THETA BLOCKS

VALERY GRITSENKO, NILS-PETER SKORUPPA, AND DON ZAGIER

Abstract. We define theta blocks as products of Jacobi theta functions divided by powers of the Dedekind eta-function and show that they give a powerful new method to construct Jacobi forms and Siegel modular forms, with applications also in lattice theory and algebraic geometry. One of the central questions is when a theta block defines a Jacobi form. It turns out that this seemingly simple question is connected to various deep problems in different fields ranging from Fourier analysis over infinite-dimensional Lie algebras to the theory of moduli spaces in algebraic geometry. We give several answers to this question.

Contents

2010 *Mathematics Subject Classification.* 11F50 (primary), 11F55 (secondary).

Key words and phrases. Jacobi forms, product expansions of Jacobi forms, Jacobi forms of lattice index, Macdonald identities.

The first author was supported by the Laboratory of Mirror Symmetry of the National Research University “Higher School of Economics” (Russian Federation Government grant, agreement no. 14.641.31.0001).

Introduction

The Jacobi theta function $\vartheta(\tau,z)$, defined for $\tau\in\mathbb{H}$, $z\in\mathbb{C}$ either as the theta series

$$(1)\qquad \vartheta(\tau,z)=\sum_{r=-\infty}^{\infty}\left(\frac{-4}{r}\right)q^{r^2/8}\,\zeta^{r/2}\qquad\big(q=e^{2\pi i\tau},\ \zeta=e^{2\pi iz}\big)$$

or else by the triple product

$$(2)\qquad \vartheta(\tau,z)=q^{1/8}\,\zeta^{1/2}\prod_{n=1}^{\infty}\big(1-q^n\big)\big(1-q^n\zeta\big)\big(1-q^{n-1}\zeta^{-1}\big),$$

is a holomorphic Jacobi form (with non-trivial character) of weight 1/2 and index 1/2. (The definitions of holomorphic Jacobi forms with character and of their weight and index are reviewed in §2.) For $a\in\mathbb{N}$ we denote by ϑ_a the Jacobi form

$$\vartheta_a(\tau,z)\ :=\ \vartheta(\tau,az)$$

of weight 1/2 and index $a^2/2$, while

$$\eta(\tau)=\sum_{r=1}^{\infty}\left(\frac{12}{r}\right)q^{r^2/24}=q^{1/24}\prod_{n=1}^{\infty}\big(1-q^n\big)$$

denotes the Dedekind eta-function. The starting point of this paper is the following observation:

Fact. *Let a and b be positive integers. Then the quotient*

$$Q_{a,b}(\tau,z)=\frac{\vartheta_a(\tau,z)\,\vartheta_b(\tau,z)\,\vartheta_{a+b}(\tau,z)}{\eta(\tau)}$$

is a holomorphic Jacobi form of weight 1 and index a^2+ab+b^2, and is a cusp form if $3g^3\mid ab(a+b)$, where $g=\gcd(a,b)$.

We will give several proofs and generalizations of this result. To do this, we first give (in §3) a general criterion for the divisibility of a holomorphic Jacobi form, and in particular of a product of ϑ_a's, by a given power of η. This will involve defining the notion of the *order* of a Jacobi form at infinity, a notion which has apparently not previously been introduced but which seems quite fundamental to the theory. This criterion will then be used to prove the holomorphy of $Q_{a,b}$ and to give many other examples—both infinite families proved theoretically and sporadic examples found by computer—of theta products divisible by high powers of η. A typical example is the family of holomorphic Jacobi forms of weight 2

$$(3)\qquad R_{a,b,c,d}=\frac{\vartheta_a\,\vartheta_b\,\vartheta_c\,\vartheta_d\,\vartheta_{a+b}\,\vartheta_{b+c}\,\vartheta_{c+d}\,\vartheta_{a+b+c}\,\vartheta_{b+c+d}\,\vartheta_{a+b+c+d}}{\eta^6}$$

where a, b, c and d are natural numbers. In many cases, including both the families $Q_{a,b}$ and $R_{a,b,c,d}$, we will also give explicit formulas for quotients of the form $\eta^{-s}\vartheta_{a_1}\cdots\vartheta_{a_N}$ as theta series of rank $N-s$. Some of these are obtained by using a general criterion (described in §3) for the divisibility of one theta series by another, while others arise by specializing the Macdonald identities (also known as Kac-Weyl denominator formulas) for suitable root systems.

A weakly holomorphic Jacobi form of the form $\vartheta_{a_1}\cdots\vartheta_{a_N}/\eta^d$ is called a theta block of length N, and it is called a holomorphic theta block if it is a Jacobi form. Its weight is equal to $(N-d)/2$, and one of the principal aims of this article is to construct explicit examples of holomorphic theta blocks whose weight is relatively small with respect to the length. For instance, the Jacobi form $R_{a,b,c,d}$ has length 10 and weight 2, and more generally in Section 8 we will construct families of length $n(n+1)/2$ and weight $n/2$. In Section 9 we will develop a general theory for constructing such families and will see many more concrete examples in the sections following it. We will be interested both in theoretical bounds for the minimal weight k for given length N (in Section 4 it is shown that the minimal weight is bounded below and above by $c_1 \log N$ and $c_2(\log N)^3$ for positive constants c_i) and in constructing explicit holomorphic theta blocks of small weight.

The special families that we construct turn out to give a very useful way of constructing Jacobi forms, especially Jacobi forms of low weight. For instance, both the first Jacobi form and the first Jacobi cusp form of weight 2 and trivial character, which have index 25 and 37, respectively and were constructed with some effort in [EZ85], are now obtained immediately as the two first cases $(a,b,c,d)=(1,1,1,1)$ and $(1,1,1,2)$ of the family $R_{a,b,c,d}$, and many other interesting examples of Jacobi forms of low weight and given character can be obtained as special cases of products of the functions $Q_{a,b}$ or of the other families. Such forms have several applications, e.g. to questions concerning the classification of moduli spaces of polarized abelian surfaces or of K3-surfaces. We will describe these applications and give some general discussion of the situation for small weight. In particular, we shall see that all holomorphic Jacobi forms of weight 1/2 and weight 1 and arbitrary character can be obtained as theta quotients $\eta^{-s}\vartheta_{a_1}^{\pm}\cdots\vartheta_{a_N}^{\pm 1}$, and will give conjectures and partial results for higher weights. We expect that the spaces of Jacobi forms of small weight and arbitrary index and character on the full modular group are in fact spanned by theta quotients. As we shall see in Section 3 this statement is, however, false for large weights.

We finally mention a side result of our studies of theta blocks, namely a rather short proof (in §10) of the Macdonald identities based on Jacobi forms of lattice index.

Part I: Basic Theory

1. Review of Jacobi forms

We first recall the definition of Jacobi forms as given in [EZ85]. Let k and m be non-negative integers. Then a *holomorphic Jacobi form of weight k and index m* (on the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, or more precisely on the full Jacobi group $\Gamma^{\mathrm{J}} = \Gamma \ltimes \mathbb{Z}^2$) is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ which satisfies the two transformation equations

$$\phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (c\tau+d)^k \,\mathbf{e}\Big(\frac{mcz^2}{c\tau+d}\Big)\,\phi(\tau,z) \tag{4}$$

and

$$\phi\big(\tau,\, z+\lambda\tau+\mu\big) = \mathbf{e}\big(-m(\lambda^2\tau+2\lambda z)\big)\,\phi(\tau,z) \tag{5}$$

for all $\tau \in \mathbb{H}$, $z \in \mathbb{C}$, $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$ and $\left(\begin{smallmatrix} \lambda \\ \mu\end{smallmatrix}\right) \in \mathbb{Z}^2$ (here $\mathbf{e}(x) = e^{2\pi i x}$ as usual) and which has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}\\ n\geq 0}} \sum_{\substack{r\in\mathbb{Z}\\ r^2\leq 4mn}} c(n,r)\, q^n\, \zeta^r\,, \tag{6}$$

where q and ζ denote $\mathbf{e}(\tau)$ and $\mathbf{e}(z)$, respectively. The Fourier coefficients $c(n,r)$ then automatically satisfy the periodicity property

$$c(n,r) = c(n+\lambda r+\lambda^2 m,\, r+2\lambda m) \qquad \text{for all } \lambda\in\mathbb{Z} \tag{7}$$

(this is equivalent to (5)), so that $c(n,r)$ is actually only a function of the numbers $d = 4nm - r^2$ and $r \bmod 2m$ in $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}/2m\mathbb{Z}$. A *Jacobi cusp form* of weight k and index m is a holomorphic Jacobi form in which the condition $4nm - r^2 \geq 0$ in (6) is strengthened to $4nm - r^2 > 0$, while a *weak Jacobi form* is defined like a holomorphic Jacobi form but with the condition $4nm - r^2 \geq 0$ in (6) dropped entirely; the periodicity property (7) then implies that $c(n,r) = 0$ unless $m\lambda^2 + r\lambda + n \geq 0$ for all $\lambda \in \mathbb{Z}$ and hence that $|r|$ is still bounded (by $\sqrt{4nm+m^2}$) for each n, so that ϕ still belongs to $\mathbb{C}[\zeta,\zeta^{-1}][[q]]$. Finally, a *weakly holomorphic Jacobi form* of weight k and index m is a holomorphic function $\phi : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ satisfying (4), (5) and (6) but without the condition $4nm \geq r^2$ in (6) and with the condition $n \geq 0$ weakened to $n \geq n_0$ for some $n_0 \in \mathbb{Z}$. An equivalent definition is that $\Delta(\tau)^h\phi(\tau,z)$ is a holomorphic Jacobi form (of weight $k+12h$ and index m) for some $h\in\mathbb{Z}$, where $\Delta = \eta^{24} \in S_{12}(\Gamma)$. Such a form has a Fourier expansion in $\mathbb{C}[\zeta,\zeta^{-1}][\![q^{-1},q]\!]$, the ring of Laurent series in q with coefficients which are Laurent polynomials in ζ.

The spaces of holomorphic Jacobi forms, Jacobi cusp forms, weak Jacobi forms and weakly holomorphic Jacobi forms are denoted by $J_{k,m}(1)$, $J_{k,m}^{\mathrm{cusp}}(1)$, $J_{k,m}^{\mathrm{weak}}(1)$ and $J_{k,m}^{!}(1)$, respectively, the latter in analogy with the more standard notation $M_k^! = M_k^!(\Gamma)$ for the space of weakly holomorphic modular forms of weight k on Γ (= holomorphic functions in $\mathbb{H}$ which transform like modular forms of weight k but are allowed to grow like a negative power of q as $\Im(\tau) \to \infty$). The "1" in parentheses, which was not used in [EZ85], means that the Jacobi forms under consideration have trivial character, and will be dropped when forms with arbitrary character are permitted. For $m = 0$ the Jacobi forms are independent of z, so that we have $J_{k,0}(1) = J_{k,0}^{\text{weak}}(1) = M_k(\Gamma)$, $J_{k,0}^{\text{cusp}}(1) = S_k(\Gamma)$, and $J_{k,0}^{!}(1) = M_k^{!}(\Gamma)$. We also have $J_{k,m}(1)J_{k',m'}(1) \subseteq J_{k+k',m+m'}(1)$, so that the vector space $J_{*,*}(1) = \bigoplus_{k,m\geq 0} J_{k,m}(1)$ is a bigraded ring. Note that the weights of weak or weakly holomorphic Jacobi forms can be negative, although in the case of weak Jacobi forms they are bounded below by $-2m$.

In this paper we will still consider Jacobi forms on the full modular group, but will allow rational weights and indices. For such forms the transformation equations (4) and (5) are true only up to certain roots of unity (of bounded order) depending on $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix}\right)$, and the exponents n and r in (6) can be rational (though again with bounded denominator). The quickest way to give a definition is simply to say that $\phi(\tau, z)^N$ is a holomorphic (or cuspidal, or weak, or weakly holomorphic) Jacobi form of weight Nk and index Nm for some positive integer N. The explicit formulas for the roots of unity occurring in the transformation equations with respect to the action of Γ and $\mathbb{Z}^2$ (multiplier system) are quite complicated, but we do not have to give them explicitly because there is an easy implicit description which suffices for the cases we are interested in (products of the functions $\vartheta_a(\tau, z)$ and of rational powers of $\eta(\tau)$). We use the symbol ε to denote the multiplier system of the function $\eta(\tau)$, and more generally ε^h for any $h \in \mathbb{Q}$ to denote the multiplier system of (any branch of) the function $\eta(\tau)^h$. (Note that the quotient of two branches of $\eta(\tau)^h$ is a constant, so that ε^h is in fact independent of the choice of branch.) We also note that the index m of any Jacobi form ϕ, even a weakly holomorphic one or one with arbitrary character, is always a non-negative half-integer, because $2m$ is the number of zeros of the function $z \mapsto \phi(\tau, z)$ in a fundamental domain for the action of the group $\mathbb{Z}\tau + \mathbb{Z}$ of translations of $\mathbb{C}$. For m integral and k, $h \in \mathbb{Q}$ we will say that a Jacobi form ϕ of weight k and index m has *character* ε^h if $\eta(\tau)^{-h}\phi(\tau, z)$ is a (weakly holomorphic) Jacobi form in the usual sense, i.e., if $k - h/2 \in \mathbb{Z}$ and $\eta^{-h}\phi \in J_{k-h/2,m}^{!}(1)$. For half-integral index we observe that the square of the Jacobi theta function $\vartheta(\tau, z)^2$ is a holomorphic Jacobi form of weight 1, index 1 and character ε^6 in the above sense, so we simply *define* its character to be ε^3; then for $m \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ and k, $h \in \mathbb{Q}$ we define a Jacobi form of weight k, index m and character ε^h by the requirement that $\eta(\tau)^{-h-3}\vartheta(\tau, z)\phi(\tau, z)$ belong to $J_{k-1-h/2,m+1/2}^{!}(1)$. The definitions in both cases depend only on h modulo 24, so we get spaces

$J_{k,m}(\varepsilon^h)$, $J^{\text{cusp}}_{k,m}(\varepsilon^h)$, $J^{\text{weak}}_{k,m}(\varepsilon^h)$ and $J^!_{k,m}(\varepsilon^h)$ for all $m \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, $k \in \mathbb{Q}$ and $h \in \mathbb{Q}/24\mathbb{Z}$ with $2k \equiv h \bmod 2$. The formulas (4), (5) and (6) imply that $\phi \in q^{h/24}\,\zeta^m\,\mathbb{C}[\zeta,\zeta^{-1}][\![q^{-1},q]\!]$ for ϕ belonging to any of these spaces. We clearly have $J_{k,m}(\varepsilon^h)J_{k',m'}(\varepsilon^{h'}) \subseteq J_{k+k',m+m'}(\varepsilon^{h+h'})$ and also $\phi_a \in J_{k,a^2m}(\varepsilon^h)$ if $\phi \in J_{k,m}(\varepsilon^h)$, where $\phi_a(\tau,z)$ denotes the Jacobi form $\phi(\tau,az)$. In particular we have $\vartheta_a \in J_{1/2,a^2/2}(\varepsilon^3)$ and more generally

$$\vartheta_{\mathbf{a}} := \prod_{j=1}^{N} \vartheta_{a_j} \in J_{N/2,A/2}(\varepsilon^{3N})$$

for $\mathbf{a} = (a_1,\dots,a_N) \in \mathbb{Z}^N$, $A = \sum_{j=1}^N a_j^2$.

It is not hard to show that every function whose N-th power, for some positive integer N, is a (weak or weakly holomorphic) Jacobi form of intgral weight and index with trivial character is indeed in $J_{k,m}(\varepsilon^h)$ (or $J^{\text{weak}}_{k,m}(\varepsilon^h)$ or $J^!_{k,m}(\varepsilon^h)$) for suitable rational k and h. Moreover, it is easily verified that, for any index m in $\frac{1}{2}\mathbb{Z}$, the transformation formula (5) remains true if one multiplies the right-hand side by the factor $\mathbf{e}\big(m(\lambda+\mu)\big)$. Note also, that for any rational k and h and half-integral m every element ϕ in $J^!_{k,m}(\varepsilon^h)$ has still a Fourier expansion of the form (6), where, however, r runs through $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$ accordingly as m is integral or not, and n runs through all rational numbers $n \geq n_o$ which are in $h/24 + \mathbb{Z}$. The (modified) transformation formula (5) implies that, for any integer λ,

$$C_\phi(\Delta,r) = \mathbf{e}(m\lambda)C_\phi(\Delta,r+2m\lambda), \tag{8}$$

where $C(\Delta,r) = c\big(\frac{r^2-\Delta}{4m},r\big)$.

Finally, we mention another special Jacobi form:

$$\vartheta^*(\tau,z) = \sum_{r\in\mathbb{Z}} \left(\frac{12}{r}\right) q^{r^2/24}\zeta^{r/2}, \tag{9}$$

which appears also in the famous Watson quintuple product identity

$$\vartheta^*(\tau,z) = \eta(\tau)\frac{\vartheta(\tau,2z)}{\vartheta(\tau,z)} = q^{1/24}\zeta^{1/2}$$
$$\times \prod_{n=1}^{\infty}\big(1-q^n\big)\big(1+q^n\zeta\big)\big(1+q^{n-1}\zeta^{-1}\big)\big(1-q^{2n-1}\zeta^2\big)\big(1-q^{2n-1}\zeta^{-2}\big).$$

The Jacobi form ϑ^* has weight $1/2$, index $3/2$ and multiplier system ε. For an integer a, we will use the notation ϑ^*_a for the Jacobi form $\vartheta^*(\tau,az)$.

2. The order of a weakly holomorphic Jacobi form at infinity

Let ϕ be a weakly holomorphic non-zero Jacobi form ϕ of index m with Fourier coefficients $c_\phi(n,r)$. We associate to ϕ a function $\operatorname{ord}(\phi,x)$

of a real variable x by setting

$$\operatorname{ord}(\phi, x) = \min\Big\{n + rx + mx^2 : n, r \text{ such that } c_\phi(n, r) \neq 0\Big\}. \tag{10}$$

This function has several remarkable properties and it will play a key role in the construction of theta blocks of small weights. In particular, as the following theorem shows, the map $\phi \mapsto \operatorname{ord}(\phi, \cdot)$ defines a valuation with values in the (additive) group of continuous functions on $\mathbb{R}/\mathbb{Z}$. A similar valuation could be associated to other cusps if one had to consider Jacobi forms on subgroups of $\mathrm{SL}(2, \mathbb{Z})$ which have more than one cusp, which justifies calling $\operatorname{ord}(\phi, \cdot)$ the *order of ϕ at infinity.*

Theorem 2.1. *The function $\operatorname{ord}_\phi = \operatorname{ord}(\phi, \cdot)$ defined by (10) has the following properties:*

(1) *It is continuous, piecewise quadratic, and periodic with period 1.*
(2) *If ϕ is of index $m = 0$ (i.e. if ϕ is a weakly holomorphic elliptic modular form, independent of z), then ord_ϕ is constant and equals the usual order of ϕ at the cusp infinity.*
(3) *For fixed any real u, x and y, there is a constant $C = C(u, x, y)$ such that one has*

$$\phi(\tau, x\tau + y)\, \mathbf{e}(mx^2\tau) = \big(C + o(1)\big)e^{-2\pi \operatorname{ord}(\phi,x)v}$$

as $\tau = u + iv$ and v tends to infinity. The constant C depends only on u, x, y modulo $N\mathbb{Z}$ for a suitable integer $N \geq 1$ and is different from zero for almost all u, x, y.
(4) *For any two weakly holomorphic Jacobi forms ϕ and ψ one has*

$$\operatorname{ord}_{\phi\psi} = \operatorname{ord}_\phi + \operatorname{ord}_\psi .$$

(5) *Let ϕ in $J^!_{k,m}(\varepsilon^h)$. Then ϕ is in $J_{k,m}(\varepsilon^h)$ if and only if $\operatorname{ord}_\phi \geq 0$, and ϕ is in $J^{cusp}_{k,m}(\varepsilon^h)$ if and only if $\operatorname{ord}_\phi > 0$.*
(6) *For any integer l and any weakly holomorphic Jacobi form ϕ, one has $\operatorname{ord}_{U_l\phi}(x) = \operatorname{ord}_\phi(lx)$, where U_l denotes the operator $(U_l\phi)(\tau, z) = \phi(\tau, lz)$.*

Proof. For proving (1) we note that ord_ϕ is locally equal to the minimum of *finitely* many quadratic polynomials, hence is continuous and piecewise quadratic. If we write

$$\operatorname{ord}(\phi, x) = \min\Big\{\tfrac{(r+2mx)^2-\Delta}{4m} : \Delta, r, \text{ such that } c_\phi\big(\tfrac{r^2-\Delta}{4m}, r\big) \neq 0\Big\},$$

we see that the periodicity is an immediate consequence of the identity (8).

The statements (2) and (6) are obvious, and (4) follows immediately from (3).

For (3) we observe that the left-hand side of the claimed identity equals

$$\sum_{n,r} c_\phi(n,r)\,\mathbf{e}\big((n+rx+mx^2)\tau+ry\big)$$
$$= \sum_{(n,r)\in S} c_\phi(n,r)\,\mathbf{e}(\mathrm{ord}(\phi,x)\tau+ry) \;+\; \mathrm{o}\big(e^{-2\pi\,\mathrm{ord}(\phi,x)v}\big),$$

where S is the (finite) set of pairs (n,r) of rational numbers such that $n+rx+mx^2=\mathrm{ord}(\phi,x)$ and $c_\phi(n,r)\neq 0$.

Finally, for (5) we note that ϕ is a holomorphic (cusp) form if and only if $c_\phi(n,r)=0$ unless the discriminant r^2-4mn of the quadratic polynomial $f(x)=n+rx+mx^2$ is (strictly) negative, i.e. unless $f(x)$ is (strictly) positive for all x. This proves the theorem. □

The order of Jacobi's theta function $\vartheta(\tau,z)$ (introduced in (1)) will play an important role in the following. We shall use the letter B for it, i.e. we set $B(x)=\mathrm{ord}(\vartheta,x)$. From the Fourier development (1) of ϑ, i.e. from the property that $c_\vartheta(n,s)\neq 0$ if and only if $n=r^2/8$ and $s=r/2$ for an odd integer r, we see that $\mathrm{ord}(\vartheta,x)$ equals the minimum of $\frac18 r^2+\frac12 rx+\frac12 x^2=\frac12(x+\frac r2)^2$, where r ranges through the odd integers. In other words

$$B(x) := \mathrm{ord}(\vartheta,x) = \min_{k\in\mathbb{Z}} \tfrac12(x-\tfrac12+k)^2 = \tfrac12\big(x-\lfloor x\rfloor-\tfrac12\big)^2. \tag{11}$$

Note also that

$$\mathrm{ord}(\vartheta/\eta,x)=\tfrac12\mathbb{B}(x),$$

where $\mathbb{B}(x)$ is the periodic function with period 1 which, for $0\le x\le 1$, equals the second Bernoulli polynomial $x^2-x+\frac16$. Indeed, for η, viewed as Jacobi form of index 0 we have immediately from the definition of ord that $\mathrm{ord}(\eta,x)=\frac{1}{24}$, so that with Theorem 2.1, (4) we obtain $\mathrm{ord}(\vartheta/\eta,x)=\mathrm{ord}(\vartheta,x)-\frac1{24}=B(x)-\frac1{24}=\frac12\mathbb{B}(x)$, by (11).

3. Theta blocks

Recall from Section 1 that $\vartheta_a(\tau,z)=\vartheta(\tau,az)$ defines an element of $J_{1/2,a^2/2}(\varepsilon^3)$. From Theorem 2.1 we deduce that $\mathrm{ord}(\vartheta_a,x)=B(ax)$ with the function $B(x)$ defined in (11). From the product expansion (2) of ϑ we deduce that, for fixed τ, the set of zeros of $\vartheta(\tau,\cdot)$ coincides with the lattice $\mathbb{Z}\tau+\mathbb{Z}$. Accordingly, we find that the zeros of $\vartheta_a(\tau,z)$ are all simple and are given by the a-division points of the lattice $\mathbb{Z}\tau+\mathbb{Z}$., i.e. by the points of the lattice $\frac1a(\mathbb{Z}\tau+\mathbb{Z})$.

Definition. A *theta block* of *length* r is a function of the form

$$\vartheta_{a_1}\vartheta_{a_2}\cdots\vartheta_{a_r}\eta^n, \tag{12}$$

where n is an integer and the a_j are integers different from zero. A *generalized theta block* is a holomorphic function in $\mathbb{H} \times \mathbb{C}$ of the form

$$\frac{\vartheta_{a_1}\vartheta_{a_2}\cdots\vartheta_{a_r}}{\vartheta_{b_1}\vartheta_{b_2}\cdots\vartheta_{b_s}}\eta^n, \tag{13}$$

where n is an integer and the a_j, b_j are non-zero integers. We call a theta block or generalized theta block *holomorphic* if it is holomorphic also at infinity, i.e., if it is a Jacobi form. Conversely, an arbitrary function of the form (13), without the requirement of holomorphy in $\mathbb{H} \times \mathbb{C}$, is called a *theta quotient*.

We note that the length r of a theta block, i.e. the number of ϑ-factors in (12) is indeed uniquely determined by the theta block as follows for example from Theorem 3.6 below.

