}_{g_2}}{\partial x^{2k}}\,. \tag{64}$$

Solving (61)–(62) we find that $v=v(x)\coloneqq\widetilde{V}^{\{3\}}_0(x)$ and $w=w(x)\coloneqq W^{\{3\}}_0(x)$ satisfy (6) and (7), which imply (8) as well as

$$v=\frac{1-\sqrt{1-72w}}{6}\,. \tag{65}$$

From (8) we get

$$\partial_x=\frac{\sqrt{1-72\,w}}{1-108\,w}\,\partial_w\,. \tag{66}$$

Lemma 1. *For $k\geq 1$,*

$$\partial_x^{2k}=\sum_{i=1}^{2k}\frac{T_{k,i}(w)}{(1-108w)^{4k-i}}\partial_w^i \tag{67}$$

for some polynomials $T_{k,1}(w),\dots,T_{k,2k}(w)$. Moreover, $\deg T_{k,i}(w)\leq k$ *for $i=1,\dots,2k$.*

Proof. For $k=1$, from (66) we obtain

$$\partial_x^2 = \frac{72(1-54w)}{(1-108w)^3}\partial_w + \frac{1-72w}{(1-108w)^2}\partial_w^2\,, \tag{68}$$

thus (67) holds. Suppose that (67) is true for $k=m$ with $\deg T_{m,i}(w) \le m$. Then for $k=m+1$ we have

$$\begin{aligned}\partial_x^{2m+2} = \partial_x^2 \circ \partial_x^{2m} &= \left(\frac{72(1-54w)}{(1-108w)^3}\partial_w + \frac{(1-72w)}{(1-108w)^2}\partial_w^2\right) \circ \sum_{i=1}^{2m} \frac{T_{m,i}(w)}{(1-108w)^{4m-i}}\partial_w^i \\ &= \sum_{i=1}^{2m+2} \frac{T_{m+1,i}(w)}{(1-108w)^{4m+4-i}}\partial_w^i\,. \end{aligned} \tag{69}$$

Here

$$\begin{aligned} T_{m+1,i} =& (1-72w)(1-108w)^2\partial_w^2 T_{m,i} + 2(1-72w)(1-108w)\partial_w T_{m,i-1} + (1-72w)T_{m,i-2} \\ &+ \big(72+216(4k-i)-(3888-15552(4k-i))w\big)(1-108w)\partial_w T_{m,i} \\ &+ \big(72+216(4k+1-i)-(3888-15552(4k+1-i))w\big)T_{m,i-1} \\ &+ 108(4k-i)\big(72+108(4k+1-i)-(3888-7776(4k+1-i))w\big)T_{m,i}\,, \end{aligned} \tag{70}$$

with $T_{m,-1}$, $T_{m,0}$, $T_{m,2m+1}$ and $T_{m,2m+2}$ defined as 0. Obviously, $\deg T_{m+1,i} \le m+1$ for $i=1,\dots,2m+2$. By mathematical induction the lemma is proved. □

Proposition 1. *For $g \ge 1$, $\widetilde{V}_g^{\{3\}}(x)$, $W_g^{\{3\}}(x)$ are given by*

$$W_g^{\{3\}}(x) = \frac{wQ_g(w)}{(1-108w)^{5g-1}}\Big|_{w=w(x)}, \quad \widetilde{V}_g^{\{3\}}(x) = \frac{\sqrt{1-72w}\,R_g(w)}{(1-108w)^{5g-1}}\Big|_{w=w(x)}, \tag{71}$$

where $Q_g(w), R_g(w)$ are polynomials of w. Moreover, $\deg Q_g(w) \le 2g-1$, $\deg R_g(w) \le 2g-1$.

Proof. For $g=1$, solving the $g=1$ equations of (63)–(64) gives

$$\widetilde{V}_1^{\{3\}}(x) = \frac{54\sqrt{1-72w}}{(1-108w)^4}\Big|_{w=w(x)}, \quad W_1^{\{3\}}(x) = \frac{162\,(5-324w)\,w}{(1-108w)^4}\Big|_{w=w(x)}. \tag{72}$$

Suppose (71) is true with $\deg Q_g(w), \deg R_g(w) \le 2g-1$ for $g \le m$. Then for $g=m+1$, by solving the $g=m+1$ equations of (63)–(64) and using (65), Lemma 1, we obtain

$$(\widetilde{V}_{m+1}^{\{3\}}(x), W_{m+1}^{\{3\}}(x)) = \Big(\frac{\sqrt{1-72w}\,R_{m+1}(w)}{(1-108w)^{5m+4}}, \frac{w\,Q_{m+1}(w)}{(1-108w)^{5m+4}}\Big)\Big|_{w=w(x)}, \tag{73}$$

where $Q_{m+1}(w)$, $R_{m+1}(w)$ are polynomials of w with $\deg Q_{m+1}(w), \deg R_{m+1}(w) \le 2m+1$. This completes the proof of the proposition by mathematical induction. □

From Proposition 1 we know that $\widetilde{V}^{\{3\}}(x,\epsilon)$, $W^{\{3\}}(x,\epsilon)$ can be written as

$$W^{\{3\}}(x,\epsilon) = w + \sum_{g\ge1}\epsilon^{2g}\sum_{\ell=3g-1}^{5g-1}\frac{A_{g,\ell}}{(1-108w)^\ell}\,, \quad \widetilde{V}^{\{3\}}(x,\epsilon) = v + \sum_{g\ge1}\epsilon^{2g}\sum_{\ell=-\infty}^{5g-1}\frac{\widetilde{B}_{g,\ell}}{(1-108w)^\ell}\,, \tag{74}$$

for some real numbers $A_{g,5g-1},\dots,A_{g,3g-1},\widetilde{B}_{g,5g-1},\dots$. Substituting (74) in (56)–(57), one can obtain that $U=2^{\frac{9}{5}}3^{\frac{11}{5}}\sum_{g\geq 0}A_{g,5g-1}(2^{\frac{2}{5}}3^{\frac{8}{5}}X)^{\frac{1-5g}{2}}$ satisfies the Painlevé I equation (13). Therefore,

$$A_{g,5g-1}=\frac{162^g}{108}C_g\,, \tag{75}$$

with C_g the constants introduced in (12).

We proceed to calculate $\mathcal{F}^{\{3\}}(x,\epsilon)$. Taking $\mathbf{s}=\mathbf{1}_3$ in (29) we have

$$\log W^{\{3\}}(x,\epsilon)=\mathcal{F}^{\{3\}}(x+\epsilon,\epsilon)\,+\,\mathcal{F}^{\{3\}}(x-\epsilon,\epsilon)\,-\,2\,\mathcal{F}^{\{3\}}(x,\epsilon)\,. \tag{76}$$

Therefore,

$$\begin{aligned}\mathcal{F}^{\{3\}}(x,\epsilon)&=\frac{1}{\epsilon^2\partial_x^2}\,\frac{\epsilon^2\partial_x^2}{e^{\epsilon\partial_x}+e^{-\epsilon\partial_x}-2}\big(\log W^{\{3\}}(x,\epsilon)\big)\\&=\frac{1}{\epsilon^2\partial_x^2}\sum_{k\geq 0}\frac{(1-2k)B_{2k}}{(2k)!}\,\epsilon^{2k}\partial_x^{2k}\big(\log W^{\{3\}}(x,\epsilon)\big)\,.\end{aligned} \tag{77}$$

It is helpful to introduce the following lemma for computing (77):

Lemma 2. *The following identity holds for $k\geq 4$:*

$$\begin{aligned}\frac{972(-2)^k}{(1-108w)^k}\frac{((k-2)!)^2}{(2k-4)!}=\partial_x^2\Bigg(&72w\sqrt{3-216w}\,\operatorname{arctanh}\big(\sqrt{3-216w}\big)\\&-72w+\sum_{i=1}^{k-4}\frac{1}{(1-108w)^i}\frac{(-2)^{i+2}}{i(i+2)}\frac{(i+2)!^2}{(2i+4)!}\Bigg)\,.\end{aligned} \tag{78}$$

Proof. Let a be a formal variable. We have

$$972\sum_{k\geq 4}\frac{((k-2)!)^2}{(2k-4)!}\frac{(-2a)^k}{(1-108w)^k}=3888\,s^2\bigg(\frac{s(s-1)}{2-s}+\frac{2\sqrt{s}\arcsin\big(\sqrt{s/2}\big)}{(2-s)^{\frac{3}{2}}}\bigg)\,, \tag{79}$$

where $s=\frac{a}{108w-1}$. Using (68) it is easy to verify that

$$\begin{aligned}&3888s^2\bigg(\frac{s(s-1)}{2-s}+\frac{2\sqrt{s}\arcsin\big(\sqrt{s/2}\big)}{(2-s)^{\frac{3}{2}}}\bigg)\\&=\partial_x^2\Bigg(\frac{a^4}{1-a}\Bigg(72w\sqrt{3-216w}\arctanh\big(\sqrt{3-216w}\big)\\&\qquad-72w+\frac{2}{3}\sqrt{1-\frac{s}{2}}\,\frac{s+1}{s}\,\frac{\arcsin\big(\sqrt{s/2}\big)}{\sqrt{s/2}}-\frac{2}{3s}\Bigg)\Bigg)\\&=\partial_x^2\Bigg(\frac{a^4}{1-a}\Bigg(72w\sqrt{3-216w}\,\operatorname{arctanh}\big(\sqrt{3-216w}\big)\\&\qquad-72w+\sum_{i\geq 0}\frac{a^i}{(1-108w)^i}\frac{(-2)^{i+2}}{i(i+2)}\frac{(i+2)!^2}{(2i+4)!}\Bigg)\Bigg)\,.\end{aligned} \tag{80}$$

In this last equality we use the identity

$$\frac{2}{3}\sqrt{1-\frac{s}{2}}\,\frac{s+1}{s}\,\frac{\arcsin\left(\sqrt{s/2}\right)}{\sqrt{s/2}} = \frac{2}{3s}+\frac{5}{9}+\sum_{i\geq 1}\frac{(-2)^{i+2}}{i(i+2)}\frac{(i+2)!^2}{(2i+4)!}s^i\,. \tag{81}$$

By taking the coefficient of a^k, $k\geq 4$, on both sides of (80) we obtain (78). □

Using Proposition 1, we have

$$\log W^{\{3\}}(x,\epsilon) = \log w + \log\left(1+\sum_{g\geq 1}\frac{\epsilon^{2g}Q_g(w)}{(1-108w)^{5g-1}}\right). \tag{82}$$

By substituting (82) in (77), and using Lemma 1 and Lemma 2, we obtain for $g\geq 2$,

$$\begin{aligned}
\mathcal{F}_g^{\{3\}} =& \frac{1-2g}{(2g)!}B_{2g}\partial_x^{2g-2}(\log w)\\
&+\sum_{k=1}^{g-1}\frac{1-2k}{(2k)!}B_{2k}\partial_x^{2k-2}\left(\sum_{\ell=1}^{g-k}\frac{(-1)^{\ell-1}}{\ell(1-108w)^{5g-5k-\ell}}\sum_{\substack{m_1,\dots,m_\ell\geq 1\\ m_1+\cdots+m_\ell=g-k}}\prod_{i=1}^{\ell}Q_{m_i}(w)\right)\\
&+\partial_x^{-2}\left(\sum_{\ell=1}^{g}\frac{(-1)^{\ell-1}}{\ell(1-108w)^{5g-\ell}}\sum_{\substack{m_1,\dots,m_\ell\geq 1\\ m_1+\cdots+m_\ell=g}}\prod_{i=1}^{\ell}Q_{m_i}(w)\right) \qquad (83)\\
=&\frac{1-2g}{(2g)!}B_{2g}\partial_x^{2g-2}(\log w)+\sum_{\ell=3g-3}^{5g-5}\frac{a_{g,\ell}}{(1-108w)^\ell}+\alpha_g w\sqrt{1-72w}+\beta_g\\
&+\gamma_g\cdot\left(72w\sqrt{3-216w}\,\operatorname{arctanh}(\sqrt{3-216w})\right.\\
&\left.-72z+\sum_{i=1}^{3g-4}\frac{1}{(1-108w)^i}\frac{(-2)^{i+2}}{i(i+2)}\frac{(i+2)!^2}{(2i+4)!}\right). \qquad (84)
\end{aligned}$$

Here α_g and β_g are integration constants, and $a_{g,3g-3},\dots,a_{g,5g-5},\gamma_g$ are rational numbers.