Occasionally we will also allow rational values for n, and will then call the corresponding function a theta block, generalized theta block or theta quotient *with fractional eta-power*. Clearly, any such function is a meromorphic Jacobi form in $J^{\mathrm{mer}}_{k,M/2}(\varepsilon^h)$, where

$$k = \tfrac{r-s+n}{2}, \quad M = \sum_{j=1}^{r} a_j^2 - \sum_{j=1}^{s} b_j^2, \quad h = 3r - 3s + n,$$

If f is a generalized theta block (with integral or fractional eta-power), then f is a weakly holomorphic Jacobi form in $J^{!}_{k,M/2}(\varepsilon^h)$.

Example 3.1. The function $\vartheta^*(\tau, z)$ defined in (9) is a generalized theta block. More generally, for every positive integer a we have the generalized theta block

$$S_a = \prod_{d|a} \vartheta_d^{\mu(a/d)}, \tag{14}$$

where μ denotes the Möbius function. Note that S_a is holomorphic in $\mathbb{H} \times \mathbb{C}$, its zeros, as function of z for fixed τ, are simple and are given by the primitive a-division points of the lattice $L_\tau = \mathbb{Z}\tau + \mathbb{Z}$, i.e. by those points of $\frac{1}{a}L_\tau$ whose images in $\frac{1}{a}L_\tau/L_\tau$ have exact order a. Hence S_a defines an element of $J^{!}_{0,\psi(a)\varphi(a)/2}(1)$ for $a \geq 2$ (whereas, for $a = 1$, we have $S_1 = \vartheta$), where $\varphi(a)$ is the Euler φ-function and $\psi(a)$ denotes the sum of all positive divisors d of a such that d/a is squarefree. Its order at infinity is given by

$$\operatorname{ord}(S_a, x) = \sum_{d|a} \mu(a/d)\, B(dx).$$

Note that the theta blocks form a semigroup with respect to the usual multiplication of functions. We shall denote this semigroup by $\mathfrak{B}$. Similarly, the generalized theta blocks form a semigroup which we shall denote by $\mathfrak{B}^*$. The theta quotients, finally, form a group denoted by $G(\mathfrak{B})$. We shall determine the structure of this group.

For a fixed τ, the divisor of a theta block $f(\tau, z)$, viewed as a theta function on $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$, is of the form $\sum_a n_a \Pi_a$, where a runs through $\mathbb{Z}_{>0}$, where the integers n_a vanish for almost all a, and where Π_a is the (formal) sum of the primitive a-division points of $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$. The formal sum

$$\mathrm{Div}(f) := \sum_a n_a(a) \in \mathbb{Z}[\mathbb{Z}_{>0}]$$

does not depend on τ. Moreover, the map $f \mapsto \mathrm{Div}(f)$ defines a group homomorphism. Using this map the structure of $G(\mathfrak{B})$ can be described as follows.

Theorem 3.2. *The map $f \mapsto \mathrm{Div}(f)$ defines an exact sequence*

$$1 \to \eta^{\mathbb{Z}} \to G(\mathfrak{B}) \xrightarrow{\mathrm{Div}} \mathbb{Z}[\mathbb{Z}_{>0}] \to 1.$$

The sequence splits via the map $D = \sum n_a(a) \mapsto S_D := \prod_a S_a^{n_a}$

Proof. From the discussion in Example 3.1 it is clear that $D \mapsto S_D$ defines a section of the map Div, which is then, in particular, surjective. If a theta block, for each fixed τ, has no zeroes or poles in $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$, then it is of index 0, hence an elliptic modular form without zeroes in the upper half plan (but with possibly a pole at the cusp infinity), hence a power of η. $\square$

There are two immediate consequences of the theorem.

Corollary 3.3. *A theta quotient is a generalized theta block (i.e. weakly holomorphic) if and only if it equals a product of the functions S_a and a power of η.*

Theorem 3.4. *For any positive integer or half-integer m, the number of generalized theta blocks of index m, counted up to multiples of powers of η, is finite. It equals the coefficient of q^{2m} in the power series expansion of $1/\prod_{a=1}^{\infty} \big(1 - q^{\varphi(a)\psi(a)}\big)$.*

Proof. Indeed, according to the theorem the number in question equals the number of elements $D = \sum_a n_a(a)$ in $\mathbb{Z}[\mathbb{Z}_{>0}]$ such that all n_a are non-negative and $m = \frac{1}{2}\sum_a n_a \varphi(a)\psi(a)$. But this number is finite since

$$\varphi(a)\psi(a) = a^2 \prod_{p|a} \big(1 - \frac{1}{p^2}\big) > a^2 \prod_{p} \big(1 - \frac{1}{p^2}\big) = 6a^2/\pi^2,$$

which is bigger than $2m$ for a large. $\square$

Remark. It is known [Sko08, Thm. 6] that, for fixed m and h, as k tends to infinity one has $\dim J_{k,m}\big(\varepsilon^h\big) = c \cdot k + O(1)$, where c is a constant depending on m and h. In particular, we see that generalized theta blocks of a given index m can never span the whole space of Jacobi forms of weight k, index m and given character if k is sufficiently large. Table 1 lists, for small indices m, all generalized theta blocks of index

TABLE 1. For small index m the sets S_m of all non-cuspidal generalized theta blocks with fractional η-power in $\bigoplus_{k,h\in\mathbb{Q}} J_{k,m}(\varepsilon^h)$

m	S_m
$\frac{1}{2}$	ϑ
1	ϑ^2
$\frac{3}{2}$	ϑ^3, $\frac{\vartheta_2}{\vartheta}\eta$
2	ϑ^4, ϑ_2
$\frac{5}{2}$	ϑ^5, $\vartheta\vartheta_2\eta^{-\frac{3}{5}}$
3	ϑ^6, $\vartheta^2\vartheta_2\eta^{-1}$, $\frac{\vartheta_2^2}{\vartheta^2}\eta^2$
$\frac{7}{2}$	ϑ^7, $\vartheta^3\vartheta_2\eta^{-\frac{9}{7}}$, $\frac{\vartheta_2^2}{\vartheta}\eta^{\frac{6}{7}}$
4	ϑ^8, $\vartheta^4\vartheta_2\eta^{-\frac{3}{2}}$, ϑ_2^2, $\frac{\vartheta_3}{\vartheta}\eta^{\frac{3}{2}}$
$\frac{9}{2}$	ϑ^9, $\vartheta^5\vartheta_2\eta^{-\frac{5}{3}}$, $\vartheta\vartheta_2^2\eta^{-\frac{2}{3}}$, $\frac{\vartheta_2^3}{\vartheta^3}\eta^3$, ϑ_3
5	ϑ^{10}, $\vartheta^6\vartheta_2\eta^{-\frac{9}{5}}$, $\vartheta^2\vartheta_2^2\eta^{-\frac{6}{5}}$, $\frac{\vartheta_2^3}{\vartheta^2}\eta^{\frac{9}{5}}$, $\vartheta\vartheta_3$
$\frac{11}{2}$	ϑ^{11}, $\vartheta^7\vartheta_2\eta^{-\frac{21}{11}}$, $\vartheta^3\vartheta_2^2\eta^{-\frac{18}{11}}$, $\frac{\vartheta_2^3}{\vartheta}\eta^{\frac{9}{11}}$, $\vartheta^2\vartheta_3$, $\frac{\vartheta_2\vartheta_3}{\vartheta^2}\eta^{\frac{27}{11}}$
6	ϑ^{12}, $\vartheta^8\vartheta_2\eta^{-2}$, $\vartheta^4\vartheta_2^2\eta^{-2}$, ϑ_2^3, $\frac{\vartheta_2^4}{\vartheta^4}\eta^4$, $\vartheta^3\vartheta_3$, $\frac{\vartheta_2\vartheta_3}{\vartheta}\eta$, $\frac{\vartheta_4}{\vartheta_2}\eta$

m, up to powers of η, normalized by a fractional η-power so that the minimum of their order at infinity becomes zero, i.e. so that they are holomorphic but not cuspidal.

As we have seen, it is easy to decide whether a theta quotient is weakly holomorphic. It remains to analyze the behaviors of a general theta block at infinity. We shall discuss this question from various points of view in the next sections. Here we confine ourselves to the study of the map which associates to a theta quotient its order at infinity. For this we note that $\operatorname{ord}(f,\cdot)$, for a theta quotient f, is an element of the additive group of real valued functions on the real line which is spanned by the functions $B(ax)$ ($a \in \mathbb{Z}_{>0}$) and $\frac{1}{24}$. It is a somewhat surprising fact that the order at infinity already determines the theta quotient. Namely, we shall prove

Theorem 3.5. *The map $f \mapsto \operatorname{ord}(f,\cdot)$ defines an isomorphism between the group of theta quotients $G(\mathfrak{B})$ and the additive group $\mathfrak{W}$ of functions spanned by the $B(ax)$ ($a \in \mathbb{Z}_{>0}$) and the constant function $\frac{1}{24}$.*

Proof. We shall prove in a moment that the functions $B(ax)$ and $1/24$ are linearly independent over $\mathbb{Z}$ (and even over $\mathbb{C}$). Hence from the order at infinity $\operatorname{ord}(f,\cdot)$ of a theta quotient f as in (13) we can read off the numbers a_j, b_j and n, which proves the theorem.

The claimed linear independence of the $B(ax)$ and $1/24$ becomes obvious if one expands $B(x)$ into its Fourier series:

$$B(x) = \frac{1}{4\pi^2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{e^{2\pi i n x}}{n^2} + \frac{1}{24}.$$

Hence, if $b(x) = \sum_{l\geq 1} c_l B(lx) + c_0/24$ with integers c_l, almost all equal to zero, then

$$b(x) = \frac{1}{4\pi^2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{p(e^{2\pi i n x}) - p(0) + p(1)/2}{n^2}, \tag{15}$$

where $p(t)$ denotes the polynomial[1] $p(t) = \sum_{l\geq 0} c_l t^l$. (For the identity we used also $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.) By the uniqueness of the Fourier expansion of $b(x)$ the polynomial p is uniquely determined by p, i.e. we have a map $b \mapsto p$, which defines an isomorphism of $\mathfrak{W}$ with the group of polynomials over $\mathbb{Z}$. This implies the claimed linear independence. □

It is worthwhile to summarize the discussion of this section in terms of the composition of the isomorphism $f \mapsto \mathrm{ord}(f, \cdot)$ with the isomorphism of $\mathfrak{W}$ and the group of polynomials over $\mathbb{Z}$ used in the preceding proof.

Theorem 3.6. *The map*

$$p(t) = \sum_{l\geq 0} c_l t^l \mapsto \vartheta_p = \eta^{2c_0} \prod_{l\geq 1} \left(\vartheta_l/\eta\right)^{c_l}$$

defines an isomorphism of the (additive) group $\frac{1}{2}\mathbb{Z} + t\mathbb{Z}[t]$ and the group $G(\mathfrak{B})$ of theta quotients. The theta quotient ϑ_p defines a meromorphic Jacobi form of weight $k = p(0)$, index $m = \frac{1}{2}\big(p'(1) + p''(1)\big)$ and character ε^h with $h = 2p(1)$. It is weakly holomorphic if and only if, for all positive integers N,

$$\frac{1}{N} \sum_{\zeta^N = 1} p(\zeta) \geq c_0 \tag{16}$$

(the sum is over all N-th roots of unity). Its order at infinity $\mathrm{ord}(\vartheta_p, \cdot)$ *is given by*

$$\mathrm{ord}(\vartheta_p, x) = \frac{1}{4\pi^2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{p(e^{2\pi i x n})}{n^2}. \tag{17}$$

[1] These formulas could be written more smoothly if we had defined ϑ_a as the quotient $\vartheta(\tau, az)/\eta(\tau)$, whose order is $\frac{1}{2}\mathbb{B}_2(ax)$, where $\mathbb{B}_2(x) = y^2 - y + \frac{1}{6}$ ($y =$ fractional part of x) is the periodically continued second Bernoulli polynomial.

Proof. The statements concerning the weight, index and character are obvious. (See the discussion at the beginning of this section.) The formula for the order at infinity is a restatement of (15). Finally, if we write $\vartheta_p = \eta^c \prod_a S_a^{n_a}$, then $p(t) = c_0 + \sum_l \sum_a n_a \mu(a/l) t^l$ (where $\mu(a/l) = 0$ if a is not a multiple of l). Accordingly we find $n_a + c_0 = \frac{1}{a} \sum_{\zeta^a=1} p(\zeta)$, and we recognize the stated criterion for being weakly holomorphic as a restatement of the first corollary of Theorem 3.2. $\square$

The construction of holomorphic generalized theta blocks, i.e., of theta quotients which define Jacobi forms, therefore amounts to the construction of polynomials $p(t)$ whose coefficients apart from the constant term are integers, that satisfy (16), and such that the right-hand side of (17) is non-negative for all $x \in \mathbb{R}$. We come back to this question in the following section.

We end this section by a criterion for a Jacobi form to be a generalized theta block.

Theorem 3.7. *A weakly holomorphic Jacobi form ϕ on the full modular group is a generalized theta block if and only if, for every τ, the function $z \mapsto \phi(\tau, z)$ has at most division points of $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ as zeroes.*

Proof. As we saw at the beginning of this section the divisor of a theta quotient consists, for fixed τ, of division points of $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$.

Assume vice versa, that ϕ is a meromorphic Jacobi form of weight k in $\frac{1}{2}\mathbb{Z}$, index m and character ε^h and assume, that, for some τ_0, the function $\phi(\tau_0, \_)$ has a zero in $r\tau_0 + s$ Then, by assumption, $r\tau_0 + s$ is a point of, say, order a in $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$. Moreover, if the zero is simple there are small neighborhoods U and V of τ_0 and $r\tau_0 + s$ such that the set of zeros of $\phi(\tau, z)$ in $U \times V$ is of the form $(\tau, \nu(\tau))$ with a holomorphic function $\nu(\tau)$ on U. Since $\nu(\tau)$ must be a division point for every τ in U, we conclude $\nu(\tau) = r\tau + s$. But then $\phi(\tau, r\tau + s)$, vanishing identically in U, is identically zero. The same holds still true if $r\tau_0 + s$ is a zero of order $n > 1$ as one sees by applying the preceding argument to ϕ/S_a^{n-1} instead of ϕ (with S_a as in (14)), which shows then that $\phi(\tau, \_)$ has a zero of order n in $r\tau + s$ for any τ.

The transformation law of ϕ under $\mathrm{SL}(2, \mathbb{Z})$ shows that $\phi(\tau, r\tau + s) = 0$ implies $\phi(\tau, r'\tau + s') = 0$ for any A in $\mathrm{SL}(2, \mathbb{Z})$, where $(r', s') = (r, s)A$ (since $\phi_{r,s}(\tau) := \phi(\tau, r\tau + s) e\left(mr^2\tau\right)$ satisfies $\phi_{r,s}|_k A = \phi_{(r,s)A}$ for any A in $\mathrm{SL}(2, \mathbb{Z})$).

It follows that there is a finite set I of positive integers and a sequence n_a $(a \in I)$ of integers such that, for every τ, the zero divisor of the theta function $\phi(\tau, \_)$ on $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ is of the form $\sum_{a \in I} n_a \Pi_a(\tau)$ with $\Pi_a(\tau)$ denoting the formal sum of primitive a-division points of $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$. But then $\phi / \prod_{a \in S} S_a^{n_a}$ has no zeros.

Applying the same argument as before to $1/\phi$ shows finally that there is a theta quotient f such that ϕ/f has no zeros and no poles in $\mathbb{H} \times \mathbb{C}$. A standard argument shows then that ϕ/f has index 0 and is independent of z, and finally, that it is a power of η. □

4. Long theta blocks of low weight

In this and the next section we shall be interested in constructing Jacobi forms of low weight as theta blocks (with fractional eta power). There are at least two reasons for studying holomorphic theta blocks, i.e., theta blocks that are holomorphic at infinity, of low weight. First, in applications one is usually interested in Jacobi forms of low weight and there is a good chance that a Jacobi form of low weight can be represented by a theta block, whereas this becomes more and more unlikely for higher weight. (Cf. the remark after Theorem 3.4.) Secondly, it turns out to be quite hard to construct theta blocks of low weight, which raises some interesting questions. In this section we answer the question for theoretical bounds for the lowest weight that one can obtain if one fixes the length of a holomorphic theta block. In the next section we shall present various infinite families of holomorphic theta blocks of low weight.

We are interested in the growth of the function

$$\mathrm{wt}(N) := \tfrac{N}{2} - 12\max(N), \tag{18}$$

where

$$\max(N) = \sup\Big\{ \min_x \Big(\textstyle\sum_{j=1}^N B(a_j x)\Big) : a_1, \ldots, a_N \in \mathbb{Z}_{\geq 1}\Big\}.$$

Recall that η^n with $n = 24 \min_x \sum_{j=1}^N B(a_j x)$ is the largest (fractional) eta power by which we can divide $\vartheta_{a_1} \cdots \vartheta_{a_N}$ and still obtain a theta block that is holomorphic at infinity (cf. (11) and Theorem 2.1). Accordingly, the quantity $\mathrm{wt}(N)$ measures the lowest weight for which there exists a Jacobi form which is a theta block built from exactly N factors ϑ_a.

Clearly, $\max(N) < N/24$ (since $\min_x B(ax) < \int_0^1 B(ax)\,dx = 1/24$) and therefore $\mathrm{wt}(N) > 0$. Alternatively, this inequality also follows from the fact that a non-constant holomorphic Jacobi form has positive weight.

As already remarked after Theorem 3.6 the construction of holomorphic theta blocks of low weight amounts to the construction of polynomials $p(t)$ in $\mathbb{R} + t\mathbb{Z}[t]$ whose coefficients apart from the first one are non-negative, such that $p(1)$ is large but $p(0)$ is at the same time small, and such that the right-hand side of (17) is non-negative. This will be the starting point for obtaining bounds for $\mathrm{wt}(N)$.

More precisely, as a consequence of Theorem 3.6, we can relate our problem to one that is well studied in the literature in the context of trigonometric polynomials as we shall see in a moment.

Lemma 4.1. *Let T_N denote the set of polynomials $p(t)$ in $\mathbb{R} + t\mathbb{Z}_{\geq 0}[t]$ such that $p(e^{2\pi ix}) + p(e^{-2\pi ix}) \geq 0$ for all real x, and whose sum of non-constant coefficients equals N. One has*

$$\mathrm{wt}(N) \leq \inf\left\{p(0) : p \in T_N\right\}.$$

Proof. The inequality results from the fact that the image of T_N under the map of Theorem 3.6 is contained in the set $\widetilde{T}_N$ of theta blocks (with fractional eta power) whose order function is non-negative. □

We do not know whether the image of T_N equals $\widetilde{T}_N$. If this held true then the inequality of the lemma would in fact be an equality.

The asymptotic behavior of

$$\mathrm{ct}(N) := \inf\left\{p(0) : p \in T_N\right\}$$

was studied in [Odl82], [Kol94], [BK96] et. al. In the first two of these articles it was shown that $\mathrm{ct}(N)$ does not grow faster than $\log(n)\, n^{1/3}$ and $n^{1/3}$, respectively. The so far strongest result (to the best of our knowledge) is $\mathrm{ct}(N) \ll \log^3 N$ for $N \geq 2$ (see [BK96, Thm. 0.5]). More precisely, one has

Theorem ([BK96, Cor. 5.4])**.** *For all N one has*

$$\mathrm{ct}^{\downarrow} := \inf\left\{p(0) : p \in T_N^{\downarrow}\right\} \leq 45\,000(1 + (\log N)^3),$$

where $T_N^{\downarrow}$ is the subset of polynomials $p(t) = a_0 + a_1 t + a_2 + \cdots$ in T_N whose non-constant coefficients a_1, a_2, …form a decreasing sequence.

Note that the right-hand side is also an upper bound for $\mathrm{ct}(N)$ since $T_N^{\downarrow}$ is a subset of T_N. The same paper ([BK96, Thm. 0.5]) also gives an estimate of $\mathrm{ct}^{\downarrow}(N)$ from below, namely

$$\tfrac{\log^2 N}{\log\log N} \ll \mathrm{ct}^{\downarrow}(N).$$

However, since we know neither the exact relation between $\mathrm{wt}(N)$ and $\mathrm{ct}(N)$ nor between the latter and $\mathrm{ct}^{\downarrow}(N)$, the last estimate is not useful for us. It might give an indication for a lower bound of $\mathrm{wt}(N)$ though.

We can indeed prove a similar estimate for $\mathrm{wt}(N)$ from below by relating $\mathrm{wt}(N)$ to another well-studied problem, namely the determination of the quantity

$$A(N) := \inf\left\{\int_0^1 \left|\operatorname{Re} p(e^{2\pi ix})\right| dx : p(t) \in t\mathbb{Z}_{\geq 0}[t],\ p(1) = N\right\}.$$

We thank Danylo Radchenko who pointed out this connection to us and also found and proved the following Lemma.

Lemma 4.2. *For all $N \geq 1$, one has*

$$\mathrm{wt}(N) \geq \left(\frac{6}{\pi^2} - \frac{1}{2}\right) \cdot A(N).$$

Proof. Let $p(t)$ be in $t\mathbb{Z}[t]$, and let c_0 be a real number such that $c_0/12 + \mathrm{ord}(\vartheta_p, x) = \mathrm{ord}(\vartheta_{c_0+p}, x) \geq 0$ for all x. Then $c_0/12$ is an upper bound for -$\int_0^1 \min(\mathrm{ord}(\vartheta_p, x), 0)\,dx$. But the latter integral equals $\frac{1}{2}\int_0^1 |\,\mathrm{ord}(\vartheta_p, x)|\,dx$ (since $\int_0^1 \mathrm{ord}(\vartheta_p, x)\,dx$ equals 0). From (17) we therefore obtain, setting $I_n = \int_0^1 |\,\mathrm{Re}\,p(e^{2\pi i n x})|\,dx$,

$$c_0 \geq 6\int_0^1 |\,\mathrm{ord}(\vartheta_p, x)|\,dx \geq \frac{3}{\pi^2}\Big(I_1 - \sum_{n\geq 2}\frac{I_n}{n^2}\Big) = \Big(\frac{6}{\pi^2} - \frac{1}{2}\Big)I_1,$$

where for the last equality we used that the $p(e^{2\pi i n x})$ all have the same L^1-norm. The lemma is now obvious. $\square$

Lower bounds for the left-hand side of the inequality of the last lemma have been studied in a different context (norms of exponential sums) in [MPS81]. In particular, the results given there imply

Theorem ([MPS81, Thm. 2]). *For all $N \geq 1$ one has*

$$A(N) \geq \frac{H_{2N}}{60},$$

where $H_{2N} = \sum_{n=1}^{2N}\frac{1}{n}$ denote the $2N$th harmonic number.

This theorem as stated here is not exactly identical to [MPS81, Thm. 2]. In fact, they prove, for any sequence of integers $a_1 < a_2 < \cdots < a_n$ and any sequence of complex numbers $\lambda_1, \ldots, \lambda_N$, the inequality

$$\int_0^1 \Big|\sum_{j=1}^{n}\lambda_j e^{2\pi i a_j x}\Big| \geq \frac{1}{60}\sum_{j=1}^{n}\frac{|\lambda_j|}{j}.$$

The preceding theorem is an obvious consequence.

Summarizing the preceding discussion we obtain

Theorem 4.3. *The quantity* $\mathrm{wt(N)}$ *in* (18) *satisfies*

$$\frac{H_{2N}}{555.930\ldots} \leq \mathrm{wt}(N) \leq 45000(1 + \log^3 N)$$

for all $N \geq 1$.

In particular, $\mathrm{wt}(N)$ grows at least like a constant times $\log N$ and at most like a constant times $\log^3 N$ as N goes to infinity. Note, however, that the bounds given in the theorem are very poor for N of intermediate size. For instance, for $N = 50$ these bounds are

$$0.00933 \;\leq\; \mathrm{wt}(50) \;\leq\; 2.74 \times 10^6\,,$$

whereas Table 2 in the next section shows that in fact $\mathrm{wt}(50) < 2.224$.

As we see from Theorem 4.3 there exist theta blocks with an arbitrary high number N of ϑ-factors which are Jacobi forms but have relatively small weight $\ll \log^3 N$. It is challenging to construct such theta blocks explicitly. The rest of this article will somehow pivot around this subject. In particular, we shall construct infinite families of theta blocks with a high number of ϑ_a-factors, fairly small weight and yet holomorphic at infinity. We shall even develop a theory that will permit to construct such families systematically. In the next section, however, we confine ourselves to describing the results of our direct search for interesting theta blocks.