For further simplification of (84), introduce

$$\mathcal{G}^{\{3\}}(x,\epsilon) := 6\left(\frac{1}{4}-\frac{x^2}{\epsilon^2}+3\left(\epsilon\partial_\epsilon+x\partial_x\right)\left(\mathcal{F}^{\{3\}}(x,\epsilon)\right)\right). \tag{85}$$

It has the genus expansion $\mathcal{G}^{\{3\}}(x,\epsilon)=\sum_{g\geq 0}\epsilon^{2g-2}\mathcal{G}_g^{\{3\}}(x)$, and for $g\geq 0$,

$$\mathcal{G}_g^{\{3\}} = -6x^2\delta_{g,0}+\frac{3}{2}\delta_{g,1}+18(2g-2)\mathcal{F}_g^{\{3\}}+18x\partial_x(\mathcal{F}_g^{\{3\}})\,. \tag{86}$$

By taking $\mathbf{s}=\mathbf{1}_3$ in equations (24), (25) and (28), we find $\mathcal{G}^{\{3\}}(x,\epsilon)=\frac{\partial Z}{\partial s_1}$, so from (29) we know

$$\frac{1}{\epsilon}\widetilde{V}^{\{3\}}(x,\epsilon)=\mathcal{G}^{\{3\}}(x+\epsilon/2,\epsilon)-\mathcal{G}^{\{3\}}(x-\epsilon/2,\epsilon)\,. \tag{87}$$

Therefore

$$\mathcal{G}^{\{3\}}(x,\epsilon)=\frac{1}{\epsilon^2\partial_x}\frac{\epsilon\partial_x}{(\Lambda^{1/2}-\Lambda^{-1/2})}(\widetilde{V}^{\{3\}}(x,\epsilon))=\partial_x^{-1}\sum_{g\geq 0}\epsilon^{2g-2}\sum_{k+g_1=g}\frac{(-1)^kB_{2k}(\frac{1}{2})}{(2k)!}\partial_x^{2k}(\widetilde{V}_{g_1}^{\{3\}}(x))\,, \tag{88}$$

and

$$\mathcal{G}_g^{\{3\}}(x,\epsilon) = \partial_x^{-1}(\widetilde{V}_g^{\{3\}}(x)) + \sum_{k=1}^{g} \frac{(-1)^k B_{2k}(\frac{1}{2})}{(2k)!} \partial_x^{2k-1}(\widetilde{V}_{g-k}^{\{3\}}(x))\,, \quad g \geq 0\,. \tag{89}$$

Here $B_k(p)$ denotes the kth Bernoulli polynomial.

Proposition 2. *For $g \geq 2$, $\mathcal{G}_g^{\{3\}}(x)$ is given by*

$$\mathcal{G}_g^{\{3\}}(x) = \frac{S_g(w)}{(1-108w)^{5g-3}}\Big|_{w=w(x)} + \tilde{\lambda}_g\,, \tag{90}$$

for a polynomial $S_g(w)$ and some integration constant $\tilde{\lambda}_g$. Moreover, $\deg S_g(w) \leq 2g-1$.

Proof. From (8) we know that $\partial_x^{-1}(*) = \partial_w^{-1}\big(\frac{1-108w}{\sqrt{1-72w}}\,*\big)$. Using this and Proposition 1, Lemma 1 in (89), we find expression (90) and that $\deg S_g(w) \leq 2g-1$. □

We are ready to prove Theorem 1.

Proof of Theorem 1. The $g=0$ case has been given above (cf. also (41) and [17]). For $g=1$, the statement can be verified directly using (63)–(64) (cf. also (42)–(43) and [17]). For $g \geq 2$, by comparing (10) and (84), it suffices to prove that the constants $\alpha_g, \beta_g, \gamma_g$ appearing in (84) all vanish. From (68) we know that $\partial_x^{2g-2}(\log w)$ is a rational function of w. Putting (84) in (86) and comparing with (90) in Proposition 2, by the vanishing of $\operatorname{arctanh}(\sqrt{3-216w})$ in $\mathcal{G}_g^{\{3\}}$ we find $\gamma_g = 0$. Since $\partial_x(\log w) = \frac{\sqrt{1-72w}}{(1-108w)w}$, using (65)–(66) we find that for $k \geq 1$,

$$\partial_x^k(v) = O(w^{\frac{1}{2}-\frac{3}{2}k})\,, \quad \partial_x^k(\log w) = O(w^{-\frac{3}{2}k})\,, \quad w \to \infty\,. \tag{91}$$

Then from (42) (cf. [49]) we know that $\mathcal{F}_g^{\{3\}} \to 0$ as $w \to \infty$. Thus $\alpha_g = \beta_g = 0$. □

We end this section by presenting several consequences of Theorem 1. First of all, note that, using Lemmas 1, 2, (74), (83), (84) and Theorem 1, we can achieve a new self-contained proof of the expression (11) of $a_{g,5g-5}$, which in turn implies (12).

Multiplying both sides of (10) by dx/x^{3-2g+j} and taking the residue at $x=0$, we obtain

$$n_g(3^{2j}) = \sigma_{g,j} + 72^{j-2g+2}(2j)! \sum_{m=0}^{j-2g+2} \binom{\frac{3j}{2} - 3g + 3 - m}{\frac{j}{2} - g + 1} \sum_{\ell=3g-3}^{5g-5} (\ell-1)_m \frac{3^m}{2^m m!} a_{g,\ell}\,, \tag{92}$$

where $\sigma_{g,j} = (2j)! \operatorname*{res}_{w=0} \frac{(1-108w)dw}{w^{j-2g+3}(1-72w)^{\frac{j}{2}-g+2}} \frac{(1-2g)B_{2g}}{2g!} \partial_x^{2g-2}(\log w)$ and $(a)_m \coloneqq a(a+1)\cdots(a+m-1)$ is the Pochhammer symbol.