Part II: Examples

5. Experimental search for long theta blocks

As we explained in the last section we are interested in long theta blocks of low weight which are holomorphic at infinity. For this we need, first of all, to describe an efficient method to calculate the minimum of the order of a theta block. For $\mathbf{a} = (a_1, \ldots, a_N)$ in $\mathbb{Z}^N$, set

$$\vartheta_{\mathbf{a}} := \prod_{j=1}^{N} \vartheta_{a_j},$$

$$B_{\mathbf{a}}(x) := \sum_{j=1}^{N} B(a_j x), \quad s_{\mathbf{a}} = 24 \min_x B_{\mathbf{a}}(x).$$

Recall that the theta block $\vartheta_{\mathbf{a}}$ has $B_{\mathbf{a}}$ as order at infinity. Hence $s_{\mathbf{a}}$ is the maximal fractional power of η by which we can divide $\vartheta_{\mathbf{a}}$ and still have a Jacobi form. The weight of the resulting form is

$$k_{\mathbf{a}} = \tfrac{1}{2}(N - s_{\mathbf{a}}).$$

Note that $B(x)$ is one half of the square of the distance of x to the closest point in $\frac{1}{2} + \mathbb{Z}$. Accordingly, $B_{\mathbf{a}}(x)$, for a given x, is one half of the square of the Euclidean distance of $x\mathbf{a}$ to the closest point in $\frac{\mathbf{1}}{\mathbf{2}} + \mathbb{Z}^N$, where $\frac{\mathbf{1}}{\mathbf{2}} = (\frac{1}{2}, \ldots, \frac{1}{2})$. In other words, $s_{\mathbf{a}}/24$ is one half of the square distance of the line $\mathbb{R} \cdot \mathbf{a}$ to the set $\frac{\mathbf{1}}{\mathbf{2}} + \mathbb{Z}^N$. If $\frac{\mathbf{n}}{2}$ is a point in $\frac{\mathbf{1}}{\mathbf{2}} + \mathbb{Z}^N$ then its square distance to $\mathbb{R} \cdot \mathbf{a}$ equals the square length of its orthogonal projection onto the orthogonal complement of $R \cdot \mathbf{a}$, i.e. it equals $\frac{1}{4}\left(\mathbf{n}^2 - (\mathbf{n} \cdot \mathbf{a})^2/\mathbf{a}^2\right)$. If we set

$$S_{\mathbf{a}}(\mathbf{n}) := \mathbf{n}^2 \cdot \mathbf{a}^2 - (\mathbf{n} \cdot \mathbf{a})^2, \tag{19}$$

then we can summarize

$$s_{\mathbf{a}} = 24 \min_x B_{\mathbf{a}}(x) = 3 \min_{\mathbf{n} \in \mathbf{1} + 2\mathbb{Z}^N} S_{\mathbf{a}}(\mathbf{n})/\mathbf{a}^2, \tag{20}$$

where $\mathbf{1} = (1, \ldots, 1)$. This formula has several consequences.

First of all, if $\mathbf{n}_0$ in $\mathbf{1} + 2\mathbb{Z}^N$ minimizes $S_{\mathbf{a}}(\mathbf{n})$, then the minimum of $B_{\mathbf{a}}(x)$ is assumed at $x = \mathbf{n}_0 \cdot \mathbf{a}/(2\mathbf{a}^2)$. Note that x is a rational number with denominator $2\mathbf{a}^2$. Summing up we have proved:

Proposition 5.1. *$B_{\mathbf{a}}(x)$ assumes its minimum at one of the points*

$$x = \frac{s}{2M} + \frac{k}{M} \quad (0 \le k < M), \tag{21}$$

where $s = \sum_{j=1}^{N} a_j$ and $M = \sum_{j=1}^{N} a_j^2$.

Remark. The proposition tells us in particular that we can determine the minimum of $B_{\mathbf{a}}(x)$ for a given $\mathbf{a}$ by trying all the M values x as in the theorem, which needs $M = \sum_{j=1}^{N} a_j^2$ steps.

Secondly, (20) implies the following criterion for $\vartheta_{\mathbf{a}}/\eta^d$ defining a Jacobi form.

Proposition 5.2. *The quotient $\vartheta_{\mathbf{a}}/\eta^d$ is holomorphic at infinity if and only if*

$$S_{\mathbf{a}}(\mathbf{n}) \ge \frac{d}{3}\,\mathbf{a}^2$$

for all vectors $\mathbf{n}$ in $\mathbf{1} + 2\mathbb{Z}^N$ (with $S_a(\mathbf{n})$ as in (19)). It is a cusp form if and only if the inequality is strict for all $\mathbf{n}$ in $\mathbf{1} + 2\mathbb{Z}^N$.

Remark. As we shall see below it is sometimes useful to write $S_{\mathbf{a}}(\mathbf{n})$ in a slightly different form. Namely, as a simple computation shows, one has

$$S_{\mathbf{a}}(\mathbf{n}) = \sum_{1 \le i < j \le N} (a_i n_j - a_j n_i)^2 ,$$

where we used $\mathbf{n} = (n_1, \ldots, n_N)$.

For minimizing $S_{\mathbf{a}}(\mathbf{n})$ for a given $\mathbf{a}$ the following formula is sometimes useful.

Proposition 5.3. *Let $\mathbf{u}_j$ $(1 \le j \le r)$ be linearly independent vectors in $\mathbb{Z}^N$ spanning the orthogonal complement of $\mathbf{a}$, and let $G = (\mathbf{u}_i \cdot \mathbf{u}_j)_{1 \le i,j \le r}$ be the Gram matrix of the $\mathbf{u}_j$. Then*

$$S_{\mathbf{a}}(\mathbf{n}) = \left(\mathbf{x} G^{-1} \mathbf{x}^t\right) \mathbf{a}^2,$$

where $\mathbf{x} = (\mathbf{n} \cdot \mathbf{u}_1, \ldots, \mathbf{n} \cdot \mathbf{u}_r)$.

Remark. If the $\mathbf{u}_j$ of the proposition do not span the orthogonal complement of $\mathbf{a}$ but are still orthogonal to $\mathbf{a}$, then we have still

$$S_{\mathbf{a}}(\mathbf{n}) \ge \left(\mathbf{x} G^{-1} \mathbf{x}^t\right) \mathbf{a}^2$$

for all vectors $\mathbf{n}$ in $\mathbf{1} + 2\mathbb{Z}^N$, where $\mathbf{x} = (\mathbf{n} \cdot \mathbf{u}_1, \ldots, \mathbf{n} \cdot \mathbf{u}_r)$ (as one easily sees by complementing the $\mathbf{u}_j$ to a full basis of the orthogonal comeplement of $\mathbf{a}$ by integral vectors which are orthogonal to the $\mathbf{u}_j$.)

Assume that $\mathbf{u}_j^2$ is odd for all j. Then $\mathbf{n} \cdot \mathbf{u}_j \equiv \mathbf{u}_j^2 \equiv 1 \bmod 2$ for $\mathbf{n}$ in $\mathbf{1} + 2\mathbb{Z}^N$, i.e. $\mathbf{x} \in \mathbf{1} + 2\mathbb{Z}^r$, and hence

$$S_{\mathbf{a}}(\mathbf{n}) \ge \left(\min_{\mathbf{x} \in \mathbf{1} + 2\mathbb{Z}^r} \mathbf{x} G^{-1} \mathbf{x}^t\right) \mathbf{a}^2$$

If the $\mathbf{u}_i$ are in addition pairwise orthogonal, so that G^{-1} is the diagonal matrix with $1/\mathbf{u}_j^2$ $(1 \le j \le r)$ as diagonal elements, we conclude (using $(\mathbf{n} \cdot u_j) \ge 1$)

$$S_{\mathbf{a}}(\mathbf{n}) \ge \left(\sum_{j=1}^{r} \frac{1}{\mathbf{u}_j^2} \right) \mathbf{a}^2$$

for all vectors $\mathbf{n}$ in $\mathbf{1} + 2\mathbb{Z}^N$.

Proof of Proposition 5.3. Indeed, for $\mathbf{x}$ in $\mathbb{R}^N$ let $\mathbf{x}_\perp$ the orthogonal projection of $\mathbf{x}$ onto the space spanned by the $\mathbf{u}_j$. Then $\mathbf{n}^2 = \mathbf{n}_\perp^2 + (\mathbf{n} - \mathbf{n}_\perp)^2$ and $\mathbf{n}_\perp \cdot \mathbf{a} = 0$, and hence

$$S_{\mathbf{a}}(\mathbf{n}) = \mathbf{n}_\perp^2 \cdot \mathbf{a}^2 + (\mathbf{n} - \mathbf{n}_\perp)^2 \cdot \mathbf{a}^2 - \left((\mathbf{n} - \mathbf{n}_\perp) \cdot \mathbf{a}\right)^2 \ge \mathbf{n}_\perp^2 \cdot \mathbf{a}^2.$$

But $\mathbf{n}_\perp = \sum_{j=1}^{r} (\mathbf{n} \cdot \mathbf{u}_j) \mathbf{u}_j^*$, where $\mathbf{u}_j^*$ are the vectors of the dual basis of $\mathbf{u}_j$ $(j = 1, \ldots, r)$ in the space spanned by the $\mathbf{u}_j$. Therefore $\mathbf{n}_\perp^2 = \mathbf{x} H \mathbf{x}^t$ with $\mathbf{x} = (\mathbf{n} \cdot \mathbf{u}_1, \ldots, \mathbf{n} \cdot \mathbf{u}_r)$ and $H = \left(\mathbf{u}_i^* \cdot \mathbf{u}_j^*\right)_{i,j}$. Since $H = G^{-1}$, the proposition is now obvious. $\square$

We are interested in the behavior of $s_{\mathbf{a}}$ (or $k_{\mathbf{a}}$) as a function of $\mathbf{a}$, and, in particular, to find $\mathbf{a}$ in $\mathbb{Z}^N$ for big N but with $s_{\mathbf{a}}$ as big as possible, or, equivalently, with $k_{\mathbf{a}}$ as small as possible. As is clear from the definition of $s_{\mathbf{a}}$ its value does not change if we divide $\mathbf{a}$ by the gcd of its entries. When looking for $\mathbf{a}$ with best $s_{\mathbf{a}}$ we can therefore assume that $\mathbf{a}$ is primitive. Except for the first few N, we do not know any method to determine, for a given N, the smallest possible weight $k_{\mathbf{a}}$, when $\mathbf{a}$ runs though all integral vectors (with positive entries) of length N. For $N = 1$ the minimum s_a of $B(ax)$, for any integral a, is 0, which is assumed by $\vartheta(\tau, z)$.

Already for $N = 2$ it is not completely evident to determine $s_{\mathbf{a}}$ for a given (primitive) $\mathbf{a} = (a, b)$. A simple calculation shows $S_{(a,b)}(r, s) = (as - br)^2$. Writing $r = -1 - 2k$ and $s = 1 + 2l$, we have $S_{(a,b)}(r, s) = \left(a + b + 2(al + bk)\right)^2$. The minimum over all integers k and l equals obviously the rest s of $a + b$ modulo 2, whence $s_{(a,b)} = 3/(a^2 + b^2)$ if $a + b$ is odd, and $s_{(a,b)} = 0$ otherwise. The maximal $s_{(a,b)}$ is therefore assumed for $a, b = 1, 2$, for which we have $s_{(a,b)} = 3/5$.

For larger N we did searches by trial and error to find $\mathbf{a}$ with small $k_{\mathbf{a}}$. Our best results are listed in Table 2. We do not know how far off our $k_{\mathbf{a}}$ are from the true minima. Note that, for small N, Theorem 4.3 does not give any useful hint in this respect.

TABLE 2. Best experimental values of $k_{\mathbf{a}}$ for $N \leq 50$. The first three rows give the true best values. ($\underline{a}$ stands for the vector $(1, 2, \ldots, a)$, a for the vector (a) and '$\cdot$' for concatenation; hence $\underline{5} \cdot 2 \cdot 7 = (1, 2, 3, 4, 5, 2, 7)$.)

N	$k_{\mathbf{a}}$	$\mathbf{a}^2$	$\mathbf{a}$
1	$1/2 = 0.500$	1	1
2	$7/10 = 0.700$	5	$\underline{2}$
3	$6/7 = 0.857$	14	$\underline{3}$
4	$9/10 = 0.900$	15	$\underline{3} \cdot 1$
5	$25/22 = 1.136$	55	$\underline{5}$
6	$27/28 = 0.964$	56	$\underline{5} \cdot 1$
7	$11/9 = 1.222$	108	$\underline{5} \cdot 2 \cdot 7$
8	$49/40 = 1.225$	240	$\underline{7} \cdot 10$
9	$37/28 = 1.321$	168	$\underline{6} \cdot 2 \cdot 3 \cdot 8$
10	$13/11 = 1.182$	286	$\underline{9} \cdot 1$
11	$289/193 = 1.497$	386	$\underline{10} \cdot 1$
12	$465/338 = 1.376$	507	$\underline{11} \cdot 1$
13	$589/398 = 1.480$	796	$\underline{11} \cdot 1 \cdot 17$
14	$304/205 = 1.483$	820	$\underline{13} \cdot 1$
15	$1917/1210 = 1.584$	605	$\underline{10} \cdot 1 \cdot 3 \cdot 4 \cdot 5 \cdot 13$
16	$281/172 = 1.634$	1032	$\underline{14} \cdot 1 \cdot 4$
17	$175/107 = 1.636$	1284	$\underline{14} \cdot 2 \cdot 3 \cdot 16$
18	$5007/3002 = 1.668$	1501	$\underline{16} \cdot \underline{2}$
19	$1463/895 = 1.635$	1790	$\underline{17} \cdot \underline{2}$
20	$256/151 = 1.695$	2114	$\underline{18} \cdot \underline{2}$
21	$2839/1650 = 1.721$	2475	$\underline{19} \cdot \underline{2}$
22	$9607/5750 = 1.671$	2875	$\underline{20} \cdot \underline{2}$
23	$2933/1658 = 1.769$	3316	$\underline{21} \cdot \underline{2}$
24	$2391/1339 = 1.786$	2678	$\underline{19} \cdot \underline{3} \cdot 5 \cdot 13$
25	$13961/7618 = 1.833$	3809	$\underline{22} \cdot \underline{3}$
26	$54/29 = 1.862$	4350	$\underline{23} \cdot 1 \cdot 3 \cdot 4$
27	$18441/9926 = 1.858$	4963	$\underline{23} \cdot \underline{3} \cdot 25$
28	$20515/11078 = 1.852$	5539	$\underline{25} \cdot \underline{3}$
29	$4577/2486 = 1.841$	6215	$\underline{26} \cdot \underline{3}$
30	$6459/3472 = 1.860$	6944	$\underline{27} \cdot \underline{3}$
31	$9679/5190 = 1.865$	7785	$\underline{27} \cdot \underline{3} \cdot 29$
32	$427/220 = 1.941$	7040	$\underline{26} \cdot \underline{5} \cdot 28$
33	$8187/4285 = 1.911$	8570	$\underline{29} \cdot \underline{3} \cdot 1$
34	$34583/17338 = 1.995$	8669	$\underline{28} \cdot \underline{5} \cdot 30$
35	$13259/6970 = 1.902$	10455	$\underline{31} \cdot \underline{3} \cdot 5$
36	$42723/21038 = 2.031$	10519	$\underline{31} \cdot \underline{3} \cdot 5 \cdot 8$
37	$12403/6272 = 1.978$	12544	$\underline{33} \cdot \underline{3} \cdot 1$
38	$1002/479 = 2.092$	11496	$\underline{32} \cdot \underline{5} \cdot 1$
39	$3371/1678 = 2.009$	12585	$\underline{33} \cdot \underline{5} \cdot 1$
40	$63307/30026 = 2.108$	15013	$\underline{35} \cdot \underline{3} \cdot 5 \cdot 8$
41	$18392/8795 = 2.091$	17590	$\underline{37} \cdot \underline{3} \cdot 1$
42	$17649/8131 = 2.171$	16262	$\underline{36} \cdot \underline{5} \cdot 1$
43	$2763/1306 = 2.116$	17631	$\underline{37} \cdot \underline{5} \cdot 1$
44	$29753/13714 = 2.170$	20571	$\underline{39} \cdot \underline{4} \cdot 1$
45	$21777/10298 = 2.115$	20596	$\underline{39} \cdot \underline{5} \cdot 1$
46	$25033/11105 = 2.254$	22210	$\underline{40} \cdot \underline{4} \cdot 2 \cdot 6$
47	$11381/5306 = 2.145$	23877	$\underline{41} \cdot \underline{5} \cdot 1$
48	$40449/18310 = 2.209$	27465	$\underline{43} \cdot \underline{4} \cdot 1$
49	$126745/58802 = 2.155$	29401	$\underline{44} \cdot \underline{4} \cdot 1$
50	$34937/15713 = 2.223$	31426	$\underline{45} \cdot \underline{4} \cdot 1$

6. Theta quarks

It turns out that there are infinite families of theta blocks which are holomorphic Jacobi forms. An explanation for this will be given by the theory which we shall develop in Section 9. In this section we discuss the first non-trivial example of such a family, the family of *theta quarks*, which was already introduced in the introduction. Recall that this family is given by

$$Q_{a,b} = \vartheta_a \vartheta_b \vartheta_{a+b}/\eta \qquad (a, b \in \mathbb{Z}_{>0}).$$

We use the word "quark" for these functions because the product of any three of them is a Jacobi form without character on the full modular group. We shall give six different proofs for the fact that $Q_{a,b}$, for any pair of positive integers a, b is indeed holomorphic at infinity.

Theorem 6.1. *For any pair of positive integers a and b, the function $Q_{a,b}$ defines a holomorphic Jacobi form of weight 1, index $a^2 + ab + b^2$ and character ε^3. It is a cusp form if and only if $3 \mid a'b'(a' + b')$ where $a' = a/g$ and $b' = b/g$ with g denoting the greatest common divisor of a and b.*

Remark. Note the the condition $3 \mid a'b'(a' + b')$ is equivalent to $a' \not\equiv b' \bmod 3$ as we shall occasionally use in the following proofs.

First proof. According to Theorem 2.1 we have to show that

$$\min_x \operatorname{ord}(Q_{a,b}, x) \geq 0$$

with equality if and only if $3g$ divides $a - b$. For this recall

$$\operatorname{ord}(Q_{a,b}, x) = B(ax) + B(bx) + B(-(a + b)x) - 1/24$$

(where we used that $B(x)$ is an even function), so that

$$\min_x \operatorname{ord}(Q_{a,b}, x) \geq \min_{(x,y,z)\in H} (B(x) + B(y) + B(z)) - 1/24.$$

where H denotes the hyperplane $x + y + z = 0$. If x, y or z is an integer the right-hand side is greater or equal to $B(0) = 1/8 > 1/24$. Otherwise the right-hand side is differentiable in small neighborhood of (x, y, z) and we can apply the method of Lagrangian multipliers: if (x, y, z) is a local minimum then $(\overline{x}, \overline{y}, \overline{z}) = \lambda(1, 1, 1)$ for some λ, where $\overline{x}, \overline{y}, \dots$ denote the fractional parts of $x, y, \dots$. The minimum of $B(x) + B(y) + B(z)$ on H is therefore taken on at $\overline{x} = \overline{y} = \overline{z} = 1/3$ or $\overline{x} = \overline{y} = \overline{z} = 2/3$, and it equals in either case $1/24$.

We leave it to the reader to work out when $Q_{a,b}$ is a cusp form. $\square$

Second proof. For this proof we use the criterion of Proposition 5.2. In the notations of the preceding section, we have $Q_{a,b} = \vartheta_{\mathbf{a}}/\eta$, where

$\mathbf{a} = (a, b, a+b)$. The vector $\mathbf{u} = (1, 1, -1)$ is perpendicular to $\mathbf{a} = (a, b, a+b)$, and hence by the remark after Proposition 5.3

$$S_{\mathbf{a}}(\mathbf{n}) \geq \frac{1}{\mathbf{u}^2}\mathbf{a}^2 = \frac{1}{3}\mathbf{a}^2$$

fo all $\mathbf{n}$ in $\mathbf{1} + 2\mathbb{Z}^3$. According to Proposition 5.2, the Jacobi form $Q_{a,b}$ is therefore holomorphic at infinity, and it is a cusp form if and only if the last inequality is strict for all $\mathbf{n}$. $\square$

Third proof. The holomorphy of $Q_{a,b}$ also follows from the following explicit formula for its Fourier expansion.

Theorem 6.2. *One has*

$$Q_{a,b} = -\sum_{r,s\in\mathbb{Z}} \left(\frac{s}{3}\right) q^{r^2+rs+s^2/3}\zeta^{(a-b)r+as}. \tag{22}$$

Proof. We have an isomorphism of $\mathbb{Z}$-lattices

$$\big\{(l,m,n) \in \mathbb{Z}^3 \mid l \equiv m \equiv n \bmod 2\big\} \cong \big\{(r,s,t) \in \mathbb{Z}^3 \mid s \equiv t \bmod 3\big\}$$

$$(l,\, m,\, n) \mapsto \Big(\frac{n-m}{2}\,,\ \frac{l+m}{2} - n\,,\ -l-m-n\Big)$$

with respect to which $\big(\frac{-4}{lmn}\big) = \big(\frac{-4}{t}\big)$. Hence

$$\begin{aligned} -\vartheta_a\,\vartheta_b\,\vartheta_{a+b} &= \sum_{l,\,m,\,n\,\in\,\mathbb{Z}} \left(\frac{-4}{lmn}\right) q^{\frac{l^2+m^2+n^2}{8}}\,\zeta^{\frac{al+bm-(a+b)n}{2}} \\ &= \sum_{\substack{r,\,s,\,t\,\in\,\mathbb{Z}\\ s\equiv t \bmod 3}} \left(\frac{-4}{t}\right) q^{r^2+rs+s^2/3\,+\,t^2/24}\,\zeta^{(a-b)r+as}\,, \end{aligned}$$

and (22) follows because $\sum_{t\equiv s \bmod 3} \big(\frac{-4}{t}\big)\, q^{t^2/24} = \big(\frac{s}{3}\big)\,\eta(\tau)$ for all s. $\square$

Remark. The above isomorphism of lattices is $\mathfrak{S}_3$-equivariant if we introduce new coordinates (u, v, w) with $u + v + w = 0$, $u \equiv v \equiv w \bmod 3$ which are related to r and s by $(u, v, w) = (-3r-2s,\ 3r+s,\ s)$. Then (22) can be symmetrically written in terms of the three integers a, b and $c = -a - b$ with sum 0 by

$$Q_{a,b} = \sum_{\substack{u+v+w=0\\ u\equiv v\equiv w \bmod 3}} \left(\frac{u}{3}\right) q^{(u^2+v^2+w^2)/18}\,\zeta^{-(au+bv+cw)/3}\,, \tag{23}$$

and the proof for this follows by using the equivariant isomorphism from the lattice $\{(t,u,v,w) \in \mathbb{Z}^4 \mid t \equiv u \equiv v \equiv w \bmod 3,\ u+v+w = 0\}$ to the lattice $\{(l,m,n) \in \mathbb{Z}^3 \mid l \equiv m \equiv n \bmod 2\}$ given by $(l,m,n) = -\frac{1}{3}(t+2u, t+2v, t+2w)$.

Fourth proof. Using the formula (23) we have to show

$$\mathbf{a}^2 \cdot \mathbf{x}^2 - (\mathbf{a}\cdot\mathbf{x})^2 \geq 0,$$

where $\mathbf{a} = (a, b, c)$ and $\mathbf{x} = (u, v, w)/3$; but this is the Cauchy-Schwarz inequality. Recall that the Cauchy-Schwarz inequality is strict unless $\mathbf{a}$ is a multiple of $\mathbf{x}$. In other words, $Q_{a,b}$ is a cusp form if and only if $\mathbf{a} = (a, b, -(a+b))$ is never proportional to a vector $\mathbf{x} = (u, v, w)$ in $\mathbb{Z}^3$ with $u \equiv v \equiv w \bmod 3$. □

Fifth proof. As we shall see in Section 9 we can obtain the theta quarks as *pullbacks* of the function ϑ_{A_2} defined by the Macdonald identity (also known as Kac-Weyl denominator formula) for the affine Lie algebra with positive root system A_2. The theory of affine Kac-Moody algebras gives in particular a formula for the Fourier expansion of this function, which shows that the pullbacks are indeed holomorphic at infinity (see [Mac72], [KP84]). More details will be given in Part III (see Example 10.2), where we shall also give a new proof of the Macdonald identities which does not make any use of affine Lie algebras. □

Sixth proof. In Section 13 we shall see that the function ϑ_{A_2}, which the fifth proof is based on, is the first Fourier-Jacobi coefficient of a holomorphic Borcherds product (see (41)), and hence its pullbacks to theta quarks are in particular holomorphic at infinity. For details we refer the reader to the proof of Theorem 13.5 and the subsequent remark. □

7. Other families of low weight

The series of theta quarks of the preceding section is not the only infinite family of theta blocks of low weight. In fact, as we shall see in Part III there are infinitely many such families. In this section we discuss various of these families which have low weight. More precisely we shall discuss families of weight 1, 3/2 and 2. Recall that a theta block of weight k consists of N functions ϑ_a divided by η^{N-k}. If the character is ε^h then $2N + 2k \equiv h \bmod 24$, hence the length of the theta block occurs in the arithmetic progression

$$N = -k + h/2 + 12d \quad (d = 0, 1, 2 \dots).$$

In Table 3 we list various families of theta blocks of low weight. For systematic reasons, which shall become clear in Part III we included also the family $Q_{a,b}$ of the last section and renamed the function $R_{a,b,c,d}$ of (3) to $\mathfrak{A}_{4,a,b,c,d}$.

Most remarkable is the series $\mathfrak{A}_{4;a,b,c,d}$, which, for given a, b, c, d, yields a Jacobi form in $J_{2,m}$ with

$$\begin{aligned} 2m = a^2 + (a+b)^2 + (a+b+c)^2 + (a+b+c+d)^2 \\ + b^2 + (b+c)^2 + (b+c+d)^2 + c^2 + (c+d)^2 + d^2. \end{aligned}$$

In particular we have $\mathfrak{A}_{4;1,1,1,2}$ in $J_{2,37}$. The latter space is one-dimensional and contains only one cusp form, which is in fact the cusp form of smallest index in weight 2 with trivial character. The first few coefficients of this cusp form were computed laboriously in [EZ85, p.145]. Here $\mathfrak{A}_{4;1,1,1,2}$ provides a closed formula.

TABLE 3. Families of theta blocks of low weight.