Following [21] (cf. [26]), introduce

$$q = \frac{1}{1-72w}\,. \tag{93}$$

Formulas (9), (14) are then translated to

$$\mathcal{F}_1^{\{3\}} = -\frac{1}{12}\log x - \frac{1}{24}\log\left(\frac{3-q}{2}\right) + \zeta'(-1)\,, \tag{94}$$

$$\mathcal{F}_2^{\{3\}} = \frac{1}{x^2}\left(\frac{3}{64\,(3-q)} - \frac{29}{32\,(3-q)^2} + \frac{191}{48\,(3-q)^3} - \frac{55}{8\,(3-q)^4} + \frac{21}{5\,(3-q)^5}\right), \tag{95}$$

$$\begin{aligned}\mathcal{F}_3^{\{3\}} = \frac{1}{x^4}\Bigg(&\frac{63}{256\,(3-q)^2} - \frac{22765}{1152\,(3-q)^3} + \frac{7925}{24\,(3-q)^4} - \frac{39311}{16\,(3-q)^5} \\ &+ \frac{1443995}{144\,(3-q)^6} - \frac{4055053}{168\,(3-q)^7} + \frac{68625}{2\,(3-q)^8} - \frac{26730}{(3-q)^9} + \frac{8820}{(3-q)^{10}}\Bigg).\end{aligned} \tag{96}$$

Formula (95) was given in [22] (cf. [26]), and (96) was given in [26]. It was proved by Eynard [26] (using a slightly different notation) that for $g \geq 2$,

$$x^{2g-2}\mathcal{F}_g^{\{3\}}(x) = \sum_{k=g-1}^{5g-5} \frac{r_{g,k}}{(3-q)^k}\,, \tag{97}$$

where $r_{g,g-1}, \dots, r_{g,5g-5}$ are rational numbers. Noting that

$$x^{2k}\partial_x^{2k}(\log w) = \sum_{\ell=0}^{3k-1} \frac{H_{k,\ell}}{(3-q)^{k+\ell}}\,, \quad k \geq 1\,,\ H_{k,0}, \dots, H_{k,3k-1} \in \mathbb{Q}\,, \tag{98}$$

and using Theorem 1 and formula (8), we can achieve a new proof of (97). Multiplying (97) by dx/x^{j+1} and taking the residue at $x = 0$, we obtain

$$n_g(3^{2j}) = 72^j (2j)! \sum_{m=0}^{j} \sum_{\ell=g-1}^{5g-5} \frac{(\ell-1)_m}{2^{m+\ell} m!} \binom{\frac{3j}{2}-1}{j-m} r_{g,\ell}\,. \tag{99}$$

Another direct consequence of Theorem 1 is that for $g \geq 2$, the genus g free energy $\mathcal{F}_g^{\{3\}}(x)$ can be written in the form

$$\mathcal{F}_g^{\{3\}}(x) = \sum_{k=1}^{5g-5} \frac{b_{g,k}}{(1-108w)^k} + \sum_{\ell=1}^{2g-2} \frac{b'_{g,k}}{w^k}\,, \tag{100}$$

where $b_{g,1}, \dots, b_{g,5g-5}, b'_{g,1}, \dots, b'_{g,2g-2}$ are rational numbers.

Finally, introduce $p = p(x)$ by

$$p = \frac{108w}{1-108w}\,. \tag{101}$$

Corollary 1. *For $g = 1$,*

$$\mathcal{F}_1^{\{3\}} = -\frac{1}{12}\log x + \frac{1}{24}\log\left(\frac{p+3}{3}\right) + \zeta'(-1)\,. \tag{102}$$

For $g \geq 2$, $\mathcal{F}_g^{\{3\}}(x)$ admits an expression of the form

$$x^{2g-2}\mathcal{F}_g^{\{3\}}(x) = \frac{B_{2g}}{4g(g-1)} + \sum_{m=1}^{5g-5} c_{g,m}\, p^m\,, \tag{103}$$

where $c_{g,1}, \dots, c_{g,5g-5}$ are rational numbers; moreover, $c_{g,1} = \cdots = c_{g,2g-2} = 0$.

Proof. By using (66) and (101), one can prove

$$x^k\partial_x^k(\log w) = -(k-1)! + \sum_{\ell=0}^{k-1} G_{k,\ell}p^{k+\ell}\,, \quad k\geq 1\,,\ G_{k,0},\dots,G_{k,k-1}\in\mathbb{Q}\,. \tag{104}$$

Then by using Theorem 1 and (8), we obtain (102), (103). For $g\geq 2$, recalling $x^{2g-2}\mathcal{F}_g^{\{3\}} - \frac{B_{2g}}{4g(g-1)} \in x^{2g-1}\mathbb{Q}[[x]]$ and noticing $p\in x\mathbb{Q}[[x]]$, we find $c_{g,1}=\cdots=c_{g,2g-2}=0$. □

Explicitly,

$$\mathcal{F}_2^{\{3\}} = \frac{1}{x^2}\left(-\frac{1}{240}+\frac{7p^5}{12960}+\frac{29p^4}{10368}+\frac{35p^3}{10368}\right),$$
$$\mathcal{F}_3^{\{3\}} = \frac{1}{x^4}\left(\frac{1}{1008}+\frac{245p^{10}}{1679616}+\frac{965p^9}{559872}+\frac{2945p^8}{373248}+\frac{813587p^7}{47029248}+\frac{29969p^6}{1679616}+\frac{5005p^5}{746496}\right).$$

Multiplying (103) by dx/x^{j+1} and taking the residue at $x=0$, we obtain for $g\geq 2$

$$n_g(3^{2j}) = 108^j(2j)!\sum_{\ell=2g-1}^{j} c_{g,\ell}\sum_{m=0}^{j-\ell}\frac{(\frac{3j}{2}-m)_m}{m!}\frac{(\frac{j}{2}+1)_{j-\ell-m}}{(j-\ell-m)!}(-3)^{-j+\ell+m} \tag{105}$$

($c_{g,\ell}$ are defined as 0 if $\ell>5g-5$).

4. On quadrangulations and (2ν)-angulations

In this section we prove Theorem 2.

Taking $\mathbf{s}=\mathbf{1}_4$ in (38) and (40), we have

$$V^{\{4\}}(x,\epsilon)\equiv 0\,, \tag{106}$$
$$4\,W^{\{4\}}(x,\epsilon)(W^{\{4\}}(x,\epsilon)+W^{\{4\}}(x+\epsilon,\epsilon)+W^{\{4\}}(x-\epsilon,\epsilon)) - W^{\{4\}}(x,\epsilon)+x=0 \tag{107}$$

(cf. also e.g. [8, 9, 29]). Substituting (34) in (107), we find $w=W_0^{\{4\}}(x)$ satisfies (16), and for $g\geq 1$,

$$W_g^{\{4\}} = \frac{4}{1-24w}\left(\sum_{g_2=1}^{g-1} W_{g_2}^{\{4\}}W_{g-g_2}^{\{4\}} + 2\sum_{\substack{0\leq g_1,g_2\leq g-1,\ j\geq 0\\ g_1+g_2+j=g}}\frac{1}{(2j)!}W_{g_2}^{\{4\}}\partial_x^{2j}(W_{g_1}^{\{4\}})\right). \tag{108}$$

From equation (16) we get

$$\partial_x = \frac{1}{1-24w}\partial_w\,. \tag{109}$$

Lemma 3. *For $k\geq 1$,*

$$\partial_x^k = \sum_{i=1}^{k}\frac{\widetilde{T}_{k,i}}{(1-24w)^{2k-i}}\partial_w^i\,, \tag{110}$$

where $\widetilde{T}_{k,1},\dots,\widetilde{T}_{k,k}$ are rational numbers.

Proof. For $k=1$, formula (110) is given by (109). Suppose that (110) is true for $k=m$ with $\widetilde{T}_{k,i}$ being rational numbers. Then for $k=m+1$, we have

$$\partial_x^{m+1} = \partial_x\circ\partial_x^m = \frac{1}{1-24w}\partial_w\circ\sum_{i=1}^{m}\frac{\widetilde{T}_{m,i}}{(1-24w)^{2m-i}}\partial_w^i = \sum_{i=1}^{m+1}\frac{\widetilde{T}_{m+1,i}}{(1-24w)^{2m+2-i}}\partial_w^i\,.$$

Here

$$\widetilde{T}_{m,i} = 24(2m-i)\widetilde{T}_{m,i} + \widetilde{T}_{m,i-1}\,, \tag{111}$$

where $\widetilde{T}_{m,0}$ and $\widetilde{T}_{m,m+1}$ are defined as 0. Obviously $\widetilde{T}_{m,i}$ are rational numbers. By mathematical induction the lemma is proved. □

Proposition 3. *For $g \geq 1$, $W_g^{\{4\}}(x)$ is given by*

$$W_g^{\{4\}}(x) = \frac{w\,P_g(w)}{(1-24w)^{5g-1}}\Big|_{w=w(x)}\,, \tag{112}$$

for some polynomial $P_g(z)$. Moreover, $\deg P_g(w) \leq g-1$.

Proof. For $g=1$, formula (108) gives

$$W_1^{\{4\}}(x) = 4\frac{W_0^{\{4\}}(x)\,\partial_x^2(W_0^{\{4\}}(x))}{1-24W_0^{\{4\}}(x)} = \frac{96\,w}{(1-24w)^4}\Big|_{w=w(x)}\,, \tag{113}$$

thus (112) holds. Suppose that (112) holds for $g \leq m$ with $\deg P_g(w) \leq g-1$. Then for $g=m+1$, by using Lemma 3 in the $g=m+1$ equation of (108), we obtain

$$W_{m+1}^{\{4\}}(x) = \frac{w\,P_{m+1}(w)}{(1-24w)^{5m+4}}\Big|_{w=w(x)}\,, \tag{114}$$

where $P_{m+1}(w)$ is a polynomial of w with $\deg P_{m+1}(w) \leq g-1$. By mathematical induction the proposition is proved. □

Proposition 3 says that

$$W^{\{4\}}(x,\epsilon) = w + \sum_{g\geq 1}\epsilon^{2g}\sum_{\ell=4g-1}^{5g-1}\frac{\widetilde{A}_{g,\ell}}{(1-24w)^{\ell}}\,, \tag{115}$$

for some rational numbers $\widetilde{A}_{g,4g-2},\dots,\widetilde{A}_{g,5g-1}$. Substituting (115) in (107), we find that $U = 2^{\frac{11}{5}}3^{\frac{4}{5}}\sum_{g\geq 0}\widetilde{A}_{g,5g-1}(2^{\frac{8}{5}}3^{\frac{2}{5}}X)^{\frac{1-5g}{2}}$ again satisfies the Painlevé I equation (13). Therefore,

$$\widetilde{A}_{g,5g-1} = \frac{48^g}{24}C_g\,, \quad g\geq 1\,, \tag{116}$$

where C_g is again the same universal constant as we found for triangulations (75).