Wt	Char.	Family
	8	$Q_{a,b} = \mathfrak{A}_{2;a,b} = \mathfrak{D}_{2;a,b} = \eta^{-1}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_b$
1	10	$\mathfrak{B}_{2;a,b} = \mathfrak{C}_{2;a,b} = \eta^{-2}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_{a+2b}\,\vartheta_b$
	14	$\mathfrak{G}_{2,a,b} = \eta^{-4}\,\vartheta_a\,\vartheta_{3a+b}\,\vartheta_{3a+2b}\,\vartheta_{2a+b}\,\vartheta_{a+b}\,\vartheta_b$
	15	$\mathfrak{A}_{3;a,b,c} = \mathfrak{D}_{3;a,b,c} = \eta^{-3}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_{a+b+c}\,\vartheta_b\,\vartheta_{b+c}\,\vartheta_c$
$\frac{3}{2}$	21	$\mathfrak{B}_{3;a,b,c} = \eta^{-6}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_{a+2b+2c}\,\vartheta_{a+b+c}\,\vartheta_{a+b+2c}\,\vartheta_b\,\vartheta_{b+c}\,\vartheta_{b+2c}\,\vartheta_c$
	21	$\mathfrak{C}_{3;a,b,c} = \eta^{-6}\,\vartheta_a\,\vartheta_{2a+2b+c}\,\vartheta_{a+b}\,\vartheta_{a+2b+c}\,\vartheta_{a+b+c}\,\vartheta_b\,\vartheta_{2b+c}\,\vartheta_{b+c}\,\vartheta_c$
	0	$\mathfrak{A}_{4;a,b,c,d} = \eta^{-6}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_{a+b+c}\,\vartheta_{a+b+c+d}\,\vartheta_b\,\vartheta_{b+c}\,\vartheta_{b+c+d}\,\vartheta_c\,\vartheta_{c+d}\,\vartheta_d$
	12	$\mathfrak{B}_{4;a,b,c,d} = \eta^{-12}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_{a+2b+2c+2d}\,\vartheta_{a+b+c}\,\vartheta_{a+b+2c+2d}\,\vartheta_{a+b+c+d}$
		$\cdot\,\vartheta_{a+b+c+2d}\,\vartheta_b\,\vartheta_{b+c}\,\vartheta_{b+2c+2d}\,\vartheta_{b+c+d}\,\vartheta_{b+c+2d}\,\vartheta_c\,\vartheta_{c+d}\,\vartheta_{c+2d}\,\vartheta_d$
	12	$\mathfrak{C}_{4;a,b,c,d} = \eta^{-12}\,\vartheta_a\,\vartheta_{2a+2b+2c+d}\,\vartheta_{a+b}\,\vartheta_{a+2b+2c+d}\,\vartheta_{a+b+c}\,\vartheta_{a+b+2c+d}$
		$\cdot\,\vartheta_{a+b+c+d}\,\vartheta_b\,\vartheta_{2b+2c+d}\,\vartheta_{b+c}\,\vartheta_{b+2c+d}\,\vartheta_{b+c+d}\,\vartheta_c\,\vartheta_{2c+d}\,\vartheta_{c+d}\,\vartheta_d$
2	4	$\mathfrak{D}_{4;a,b,c,d} = \eta^{-8}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_{a+2b+c+d}\,\vartheta_{a+b+c}\,\vartheta_{a+b+c+d}\,\vartheta_{a+b+d}\,\vartheta_b\,\vartheta_{b+c}$
		$\cdot\,\vartheta_{b+c+d}\,\vartheta_{b+d}\,\vartheta_c\,\vartheta_d$
	4	$\mathfrak{F}_{4;a,b,c,d} = \eta^{-20}\,\vartheta_a\,\vartheta_{2a+3b+4c+2d}\,\vartheta_{a+b}\,\vartheta_{a+3b+4c+2d}\,\vartheta_{a+2b+2c}$
		$\cdot\,\vartheta_{a+2b+4c+2d}\,\vartheta_{a+2b+3c+d}\,\vartheta_{a+2b+3c+2d}\,\vartheta_{a+2b+2c+d}\,\vartheta_{a+2b+2c+2d}$
		$\cdot\,\vartheta_{a+b+c}\,\vartheta_{a+b+2c}\,\vartheta_{a+b+2c+d}\,\vartheta_{a+b+2c+2d}\,\vartheta_{a+b+c+d}\,\vartheta_b\,\vartheta_{b+c}\,\vartheta_{b+2c}$
		$\cdot\,\vartheta_{b+2c+d}\,\vartheta_{b+2c+2d}\,\vartheta_{b+c+d}\,\vartheta_c\,\vartheta_{c+d}\,\vartheta_d$

A courageous reader might like to verify that the given families are indeed holomorphic at infinity. In principle this can be done along the lines of the first two proofs for the family of theta quarks as in the preceding section. Here we confine ourselves to the families $\mathfrak{B}_{2;a,b}$ and $\mathfrak{G}_{2,a,b}$. However, for weights $3/2$ and 2, a straightforward verification becomes rather tedious. A more conceptual proof that these families are holomorphic at infinity will be given in Part III (cf. Theorem 10.1). The family $\mathfrak{A}_{4;a,b,d}$ will be discussed in the next section as one instance of a natural infinite collection of infinite families of theta blocks.

Proposition 7.1. *The function*

$$\mathfrak{B}_{2;a,b} = \frac{\vartheta_a\,\vartheta_{a+b}\,\vartheta_{a+2b}\,\vartheta_b}{\eta^2}$$

is a holomorphic Jacobi form of weight 1 *and (integral or half integral) index* $3a^2/2 + 3ab + 3b^2$ *with character* η^{10}*. For coprime* a *and* b *it is a cusp form if and only if* a *is odd or* $3 \nmid b(a+b)$*.*

Proof. We analyze the theta block $\vartheta_{\mathbf{a}}/\eta^2$ for $\mathbf{a} = (a, a+b, a+2b, b)$ (notation as in §5). According to Proposition 5.2, we have to prove that

$$S_{\mathbf{a}}(\mathbf{n}) \geq \frac{2}{3}\mathbf{a}^2$$

for all $\mathbf{n}$ in $\mathbf{1} + 2\mathbb{Z}^4$. For this we use the remark after Proposition 5.3: The vectors $\mathbf{u}_1 = (0, 1, -1, 1)$ and $\mathbf{u}_2 = (1, -1, 0, 1)$ and $\mathbf{a}$ are pairwise orthogonal, and $\mathbf{u}_1^2 = \mathbf{u}_2^2 = 3$, and hence the claimed inequality follows. We leave the proof of the cusp condition to the reader. □

The case of the family $\mathfrak{G}_{2;a,b}$ of "six theta over four eta" can be treated similarly.

Proposition 7.2. *The function*

$$\mathfrak{G}_{2;a,b} = \frac{\vartheta_a\,\vartheta_{3a+b}\,\vartheta_{3a+2b}\,\vartheta_{2a+b}\,\vartheta_{a+b}\,\vartheta_b}{\eta^4}$$

is a holomorphic Jacobi form of weight 1 *and index* $4(3a^2 + 3ab + b^2)$ *with character* η^{14}*.*

Proof. We proceed as in the preceding proof. Setting

$$\mathbf{a} = (a, 3a+b, 3a+2b, 2a+b, a+b, b),$$

we have to prove

$$S_{\mathbf{a}}(\mathbf{n}) \geq \frac{4}{3}\mathbf{a}^2$$

for all $\mathbf{n}$ in $\mathbf{1} + 2\mathbb{Z}^6$. For this we apply Proposition 5.3 to the vectors $\mathbf{u}_j$ given by

$$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

It is quickly checked that they are orthogonal to $\mathbf{a}$, and that the Gram matrix $G = (\mathbf{u}_i \cdot \mathbf{u}_j)$ satisfies

$$4G^{-1} = \begin{pmatrix} 2 & -1 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & \frac{4}{3} \end{pmatrix}.$$

Using $\mathbf{n} \cdot \mathbf{u}_j \equiv 1 \bmod 2$ for $\mathbf{n} \in \mathbf{1} + 2\mathbb{Z}^6$, we deduce from Proposition 5.3

$$S_{\mathbf{a}}(\mathbf{n})/\mathbf{a}^2 \geq \frac{1}{4} \min_{\mathbf{x} \in \mathbf{1}+2\mathbb{Z}^3} \mathbf{x}^t K \mathbf{x} + \frac{1}{3},$$

where

$$K = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

The minimum in question must be an even integer (since K is even). It must be ≥ 4 since for odd x, y, z, we have $\frac{1}{2}(x, y, z)K(x, y, z)^t = x^2 - xy + xz + y^2 - yz + z^2 \equiv 0 \bmod 2$; in fact, it is 4 as one sees for $x = y = z = 1$. The desired estimate is now obvious. □

8. An infinite collection of families

In the previous section we saw various infinite families of theta blocks. In Part III we shall propose a general theory which explains the existence of these families and generates even more. More specifically, we shall associate an infinite family to every root system. The infinite families which we shall propose in this section turn out to be those attached to the root systems A_n. However, we include this section in the hope that the reader might find it profitable to study the latter families here using elementary arguments without having to go through the details of the theory developed in Part III.

For the rest of this section we fix an integer $n \geq 2$. For any integral vector $\mathbf{a} = (a_0, \ldots, a_n)$ with pairwise different entries, we set

$$\Theta_{\mathbf{a}} := \eta^{-n(n-1)/2} \prod_{0 \leq i < j \leq n} \vartheta_{a_i - a_j}. \tag{24}$$

Clearly, $\Theta_{\mathbf{a}}$ depends only on the coset of $\mathbf{a}$ in $\mathbb{Z}^{n+1}/\mathbb{Z} \cdot \mathbf{1}$, where as before $\mathbf{1} = (1, 1, \cdots, 1)$. Moreover, changing the signs of any entries or the order of the entries of $\mathbf{a}$ leaves $\Theta_{\mathbf{a}}$ invariant up to sign. The assumption that the a_j are pairwise different ensures that $\Theta_{\mathbf{a}}$ does not vanish identically. Note that for $n = 2$, we have

$$\Theta_{a+b,b,0} = \eta^{-1} \vartheta_a \vartheta_{a+b} \vartheta_b,$$

which is the family of theta quarks, and similarly

$$\Theta_{(0,a,a+b,a+b+c,a+b+c+d)} = R_{a,b,c,d}.$$

We also define a quadratic form Q by

$$Q(\mathbf{a}) := \frac{1}{2} \sum_{0 \leq i < j \leq n} (a_i - a_j)^2 = \frac{n+1}{2} \left(\sum_{i=0}^{n} a_i^2 \right) - \frac{1}{2} \left(\sum_{i=0}^{n} a_i \right)^2.$$

Again we recognize that $Q(\mathbf{a})$ depends only on $\mathbf{a} \bmod \mathbb{Z} \cdot \mathbf{1}$.

In this section we shall prove that the functions $\Theta_{\mathbf{a}}$, with $\mathbf{a}$ as above a vector in $\mathbb{Z}^{n+1}$ having pairwise distinct entries, are theta blocks. More precisely, we shall prove:

Theorem 8.1. *The function $\Theta_{\mathbf{a}}$ defined in (24) is a theta block of length $n(n+1)/2$ and weight $n/2$. More precisely, $\Theta_{\mathbf{a}}$ belongs to the space $J_{\frac{n}{2}, Q(\mathbf{a})}\big(\varepsilon^{n(n+2)}\big)$. In particular, if $n+1$ is relatively prime to 6, it belongs to $J_{\frac{n}{2}, Q(\mathbf{a})}$.*

The first case where the character $\varepsilon^{n(n+2)}$ is trivial occurs for $n=4$, when the $\Theta_{\mathbf{a}}$ define Jacobi forms in $J_{2,Q(\mathbf{a})}$. In fact, the family $\Theta_{\mathbf{a}}$ equals the family $R_{a,b,c,d} = \mathfrak{A}_{4;a,b,c,d}$ mentioned in the introduction and in Table 3. There are at least three more infinite families (all being products of ten theta functions divided by η^6) which yield Jacobi forms of weight 2 without character (see Table 5 in Section 10).

Finally, one may ask when $\Theta_{\mathbf{a}}$ is a cusp form. The answer, whose proof can be found at the end of the proof of Theorem 8.1 below, is as follows.

Supplement 8.2. *Let g denote the gcd of the differences $a_i - a_j$. Then $\Theta_{\mathbf{a}}$ is a cusp form if and only if there exists $0 \le i < j \le n$ such that $(a_i - a_j)/g$ is divisible by $n+1$.*

Just as for the family $Q_{a,b}$ of theta quarks in Theorem 6.2, one can describe the Fourier expansion of $\Theta_{\mathbf{a}}$ in closed form.

Theorem 8.3. *For the theta block $\Theta_{\mathbf{a}}$ defined in (24), one has*

$$\Theta_{\mathbf{a}} = \sum_{\substack{\mathbf{x}\in(\frac{n}{2}+\mathbb{Z})^{n+1}\\ \mathbf{x}\cdot\mathbf{1}=0}} \sigma(\mathbf{x})\, q^{\mathbf{x}^2/2(n+1)}\, \zeta^{\mathbf{a}\cdot\mathbf{x}},$$

where $\sigma(x) = \operatorname{sig}(\pi)$ if there is a permutation π of $\{0,\dots,n\}$ such that $\mathbf{x} \equiv -\frac{n}{2}\mathbf{1} + (\pi(0),\pi(1),\dots,\pi(n)) \bmod (n+1)\mathbb{Z}$, and $\sigma(x) = 0$ otherwise.

A proof of this identity will be given in a more general context in Part III. It can easily be deduced by from Theorem 10.5 in Part III applied to the root system A_n. Alternatively it can also be obtained directly, without referring to root systems, by restriction to one variable of a more general identity for many variables discussed in Theorem 9.1 below. More precisely, our identity is obtained by applying Theorem 9.1 to (in the notations of that theorem)

$$\begin{aligned}
&\underline{L} = \left(L, (\mathbf{x},\mathbf{y}) \mapsto \tfrac{\mathbf{x}\cdot\mathbf{y}}{n+1}\right), \quad s : e_i - e_j \ (0 \le i < j \le n),\\
&G = \text{permutations of the entries of vectors in } L,\\
&w = \left(-\tfrac{n}{2}, -\tfrac{n}{2}+1, \dots, \tfrac{n}{2}\right),
\end{aligned}$$

where L denotes the lattice of all vectors $\mathbf{x}$ in $\mathbb{Z}^{n+1}$ which satisfy $\sum_{j=1}^{n+1} x_j = 0$ and $x_h \equiv x_j \bmod (n+1)$ for all $0 \le h, j \le n$, and where e_j is the vector of length $n+1$ with $+1$ at the jth place and 0 at all other places. To obtain literally our theorem one has then, first of all, to replace the variable $z \in \mathbb{C}\otimes L$ used in Theorem 9.1 by $\mathbf{a}z$ where now z runs through the complex numbers. Secondly, one has to use that $\underline{L}$ is isometric to the lattice $\mathbb{Z}^{n+1}/\mathbb{Z}$ equipped with the quadratic form Q from the beginning of this section via the map (from right to left) $\mathbf{a} \mapsto (n+1)\mathbf{a} - (\mathbf{a}\cdot\mathbf{1})\mathbf{1}$. We leave the details to the interested reader.

It might be amusing to look for a combinatorial proof of the identity of Theorem 8.3 along the lines of the proof of the special case Theorem 6.2. We finally mention a nice restatement of Theorem 8.3, which is as follows.

Theorem 8.4. *One has*

$$\Theta_{\mathbf{a}}(\tau, z) = \int_0^1 \det \begin{bmatrix} \vartheta_0^*(\tau, za_0 + w) & \cdots & \vartheta_0^*(\tau, za_n + w) \\ \vdots & & \vdots \\ \vartheta_n^*(\tau, za_0 + w) & \cdots & \vartheta_n^*(\tau, za_n + w) \end{bmatrix} dw, \tag{25}$$

where

$$\vartheta_j^* = \sum_{s \in j - \frac{n}{2} + (n+1)\mathbb{Z}} q^{\frac{s^2}{2(n+1)}} \zeta^s.$$

This is indeed merely a restatement of the preceding Theorem. To recognize this, write the determinant after the integral in the form

$$\begin{aligned} &\sum_{\pi} \operatorname{sig}(\pi) \prod_{j=0}^{n} \vartheta_{\pi(j)}^*(\tau, a_j z + w) \\ &\qquad = \sum_{\pi \in S_N} \operatorname{sig}(\pi) \prod_{j=0}^{n} \sum_{x_j \in \pi(j) - \frac{n}{2} + (n+1)\mathbb{Z}} q^{x_j^2/2(n+1)}\, e\left((a_j z + w)x_j\right), \end{aligned}$$

where π runs through the group of permutations of $\{0, \ldots, n\}$. Writing the product as an $(n+1)$-ary theta series, and integrating in w from 0 to 1 yields the Fourier expansion of $\Theta_{\mathbf{a}}$ as given in Theorem 8.3.

Note that (25) suggests an elementary proof. Namely, it is obvious that, for any fixed τ, the right-hand side $I_{\mathbf{a}}$ vanishes at the $(a_i - a_j)$-division points of $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ (as it should in view of the claimed identity and the zeros of $\Theta_{\mathbf{a}}$). Indeed, if we replace z by $(\tau\lambda + \mu)\mu/(a_i - a_j)$ with any integers μ, λ then the determinant on the right-hand side of (25) becomes zero since the ith and jth row become equal up to multiplication by a constant (since $\frac{a_i}{a_i - a_j} = \frac{a_j}{a_i - a_j} + 1$). Unfortunately, this still does not prove that the divisors of $I_{\mathbf{a}}(\tau, \_)$ and $\Theta_{\mathbf{a}}(\tau, \_)$, viewed as theta functions of the elliptic curve $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$, coincide; for this we would have to consider also multiplicities. However, if we could prove that the divisors coincide (or at least one is contained in the other) and that $I_{\mathbf{a}}$ is also in $J_{\frac{n}{2}, Q(\mathbf{a})}\left(\varepsilon^{n(n+2)}\right)$ (note that the transformation law with respect to $z \mapsto z + \lambda\tau + \nu$ with integral λ, μ is obvious) then we could conclude that $I_{\mathbf{a}}$ and $\Theta_{\mathbf{a}}$ are equal up to multiplication by a holomorphic modular function f of weight 0 on $\mathrm{SL}(2, \mathbb{Z})$. Comparing the non-zero terms with lowest q-power, i.e. verifying

$$q^{-(n(n-1)/48} \prod_{0\le i<j\le n} q^{1/8}\zeta^{(a_i-a_j)/2}\left(1-\zeta^{-(a_i-a_j)/2}\right) \\ = q^{w^2/2(n+1)} \sum_{\pi} \mathrm{sig}(\pi)\zeta^{w\cdot\left(a_{\pi(0)},\dots,a_{\pi(n)}\right)},$$

shows then that f is also holomorphic at infinity, whence constant (and equal to 1).

Proof of Theorem 8.1. We have to show that

$$f(x) := \sum_{0\le i<j\le n} B\left((a_i-a_j)x\right) \ge \frac{n(n-1)}{24}$$

for all x in $\mathbb{R}$. For this we replace $a_i x$ by x_i ($i=0,\dots,n$) and show that, more generally,

$$\sum_{0\le i<j\le n} B\left(x_i-x_j\right) \ge \frac{n(n-1)}{24}$$

for any $\mathbf{x}=(x_0,\dots,x_n)$ in $\mathbb{R}^{n+1}$.

Since the function in question is symmetric and periodic in each variable, we can assume that $0\le x_0\le\cdots\le x_n\le 1$, in which case $B(x_i-x_j)=\frac{1}{2}\left(x_i-x_j+\frac{1}{2}\right)^2$ for $0\le i<j\le n$, so we need only find the minimum of

$$S := \sum_{0\le i<j\le n}\left(x_i-x_j+\frac{1}{2}\right)^2$$

over $\mathbb{R}^{n+1}/\mathbb{R}\cdot\mathbf{1}$. Restricting to $\mathbf{x}$ with $\sum_i x_i=0$ (i.e. the orthogonal complement of $\mathbb{R}\cdot\mathbf{1}$ in $\mathbb{R}^{n+1}$) and minimizing S using Lagrange multipliers shows that S assumes its local minima where the partial derivatives $\frac{\partial S}{\partial x_k}$ ($0\le k\le n$) are independent of k. Since we have $\frac{1}{2}\frac{\partial S}{\partial x_k}=(n+1)x_i+\frac{1}{2}(n-2k)$ (for $\sum_i x_i=0$) the latter condition is $x_k=\frac{1}{2(n+1)}(2k-n)$, and then

$$S=\sum_{0\le i<j\le n}\left(\frac{i-j}{n+1}+\frac{1}{2}\right)^2=\frac{n(n-1)}{24},$$

which proves the theorem.

Note that we the preceding proof also shows that $f(x)=\frac{n(n-1)}{24}$ if and only if the differences $(a_i-a_j)x$ are in $\frac{1}{n+1}\mathbb{Z}$ but not integral ($0\le i<j\le n$). From this the supplement to the theorem is obvious. $\square$

Part III: General Theory

9. Infinite families and Jacobi forms of lattice index

In this section we describe a general principle for constructing infinite families of theta blocks which are proper Jacobi forms. This principle is summarized in Theorem 9.1. As we shall see in the next section, all of the infinite series of theta blocks that we studied in the previous sections can in fact be obtained using this principle. To explain our construction we need to consider a more general type of Jacobi form, namely Jacobi forms whose index is a lattice. We explain these in the following paragraphs before we state the aforementioned construction. A more thorough theory of lattice index Jacobi forms is developed in [BS13a], [Gri94a], [CG13] and various other articles. We recall here the basics of the theory of Jacobi forms of lattice index as developed in [BS13a].

Let $\underline{L} = (L, \beta)$ be an integral lattice. Hence L is a free $\mathbb{Z}$-module of, say, rank n and $\beta : L \times L \to \mathbb{Z}$ is a symmetric non-degenerate bilinear form. If U is a $\mathbb{Z}$-submodule of full rank in $\mathbb{Q} \otimes L$ we denote by $U^\sharp$ its dual subgroup, i.e. the subgroup of all elements y in $\mathbb{Q} \otimes L$ such that $\beta(y, x)$ takes integral values for all $x \in U$. We shall use in the following $\beta(x) = \frac{1}{2}\beta(x, x)$. Note that $\beta(x)$ is not necessarily integral. If it is we call $\underline{L}$ *even*, otherwise *odd*. In any case, the map $x \mapsto \beta(x)$ defines an element of order 1 or 2 in the dual group $\mathrm{Hom}(L, \mathbb{Q}/\mathbb{Z})$ of L. The kernel L_{ev} of this homomorphism defines an even sublattice of index 2 in L if $\underline{L}$ is an odd lattice, and otherwise $L_{\mathrm{ev}} = L$. Since β is non-degenerate there exists an element r in $\mathbb{Q} \otimes L$ such that $\beta(x) \equiv \beta(r, x) \bmod \mathbb{Z}$ for x in L. We set

$$L^\bullet := \big\{ r \in \mathbb{Q} \otimes L : \beta(x) \equiv \beta(r, x) \bmod \mathbb{Z} \text{ for all } x \text{ in } L \big\},$$

and following the literature we call $L^\bullet$ on lattices the *shadow of* $\underline{L}$, and we call the elements of $L^\bullet$ *shadow vectors of* $\underline{L}$. Clearly, for an even $\underline{L}$, we have $L^\bullet = L^\sharp$, and, for an odd $\underline{L}$, we have ${L_{\mathrm{ev}}}^\sharp = L^\sharp \cup L^\bullet$ (i.e. $L^\bullet$ is the non-trivial coset in ${L_{\mathrm{ev}}}^\sharp / L^\sharp$).

Recall from Section 1 that ε^h denotes the $\mathrm{SL}(2, \mathbb{Z})$-cocycle defined by $\varepsilon^h(A) = f(A\tau)/f(\tau)$, where $f(\tau)$ denotes any (fixed) branch of the function $\eta(\tau)^h$. By slight abuse of language we occasionally call the multiplier system ε^h a *character*.

Let k and h be rational numbers such that $k \equiv h/2 \bmod \mathbb{Z}$.

Definition. A *Jacobi form of weight k, index $\underline{L}$ and character ε^h* is a holomorphic function $\phi(\tau, z)$ of a variable $\tau \in \mathbb{H}$ and a variable $z \in \mathbb{C} \otimes L$ which satisfies the following properties:

(i) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}(2,\mathbb{Z})$ one has

$$\phi\left(A\tau, \frac{z}{c\tau+d}\right) = e\left(\frac{c\,\beta(z)}{c\tau+d}\right)(c\tau+d)^{k-h/2}\,\varepsilon^h(A)\,\phi(\tau,z). \tag{26}$$

(ii) For all $x, y \in L$ one has

$$\phi(\tau, z + x\tau + y) = e\big(\beta(x+y)\big)\,e\big(-\tau\beta(x) - \beta(x,z)\big)\,\phi(\tau,z).$$

(iii) The Fourier development of ϕ is of the form

$$\phi(\tau,z) = \sum_{n\in\frac{h}{24}+\mathbb{Z}} \sum_{\substack{r\in L^\bullet \\ n\geq\beta(r)}} c(n,r)\,q^n\,e\big(\beta(r,z)\big). \tag{27}$$

The space of Jacobi forms of weight k, index $\underline{L}$ and character ε^r is denoted by $J_{k,\underline{L}}\left(\varepsilon^h\right)$.

Note that the crucial point in (iii) is the condition $n \geq \beta(r)$. That ϕ has Fourier expansion with n and r in the range described by the first conditions below the sum signs holds true for any holomorphic $\phi(\tau,z)$ satisfying the transformation laws (i) and (ii) (as one easily sees by applying these transformations laws to $\tau \mapsto \tau+1$ and, for all μ in L, to $z \mapsto z+\mu$). Note also that the factor $e\big(\beta(x+y)\big)$ in (ii) defines a linear character of the group $L \times L$. It is trivial if $\underline{L}$ is even. A priori, for the transformation formula (ii), one could consider also other characters of $L \times L$. However, it can be shown [BS13a] that, for a character different from the given one, there are no non-trivial functions satisfying (i) and (ii).

Note also that $J_{k,\underline{L}}\left(\varepsilon^h\right)$ depends only on the coset $h+24\mathbb{Z}$, as follows from $\varepsilon^{h+24k}(A) = (c\tau+d)^{12k}\varepsilon^h(A)$ $(A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$.

If we fix a $\mathbb{Z}$-basis $\{a_p\}$ for L we can identify L and $\mathbb{C}\otimes L$ with $\mathbb{Z}^n$ and $\mathbb{C}^n$, respectively, and Jacobi forms of index $\underline{L}$ can be considered as holomorphic functions on $\mathbb{H}\times\mathbb{C}^n$. In fact, if $\underline{L}$ is an even lattice, so that the Gram matrix $F = \frac{1}{2}\left(\beta(a_p,a_q)\right)$ is half-integral, and if $h=0$, the space $J_{k,\underline{L}}(\varepsilon^h)$ then becomes what in the literature [Sko08] is usually called the space of Jacobi forms of weight k and matrix index F and which is denoted by $J_{k,F}$. Moreover, if $\underline{L}$ is of rank 1 with determinant $m = |L^\sharp/L|$, then $J_{k,\underline{L}}\left(\varepsilon^h\right)$ is nothing other than the space $J_{k,\frac{m}{2}}(\varepsilon^h)$ that was introduces in Section 1.

There is a family of natural maps between all these spaces of Jacobi forms. Namely, if $\alpha : \underline{L} \to \underline{M}$ is an isometric embedding then the map $(\alpha^*\phi)(\tau,z) = \phi(\tau,\alpha z)$ defines a map

$$\alpha^* : J_{k,\underline{M}}(\varepsilon^h) \to J_{k,\underline{L}}(\varepsilon^h). \tag{28}$$

This follows immediately from the definition of our Jacobi forms.

There are two particular cases where such embeddings are of special interest for our considerations. The first case occurs when a lattice $\underline{L} = (L,\beta)$ can be isometrically embedded into the lattice $\underline{\mathbb{Z}}^N := (\mathbb{Z}^N,\cdot)$ (where the dot denotes the standard scalar product of column vectors).

Such an embedding permits to construct Jacobi forms of index $\underline{L}$ in a simple way. Namely, let α_j be the coordinate functions of this embedding, so that $\beta(x,x) = \sum_j \alpha_j(x)^2$. Then

$$\prod_{j=1}^{N} \vartheta\left(\tau, \alpha_j(z)\right) \in J_{\frac{N}{2},\underline{L}}(\varepsilon^{3N}).$$

Vice versa, if such a product defines a Jacobi form of index $\underline{L}$ then necessarily $\beta(x,x) = \sum_j \alpha_j(x)^2$, and the α_j define an isometric embedding of $\underline{L}$ into $\underline{\mathbb{Z}}^N$.

The other interesting embedding is of the form

$$s_x : (\mathbb{Z}, (u,v) \mapsto muv) \to \underline{L} = (L, \beta), \quad s_x(u) \mapsto ux,$$

where x is a non-zero element in L and $m = \beta(x,x)$. Here we obtain maps

$$s_x^* : J_{k,\underline{L}}\left(\varepsilon^h\right) \to J_{k,\frac{m}{2}}\left(\varepsilon^h\right), \quad \phi(\tau, z) \mapsto \phi(\tau, xw) \quad (w \in \mathbb{C}).$$

In fact, all the families of theta blocks that we found so far are of the form $\{s_x^*\phi\}_{x\in L}$ for suitable lattices $\underline{L}$ and special Jacobi forms ϕ in $J_{k,\underline{L}}\left(\varepsilon^h\right)$. Moreover, these special Jacobi forms ϕ are always obtained via the first construction, i.e. via an embedding of $\underline{L}$ into $\underline{\mathbb{Z}}^N$ for a suitable N. In all these examples the weight k of the special Jacobi form equals $n/2$, where n is the rank of $\underline{L}$. This is due to the fact that in those cases we can divide by a power of η. In general a division by a power of η will not yield a proper Jacobi form since the condition (iii) in the definition of Jacobi forms is not invariant under such a division. However, a special situation which makes such a division possible, and which applies to all our examples, is described by Theorem 9.1 below.

For the statement of the theorem we need some preparations. By a *eutactic star (of rank N) on a lattice $\underline{L} = (L, \beta)$* we understand a family s of non-zero vectors s_j in $L^\sharp$ ($1 \le j \le N$) such that

$$x = \sum_{j=1}^{N} \beta(s_j, x) s_j$$

for all x in $\mathbb{Q} \otimes L$. For a eutactic star s, one has

$$\beta(x,x) = \sum_j \beta(s_j, x)^2$$

for all x, i.e. the map $x \mapsto \big(\beta(s_1,x), \dots, \beta(s_N,x)\big)$ defines an isometric embedding $\alpha_s : \underline{L} \to \underline{\mathbb{Z}}^N$. Vice versa, if α is such an embedding, then, since β is non-degenerate, there exist vectors s_j such that the jth coordinate function of α is given by $\beta(s_j, x)$. It is easy to show that the family s_j (omitting the possible zero vectors) is a eutactic star.

For a eutactic star s on $\underline{L}$, we set

$$\vartheta_s(\tau, z) = \prod_{j=1}^{N} \vartheta\big(\tau, \beta(s_j, z)\big) \quad (z \in \mathbb{C} \otimes L).$$

From our previous discussion we know that the function ϑ_s defines a non-zero (holomorphic) Jacobi form of weight $N/2$ and index $\underline{L}$. We are interested to find eutactic stars s such that the ϑ_s can be divided by a high power of η and still remains holomorphic at infinity (i.e. satisfies the condition $n \geq \beta(r)$ in the Fourier expansion (iii) in the definition of Jacobi forms). It is not hard to see that the weight of a non-zero Jacobi form of index $\underline{L}$ which has rank n is $\geq n/2$. Thus the highest power of η by which we are allowed to divide ϑ_s is η^{N-n}. We shall not discuss here the question of determining the exact power but refer the reader to [BS13a]. Instead we describe here one situation where $\eta^{n-N}\vartheta_s(\tau, z)$ is in fact a holomorphic Jacobi form.

For this let G be a subgroup of the orthogonal group $O(\underline{L})$ that leaves s invariant up to signs, i.e. such that for each g in G there exists a permutation σ of the indices $1 \leq j \leq N$ and signs $\epsilon_j \in \{\pm 1\}$ such that $gs_j = \epsilon_j s_{\sigma(j)}$ for all j. We set

$$\mathrm{sn}(g) = \prod_j \epsilon_j.$$

Note that $\mathrm{sn}(g)$ does not depend on the choice of σ. It follows that $g \mapsto \mathrm{sn}(g)$ defines a linear character $\mathrm{sn} : G \to \{\pm 1\}$.

The group G acts naturally on $L^\bullet/L_{\mathrm{ev}}$. We call the eutactic star s *G-extremal on $\underline{L}$* if there is exactly one G-orbit in $L^\bullet/L_{\mathrm{ev}}$ whose elements have their stabilizers in the kernel of sn.

Theorem 9.1. *Let $\underline{L} = (L, \beta)$ be an integral lattice of rank n, let s be a G-extremal eutactic star of rank N on $\underline{L}$. Then there is a constant γ and a vector w in $L^\bullet$ such that*

$$\eta^{n-N} \prod_{j=1}^{N} \vartheta\big(\tau, \beta(s_j, z)\big) = \gamma \sum_{x \in w + L_{\mathrm{ev}}} q^{\beta(x)} \sum_{g \in G} \mathrm{sn}(g)\, e\big(\beta(gx, z)\big). \tag{29}$$

In particular, the product on the left defines an element of the space of Jacobi forms $J_{n/2,\underline{L}}(\varepsilon^{n+2N})$.

Remark. Let x be an element of $\mathbb{R} \otimes L$ such that $\beta(s_j, x) \neq 0$ for all j. (Such x exist since the s_j span $\mathbb{R} \otimes L$ and therefore cannot be contained in any hyperplane.) The identity (29) then holds true with w replaced by

$$w_0 = \frac{1}{2}\left(\epsilon_1 s_1 + \epsilon_2 s_2 + \cdots + \epsilon_N s_N\right),$$

where ϵ_j denotes the sign of $\beta(s_j, x)$. Indeed, comparing the coefficients of the smallest q-power on both sides of (29) one finds that

$$\prod_{j=1}^{N} \left(e\left(\tfrac{1}{2}\beta(s_j, z)\right) - e\left(\tfrac{1}{2}\beta(-s_j, z)\right)\right) = \gamma \sum_{x,g} \mathrm{sn}(g) e\left(\beta(g(w+x), z)\right),$$

where the sum on the right is over all g in G and all x in L_{ev} such that $\beta(w+x) = (n+2N)/24$. The left-hand side equals the sum $\sum_v \pm e\left(\beta(v, z)\right)$, where v runs through all vectors v of the form $v = v_\sigma = \frac{1}{2}\sum_{j=1}^{N} \sigma_j s_j$ with $\sigma_j = \pm 1$. From this we see that we can replace w by any v_{σ_0} among these v which is different from 0 and different from all v_σ with $\sigma \neq \sigma_0$. But $w_0 = v_\epsilon$ is such a v_{σ_0} since $\beta(w_0, x) > 0$ and $\beta(w_0 - v_\sigma, x) > 0$ for all $\sigma \neq \epsilon$.

Note also that it follows that $q^{\beta(w_0)}$ is the smallest q-power occurring on both sides of (29). In other words

$$\beta(w_0) = \frac{n+2N}{24}.$$

Proof of Theorem 9.1. As before denote the product on the left-hand side of the claimed identity (without the η-power) by ϑ_s. It is clear that ϑ_s is an element of $J_{N/2,\underline{L}}(\varepsilon^{3N})$. However, ϑ_s satisfies in addition $g^*\vartheta_s = \mathrm{sn}(g)\,\vartheta_s$ for all g in G, as follows from the very definition of sn and the identity $\vartheta(\tau, -z) = -\vartheta(\tau, z)$.

For an integer h and for k in $\frac{h}{2} + \mathbb{Z}$, let $V_k(\varepsilon^h)$ be the subspace of all Jacobi forms ϕ in $J_{k,\underline{L}}(\varepsilon^h)$ that satisfy

$$g^*\phi = \mathrm{sn}(g)\,\phi \quad \text{for all } g \in G. \tag{30}$$

Denote the function on the right-hand side of the claimed identity (29) by ϕ_s. We shall show in a moment that, for all integers h and all k in $\frac{h}{2} + \mathbb{Z}$, we have

$$V_k(\varepsilon^h) = M_{k-(r+n)/2}\,\eta^r\phi_s, \tag{31}$$

where $0 \le r < 24$, $h \equiv r + n + 2N \bmod 24$. Here, for any l, we use M_l for the space of elliptic modular forms of weight l on $\mathrm{SL}(2,\mathbb{Z})$ (which is trivial unless l is an even integer).

But then the claimed identity (29) is immediate. Namely, from (31), we deduce $\vartheta_s = f\eta^r\phi_s$ for some modular form f of level one. If f had a zero at a point τ_0 in the upper half-plane, then $\vartheta_s(\tau_0, z)$ would vanish identically as function of z. However, this is impossible as the product expansion for $\vartheta(\tau, z)$ shows. We conclude that f must itself be a power of η. Comparing weights then proves the claimed formula.

Note that we used here only that the left-hand side of in (31) is contained in the right-hand side. It follows from $\vartheta_s = \eta^{n-N}\phi_s$ that ϕ_s is an element of $V_{n/2}(\varepsilon^{n+2N})$, whence that the right-hand side of (31) is contained in the left-hand side.

It remains to prove that the left-hand side of (31) is contained in the right-hand side. Applying the transformation law (ii) for Jacobi forms to $z \mapsto z + x\tau$ ($x \in L$) we obtain, for the Fourier coefficients $c(n,r)$ of a Jacobi form ϕ in $J_{k,\underline{L}}(\varepsilon^h)$, the identities

$$c\big(n + \beta(r+x) - \beta(r), r+x\big) = c(n,r)\, e\big(\beta(x)\big).$$

Hence, if we set

$$C(D,r) := c\big(D + \beta(r), r\big),$$

then $C(D, r+x) = C(D,r)\, e\big(\beta(x)\big)$ for all x in L. In particular, we recognize that $r \mapsto C(D,r)$, for fixed D, factors through a map on $L^\bullet/L_{\mathrm{ev}}$.

Now assume that ϕ is contained in the left-hand side of (31). Then $g^*\phi = \mathrm{sn}(g)\phi$ for all g, from which we deduce

$$C(D, g^{-1}r) = \mathrm{sn}(g)\, C(D,r).$$

Since s is extremal this implies $C(D,r) = 0$ unless the stabilizer of $r + L_{\mathrm{ev}}$ in G is contained in the kernel of sn. By assumption there is exactly one G-orbit in $L^\bullet/L_{\mathrm{ev}}$ whose elements have stabilizer in the kernel of sn. Let $w + L_{\mathrm{ev}}$ an element of this orbit. The Fourier expansion (iii) of ϕ can then be written in the form

$$\begin{aligned}\phi(\tau,z) &= \sum_{r\in L^\bullet} \sum_{\substack{D \in -\beta(r) + \frac{h}{24} + \mathbb{Z}\\ D\geq 0}} C(D,r)\, q^{D+\beta(r)}\, e\big(\beta(r,z)\big)\\ &= \nu \sum_{g\in G, x\in L} \sum_{\substack{D \in -\beta(w) + \frac{h}{24} + \mathbb{Z}\\ D\geq 0}} \mathrm{sn}(g)\, C(D,w)\, q^{D+\beta(w+x)}\, e\big(\beta(g(w+x),z)\big),\end{aligned}$$

where $1/\nu$ is the order of the stabilizer of $w + L_{\mathrm{ev}}$ in the group G. We therefore find $\phi = f\,\phi_s$, where $f = \nu \sum_D C(D,w)\, q^D$. From the usual theory of transformation laws for theta functions one can easily deduce that ϕ_s defines an element of $J_{n/2,\underline{L}}(\varepsilon^{n+2N})$ (for details we refer the reader to [BS13a]). It follows that f is a modular form on $\mathrm{SL}(2,\mathbb{Z})$ of weight $k - n/2$ with multiplier system ε^r. But this space of modular forms equals $M_{k-(r+n)/2}\eta^r$, which proves that ϕ lies in the the right-hand side of (31). This proves the theorem. $\square$

Example 9.2 (Jacobi triple product identity)**.** The simplest non-trivial example for the situation described in Theorem 9.1 is given by the eutactic star s on $\underline{\mathbb{Z}}$ consisting of the single vector $s_1 = 1$ in $\mathbb{Z}$. Here s is G-extremal, where G is generated by $[-1]$ (multiplication by -1), and where $\mathrm{sn}([-1]) = -1$. The discriminant module of ${\mathbb{Z}_{\mathrm{ev}}}^\sharp/\mathbb{Z}_{\mathrm{ev}} = (\frac{1}{2}\mathbb{Z})/2\mathbb{Z}$ decomposes into the three G-orbits $\{\widetilde{1/2}, \widetilde{3/2}\}$, $\{\widetilde{1}\}$ and $\{\widetilde{0}\}$ (where $\widetilde{x}$ denotes the coset of x modulo $2\mathbb{Z}$). Only the stabilizer of the first one is trivial. In this case the resulting identity (29) takes the form (1), which is the Jacobi triple product identity.

10. Theta blocks constructed from root systems

The theorem of the preceding section described a general principle for constructing from special lattices Jacobi forms in several variables that generate infinite families of holomorphic theta blocks (i.e. theta blocks that are holomorphic at infinity). In this section, we show that there are indeed infinitely many lattices to which the theorem can be applied, namely, lattices constructed from root systems. Example 9.2 is the most basic example for this theory. The corresponding infinite families of theta blocks that will arise from our construction in fact include all the examples of families that were introduced in the previous sections.

The main result of this section can be summarized as follows.

Theorem 10.1. *Let R be a root system*[2] *of dimension n, let R^+ be a system of positive roots of R and let F denote the subset of simple roots in R^+. For r in R^+ and f in F, let $\gamma_{r,f}$ be the (non-negative) integers such that $r = \sum_{f\in F} \gamma_{r,f} f$. The function*

$$\vartheta_R(\tau, z) := \eta(\tau)^{n-|R^+|} \prod_{r\in R^+} \vartheta\big(\tau, \sum_{f\in F} \gamma_{r,f} z_f\big).$$

($\tau \in \mathbb{H}$, $z = \{z_f\}_{f\in F} \in \mathbb{C}^F$) defines a Jacobi form in $J_{n/2,\underline{R}}\big(\varepsilon^{n+2N}\big)$. Here the lattice $\underline{R}$ equals $\mathbb{Z}^F$ equipped with the quadratic form $Q(z) := \frac{1}{2}\sum_{r\in R^+} \big(\sum_f \gamma_{r,f} z_f\big)^2$.

Table 4. The Jacobi form ϑ_R associated to the irreducible root system R consists of $|R^+|$ many ϑ's multiplied by $\eta^{-\nu}$ and has weight k and character ε^l.

R	$\lvert R^+\rvert$	ν	k	l
A_n	$n(n+1)/2$	$n(n-1)/2$	$n/2$	$n(n+2)$
B_n	n^2	$n(n-1)$	$n/2$	$n(2n+1)$
C_n	n^2	$n(n-1)$	$n/2$	$n(2n+1)$
D_n	$n(n-1)$	$n(n-2)$	$n/2$	$n(2n-1)$
E_6	36	30	3	6
E_7	63	56	7/2	13
E_8	120	114	4	8
F_4	24	20	2	4
G_2	6	4	1	14

[2]All *root systems* considered here are to be understood in the strict sense (see [Hum78, § 9.2]), i.e. any root system can be partitioned into the union of pairwise orthogonal sets each of which is a root system in the Euclidean space generated by its elements and as such isomorphic to one of the irreducible root systems A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2.

Remark. 1. We remark that the matrix $C := \left(\gamma_{r,f}\right)$ that defines ϑ_R does not depend on the choice of the set of positive roots (up to permutations of its rows or columns). Indeed, the Weyl group of R acts transitively on the collection of possible sets of positive roots, at the same time permuting the respective subsets of simple roots. It is not difficult to calculate the matrix C directly from the Dynkin diagram or Cartan matrix of R (see e.g. [FH91, §21.3]).

2. Obviously it suffices to prove the theorem for irreducible root systems since for any two root systems R and R' with ambient Euclidean spaces E and E' one has $\vartheta_{R\oplus R'} = \vartheta_R\vartheta_{R'}$, where $R\oplus R'$ denotes the root system $(R\times\{0\})\cup(\{0\}\times R')$ in $E\times E'$.

3. As already explained in the previous section every choice of integer vectors $a = \{a_f\}_{f\in F}$ such that $a_f \neq 0$ for all f yields a theta block

$$\eta(\tau)^{n-N}\prod_{r\in R^+}\vartheta\big(\tau, w\sum_{f\in F}\gamma_{r,f}a_f\big) \in J_{n/2,Q(a)}\big(\varepsilon^{n+2N}\big)$$

in the variables τ, w in $\mathbb{H}\times\mathbb{C}$. Note that we can even assume that $a_f > 0$ for all f. For this let $(\cdot,\cdot)$ denote the scalar product of the ambient Euclidean space E of the root system R. For a given a let x be the element of E such that $(x, f) = a_f$ for all f. It follows $\sum_f \gamma_{r,f}a_f = (x, r)$ for all r in R^+. The general theory of root systems shows that there is a g in the Weyl group of R which maps x into the fundamental Weyl chamber, i.e. such that $a'_f := (gx, f) > 0$ for all f. But there is a permutation $r \mapsto r'$ of R^+ such that $gr = \pm r'$, and we have $\sum_f \gamma_{r',f}a'_f = \pm(gx, gr) = \pm(x, r) = \pm\sum_f \gamma_{r,f}a_f$. Therefore, $\{a_f\}_f$ and $\{a'_f\}$ yield the same theta block up to a sign.

Example 10.2. The only root system of rank 1 is A_1, and we have $\vartheta_{A_1} = \vartheta$ (see Example 9.2). If we choose for R the root system A_n, then any chosen simple roots f_j $(1 \leq j \leq n)$ can be ordered such that the positive roots are the sums of consecutive roots $f_i + f_{i+1} + \cdots + f_j$ $(1 \leq i \leq j \leq n)$. Accordingly

$$\vartheta_{A_n}(\tau, z) = \eta(\tau)^{-n(n-1)/2}\prod_{1\leq i\leq j\leq n}\vartheta\big(\tau, z_i + \cdots + z_j\big) \in J_{n/2,\underline{A_n}}\big(\varepsilon^{n(n+2)}\big).$$

(we write z_i for z_{f_i}). For $n = 2$, we obtain the function

$$\vartheta_{A_2}(\tau, z) = \vartheta(\tau, z_1)\vartheta(\tau, z_2)\vartheta(\tau, z_1 + z_2)/\eta(\tau),$$

which under specialization yields the infinite family of theta quarks.

Example 10.3. The spaces of Jacobi forms $J_{2,m}$ are, for integral m, deeply connected to the arithmetic of the modular forms of weight 2 in $\Gamma_0(m)$ (see e.g. [SZ88]). There are four infinite families of theta blocks of weight 2 with trivial character that we can deduce from the ϑ_R, as is easily inferred from Table 4. These are the families of theta blocks associated to

$$\vartheta_{A_4},\ \vartheta_{G_2}\vartheta_{B_2} = \vartheta_{G_2}\vartheta_{C_2},\ \vartheta\vartheta_{B_3},\ \vartheta\vartheta_{C_3}.$$

(Recall that B_2 is isomorphic to C_2.) The members of these families consist in each case of 10 ϑ's over η^6 (see Table 5).

TABLE 5. The four infinite families $\vartheta_R(\tau,(a,b,c,d)z)$ of theta blocks of weight 2 and trivial character associated to root systems (We write ϑ_n for the function $\vartheta(\tau,nz)$.)

R	$\vartheta_R(\tau,(a,b,c,d)z)$
A_4	$\eta^{-6}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_{a+b+c}\,\vartheta_{a+b+c+d}\,\vartheta_b\,\vartheta_{b+c}\,\vartheta_{b+c+d}\,\vartheta_c\,\vartheta_{c+d}\,\vartheta_d$
$G_2\oplus B_2$	$\eta^{-6}\,\vartheta_a\,\vartheta_{3a+b}\,\vartheta_{3a+2b}\,\vartheta_{2a+b}\,\vartheta_{a+b}\,\vartheta_b\,\vartheta_c\,\vartheta_{c+d}\,\vartheta_{c+2d}\,\vartheta_d$
$A_1\oplus B_3$	$\eta^{-6}\,\vartheta_a\,\vartheta_b\,\vartheta_{b+c}\,\vartheta_{b+2c+2d}\,\vartheta_{b+c+d}\,\vartheta_{b+c+2d}\,\vartheta_c\,\vartheta_{c+d}\,\vartheta_{c+2d}\,\vartheta_d$
$A_1\oplus C_3$	$\eta^{-6}\,\vartheta_a\,\vartheta_b\,\vartheta_{2b+2c+d}\,\vartheta_{b+c}\,\vartheta_{b+2c+d}\,\vartheta_{b+c+d}\,\vartheta_c\,\vartheta_{2c+d}\,\vartheta_{c+d}\,\vartheta_d$

We now explain how Theorem 10.1 follows from the general principle explained in Theorem 9.1 of the preceding section. In the course of its proof we shall redefine ϑ_R and $\underline{R}$, but shall eventually see that the new and old definitions in fact define the same objects. As already pointed out in the remark after the theorem we can assume without loss of generality that the root system R is irreducible.

So let R be an irreducible root system of dimension n, let R^+ be a system of positive roots of R, let N be the number of positive roots, and let[3]

$$h=\frac{1}{n}\sum_{r\in R^+}(r,r), \tag{32}$$

where $(\cdot,\cdot)$ denotes the Euclidean inner product of the ambient Euclidean vector space E of the root system R. We let W be the lattice

$$W=\big\{x\in E:(x,r)/h\in\mathbb{Z}\text{ for all }r\in R\big\},$$

and we set

$$\underline{R}=\big(W,(\cdot,\cdot)/h\big). \tag{33}$$

The dual lattice $W^\sharp$ (with respect to the scalar product $(\cdot,\cdot)/h$) equals the lattice Λ spanned by the roots r in R.

Lemma 10.4. *One has*

$$h(z,z)=\sum_{r\in R^+}(r,z)^2, \tag{34}$$

for all z in E.

Proof. The bilinear form $\beta(x,y):=\sum_{r\in R^+}(r,x)(r,y)$ is symmetric and positive definite (since the roots r span E). There hence exists an automorphism λ of the real vector space E such that $\beta(x,y)=(\lambda(x),y)$. The Weyl group of R permutes the roots, and therefore $\beta(x,y)$ is

[3]If all roots have square length 2 then h coincides with the *Coxeter number* of the given root system, otherwise it is different.

invariant under the Weyl group. This in turn implies that λ commutes with the elements of the Weyl group. However, the latter is known to act irreducibly on E (see e.g. [Hum78, §10.4, Lemma B]). By Schur's lemma we then conclude that λ is multiplication by a scalar c, whence $\beta(x, y) = c(x, y)$. It remains to show $c = h$. For this choose an orthonormal basis e_j of E. Then, using Parseval's identity, we find $cn = \sum_j c(e_j, e_j) = \sum_{r,j}(r, e_j)^2 = \sum_r (r, r)$, which proves the lemma. $\square$

The lemma implies that $\underline{R}$ is an integral lattice and, in particular, that W is contained in its dual, which is Λ.