Taking $\mathbf{s}=\mathbf{1}_4$ in (29), we have

$$\log W^{\{4\}}(x,\epsilon) = \mathcal{F}^{\{4\}}(x+\epsilon,\epsilon) + \mathcal{F}^{\{4\}}(x-\epsilon,\epsilon) \,-\, 2\,\mathcal{F}^{\{4\}}(x,\epsilon)\,. \tag{117}$$

Similar to (77), we have

$$\mathcal{F}^{\{4\}}(x,\epsilon) = \frac{1}{\epsilon^2\,\partial_x^2}\sum_{k\geq 0}\frac{(1-2k)B_{2k}}{(2k)!}\,\epsilon^{2k}\partial_x^{2k}\big(\log W^{\{4\}}(x,\epsilon)\big)\,. \tag{118}$$

Proof of Theorem 2. The $g=0$ case has been proved above (cf. also (41), [17] or [18, Proposition 3, Theorem 5]). For $g=1$, the statement can be verified directly using (108)

(cf. also (42)–(43) or (52)). For $g \geq 2$, substituting (34) in (118) and using (112), we obtain

$$\begin{aligned}\mathcal{F}_g^{\{4\}} =& \frac{1-2g}{(2g)!}B_{2g}\partial_x^{2g-2}(\log w)\\ &+\sum_{k=1}^{g-1}\frac{1-2k}{(2k)!}B_{2k}\partial_x^{2k-2}\bigg(\sum_{\ell=1}^{g-k}\frac{(-1)^{\ell-1}}{\ell(1-24w)^{5g-5k-\ell}}\sum_{\substack{m_1,\dots,m_\ell\geq 1\\ m_1+\cdots+m_\ell=g-k}}\prod_{i=1}^{\ell}P_{m_i}(w)\bigg)\\ &+\partial_x^{-2}\bigg(\sum_{\ell=1}^{g}\frac{(-1)^{\ell-1}}{\ell(1-24w)^{5g-\ell}}\sum_{\substack{m_1,\dots,m_\ell\geq 1\\ m_1+\cdots+m_\ell=g}}\prod_{i=1}^{\ell}P_{m_i}(w)\bigg)\\ =&\frac{1-2g}{(2g)!}B_{2g}\partial_x^{2g-2}(\log w)+\sum_{\ell=4g-4}^{5g-5}\frac{\tilde{a}_{g,\ell}}{(1-24w)^{\ell}}+\alpha_g w(1-12w)+\beta_g\,. \end{aligned}\tag{119}$$

Here $\tilde{a}_{g,4g-4},\dots,\tilde{a}_{g,5g-5}$ are rational numbers and α_g,β_g are integration constants. Using $\partial_x(\log w)=\frac{1}{(1-24w)w}$ and (109) we can prove by induction on k that for $k\geq 1$,

$$\partial_x^k(\log w)=O(w^{-2k})\,,\quad w\to\infty\,.\tag{120}$$

Then from (42) (cf. [18, Theorem 5]) we know that $\mathcal{F}_g^{\{4\}}\to 0$ as $w\to\infty$. Therefore, $\alpha_g=\beta_g=0$. □

Multiplying both sides of (18) by dx/x^{3-2g+k} and taking the residue at $x=0$, we obtain

$$n_g(4^k)=\tilde{\sigma}_{g,k}+12^{k-2g+2}k!\sum_{m=0}^{k-2g+2}\binom{2k-4g+4-m}{k-2g+2-m}\Big(\sum_{\ell=4g-4}^{5g-5}\tilde{a}_{g,\ell}(\ell-1)_m\Big)\frac{2^m}{m!}\,,\tag{121}$$

where $\tilde{\sigma}_{g,k}\coloneqq k!\operatorname*{res}_{w=0}\frac{(1-24w)dw}{w^{k-2g+3}(1-12w)^{k-2g+3}}\frac{(1-2g)B_{2g}}{(2g)!}\partial_x^{2g-2}(\log w)$.

Another direct consequence of Theorem 2 is that for $g\geq 2$, $\mathcal{F}_g^{\{4\}}(x)$ admits the form

$$\mathcal{F}_g^{\{4\}}(x)=\sum_{k=1}^{5g-5}\frac{\tilde{b}_{g,k}}{(1-24w)^k}+\sum_{k=1}^{2g-2}\frac{\tilde{b}'_{g,k}}{w^k}\,,\tag{122}$$

where $\tilde{b}_{g,1},\dots,\tilde{b}_{g,5g-5},\tilde{b}'_{g,1},\dots,\tilde{b}'_{g,2g-2}$ are rational numbers satisfying $\tilde{b}_{g,2}=\tilde{b}_{g,4}=\cdots=\tilde{b}_{g,4g-6}=0$.

Setting

$$p=\frac{24w}{1-24w}\tag{123}$$

leads to the following corollary.

Corollary 2. *For $g=1$,*

$$\mathcal{F}_1^{\{4\}}(x)=-\frac{1}{12}\log x+\frac{1}{12}\log\bigg(\frac{p+2}{2}\bigg)+\zeta'(-1)\,.\tag{124}$$

For $g\geq 2$, $\mathcal{F}_g^{\{4\}}(x)$ has the following expression:

$$x^{2g-2}\mathcal{F}_g^{\{4\}}(x)=\frac{B_{2g}}{4g(g-1)}+\sum_{m=1}^{5g-5}\tilde{c}_{g,m}\,p^m,\tag{125}$$

where $\tilde{c}_{g,1},\dots,\tilde{c}_{g,5g-5}$ *are rational numbers. Moreover,* $\tilde{c}_{g,1}=\cdots=\tilde{c}_{g,2g-2}=0$.

Proof. The proof based on Theorem 2 is similar to that of Corollary 1, so is omitted. □

Multiplying both sides of (125) by dx/x^{k+1} and taking the residue at $x=0$, we obtain

$$n_g(4^k)=24^k k!\sum_{\ell=2g-1}^{k}\tilde{c}_{g,\ell}\sum_{m=0}^{k-\ell}\frac{(2k-m)_m}{m!}\frac{(k+1)_{k-\ell-m}}{(k-\ell-m)!}\,(-2)^{-k+\ell+m} \tag{126}$$

($\tilde{c}_{g,\ell}$ are defined as 0 if $\ell>5g-5$).