From (34) we immediately have available the embedding $\underline{R} \to \underline{\mathbb{Z}}^N$ defined by $z \mapsto ((z, r_1), \ldots, (z, r_N))$, where r_j runs through R^+. In other words, R^+ is a eutactic star on $\underline{R}$. The Weyl group G of R leaves R invariant, and the character sn considered in the preceding section associates to an element g in the Weyl group the number $(-1)^{\ell(g)}$, where $\ell(g)$ is the length of g, i.e. the number of roots in R^+ such that gr is negative.

We shall prove in the next section that the eutactic star R^+ on $\underline{R}$ is extremal with respect to the Weyl group G of R (see Theorem 11.1, whose proof relies on general properties of root systems and is completely unrelated to the theory of Jacobi forms).

We can therefore apply Theorem 9.1 and the remark folloing it to the eutactic star R^+ on W and conclude (leaving the computation of the constant γ in Theorem 9.1 to the reader)

Theorem 10.5. *Let R be an irreducible root system with a choice of positive roots R^+, and let w be half the sum of the positive roots of R. Then, in the notations of the preceding paragraphs, we have*

$$\begin{aligned}(35)\quad \vartheta_R(\tau, z) &:= \eta(\tau)^{n-N} \prod_{r \in R^+} \vartheta\big(\tau, (r, z)/h\big) \\ &= \sum_{x \in w + W_{\mathrm{ev}}} q^{(x,x)/2h} \sum_{g \in G} \mathrm{sn}(g)\, e\big((gx, z)/h\big)\end{aligned}$$

for all τ is the upper half-plane and all z in $\mathbb{C} \otimes W$. In particular the function ϑ_R defines a holomorphic Jacobi form in $J_{n/2,\underline{R}}(\varepsilon^{n+2N})$.

Now let F be the set of simple roots in R^+. For any f in F and z in $\mathbb{C} \otimes W$, we set $z_f := (f, z)/h$. The map $z \mapsto \{z_f\}_{f \in F}$ defines an isomorphism of $\mathbb{C}$-vector spaces $\mathbb{C} \otimes W \to \mathbb{C}^F$, which maps W onto $\mathbb{Z}^F$. We have $(r, z)/h = \sum_{f \in F} \gamma_{r,f} z_f$ and accordingly (using (34)) $(x, x)/2h = Q(\{x_f\})$ with $\gamma_{r,f}$ and Q as in Theorem 10.1. Thus, under the map $z \mapsto \{z_f\}_{f \in F}$, the lattice $\underline{R}$ and the function $\vartheta_R(\tau, z)$ take on the form described in Theorem 10.1, which is therefore merely a weaker form of the preceding theorem.

Remark. The identities of the preceding theorem can be read as identities between formal power series by replacing $e\big((gx,z)/h\big)$ by a formal variable e^{gx}, and then can be identified with what is known as *Macdonald identities* [Mac72, (0.4))][4]. The latter were discovered in the context of infinite dimensional Lie algebras, more precisely, affine Lie algebras, and stated and proved without any reference to Jacobi forms, whose theory was only developed one decade later. Thus our proof of Theorem 10.5 provides a new proof of and new approach to the Macdonald identities.

11. A certain property of root systems

We continue the notations of Theorem 10.5. In other words

- R is an irreducible root system,
- R^+ a fixed choice of positive roots,
- Λ the lattice spanned by its roots,
- G the Weyl group of R,
- $h = \frac{1}{n}\sum_{r\in R^+}(r,r)$, $w = \frac{1}{2}\sum_{r\in R^+} r$, and
- $W = h\Lambda^\sharp$, $\underline{R} = (W, (\cdot,\cdot)/h)$ (so that Λ becomes the dual of $\underline{R}$, i.e. the dual of W with respect to $(\cdot,\cdot)/h$).

Note that w is an element of $W^\bullet$. Indeed, $(x,x)/h \equiv (2w,x)/h \bmod 2$ for all x in W, as follows from (34). Moreover, let

- α be the highest root in R^+,
- C the fundamental Weyl chamber associated to R^+, and
- $r^\vee$, for any root r, the coroot of r (i.e. $r^\vee = 2r/(r,r)$).

Recall from (34) that R^+ is a eutactic star on $\underline{R}$. The goal of this section is the proof of the following property of irreducible root systems.

Theorem 11.1. *The eutactic star R^+ on $\underline{R}$ is extremal with respect to the Weyl group G of R.*

The theorem is an immediate consequence of the properties of irreducible root systems summarized in following lemma. To the best of our knowledge, these properties have not previously been mentioned in the literature.

Lemma 11.2. *Let v be any element in $W^\bullet$ which has minimal length among all elements in $v + W_{\text{ev}}$*[5]*. Then:*

---

[4]To identify the identity of Theorem 10.5 for a given root system R with Macdonald's identity (0.4) in [Mac72] for the coroot system $R^\vee$ (i.e. the system $r^\vee = 2r/(r,r)$ ($r \in R$) one needs the formula $h = (\alpha+\rho,\alpha+\rho) - (\rho,\rho)$, where ρ and α are the Weyl vector and highest root of R (see Lemma 11.3 for a proof). Moreover, one needs to note that Macdonald's $\chi(\mu)$ is zero unless μ is in M_{ev}.

[5]In general there might be several elements of minimal length in a given coset in $W^\bullet/W_{\text{ev}}$. For instance, w has minimal length in $w + W_{\text{ev}}$ for any irreducible root system, but $(w-hf, w-hf) = (w,w)$ and $hf \in W_{\text{ev}}$ for two of the six simple roots of E_6.

(1) $(\alpha, v) \le h$.
(2) *If* $(\alpha, v) = h$ *then* $v \equiv g_\alpha(v) \bmod W_{\mathrm{ev}}$, *where* g_α *is the reflection through the hyperplane perpendicular to* α.
(3) *If* $(\alpha, v) < h$ *and* $v \in C$, *then* $v = w$.

Proof. To prove the lemma let f_i $(1 \le i \le n)$ be the simple roots of R^+, and let λ_i be the dual basis of the basis $f_i^\vee$ of E. One has $w = \lambda_1 + \cdots + \lambda_n$ (see [Hum78, §13.3, Lemma A] for a short proof).

To prove (i) note that by the very definition of root system $(\alpha^\vee, r)$ is integral for every root r, i.e. $h\alpha^\vee$ defines an element of W. Moreover, $(\alpha^\vee, w)$ is integral too (since $w = \sum_i \lambda_i$, and since the $f_i^\vee$ are simple roots for the coroot system $r^\vee$ $(r \in R)$, so that in particular $\alpha^\vee$ is an integral linear combination of the $f_i^\vee$), so $h\alpha^\vee$ defines an element of W_{ev}. Since v has minimal length in its class $v + W_{\mathrm{ev}}$ we have in particular $(v - h\alpha^\vee, v - h\alpha^\vee) \ge (v, v)$, i.e. $h \ge (\alpha, v)$.

For (ii) note that $g(v) = v - (\alpha, v)\alpha^\vee$, whence $v - g(v) = h\alpha^\vee$.

For (iii) we now suppose that v is in C and

$$h > (\alpha, v).$$

By Lemma 11.3 below we have $h = \frac{1}{2}(\alpha, \alpha) + (\alpha, w)$. It follows that

$$(\alpha^\vee, w) \ge (\alpha^\vee, v),$$

where we used that both sides of this inequality are integers (that $(\alpha^\vee, w)$ is an integer was proved above; but then the right-hand side is also integral since $v \in w + \Lambda$ and $(\alpha^\vee, \Lambda) \subseteq \mathbb{Z}$).

Since $\alpha^\vee = \sum_i (\alpha^\vee, \lambda_i) f_i^\vee$, $w = \sum_i \lambda_i$, and $v = \sum_i (v, f_i^\vee)\lambda_i$ the last inequality can be written as

$$\sum_i (\alpha^\vee, \lambda_i) \ge \sum_i (\alpha^\vee, \lambda_i)\,(v, f_i^\vee)$$

But the $(v, f_i^\vee)$ are strictly positive (since $v \in C$) and integers (since $v \in w + \Lambda$, $(w, f_i^\vee) = 1$ and $(\Lambda, f^\vee) \subseteq \mathbb{Z}$). Moreover, the $(\alpha^\vee, \lambda_i)$ are strictly positive (since $\alpha = \sum_i \alpha_i f_i$ with non-negative integers α_i, which are all strictly positive since α is the highest root, so that in particular $\alpha - f_i$ is still a linear combination in the f_i with non-negative integers). The last inequality therefore implies $(v, f_i^\vee) = 1$ for all i, i.e. $v = w$. This proves the lemma. □

Lemma 11.3. *Let* α *be the highest root in* R^+. *Then*

$$h = \frac{1}{2}\left((\alpha + w, \alpha + w) - (w, w)\right).$$

Proof. From (34) we obtain

$$2h = \sum_{r \in R^+} (r, \alpha^\vee)(r, \alpha).$$

For any positive root $r \neq \alpha$ not perpendicular to α, one has $(r, \alpha^\vee) = 1$. (Since the highest root is a long root one has $(\alpha, \alpha) \geq (r, r)$, which implies $(r, \alpha^\vee) \leq (r^\vee, \alpha)$. On the other hand, by the Cauchy-Schwartz inequality $(r, \alpha^\vee)(r^\vee, \alpha) < 4$, and since both scalar products are integers we find $(r, \alpha^\vee) = \pm 1$. But $(r, \alpha^\vee) = -1$ would imply $s_\alpha(r) = r + \alpha$, contradicting the fact that $s_\alpha(r)$ is a root and α is the highest root.) It follows that $2h = (\alpha, \alpha) + (y, \alpha)$, where y is the sum over all positive roots which are not perpendicular to α (here we use also $(\alpha, \alpha^\vee) = 2$). Obviously $(y, \alpha) = (2w, \alpha)$ so that $2h = (\alpha, \alpha) + 2(w, \alpha)$. The claimed formula now becomes obvious. □

Proof of Theorem 11.1. To finally prove Theorem 11.1, let $v + W_{\mathrm{ev}}$ be a class in $W^\bullet / W_{\mathrm{ev}}$. We can assume that v is a vector of minimal length in its class and that v is contained in the closure $\overline{C}$ (since $E = \bigcup_{g \in G} g\overline{C}$). By the lemma $(\alpha, v) \leq h$, and either $v = w$, or else $v + W_{\mathrm{ev}}$ is stabilized by g_α or v is contained in a wall of the Weyl chamber. In the latter case v is stabilized by the reflection through the hyperplane containing the wall, which is perpendicular to some fundamental root. Any reflection through a hyperplane has determinant -1. But determinant and the character $g \mapsto (-1)^{\ell(g)}$ coincide for reflections through hyperplanes perpendicular to roots, as follows from the fact that a reflection through the hyperplane perpendicular to a fundamental root has length 1 [Hum78, §10.2, Lemma B], and that the Weyl group is generated by such reflections.

It remains to prove that the Weyl group stabilizer of $w + W_{\mathrm{ev}}$ is contained in the kernel of sn. For this note that ϑ_R satisfies $g^*\vartheta_R = \mathrm{sn}(g)\vartheta_R$ (g in G). Since ϑ_R is obviously different from zero its Fourier development contains a Fourier coefficient $C(D, x) \neq 0$. Since $C\left(D, g(x)\right) = \mathrm{sn}(g) C(D, x)$ for all g in G and $C(D, x)$ depends only on $x + W_{\mathrm{ev}}$ we see that the orbit of $x + W_{\mathrm{ev}}$ is not stabilized by any g of odd length. By what we have seen $x + W_{\mathrm{ev}}$ must then be in the orbit of $w + W_{\mathrm{ev}}$. This proves Theorem 11.1. □

12. Theta blocks of weight 1/2 and weight 1

It is possible to give a complete description of the Jacobi forms of weight 1/2 and weight 1 (and scalar index). A first description of this kind was can be found in [Sko85], where it was proved that there are essentially only two Jacobi forms of weight 1/2, and that there is no non-zero Jacobi form of weight 1 and trivial character (see Theorem 12.1 below). In [Sko08] and in [BS13b] these results were extended to include Jacobi forms of weight 1 with arbitrary character.

The key for obtaining explicit formulas for Jacobi forms of weight 1/2 and weight 1 is the *theta expansion* of a Jacobi form, and the theory of Weil representations of $\mathrm{SL}(2, \mathbb{Z})$. To explain this, let $\underline{L} = (L, \beta)$ be an integral positive definite lattice of rank n. Recall from Section 9 that $L^\bullet$ denotes the shadow of L. For any linear character λ of $L^\bullet \cup L^\sharp$ that continues the character $x \mapsto e(\beta(x))$ of L define a holomorphic function $\vartheta_{\underline{L},\lambda}(\tau, z)$ of variables $\tau \in \mathbb{H}$ and $z \in \mathbb{C} \otimes L$ by

$$\vartheta_{\underline{L},\lambda}(\tau, z) = \sum_{r \in L^\bullet} \lambda(r)\, e\left(\tau\beta(r) + \beta(r, z)\right),$$

and let $\Theta(\underline{L})$ denote the complex space spanned by all the $\vartheta_{\underline{L},\lambda}$, where λ runs through all these characters. Note that $\Theta(\underline{L})$ has dimension $|L^\sharp/L|$. It can be shown that the space $\Theta(\underline{L})$ becomes a right $\mathrm{Mp}(2, \mathbb{Z})$-module via the map $(\alpha, \phi) \mapsto \phi|_{\underline{L},n/2}\alpha$. Here $\mathrm{Mp}(2, \mathbb{Z})$ is the usual twofold central extension of $\mathrm{SL}(2, \mathbb{Z})$ used in the theory of elliptic modular forms of weight $1/2$ consisting of pairs $\alpha = (A, w)$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $\mathrm{SL}(2, \mathbb{Z})$ and w is one of the two holomorphic roots of the function $c\tau + d$ ($\tau \in \mathbb{H}$). Moreover, $\phi|_{\underline{L},n/2}\alpha$ is defined as the right-hand side of (26) with the factor $(c\tau + d)^{k-h/2}\varepsilon^h(A)$ replaced by $w(\tau)^{-n}$. That $\Theta(\underline{L})$ with respect to the given action is an $\mathrm{Mp}(2, \mathbb{Z})$-module is a well-known fact for even $\underline{L}$ [Klo46]; for odd $\underline{L}$ see [BS13a]. The representations associated to the $\mathrm{Mp}(2, \mathbb{Z})$-modules $\Theta(\underline{L})$ can be characterized purely algebraically as a natural class of representations, which for even $\underline{L}$ are known as *Weil representations of* $\mathrm{SL}(2, \mathbb{Z})$.

Every Jacobi form ϕ in $J_{k,\underline{L}}(\varepsilon^h)$ has a *theta expansion*, i.e. it can be written in the form

$$\phi(\tau, z) = \sum_{\lambda} h_\lambda(\tau)\, \vartheta_{\underline{L},\lambda}(\tau, z)$$

with holomorphic functions h_λ and with λ running through the characters of $L^\bullet \cup L^\sharp$ whose restriction to L is $x \mapsto e(\beta(x))$. This follows immediately from the considerations at the end of the proof of Theorem 9.1. For integral h and integral or half-integral k, the h_λ are modular forms of weight $k - n/2$ on some congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. More precisely, there exists a natural number N such that h_λ is in $M_{k-n/2}(4N)$, where the latter denotes the space of all holomorphic functions h on $\mathbb{H}$ such that $h(A\tau) = w(\tau)^{2k-n}h(\tau)$ for all (A, w) in $\Gamma(4N)^*$ and, for each (A, w) in $\mathrm{Mp}(2, \mathbb{Z})$ the function $h(A\tau)w(\tau)^{n-2k}$ is bounded in $\Im(\tau) \geq 1$. Here $\Gamma(4N)^*$ is the section of $\Gamma(4N)$ in $\mathrm{Mp}(2, \mathbb{Z})$ consisting of all (A, w), where $w(\tau) = \theta(A\tau)/\theta(\tau)$ with $\theta(\tau) = \sum_{r\in\mathbb{Z}} e(\tau r^2)$.

Using the invariance of the $\Theta(\underline{L})$ under $\mathrm{Mp}(2, \mathbb{Z})$, we can reformulate the theta expansions of Jacobi forms of index $\underline{L}$ as a natural isomorphism

$$J_{k,\underline{L}}(\varepsilon^h) \cong \left(M_{k-n/2} \otimes \Theta(\underline{L})\right)(\varepsilon^h). \tag{36}$$

Here $M_{k-n/2}$ denotes the (infinite-dimensional) $\mathrm{Mp}(2, \mathbb{Z})$-module generated by all spaces $M_{k-n/2}(4N)$ with the $\mathrm{Mp}(2, \mathbb{Z})$-action $((A, w), h) \mapsto h(A\tau)w(\tau)^{n-2k}$. (It can be verified that the groups $\Gamma^*(4N)$ are normal in $\mathrm{Mp}(2, \mathbb{Z})$, so that $M_{k-n/2}$ is indeed invariant under the given action). Moreover, for an $\mathrm{Mp}(2, \mathbb{Z})$-module V, we let $V(\varepsilon^h)$ denote the subspace of all v such that $\alpha.v = \varepsilon(\alpha)^h v$ (where ε denotes the linear character of $\mathrm{Mp}(2,\mathbb{Z})$ which, for $\alpha = (A, w)$ is defined by $\varepsilon(\alpha) = \eta(A\tau)/w(\tau)\eta(\tau)$).

For the *singular weight* of the index $\underline{L}$, i.e. for the weight $k = n/2$, we obtain in particular

$$J_{n/2,\underline{L}}(\varepsilon^h) \cong \Theta(\underline{L})(\varepsilon^h).$$

For lattices of rank one, i.e. for the lattices

$$\underline{\mathbb{Z}}(m) = (\mathbb{Z}, x, y \mapsto mxy)$$

the $\mathrm{Mp}(2,\mathbb{Z})$-modules $\Theta(\underline{\mathbb{Z}}(m))$ were decomposed into irreducible parts in [Sko85, Satz 1.8, p. 22][6]. As corollary of the results there the following was proved (see [Sko85, Beispiele, p. 26–27]):

Theorem 12.1. ([Sko85]) *For any integer $m \geq 1$ and $0 \leq h < 24$ one has $J_{\frac{1}{2},\frac{m}{2}}(\varepsilon^h) = 0$ unless, for some integer d, one has $(m,h) = (d^2, 3)$ or $(m,h) = (3d^2, 1)$. In the latter case one has*[7]

$$J_{\frac{1}{2},\frac{d^2}{2}}(\varepsilon^3) = \mathbb{C}\cdot\vartheta_d, \qquad J_{\frac{1}{2},\frac{3d^2}{2}}(\varepsilon) = \mathbb{C}\cdot\vartheta_d^*.$$

For the *critical weight* of the index $\underline{L}$, i.e., for $k = (n+1)/2$, the isomorphism (36) involves $M_{1/2}$. Based on a theorem of Serre and Stark a complete decomposition of the $\mathrm{Mp}(2,\mathbb{Z})$-module $M_{1/2}$ into irreducible parts was given in [Sko85, Satz 5.2, p.101]. In particular, one has, for any natural number N,

$$M_{1/2}(4N) = \bigoplus_{\substack{d\mid N\\ N/d \text{ squarefree}}} \Theta^{\text{null}}\big(\underline{\mathbb{Z}}(2d)\big),$$

where the spaces on the right are the images of $\Theta\big(\underline{\mathbb{Z}}(m)\big)$ under the map $\vartheta \mapsto \vartheta(\tau, 0)$. As a result of these consideration we obtain the isomorphism [BS13a]

$$J_{\frac{n+1}{2},\underline{L}}(\varepsilon^a) \cong \bigoplus_{\substack{d\mid N\\ N/d \text{ squarefree}}} p^*\Theta\big(\underline{\mathbb{Z}}(2d)\oplus\underline{L}\big)(\varepsilon^a), \tag{37}$$

where, for any m, the map p is the isometric embedding of $\underline{L}$ into $\underline{\mathbb{Z}}(2m) \oplus \underline{L} = \big(\mathbb{Z}\oplus L, (x\oplus y, x'\oplus y') \mapsto mxx' + \beta(y,y')\big)$ given by $y \mapsto 0 \oplus y$, and where p^* is the pullback defined in (28). Moreover, for N one can take any multiple of the level of $\underline{L}$ and 24. Note that the spaces on the right-hand side of (37) are spaces of Jacobi forms of singular weight. Thus, Jacobi forms of critical weight and rank n index are always pullbacks of Jacobi forms of singular weight and of index of rank $n+1$.

[6]Actually, in loc.cit. only the $\mathrm{Mp}(2,\mathbb{Z})$-modules $\Theta(\underline{\mathbb{Z}}(2m))$ were decomposed. However, it is quickly checked that $\Theta(\underline{\mathbb{Z}}(m))$ is a $\mathrm{Mp}(2,\mathbb{Z})$-submodule of $\Theta(\underline{\mathbb{Z}}(4m))$, which allows to infer the decomposition of the former from the latter.

[7]Recall that we use ϑ_d^* for the function $\vartheta^*(\tau, dz)$, where ϑ^* is the quintuple product defined in (9).

To make (37) explicit we would need a description of the one-dimensional $\mathrm{Mp}(2,\mathbb{Z})$-submodules of $\Theta(\underline{L})$ for arbitrary L. For lattices of rank 1 such a description led to Theorem 12.1. In general we do not know how to describe the one-dimensional $\mathrm{Mp}(2,\mathbb{Z})$-submodules of $\Theta(\underline{L})$. However, for lattices of rank 2 such a description has been found in [BS13b]. As a result it was possible to prove the following theorem.

Theorem 12.2. *([BS13b]) Let m be a positive integer, and for $h = 2, 4, 6, 8, 10, 14$ let R and ϕ_R be the root system and Jacobi form as described in the row of h in Table 6. With $\underline{R}$ denoting the lattice defined in Theorem 10.1, one has the following:*

(1) *For $h = 4, 6, 8, 10, 14$ the space $J_{1,m}(\varepsilon^h)$ is spanned by the theta blocks $\phi_R(\tau, \ell z)$, where ℓ runs through all elements of $\underline{R}$ with square length $2m$.*
(2) *For $h = 2$ the space $J_{1,m}(\varepsilon^2)$ contains the theta blocks $\phi_R(\tau, \ell z)$ ($\ell \in \underline{R}$ with square length $2m$), but is in general not spanned by them.*
(3) *For all other values of h modulo 24 one has $J_{1,m}(\varepsilon^h) = 0$.*

TABLE 6. The six $\phi_R(\tau, z_1, z_2)$ which yield infinite families of theta blocks of weight 1 and character ε^h. (We use a and b for the projections onto the first respectively second coordinate, and we write ϑ_λ and ϑ^*_λ for the functions $\vartheta(\tau, \lambda(z_1, z_2))$ and $\vartheta^*(\tau, \lambda(z_1, z_2))$.)

h	R	ϕ_R
2	$A_1 \oplus A_1$	$\eta^2\,\vartheta^*_a\,\vartheta^*_b$
4	$A_1 \oplus A_1$	$\eta\,\vartheta_a\,\vartheta^*_b$
6	$A_1 \oplus A_1$	$\vartheta_a\,\vartheta_b$
8	A_2	$\eta^{-1}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_b$
10	B_2	$\eta^{-2}\,\vartheta_a\,\vartheta_{a+b}\,\vartheta_{a+2b}\,\vartheta_b$
14	G_2	$\eta^{-4}\,\vartheta_a\,\vartheta_{3a+b}\,\vartheta_{3a+2b}\,\vartheta_{2a+b}\,\vartheta_{a+b}\,\vartheta_b$

Remark. That $J_{1,m} = 0$ for all m was already proved in [Sko85, Satz 6.1, p. 113], and that $J_{1,m}(\varepsilon^{16}) = 0$ and the description of the spaces $J_{1,m}(\varepsilon^8)$ was shown in [Sko08, Thms. 11, 12].

PART IV: APPLICATIONS AND OPEN QUESTIONS

13. BORCHERDS PRODUCTS AND THETA BLOCKS

In [GN98], the authors proposed a construction of certain Borcherds products using Jacobi forms. The general theory of Borcherds products was developed in [Bor95] and [Bor98]. We recall the construction of [GN98].

For any positive integer m there is a level-raising Hecke type operator $V_m : J^!_{k,t} \to J^!_{k,mt}$ (see [EZ85, page 41]). For any ϕ in $J^!_{k,t}$, the Fourier coefficients $c_\phi(n,r)$ of ϕ and $c_{\phi|V_m}(n,r)$ of $\phi|V_m$ are related by the formula

$$c_{\phi|V_m}(n,r) = \sum_{d|n,r,m} d^{k-1} c_\phi\left(\frac{nm}{d^2}, \frac{r}{d}\right),$$

the sum being over all common positive divisors of n, r, and m. We consider the following series

$$\mathrm{Lift}(\phi)(\tau,z,\omega) := c_\phi(0,0)\, G_k(\tau) + \sum_{m\geq 1} \phi|V_m(\tau,z)\,\mathbf{e}(mt\omega), \tag{38}$$

where G_k for even $k \geq 2$ denotes the Eisenstein series

$$G_k(\tau) = \frac{1}{2}\zeta(1-k) + \sum_{n\geq 1} \sigma_{k-1}(n)\,\mathbf{e}(n\tau),$$

and where $G_k = 0$ for all other k (note that $c_\phi(0,0) = 0$ for $k = 2$). If ϕ is holomorphic at infinity this series is convergent for all $\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right)$ with positive definite imaginary part and defines an element of the space $M_k(\Gamma_t)$ of Siegel modular forms of weight k and genus 2 on the paramodular group Γ_t (see [Gri94b]). The map Lift for $t = 1$ is the lifting that was used by Maass to prove the original Saito-Kurokawa conjecture and that was discussed in detail in [EZ85, §6].

If ϕ has weight 0, i.e. if ϕ is in $J^!_{0,t}$, and if $c_\phi(n,r)$ is an integer for all n,r with $4tn - r^2 \leq 0$, then we define (see [GN98, eq. (2.7)])

$$B(\phi)(\tau,z,\omega) = Th(\phi)\,\mathbf{e}(C\omega)\,\exp\bigl(-\mathrm{Lift}(\phi)\bigr), \tag{39}$$

where $C = \frac{1}{2}\sum_{l>0} c_\phi(0,l)\, l$ and where

$$Th(\phi) = \eta(\tau)^{c_\phi(0,0)} \prod_{l\geq 1} \left(\frac{\vartheta_l(\tau,z)}{\eta(\tau)}\right)^{c_\phi(0,l)}.$$

A straightforward computation shows

$$B(\phi) = Th(\phi)\, p^C \prod_{\substack{n,l,m\in\mathbb{Z}\\ m\geq 1}} \bigl(1 - q^n\zeta^l p^{tm}\bigr)^{c_\phi(nm,l)},$$

where $p = \mathbf{e}(\omega)$. This product and the series (39) converges in a connected subdomain of the Siegel upper half-plane, it can be meromorphically continued to the whole upper half-plane, and then it becomes meromorphic modular form of weight $c_\phi(0,0)/2$ for the paramodular group Γ_t with known character and divisor (see [GN98, Thm. 2.1]). In fact, the function $B(\phi)$ is a Borcherds product in the neighborhood of a one-dimensional cusp of the paramodular group.

As an immediate corollary we obtain the following proposition.

Theorem 13.1. *Let ϕ be an element of $J^!_{0,t}$ with Fourier coefficients $c_\phi(n,l)$, and assume that $c_\phi(n,r)$ is an integer for any n,r such that $4tn - r^2 \le 0$, and that the sums $\sum_{d\ge 1} c_\phi(d^2n, dl)$ are non-negative for all n,l with $4nt - l^2 < 0$. Then the theta quotient $Th(\phi)$ is a Jacobi form (i.e. holomorphic in $\mathbb{H}\times\mathbb{C}$ and at infinity).*

Remark. In particular, if $c_\phi(n,r) \ge 0$ for all n,r with $4tn - r^2 < 0$ then the theta quotient of the proposition (which is then in fact a theta block) is holomorphic at infinity.

Proof. As suggested by the product expansion the multiplicities of all irreducible rational quadratic divisors (Humbert surfaces) of $B(\phi)$ is given by the sums in the proposition (for a proof see [GN98, Thm. 2.1]), whence $B(\phi)$ is holomorphic. The theta quotient $Th(\phi)$ is the first non-zero Fourier-Jacobi coefficient of $B(\phi)$, and is hence holomorphic (including infinity). □

Example 13.2 (The first Jacobi and paramodular cusp form of weight 3)**.** Consider the weak Jacobi form of weight 0 and index 13

$$\varphi_{0,13} = \frac{\vartheta_2\vartheta_3\vartheta_4}{\vartheta^3} = \zeta^{\pm 3} + 3\zeta^{\pm 2} + 5\zeta^{\pm 1} + 6 + O(q)$$

where $\zeta^{\pm m} = \zeta^m + \zeta^{-m}$. Note that the q^0-part contains in fact all non-zero coefficients $c_\phi(n,r)$ with $52n - r^2 < 0$. Indeed, the product

$$\eta^5\varphi_{0,13} = \big(\eta\vartheta_2/\vartheta\big)^2\big(\eta^2\vartheta_3/\vartheta\big)\big(\eta\vartheta_4/\vartheta_2\big)$$

defines a generalized theta block of weight $5/2$ (see Corollary 3.3), and it is even holomorphic at infinity (in fact, each of the three factors in the last formula is already holomorphic at infinity as can be easily checked). It follows that $52n - r^2 \ge -\frac{52\cdot 5}{24}$ for any non-zero Fourier coefficient $c_\phi(n,r)$ of $\varphi_{0,13}$. But $c_\phi(n,r)$ depends only on $52n - r^2$ and $\pm r \mod 26$. Analyzing the residues r^2 modulo 52 we see that all non-zero Fourier coefficients with $52n - r^2 < 0$ are given by $c_\phi(0,3) = 1$, $c_\phi(0,2) = 3$ and $c_\phi(0,1) = 5$. In particular, the Borcherds product $B(\varphi_{0,13})$ is holomorphic. Its first Fourier-Jacobi coefficient

$$\varphi_{3,13} = \vartheta_3\vartheta_2^3\vartheta^5/\eta^3 = Q_{1,1}^2 Q_{1,2}.$$

turns out to be a product of three theta quarks. It is among all Jacobi cusp form of weight 3 the one with smallest index. The divisor of $B(\varphi_{0,13})$ is a sum of Humbert modular surfaces:

$$\operatorname{div} B(\varphi_{0,13}) = \Gamma_{13}\langle z = 1/3\rangle + 3\cdot\Gamma_{13}\langle z = 1/2\rangle + 9\cdot\Gamma_{13}\langle z = 0\rangle.$$

This is a part of the divisor of $\mathrm{Lift}(\varphi_{3,13}) \in S_3(\Gamma_{13})$ because the lifting procedure preserves the divisor of the lifted Jacobi form ϕ (more precisely, of the function $\phi(\tau,z)\,\mathbf{e}(tw)$). The quotient $\mathrm{Lift}(\varphi_{3,13})/B(\frac{\vartheta_2\vartheta_3\vartheta_4}{\vartheta^3})$ is holomorphic on the the Siegel upper half-plane and Γ_{13}-invariant, whence constant by the Köcher principle. Comparing the first Fourier-Jacobi coefficients we have two formulas for the first paramodular cusp form $F_3^{(13)}$ of (canonical) weight 3:

$$F_3^{(13)} = \mathrm{Lift}(\vartheta_3\vartheta_2^3\vartheta^5/\eta^3) = B(\vartheta_2\vartheta_3\vartheta_4/\vartheta^3) \in S_3(\Gamma_{13}).$$

Moreover, we note that $F_3^{(13)}dZ \in H^{3,0}(\bar{\mathcal{A}}_{1,13})$ defines a canonical differential form on any smooth compact model of the moduli space of $(1,13)$-polarized abelian surfaces. The formula above determines the main part of its canonical divisor.

An effective construction of weak Jacobi forms satisfying the assumptions of Proposition 13.1 was proposed in [GPY15] [8].

Theorem 13.3 ([GPY15])**.** *Let Θ be a theta block of weight $k > 0$ and integral index t and trivial character which has integral vanishing order $v > 0$ in q. If v is odd assume that Θ is holomorphic at infinity. Then $\psi = (-1)^v\frac{\Theta|V_2}{\Theta}$ is a weakly holomorphic Jacobi form of weight 0 and index t which satisfies the assumptions of Theorem 13.1.*

Remark. The proof of the theorem can be found loc.cit., but the educated reader can also read it off from the formula

$$\Theta|V_2(\tau,z) = 4\Theta(2\tau,2z) + \tfrac{1}{2}\Theta(\tfrac{\tau}{2},z) + \tfrac{1}{2}\Theta(\tfrac{\tau+1}{2},z).$$

This formula shows in particular that the q-order of $\Theta|V_2$ equals $\lceil v/2\rceil$ and the q-order of ψ equals $-\lfloor v/2\rfloor$. For $v = 1$, the function ψ defines in particular a weak Jacobi form.

The above example of the paramodular form $F_3^{(13)}$ of weight 3 is the blue print for the following conjecture.

Conjecture 13.4 ([GPY15])**.** *Let $\Theta \in J_{k,t}$ be a theta block with trivial character and with order of vanishing 1 in q. Then* $\mathrm{Lift}(\Theta) = B(-\frac{\Theta|V_2}{\Theta})$.

The next theorem shows that a similar conjecture might be true for theta blocks with order in q smaller than 1.

Theorem 13.5.

(1) *Let $\Theta = \prod_{j=1}^3 Q_{a_j,b_j} \in J_{3,d}$ be a product of three theta quarks and set*

$$\phi := -\frac{\Theta|V_2}{\Theta}.$$

(Note that Theorem 13.3 implies that ϕ is in $J^!_{0,d}$ satisfying the assumptions of Theorem 13.1.) Then

$$\mathrm{Lift}(\Theta) = B(\phi) \in M_3(\Gamma_d).$$

This is a cusp form if at least one of the three theta quarks is a cusp form.

[8] The article [GPY15] is partly based on results of the current paper.

(2) *Let* $Q_{a,b} \in J_{3,t}(\varepsilon^8)$ *be an arbitrary theta quark, and set*

$$\phi := -\frac{Q_{a,b}|_{\varepsilon^8} V_4}{Q_{a,b}}.$$

Then ϕ *is in* $J^!_{0,3t}$, *and one has*

$$\mathrm{Lift}_{\varepsilon^8}(Q_{a,b}) = B(\phi).$$

This function defines a modular form of weight one with respect to the paramodular group Γ_{3t} *with a character* χ_3 *of order* 3.

The lift $\mathrm{Lift}_{\varepsilon^8}(\phi)$ for a Jacobi form ϕ with character ε^8, which was introduced in [GN98, Theorem 1.12], is defined as in (38) but with $\phi|V_m$ replaced by $\phi|_{\varepsilon^8}V_m$ and the summation restricted to all $m \equiv 1 \bmod 3$. The operator $|_{\varepsilon^8}V_m$ is a Hecke type operator defined similar to $|V_m$, whose precise definition is given in [GN98, 1.12]. The identity $\mathrm{Lift}(\Theta) = B(\phi)$ of (1) was already stated in [GPY15, Thm 8.3], and is in fact a corollary of [Gri18, Thm. 5.6]).

Proof of Theorem 13.5. We consider the function ϑ_{A_2} of Theorem 10.5 associated to the root system A_2, which defines a Jacobi form of weight 1 with character ε^8 and with lattice index $\underline{A_2}$ (defined in (33)). Recall that this is the function occurring in the Macdonald identity (also known as denominator function) of the affine Kac-Moody algebra $\widehat{A}_2$. The lattice $\underline{A_2}$ is a root lattice of type A_2 (i.e. its vectors of square length 2 span it and form a root system Φ of type A_2). If f_1, f_2 are primitive roots of A_2, then $\lambda_1 = f_1$ and $\lambda_2 = -f_2$ are fundamental weights of Φ (i.e. λ_1, λ_2 form a dual basis of a set of primitive roots of Φ). For z in $\mathbb{C} \otimes \underline{A_2}$, we set $z_j = (z, \lambda_j)$, where $(\_,\_)$ denotes the bilinear form of $\underline{A_2}$. Then ϑ_{A_2} becomes (see Example 10.2)

$$\vartheta_{A_2}(\tau, z) = \vartheta(\tau, z_1)\vartheta(\tau, z_2 - z_1)\vartheta(\tau, z_2)/\eta(\tau).$$

We note that $3\lambda_i$ is a reflective[9] vector of square length 6 in $\underline{A_2}$, and the divisor $z_j = 0$ is the hyperplane of the reflection σ_{λ_i}.

We need also the Jacobi form

$$\vartheta_{3A_2}(\tau, Z) = \vartheta_{A_2}(\tau, Z_1)\vartheta_{A_2}(\tau, Z_2)\vartheta_{A_2}(\tau, Z_3) \in J_{3,3\underline{A_2}}.$$

Here $3\underline{A_2}$ stands for the threefold orthogonal sum of $\underline{A_2}$; moreover, we identify $\mathbb{C} \otimes \left(3\underline{A_2}\right)$ with the threefold direct sum of $\mathbb{C} \otimes \underline{A_2}$ and write accordingly any Z in the former space as $Z = (Z_1, Z_2, Z_3)$ with Z_i in $\mathbb{C} \otimes \underline{A_2}$. We remark that ϑ_{3A_2} coincides also with the Jacobi form associated by Theorem 10.5 to the threefold orthogonal sum of the root system A_2.

[9]A vector x of a lattice $\underline{L} = (L, \beta)$ is called *reflective* if the reflection $\sigma_x(y) = y - 2x\beta(x, y)/\beta(x, x)$ defines an isometry of $\underline{L}$.

To the Jacobi forms ϑ_{A_2} and ϑ_{3A_2} we can apply a lifting construction similar to (38) above (see [Gri94b] and [CG13] for the case of Jacobi forms with characters). We obtain orthogonal modular forms

$$\mathrm{Lift}_{\varepsilon^8}(\vartheta_{A_2}) \in M_1(\widetilde{\mathrm{O}}^+(2\underline{U} \oplus \underline{A_2}(-3)), \chi_3),$$
$$\mathrm{Lift}(\vartheta_{3A_2}) \in M_3(\widetilde{\mathrm{O}}^+(2\underline{U} \oplus 3\underline{A_2}(-1))),$$

where $\underline{U}$ is the even unimodular lattice of signature $(1,1)$, $\underline{A_2}(n)$ is the lattice obtained from $\underline{A_2}$ by renormalizing its bilinear form by the factor n and $\widetilde{\mathrm{O}}^+(\dots)$ denote the stable orthogonal groups of the given lattices (which are of signature $(2,4)$ and $(2,8)$, respectively). In both cases the (reflective) divisor of the lifted Jacobi form induces a subdivisor of the lifting.

We construct a Jacobi form of weight 0 with index $3\underline{A_2}$ using again the operator V_2; we set

$$\begin{aligned}\varphi_{0,3A_2}(\tau, Z) &:= -\frac{\vartheta_{3A_2}|V_2}{\vartheta_{3A_2}} = \sum_{\substack{n\ge 0\\ \ell\in 3\underline{A_2}^\sharp}} c(n,\ell)\, q^n\, \mathbf{e}((\ell, Z)) \\ &= 6 + \sum_{j=1,3,5} (\zeta_j^{\pm 1} + \zeta_{j+1}^{\pm 1} + (\zeta_j\zeta_{j+1}^{-1})^{\pm 1}) + O(q),\end{aligned} \tag{40}$$

where $\zeta_j = \exp(2\pi i z_j)$, $\zeta_j^{\pm 1} = \zeta_j + \zeta_j^{-1}$. The action of V_2 on $\varphi_{0,3A_2}$ is given by

$$\vartheta_{3A_2}|\,V_2 = 4\vartheta_{3A_2}(2\tau, 2Z) + \tfrac{1}{2}\vartheta_{3A_2}(\tfrac{\tau}{2}, Z) + \tfrac{1}{2}\vartheta_{3A_2}(\tfrac{\tau+1}{2}, Z).$$

Using this formula and the explicitly known divisor of ϑ_{3A_2} one verifies that $\varphi_{0,\,3A_2} \in J^{\mathrm{weak}}_{0,3\underline{A_2}}$, where the superscript *weak* means that $c(n,\ell) = 0$ unless $n \ge 0$. For any $\ell \in 3\underline{A_2}^\sharp$, we therefore have $c(n,\ell) = 0$ unless

$$2n - (\ell,\ell) \ge -\min_{v\in \ell+3A_2} (v,v) \ge -2.$$

This justifies the first terms of the Fourier expansion in (40). Consequently the Borcherds product $B(\varphi_{0,3A_2})$ is a holomorphic form of weight $c(0,0)/2 = 3$ with divisors of order 1 along all $\widetilde{\mathrm{O}}^+(2\underline{U}\oplus 3\underline{A_2}(-1))$-orbits of the vectors (of square length -6) $\pm\lambda_i$, $\pm\lambda_{i+1}$ and $\pm(\lambda_i - \lambda_{i+1})$ ($i \in \{1,2,3\}$). Using the Köcher principle as in Example 13.2 we finally obtain

$$\mathrm{Lift}(\vartheta_{3A_2}) = B(\varphi_{0,\,3A_2}).$$

This identity remains true if we replace ϑ_{3A_2} and $\varphi_{0,\,3A_2}$ by its pullbacks via $\mathbb{C} \ni w \mapsto (a_1, b_1, a_2, b_2, a_3, b_3)w$, which yields the identity claimed in (1).

Via the isometric embedding $\alpha : \underline{A_2}(3) \to 3\underline{A_2}$, $x \mapsto (x,x,x)$, we obtain pullbacks

$$\alpha^* B(\varphi_{0,\,3A_2}) \in M_3(\widetilde{\mathrm{O}}^+(2\underline{U} \oplus \underline{A_2}(-3)))$$

and

$$\varphi_{0,A_2} := \tfrac{1}{3}\alpha^*\varphi_{0,3A_2} = 2 + \zeta_1^{\pm 1} + \zeta_2^{\pm 1} + (\zeta_1\zeta_2^{-1})^{\pm 1} + O(q),$$

the latter defining a Jacobi form of index $\underline{A_2}(3)$. The Borcherds lift $B(\varphi_{0,A_2}) \in M_1(\widetilde{\mathrm{O}}^+(2\underline{U} \oplus A_2(-3)), \chi_3)$ is a third root of $\alpha^* B(\varphi_{0,3A_2})$. Its divisor is determined by the reflections corresponding to the fundamental weights. Again $\mathrm{Lift}_{\varepsilon^8}(\varphi_{0,A_2})$ defines a function with the same divisor and we obtain

$$B(\varphi_{0,A_2}) = \mathrm{Lift}_{\varepsilon^8}(\vartheta_{A_2}). \tag{41}$$

The specialization to $(z_1, z_2) = (-a, b)z$ is the second identity of the theorem. □

Remark. We remark that the proof of (41) and the fact that both sides of this identity are holomorphic did not make use of the fact that ϑ_{A_2} is holomorphic at infinity. However, this is implied by (41), which yields the sixth proof of the fact that the theta quarks are holomorphic at infinity.

14. Miscellaneous observations and open questions

14.1. Jacobi-Eisenstein series and Jacobi cusp forms of small weight. The simplest theta block with trivial character is the product of eight theta series $\prod_{j=1}^{8} \vartheta_{a_j} \in J_{4,(a_1^2+\ldots+a_8^2)/2}$, where $a_1 + \ldots + a_8$ is even (and as usual $\vartheta_a(\tau, z) = \vartheta(\tau, az)$). This is a cusp form if and only if $(a_1 \cdot \ldots \cdot a_8)/d^8$ is even, where $d = \gcd(a_1, \ldots, a_8)$. A similar product of 24 quintuple products $\prod_{j=1}^{24} \eta\, \vartheta^*_{a_j} \in J_{12,\frac{3}{2}(a_1^2+\ldots+a_{24}^2)}$ (where $\vartheta^*_a = \vartheta_{2a}/\vartheta_a$) is a Jacobi cusp form if $(a_1 \cdot \ldots \cdot a_{24})/d^{24}$ is divisible by 2 or 3 (see [GH98, Lemma 1.2]).

In particular, ϑ^8 equals the Jacobi-Eisenstein series $E_{4,4,1}$ of weight 4 and index 4 (see [EZ85, p. 25]). The first Jacobi cusp form of weight 4 is $\vartheta^6\vartheta_2^2 \in J_{4,7}$.

The Fourier coefficients of the 24-fold product ϑ^8, i.e., the eighth power of the Jacobi triple product, can be calculated explicitly in terms of Cohen's numbers (see [GW18]). *It would be interesting to calculate the Fourier coefficients of the* 120*-fold product* $(\vartheta^*)^{24} \in J_{12,36}$.

The first two examples of Jacobi forms of weights 2 and 3 are the Jacobi-Eisenstein series $E^{(\chi)}_{2,25}$ and $E_{3,9,1}$ where $\chi = \left(\frac{*}{5}\right)$ is the primitive even character modulo 5 (we use the notations of [EZ85, p. 25–26]). Both series are theta blocks:

$$E^{(\chi)}_{2,25} = \eta^{-6}\vartheta^4\vartheta_2^3\vartheta_3^2\vartheta_4 \quad \text{and} \quad E_{3,9,1} = Q^3_{1,1} = \eta^{-3}\vartheta^6\vartheta_2^3.$$

It would be interesting to find explicitly their Fourier coefficients similar to [EZ85] and [GW18], which would give new identities for these 24-fold products.

The next two Jacobi forms of weight 2 and 3 are the cusp forms $\varphi_{2,37}$ and $\varphi_{3,13}$ of weight 2 and 3 and index 37 and 13, respectively. A table of Fourier coefficients of $\varphi_{2,37}$ was given in [EZ85] (see pages 118–120 and Table 4 on page 145). Now we can give explicit formulas for these two Jacobi cusp forms:

$$\varphi_{2,37} = \eta^{-6}\vartheta^3\vartheta_2^3\vartheta_3^2\vartheta_4\vartheta_5 \quad \text{and} \quad \varphi_{3,13} = \eta^{-3}\vartheta^5\vartheta_2^3\vartheta_3.$$

We note that $\varphi_{3,13}$ provides the existence of a canonical differential form on the moduli space of $(1,13)$-polarized abelian surfaces and non-triviality of the third cohomology group $H^3(\Gamma_{13},\mathbb{C})$ of the paramodular group Γ_{13} (see [Gri94b]).

14.2. Jacobi cusp forms of weight 2 and 3 with large q-order. The product of three theta quarks is a holomorphic Jacobi form of type 9-ϑ/3-η. It has q-order one. We can construct 21-ϑ/15-η theta blocks, which have then weight 3 and q-order 2. The following three examples are related to the antisymmetric Siegel paramodular forms of weight 3 (see [GPY])

$$\begin{aligned}\varphi_{3,122} &= \vartheta[-15;\ 1^5, 2^5, 3^4, 4^3, 5^2, 6, 7],\\ \varphi_{3,167} &= \vartheta[-15;\ 1^4, 2^5, 3^3, 4^3, 5^2, 6^2, 7, 8],\\ \varphi_{3,173} &= \vartheta[-15;\ 1^4, 2^4, 3^3, 4^4, 5^2, 6^2, 7, 8].\end{aligned}$$

Here we use the notation

$$\vartheta[-N;\ a^n, ..., b^m] = \eta^{-N}\vartheta_a^n \cdot \ldots \cdot \vartheta_b^m.$$

For weight 2, there are holomorphic theta blocks of type 22-ϑ/18-η, which have then q-order 2:

$$\begin{aligned}\varphi_{2,587} &= \vartheta[-18;\ (1,2,3,4,5,6,8)^2, 2, 7, 9, 10, 11, 12, 13, 14],\\ \varphi_{2,713} &= \vartheta[-18;\ (1,2,4,5,6,8)^2, 2, 3, 7, 8, 9, 10, 11, 12, 13, 15],\\ \varphi_{2,893} &= \vartheta[-18;\ 1, (2,3,4,5,6,8)^2, 7, 9, 10, 11, 12, 13, 14, 16, 19].\end{aligned}$$

The problem of constructing new Hecke paramodular forms of genus 2 is related to the question of existence of theta blocks of q-order 2. The form $\varphi_{2,587}$ is the leading Fourier-Jacobi coefficient of the unique anti-symmetric Siegel form $F^{(587)}$ of weight 2 for the paramodular group Γ_{587} (see [GPY]). The existence of $F^{(587)}$ supports the first part of the Brumer conjecture. According to its second part the Spin-L-function of $F^{(587)}$ is equal to the Hasse-Weil L-function of an abelian surface with conductor $N = 587$.

The form $\varphi_{3,122}$ is the leading Fourier-Jacobi coefficient of the anti-symmetric Siegel form of weight 3 for the paramodular group Γ_{112}. It is expected that the L-function of this paramodular form is related to a motivic L-function of Calabi-Yau treefolds.

We can give also an example of a weight 2 Jacobi form of q-order 3, namely a theta block of type 34-ϑ/30-η:

$$\varphi_{2,2p} = \vartheta[-30;\ (1,2,3,4,5)^2, 6,7,8,9,10,\dots,27,28,30].$$

Its index equals 2 times the prime $p = 8669$.

Theorem 13.3 together with 13.1 provides a method to construct a theta block which is holomorphic at infinity from a given theta block satisfying certain mild conditions. We apply this method to the 34-ϑ/30-η-block $\varphi_{2,2p}$. Its q-order is 3 and it is holomorphic at infinity. Hence we can apply the two cited theorems: setting

$$\psi_{0,2p} = \frac{\varphi_{2,2p}|V_2}{\varphi_{2,2p}} = c(0,0) + \sum_{0<l<m} c(0,l)(\zeta^l + \zeta^{-l}) + O(q),$$

the theta block

$$Th(\varphi_{2,2p}) = \eta^{c(0,0)} \prod_{l>0} \left(\frac{\vartheta_l}{\eta}\right)^{c(0,l)}$$

defines a Jacobi form. Note that the block $Th(\varphi_{2,2p})$ has weight 444, index 41888608, and q-order 2488; it is of the form 29412-ϑ/28524-η.