Following [21], introduce q by

$$w=\frac{q-1}{12q}\,. \tag{127}$$

Corollary 3. *For* $g=1$,

$$\mathcal{F}_1^{\{4\}}(x)=-\frac{1}{12}\log x-\frac{1}{12}\log(2-q)+\zeta'(-1). \tag{128}$$

For $g\geq 2$, $\mathcal{F}_g^{\{4\}}(x)$ *admits an expression of the form*

$$x^{2g-2}\mathcal{F}_g^{\{4\}}(x)=\sum_{k=2g-2}^{5g-5}\frac{\tilde{r}_{g,k}}{(2-q)^k}\,, \tag{129}$$

where $\tilde{r}_{g,2g-2},\dots,\tilde{r}_{g,5g-5}$ *are rational numbers.*

The proof is again omitted.

It was proved in [22] that, for any fixed $\nu\geq 2$, the generating function $e_g(s)\coloneqq\sum_{k\geq 1}\frac{n_g((2\nu)^k)}{k!}s^k$ admits the expression

$$e_g(s)=C^{(g)}+\sum_{\ell=2g-2}^{5g-5}\frac{\tilde{r}_{g,\ell}(\nu)}{(\nu-(\nu-1)z_0)^\ell}\,,\quad g\geq 2\,. \tag{130}$$

Here, $\tilde{r}_{g,2g-2}(\nu),\dots,\tilde{r}_{g,5g-5}(\nu)$ are some rational numbers, $C^{(g)}$ is a certain constant independent of ν, and $z_0=z_0(s)=1+\frac{(2\nu)!}{\nu!(\nu-1)!}s+\cdots$ is the unique solution to

$$z_0=1+\frac{(2\nu)!}{\nu!(\nu-1)!}s\,z_0^\nu\,, \tag{131}$$

Here, $z_0(x^{\nu-1})=q(x)$. For the case when $\nu=2$, Eynard [26] gave an expression of $e_g(s)$, which is equivalent to (130). Ercolani–Lega–Tippings [24, Conjecture 5.2] conjectured that

$$C^{(g)}=-\frac{B_{2g}}{4g(g-1)},\quad g\geq 2. \tag{132}$$

By using (4), (19), (131) and based on (130), we find that Ercolani–Lega–Tippings's conjecture (132) is equivalent to Conjecture A.

Proof of Theorem A. By comparing (20) with (129). □

We note that the topological recursion (see e.g. [26]) should also lead to a proof of Theorem A.

Now we use a formula from [22] to give a second proof of Theorem A. It was already shown in [22] that the $C^{(g)}$ for any ν satisfy the following recursion:

$$\frac{(2g-1)!}{(2g+2)!} - \frac{(2g-1)!}{12(2g)!} + \sum_{k=2}^{g} \frac{(1-2g)_{2g-2k+2}}{(2g-2k+2)!} C^{(k)} = 0\,, \quad g \geq 2\,. \tag{133}$$

A second proof of Theorem A. Define $\widetilde{C}^{(0)} := 1$, $\widetilde{C}^{(1)} := 1/6$, and $\widetilde{C}^{(g)} := -4g(g-1)C^{(g)}$ for $g \geq 2$. Then it follows from (133) that

$$\sum_{k=0}^{g} \frac{1}{(2g-2k+2)!} \frac{2k-1}{(2k)!} \widetilde{C}^{(k)} = -\frac{1}{2}\delta_{g,0}\,, \quad g \geq 0\,. \tag{134}$$

For a formal variable y, mutiplying (134) by y^g and summing over g, we find

$$\left(\sum_{\ell \geq 0} \frac{y^{2\ell+2}}{(2\ell+2)!}\right)\left(\sum_{k \geq 0} \frac{2k-1}{(2k)!} \widetilde{C}^{(k)} y^{2k-2}\right) = -\frac{1}{2}\,, \tag{135}$$

thus

$$\sum_{k \geq 0} \frac{2k-1}{(2k)!} \widetilde{C}^{(k)} y^{2k-2} = -(e^{\frac{y}{2}} - e^{-\frac{y}{2}})^{-2}\,. \tag{136}$$

Integrating (136) with respect to y and multiplying both sides by y, we find

$$\sum_{k \geq 0} \frac{\widetilde{C}^{(k)}}{(2k)!} y^{2k} = \frac{y}{2}\frac{e^y+1}{e^y-1} = \frac{1}{2}\Big(\frac{y}{e^y-1} + \frac{-y}{e^{-y}-1}\Big) = \sum_{k \geq 0} \frac{B_{2k}}{(2k)!} y^{2k}\,. \tag{137}$$

Hence $\widetilde{C}^{(g)} = B_{2g}$. Theorem A is proved. □

5. More on the $b = 2\nu$ case

In this section, we prove Theorem 3 and give some more discussions on the $b = 2\nu$ case. As a particular example of equation (47), we know that w satisfies

$$x = w - \frac{(2\nu)!}{\nu!(\nu-1)!} w^{\nu}\,. \tag{138}$$

Obviously the variable q introduced in (19) is related to w by $q = w/x$.

Lemma 4. *For $k \geq 1$, we have*

$$\partial_x^k(\log w) = \frac{1}{x^k} \sum_{\ell=0}^{k-1} \frac{(-1)^{k-\ell+1} e_{k,\ell}(\nu)}{(\nu-(\nu-1)q)^{k+\ell}}\,, \tag{139}$$

where $e_{k,\ell}(\nu) \in \mathbb{Z}[\nu]$, $\ell = 0, \ldots, k-1$, with $e_{k,k-1}(\nu) = (2k-3)!!\nu^{k-1}$ and $\deg e_{k,\ell}(\nu) \leq k-1$.