From Theorem 4.3 we know that the number N of ϑ in a theta block of weight 2 which is holomorphic at infinity is bounded; namely, one has $H_{2N}/555.960 \le 2$, which implies $N \le \frac{1}{2}e^{2\cdot 555.960}$. The q-order of a theta block of type N-ϑ/n-η equals $N/8 - n/24$. Hence the q-order v of a theta block of weight 2 and holomorphic at infinity is bounded; one has $v \le \frac{1}{16}e^{2\cdot 555.960}$. This leads to the natural question: *to find the maximal possible q-order of theta blocks of weight 2 or to find a reasonable upper bound.*

A theta block of weight 2 and trivial character needs to be of the form $(10+12d)$-ϑ/$(6+12d)$-η ($d = 0,1,2,\dots$). In Part III we found four infinite families of theta blocks holomorphic at infinity of weight 2 with trivial character of type 10-ϑ/6-η (see Table 5). In this section we saw examples of theta blocks holomorphic at infinity of weight 2 with trivial character of type 22-ϑ/18-η and 34-ϑ/30-η. This raises the question: *to find an arithmetic or representation theoretic explanation for the existence of theta blocks of types $(10+12d)$-ϑ/$(6+12d)$-η for $d \ge 1$.*

14.3. Jacobi forms of weight 2 without character. As we saw in Section 12, all spaces of Jacobi forms of weight 1/2 and weight 1 are spanned by theta blocks (with the exception of weight 1 and character ε^2). We also know from Section 3 (Remark after Theorem 3.4) that, for growing weight k and fixed index m and character ε^h, the proportion of the subspace of $J_{k,m}(\varepsilon^h)$ spanned by theta blocks becomes smaller and smaller. In view of the lifting of Jacobi forms in $J_{2,m}$ to modular forms of weight 2 and level m (see [SZ88, Thm. 5]) it is of interest *to know if all of the spaces $J_{2,m}$ are still spanned by theta blocks, or how big the subspace spanned by theta blocks is.*

TABLE 7. The table lists, for each index $1 \leq m < 200$ such that $J_{2,m} \neq 0$, the dimensions e and c of the subspace of Eisenstein series and cusp forms and the numbers te and tc of theta blocks in $J_{2,m}$ which are non-cusp forms and cusp forms, respectively.

m	e	c	te	tc	m	e	c	te	tc	m	e	c	te	tc	m	e	c	te	tc
25	1	0	1	0	100	2	0	3	0	138	0	1	0	1	170	0	3	0	7
37	0	1	0	1	101	0	1	0	1	139	0	3	0	6	171	0	2	0	3
43	0	1	0	1	102	0	1	0	1	141	0	2	0	2	172	0	3	0	6
49	2	0	3	0	103	0	2	0	3	142	0	2	0	3	173	0	4	0	11
50	1	0	1	0	106	0	2	0	2	143	0	1	0	1	174	0	1	0	0
53	0	1	0	1	107	0	2	0	3	144	1	0	1	0	175	1	2	3	2
57	0	1	0	1	109	0	3	0	6	145	0	3	0	7	176	0	2	0	3
58	0	1	0	1	111	0	1	0	1	146	0	2	0	2	177	0	4	0	9
61	0	1	0	1	112	0	1	0	1	147	2	2	8	3	178	0	3	0	5
64	1	0	1	0	113	0	3	0	7	148	0	3	0	7	179	0	3	0	5
65	0	1	0	1	114	0	1	0	1	149	0	3	0	5	181	0	5	0	14
67	0	2	0	3	115	0	2	0	3	150	1	0	1	0	182	0	2	0	3
73	0	2	0	3	116	0	1	0	1	151	0	3	0	5	183	0	3	0	5
74	0	1	0	1	117	0	1	0	1	152	0	1	0	1	184	0	3	0	6
75	1	0	1	0	118	0	1	0	1	153	0	2	0	3	185	0	4	0	11
77	0	1	0	1	121	4	1	16	1	154	0	2	0	3	186	0	2	0	2
79	0	1	0	1	122	0	2	0	3	155	0	2	0	3	187	0	5	0	20
81	2	0	3	0	123	0	2	0	3	156	0	1	0	1	188	0	2	0	2
82	0	1	0	1	124	0	1	0	1	157	0	5	0	18	189	0	2	0	3
83	0	1	0	1	125	1	2	3	3	158	0	3	0	6	190	0	2	0	3
85	0	2	0	3	127	0	3	0	7	159	0	1	0	1	191	0	2	0	1
86	0	1	0	1	128	1	1	2	1	160	0	1	0	1	192	1	1	2	1
88	0	1	0	1	129	0	2	0	2	161	0	2	0	2	193	0	7	0	33
89	0	1	0	1	130	0	2	0	3	162	2	1	5	1	194	0	3	0	4
91	0	2	0	3	131	0	1	0	1	163	0	6	0	26	195	0	1	0	1
92	0	1	0	1	133	0	4	0	12	164	0	1	0	0	196	4	1	13	1
93	0	2	0	3	134	0	2	0	3	165	0	2	0	3	197	0	6	0	27
97	0	3	0	7	135	0	1	0	1	166	0	2	0	2	198	0	2	0	3
98	2	0	3	0	136	0	1	0	1	167	0	2	0	3	199	0	4	0	8
99	0	1	0	1	137	0	4	0	11	169	5	3	45	5					

The first question can be quickly answered. A computer search for $m < 200$ shows that $J_{2,m}$ is spanned by theta blocks for all m with the exception of $m = 164$ (see Table 7). In fact, $J_{2,164}$, which is one-dimensional and contains exactly one cusp form, does not contain a theta block, not even a single generalized theta block.

However, for computational purposes this is often no serious problem. For instance, the one-dimensional space $J_{2,164}$ can be easily obtained applying the index raising operator V_2 (see [EZ85, §4]) to the single theta block in $J_{2,82}$ (which is $\vartheta_1\vartheta_2^3\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6\vartheta_7/\eta^6$). Alternatively one can try to find sufficiently many theta blocks which are not necessarily holomorphic at infinity but span a space containing a given $J_{k,m}$.

In the context of the mentioned computations it is worthwhile to mention that, for $1 \leq m < 200$, the spaces $J_{2,m}$ contain no theta quotients.

Concerning the second question we do not know of any method to determine the size of the subspace in $J_{2,m}$ spanned by theta blocks. Heuristically, however, one might expect it to be large in general. Indeed, the well-known dimension formula shows $\dim J_{2,m} \sim \frac{m+1}{24}$. On the other hand, already the four families of theta blocks from Table 5 each provide as many theta blocks of index m as there are positive integers a, b, c, d such that the sum of the squares of the indices of the theta block defining this family equals $2m$, a number whose order of magnitude is m.

14.4. Theta blocks and elliptic curves. As mentioned in the introduction, the first Jacobi cusp form of weight 2, which has index 37, is a theta block. This is of particular interest since this form corresponds to the first elliptic curve of odd rank (which has in fact rank 1 and level 37) via the Hecke equivariant lifting of $J_{2,37}$ onto the space of modular forms of weight 2 and level 37.

In general, we do not know any reason that an arbitrary theta block in $J_{2,m}$ is a Hecke eigenform except for the banal reason that $J_{2,m}$ or the subspace of cusp forms in $J_{2,m}$ is one-dimensional, so that any Jacobi form in one of these spaces is trivially an eigenform. In particular, we do not expect that the Jacobi form associated to an elliptic curve is a theta block. However, there are exactly 52 indices where $J_{2,m}$ contains only one cusp form. For 10 of these indices the corresponding Jacobi form is an old form. For each index m in the set S of the remaining 42 (see Table 8) the associated cusp Jacobi form ϕ_m corresponds via the mentioned lifting to an elliptic curve over the rationals of conductor m whose L-series $L(E, s)$ has a minus sign in its functional equation. This correspondence is given by the identities

$$\sum_{n\geq 1} \left(\tfrac{D}{n}\right) n^{-s} \sum_{n\geq 1} C_{\phi_m}(Dn^2, rn) n^{-s} = C_{\phi_m}(D, r) L(E, s),$$

valid for any negative fundamental discriminant D and integer r such that $D \equiv r^2 \bmod 4m$.

As it turns out, each of these ϕ_m with the exception of ϕ_{300} is a theta block. More precisely, we found that for each index $m \neq 300$ in S there is exactly one theta block of length 10 in $J_{2,m}$ which is a cusp form. (For $m \leq 200$ and $m = 216$ we verified in addition that there is no theta block of length strictly greater than 10 in the subspace of cusp forms of $J_{2,m}$.)

In Table 8 we give for each m in S a minimal equation for an elliptic curve over $\mathbb{Q}$ with conductor m and root number -1 (in general the isogeny classes of the given curves decompose into more than one rational isomorphism classes) and, for $m \neq 300$, the corresponding theta block. All these elliptic curves have rank 1. Except for $m \in \{89, 121\}$ the theta blocks in this table belong to one or more of the four families associated to the root systems A_4, $G_2 \oplus B_2$, $A_1 \oplus B_3$ and $A_1 \oplus C_3$ (see Table 5).

The space $J_{2,300}$ has dimension 3 and contains 5 theta blocks of length 10, which span the whole space. Here ϕ_{300} does not equal any of these 5 theta blocks. *We do not know whether it equals a theta block of length greater than* 10 *(i.e., of length* $N = 22, 34, 46, \ldots$*) or a generalized theta block.*

Concerning the question of an explicit example of a rational elliptic curve whose associated Jacobi form is not a theta block, we found $m = 91$ as the first m such that the subspace of cusp forms in $J_{2,m}$ has dimension greater than 1 and contains a Hecke eigenform with rational eigenvalues. In fact, $J_{2,91}$ contains no non-cusp forms and has dimension 2, so that both Hecke eigenforms in this space have rational eigenvalues and hence correspond to elliptic curves. The Cremona label of these elliptic curves are 91.a1 and 91.b1, 91.b2, 91.b3, and they all have rank 1. The space $J_{2,91}$ contains exactly three cuspidal generalized theta blocks (which are in fact theta blocks):

$$A = \frac{\vartheta^2\vartheta_2\vartheta_3\vartheta_4^2\vartheta_5^2\vartheta_6\vartheta_7}{\eta^6}, \quad B = \frac{\vartheta^2\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5\vartheta_7\vartheta_8}{\eta^6}, \quad C = \frac{\vartheta\vartheta_2^2\vartheta_3^2\vartheta_4^2\vartheta_5\vartheta_7^2}{\eta^6}.$$

One has $A + B = C$ (as follows for instance from the theta relations below). The Hecke eigenforms are

$$A + B = C, \quad A - B.$$

Hence one is a theta block, the other one is not.

14.5. Linear relations among theta blocks. When studying linear dependencies between sets of theta blocks one can restrict to sets whose elements have the same weight, same index and same character (since the ring of all Jacobi forms is graded by weight, index, character). Table 7 suggests many concrete examples of linear dependencies. For instance $J_{2,169}$ has dimension 8 but contains 50 theta blocks.

Using the following identity, which seems to be due to Weierstrass (see [Wei82, 1.]),

$$\begin{aligned}
&\vartheta(\tau, z_0 + z_1)\vartheta(\tau, z_0 - z_1)\vartheta(\tau, z_2 + z_3)\vartheta(\tau, z_2 - z_3)\\
&+\vartheta(\tau, z_0 + z_2)\vartheta(\tau, z_0 - z_2)\vartheta(\tau, z_3 + z_1)\vartheta(\tau, z_3 - z_1)\\
&+\vartheta(\tau, z_0 + z_3)\vartheta(\tau, z_0 - z_3)\vartheta(\tau, z_1 + z_2)\vartheta(\tau, z_1 - z_2) = 0.
\end{aligned}$$

one obtains immediately an infinite family of linear relations between theta blocks. Namely, using again $\vartheta_a(\tau, z) = \vartheta(\tau, az)$ and substituting

TABLE 8. For each m such that the subspace of cusp forms in $J_{2,m}$ is generated by a new form ϕ_m , the associated elliptic curve and a theta block representation of $\eta^6\,\phi_m$. (CL is the *Cremona label* of the respective elliptic curve.)

m	CL	Curve	Theta block
37	37a1	$y^2+y=x^3-x$	$\vartheta_1^3\vartheta_2^3\vartheta_3^2\vartheta_4\vartheta_5$
43	43a1	$y^2+y=x^3+x^2$	$\vartheta_1^3\vartheta_2^2\vartheta_3^2\vartheta_4^2\vartheta_5$
53	53a1	$y^2+xy+y=x^3-x^2$	$\vartheta_1^3\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5\vartheta_6$
57	57a1	$y^2+y=x^3-x^2-2x+2$	$\vartheta_1^2\vartheta_2^2\vartheta_3^3\vartheta_4\vartheta_5\vartheta_6$
58	58a1	$y^2+xy=x^3-x^2-x+1$	$\vartheta_1^2\vartheta_2^3\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6$
61	61a1	$y^2+xy=x^3-2x+1$	$\vartheta_1^2\vartheta_2^3\vartheta_3^2\vartheta_4\vartheta_5\vartheta_7$
65	65a1	$y^2+xy=x^3-x$	$\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5^2\vartheta_6$
77	77a1	$y^2+y=x^3+2x$	$\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5\vartheta_6\vartheta_7$
79	79a1	$y^2+xy+y=x^3+x^2-2x$	$\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5^2\vartheta_8$
82	82a1	$y^2+xy+y=x^3-2x$	$\vartheta_1\vartheta_2^3\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6\vartheta_7$
83	83a1	$y^2+xy+y=x^3+x^2+x$	$\vartheta_1^2\vartheta_2\vartheta_3^2\vartheta_4^2\vartheta_5\vartheta_6\vartheta_7$
88	88a1	$y^2=x^3-4x+4$	$\vartheta_1^2\vartheta_2^2\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6\vartheta_8$
89	89a1	$y^2+xy+y=x^3+x^2-x$	$\vartheta_1^3\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_6^2\vartheta_7$
92	92b1	$y^2=x^3-x+1$	$\vartheta_1\vartheta_2^2\vartheta_3^2\vartheta_4^2\vartheta_5\vartheta_6\vartheta_8$
99	99a1	$y^2+xy+y=x^3-x^2-2x$	$\vartheta_1^2\vartheta_2\vartheta_3^2\vartheta_4^2\vartheta_5\vartheta_6\vartheta_9$
101	101a1	$y^2+y=x^3+x^2-x-1$	$\vartheta_1^2\vartheta_2\vartheta_3\vartheta_4\vartheta_5^2\vartheta_6^2\vartheta_7$
102	102a1	$y^2+xy=x^3+x^2-2x$	$\vartheta_1\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5\vartheta_6^2\vartheta_8$
112	112a1	$y^2=x^3+x^2+4$	$\vartheta_1\vartheta_2^2\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6\vartheta_7\vartheta_8$
117	117a1	$y^2+xy+y=x^3-x^2+4x+6$	$\vartheta_1\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5\vartheta_6\vartheta_7\vartheta_9$
118	118a1	$y^2+xy=x^3+x^2+x+1$	$\vartheta_1^2\vartheta_2^2\vartheta_4\vartheta_5\vartheta_6^2\vartheta_7\vartheta_8$
121	121b1	$y^2+y=x^3-x^2-7x+10$	$\vartheta_1\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5^2\vartheta_7\vartheta_{10}$
124	124a1	$y^2=x^3+x^2-2x+1$	$\vartheta_1^2\vartheta_2\vartheta_4^2\vartheta_5\vartheta_6^2\vartheta_7\vartheta_8$
128	128a1	$y^2=x^3+x^2+x+1$	$\vartheta_1\vartheta_2^2\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6\vartheta_8\vartheta_9$
131	131a1	$y^2+y=x^3-x^2+x$	$\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5\vartheta_7\vartheta_{12}$
135	135a1	$y^2+y=x^3-3x+4$	$\vartheta_1\vartheta_2\vartheta_3^2\vartheta_4\vartheta_5^2\vartheta_6\vartheta_8\vartheta_9$
136	136a1	$y^2=x^3+x^2-4x$	$\vartheta_1\vartheta_2^2\vartheta_3\vartheta_4\vartheta_5\vartheta_6\vartheta_7\vartheta_8^2$
138	138a1	$y^2+xy=x^3+x^2-x+1$	$\vartheta_1\vartheta_2\vartheta_3^2\vartheta_4^2\vartheta_6^2\vartheta_7\vartheta_{10}$
143	143a1	$y^2+y=x^3-x^2-x-2$	$\vartheta_1^2\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_6\vartheta_7\vartheta_8\vartheta_9$
152	152a1	$y^2=x^3+x^2-x+3$	$\vartheta_1\vartheta_2\vartheta_3^2\vartheta_4^2\vartheta_6\vartheta_7\vartheta_8\vartheta_{10}$
156	156a1	$y^2=x^3-x^2-5x+6$	$\vartheta_1\vartheta_2^2\vartheta_3^2\vartheta_4\vartheta_5\vartheta_6\vartheta_8\vartheta_{12}$
160	160a1	$y^2=x^3+x^2-6x+4$	$\vartheta_1\vartheta_2\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6\vartheta_7\vartheta_8\vartheta_{10}$
162	162a1	$y^2+xy=x^3-x^2-6x+8$	$\vartheta_1\vartheta_2\vartheta_3\vartheta_4^2\vartheta_5\vartheta_6^2\vartheta_9\vartheta_{10}$
192	192a1	$y^2=x^3-x^2-4x-2$	$\vartheta_1\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_6^2\vartheta_7\vartheta_8\vartheta_{12}$
196	196a1	$y^2=x^3-x^2-2x+1$	$\vartheta_1\vartheta_2\vartheta_3\vartheta_4^2\vartheta_5\vartheta_7\vartheta_8^2\vartheta_{12}$
200	200b1	$y^2=x^3+x^2-3x-2$	$\vartheta_1\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_6\vartheta_8^2\vartheta_9\vartheta_{10}$
210	210d1	$y^2+xy=x^3+x^2-3x-3$	$\vartheta_1\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_6^2\vartheta_7\vartheta_{10}\vartheta_{12}$
216	216a1	$y^2=x^3-12x+20$	$\vartheta_2^2\vartheta_3\vartheta_4\vartheta_5\vartheta_6\vartheta_7\vartheta_8\vartheta_9\vartheta_{12}$
220	220a1	$y^2=x^3+x^2-45x+100$	$\vartheta_2^2\vartheta_3\vartheta_4\vartheta_5^2\vartheta_7\vartheta_8\vartheta_{10}\vartheta_{12}$
240	240c1	$y^2=x^3-x^2+4x$	$\vartheta_1\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_6\vartheta_8\vartheta_9\vartheta_{10}\vartheta_{12}$
252	252b1	$y^2=x^3-12x+65$	$\vartheta_1\vartheta_2\vartheta_3\vartheta_4\vartheta_6\vartheta_7\vartheta_8\vartheta_9\vartheta_{10}\vartheta_{12}$
300	300d1	$y^2=x^3-x^2-13x+22$	?
360	360e1	$y^2=x^3-18x-27$	$\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_6\vartheta_7\vartheta_9\vartheta_{10}\vartheta_{12}\vartheta_{16}$

$(z_0 + z_1, z_0 - z_1, z_2 + z_3, z_2 - z_3) = (a, b, c, d)z$ yields the relations

$$\vartheta_a\vartheta_b\vartheta_c\vartheta_d + \vartheta_{(a+b+c-d)/2}\vartheta_{(a+b-c+d)/2}\vartheta_{(a-b+c+d)/2}\vartheta_{(a-b-c-d)/2} = \vartheta_{(a+b+c+d)/2}\vartheta_{(a+b-c-d)/2}\vartheta_{(a-b+c-d)/2}\vartheta_{(a-b-c+d)/2}.$$

Here a, b, c, d denotes any quadruple of integers whose sum is even. For instance, for $a, b, c, d = 1, 4, 5, 6$ we obtain $\vartheta\vartheta_4\vartheta_5\vartheta_6 + \vartheta_2\vartheta_3\vartheta_4\vartheta_{-7} = \vartheta_8\vartheta_{-3}\vartheta_{-2}\vartheta_{-1}$, which after multiplication by $\vartheta\vartheta_2\vartheta_3\vartheta_4\vartheta_5\vartheta_7/\eta^6$ yields the identity $A + B = C$ of the preceding section.

There is also a five term relation similar to Weierstrass' three term relation, whose terms are also products of four ϑ, and which is due to Jacobi (see [Jac81, p. 507, formula (A)]). *It is an interesting question if one can develop a theory based on such relations for theta functions in several variables which explains all linear relations among theta blocks.*

References

[BK96] A. S. Belov and S. V. Konyagin. On an estimate for the free term of a nonnegative trigonometric polynomial with integer coefficients. *Izv. Ross. Akad. Nauk Ser. Mat.*, 60(6):31–90, 1996. 15

[Bor95] Richard E. Borcherds. Automorphic forms on $\mathrm{O}_{s+2,2}(\mathbf{R})$ and infinite products. *Invent. Math.*, 120(1):161–213, 1995. 45

[Bor98] Richard E. Borcherds. Automorphic forms with singularities on Grassmannians. *Invent. Math.*, 132(3):491–562, 1998. 45

[BS13a] Hatice Boylan and Nils-Peter Skoruppa. Jacobi forms of lattice index. Part I: Basic theory, shadow representations and maximal lattices. preprint, 2013. 30, 31, 33, 35, 43, 44

[BS13b] Hatice Boylan and Nils-Peter Skoruppa. Jacobi forms of lattice index. Part II: Singular and critical weight. preprint, 2013. 42, 45

[CG13] F. Cléry and V. Gritsenko. Modular forms of orthogonal type and Jacobi theta-series. *Abh. Math. Semin. Univ. Hambg.*, 83(2):187–217, 2013. 30, 50

[EZ85] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985. 3, 4, 5, 24, 46, 51, 52, 54

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics. 37

[GH98] V. Gritsenko and K. Hulek. Commutator coverings of Siegel threefolds. *Duke Math. J.*, 94(3):509–542, 1998. 51

[GN98] Valeri A. Gritsenko and Viacheslav V. Nikulin. Automorphic forms and Lorentzian Kac-Moody algebras. II. *Internat. J. Math.*, 9(2):201–275, 1998. 45, 46, 47, 49

[GPY] V. Gritsenko, C. Poor, and D. Yuen. Antisymmetric paramodular forms of weights 2 and 3. ArXiv:1609.04146, 20 pages. 52

[GPY15] Valery Gritsenko, Cris Poor, and David S. Yuen. Borcherds products everywhere. *J. Number Theory*, 148:164–195, 2015. 48, 49

[Gri94a] V. Gritsenko. Modular forms and moduli spaces of abelian and $K3$ surfaces. *Algebra i Analiz*, 6(6):65–102, 1994. 30

[Gri94b] Valeri Gritsenko. Irrationality of the moduli spaces of polarized abelian surfaces. *Internat. Math. Res. Notices*, (6):235 ff., approx. 9 pp. 1994. 46, 50, 52

[Gri18] Valery Gritsenko. Reflective modular forms and applications. *Russian Math. Surveys 73:5*, pages 797–864, 2018. 49

[GW18] Valery Gritsenko and Haowu Wang. Powers of Jacobi triple product, Cohen's numbers and the Ramanujan Δ-function. *Eur. J. Math.*, 4(2):561–584, 2018. 51

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised. 36, 39, 41, 42

[Jac81] C. G. J. Jacobi. Theorie der elliptischen Functionen aus den Eigenschaften der Thetareihen abgeleitet. In C. W. Borchardt, editor, *Gesammelte Werke*, volume Erster Band, pages 497–538. Reimer, G., Berlin, 1881. 58

[Klo46] H. D. Kloosterman. The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups. I. *Ann. of Math. (2)*, 47:317–375, 1946. 43

[Kol94] Mihail N. Kolountzakis. On nonnegative cosine polynomials with nonnegative integral coefficients. *Proc. Amer. Math. Soc.*, 120(1):157–163, 1994. 15

[KP84] Victor G. Kac and Dale H. Peterson. Infinite-dimensional Lie algebras, theta functions and modular forms. *Adv. in Math.*, 53(2):125–264, 1984. 23

[Mac72] I. G. Macdonald. Affine root systems and Dedekind's η-function. *Invent. Math.*, 15:91–143, 1972. 23, 40

[MPS81] O. Carruth McGehee, Louis Pigno, and Brent Smith. Hardy's inequality and the L^1 norm of exponential sums. *Ann. of Math. (2)*, 113(3):613–618, 1981. 16

[Odl82] A. M. Odlyzko. Minima of cosine sums and maxima of polynomials on the unit circle. *J. London Math. Soc. (2)*, 26(3):412–420, 1982. 15

[Sko85] Nils-Peter Skoruppa. *Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 159. Universität Bonn Mathematisches Institut, Bonn, 1985. Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1984. 42, 44, 45

[Sko08] Nils-Peter Skoruppa. Jacobi forms of critical weight and Weil representations. In *Modular forms on Schiermonnikoog*, pages 239–266. Cambridge Univ. Press, Cambridge, 2008. 10, 31, 42, 45

[SZ88] Nils-Peter Skoruppa and Don Zagier. Jacobi forms and a certain space of modular forms. *Invent. Math.*, 94(1):113–146, 1988. 37, 53

[Wei82] K. Weierstrass. Zur Theorie der Jacobischen Funktionen von mehreren Veränderlichen. *Sitzungsber. Königl. Preuss. Akad. Wiss.*, pages 505–508, 1882. Werke, Band 3, pp. 155–159. 56

Université de Lille, Villeneuve d'Ascq, France and National Research University "Higher School of Economics", Moscow, Russian Federation

Email address: Valery.Gritsenko@math.univ-lille1.fr

Department Mathematik, Universität Siegen, 57068 Siegen, Germany

Email address: nils.skoruppa@gmail.com

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

Email address: dbz@mpim-bonn.mpg.de