Proof. For $k = 1$, the statement can be proved easily (here we recall the convention that $(-1)!! = 1$). Suppose that the statement is true for $k = m$. Then for $k = m+1$, we have

$$\partial_x^{m+1}(\log w) = \frac{1}{x^{m+1}} \sum_{\ell=0}^{m} \frac{(-1)^{m-\ell+2} e_{m+1,\ell}(\nu)}{(\nu-(\nu-1)q)^{m+1+\ell}}\,, \tag{140}$$

where

$$e_{m+1,\ell}(\nu) = (\ell+1)e_{m,\ell+1}(\nu) + (m+\ell)(\nu+1)e_{m,\ell}(\nu) + (m+\ell-1)\nu\, e_{m,\ell-1}(\nu)\,, \tag{141}$$

with $e_{m,-2}(\nu) = e_{m,-1}(\nu) = e_{m,m}(\nu) = e_{m,m+1}(\nu) = 0$. Therefore, $e_{m+1,\ell}(\nu) \in \mathbb{Z}[\nu]$, $\deg e_{m,\ell}(\nu) \le m$ and $e_{m+1,m} = (2m-1)!!\nu^m$. The lemma is proved. □

Using (42), (43) and Lemma 4, we obtain

$$\mathcal{F}_1^{\{2\nu\}}(x) = -\frac{1}{12}\log x - \frac{1}{12}\log(\nu-(\nu-1)q) + \zeta'(-1)\,. \tag{142}$$

A third proof of Theorem A. According to the Hodge–GUE correspondence [16, 18], the function $\mathcal{F}_g^{\{2\nu\}}(x)$ is a linear combination of $\prod_{i=1}^{\ell(\lambda)}\partial_x^{\lambda_i}(\log w)/(\partial_x(\log w))^{\ell+2g-m-1-k}$ with rational coefficients, where $1 \le m \le g$, $0 \le k \le 3m-3$, $\ell \ge 1$, and λ are partitions satisfying $\ell(\lambda) = 2g-2m+\ell$, $|\lambda| = 4g-m-3-k+\ell$. From (139) we know that

$$\frac{\prod_{i=1}^{\ell(\lambda)}\partial_x^{\lambda_i}(\log w)}{(\partial_x(\log w))^{\ell+2g-m-1-k}} = \frac{(\nu-(\nu-1)q)^{2-2g}}{x^{2g-2}} \sum_{0\le s_i\le\lambda_i-1} \frac{\prod_{i=1}^{\ell(\lambda)}(-1)^{\lambda_i-s_i+1}e_{\lambda_i,s_i}(\nu)}{(\nu-(\nu-1)q)^{|s|}}\,. \tag{143}$$

Formula (20) then follows from the fact that $-(5g-5) \le 2-2g-|s| \le -(2g-2)$. □

Proof of Theorem 3. The fact that $r_{g,\ell}(\nu) \in \mathbb{Q}[\nu]$ follows from (143) and that $e_{g,\ell}(\nu) \in \mathbb{Z}[\nu]$, which we proved in Lemma 4. Since $\deg e_{k,\ell}(\nu) \le k-1$, we have

$$\deg\big(e_{\lambda_1,s_1}(\nu)\cdots e_{\lambda_{\ell(\lambda)},s_{\ell(\lambda)}}(\nu)\big) = |\lambda| - \ell(\lambda) = 2g+m-3-k \le 3g-3\,, \tag{144}$$

for $0 \le s_i \le \lambda_i$, $i = 1,\dots,\ell(\lambda)$. Hence $\deg r_{g,\ell}(\nu) \le 3g-3$. □

Similar to [19], by using Theorem A we can prove that, for $2m-2+j > 0$,

$$\partial_x^j(\mathcal{F}_m^{\{2\nu\}}(x)) = \frac{1}{x^{2m-2+j}} \sum_{\ell=2m-2+j}^{5m-5+2j} \frac{r_{m,\ell,j}(\nu)}{(\nu-(\nu-1)q)^\ell}\,. \tag{145}$$

By (29) we have

$$W_g^{\{2\nu\}}(x) = w\sum_{n=1}^{g}\frac{2^n}{n!} \sum_{\substack{k_1+m_1,\dots,k_n+m_n\ge 1\\ |k|+|m|=g}} \prod_{i=1}^{n}\frac{\partial_x^{2k_i+2}(\mathcal{F}_{m_i}^{\{2\nu\}}(x))}{(2k_i+2)!}\,, \quad g \ge 1\,. \tag{146}$$

Using (145) and (146), we obtain

$$x^{2g-1}W_g^{\{2\nu\}}(x) = q\sum_{\ell=2g}^{5g-1}\frac{t_{g,\ell}(\nu)}{(\nu-(\nu-1)q)^\ell}\,, \quad g \ge 1\,. \tag{147}$$

Using Theorem 3, we show that $r_{m,\ell,j}(\nu)$ are polynomials in ν of degree at most $3m-3+j$. Hence, we obtain:

Corollary 4. *For any $g \ge 1$, $t_{g,\ell}(\nu)$ are polynomials of ν with* $\deg t_{g,\ell}(\nu) \le 3g-1$.

We note that part of the motivation of Corollary 4 comes from [31]. Indeed, following [31], define

$$Q_{g,k}(\nu) := k![x^{1-2g+(\nu-1)k}]W_g^{\{2\nu\}}(x)/\Big(\frac{(2\nu)!}{\nu!(\nu-1)!}\Big)^k\,. \tag{148}$$

(As shown in [25], $k![x^{1-2g+(\nu-1)k}]W_g^{\{2\nu\}}(x)$ counts the number of two-legged 2ν-valent genus-g maps with k vertices.) It then follows from (19) and (147) that

$$Q_{g,k}(\nu) = k! \sum_{m=0}^{k} \binom{\nu k}{k-m} \frac{(\nu-1)^m}{m!} \sum_{\ell=2g}^{5g-1} t_{g,\ell}(\nu)(\ell-1)_m \,, \tag{149}$$

which, together with Corollary 4, implies validity of a conjectural statement in [31, Remark 2.9] that $Q_{g,k}(\nu)$ are polynomials in ν of degree $3g-1+k$.

Finally, we note that it can be deduced using Theorem A and a result of [22] that

$$r_{g,5g-5}(\nu) = \frac{\nu^{3g-3}}{12^g} \frac{C_g}{(5g-3)(5g-5)} \,, \quad g \geq 2 \,, \tag{150}$$

which can also be proved using the Hodge–GUE correspondence (cf. [16] and [47, Corollary 4.6]) and [35, (6.10)]. Here C_g are defined by (13). By (145)–(147) and (150) we have

$$t_{g,5g-1}(\nu) = \frac{\nu^{3g-1}}{12^g} C_g \,. \tag{151}$$

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