

# STRICHARTZ ESTIMATES WITH BROKEN SYMMETRIES

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ABSTRACT. In this note we study the eigenvalue problem for a quadratic form associated with Strichartz estimates for the Schrödinger equation, proving in particular a sharp Strichartz inequality for the case of odd initial data. We also describe an alternative method that is applicable to a wider class of matrix problems.

## 1. INTRODUCTION

In this paper we study the eigenvalue problem associated with the quadratic form <sup>1</sup>

$$Q(f) = \sqrt{12} \int_{\mathbb{R}} \int_{\mathbb{R}} |u_f((y, y, y), t)|^2 dy dt \quad (1)$$

restricted to the subspace of functions  $f \in L^2(\mathbb{R}^3)$  with some prescribed parity. Here  $u_f : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}$  denotes the solution of the Schrödinger equation

$$\partial_t u(x, t) = i\Delta u(x, t), \quad u(x, 0) = f(x) \quad (2)$$

with initial data  $f$  and we say that  $f(x_1, x_2, x_3)$  has parity  $\varepsilon \in \{\pm 1\}^3$  if it is even with respect to the  $x_i$  with  $\varepsilon_i = 1$  and odd with respect to the  $x_i$  with  $\varepsilon_i = -1$ . Enforcing such parity constraints on the initial data breaks the fundamental symmetries associated with the Schrödinger equation (for instance Galilean invariance). The study of such quadratic form is motivated by its intrinsic relation with Strichartz estimates, as it was used in [2, 8] to produce sharp bounds for the space-time  $L^6$ -norm of  $u_f$  in one space dimension. For a more general overview of how results of this sort are used we refer to the recent survey [4].

Since the form  $Q$  is invariant under permutation of the coordinates, we can restrict to the case where  $\varepsilon_i$  is  $-1$  for  $i \leq \kappa$  and  $1$  for  $i > \kappa$  for some  $\kappa \in \{0, 1, 2, 3\}$ , i.e., to the subspace  $L^2_{\kappa}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$  of functions that are odd with respect to the first  $\kappa$  variables and even with respect to the others. It turns out that the eigenvalues associated to the restriction of  $Q$  to each  $L^2_{\kappa}(\mathbb{R}^3)$  are all *rational* and can be given explicitly as the coefficients of certain generating functions (which was amusing and unexpected for the authors). Specifically, define four power series  $G_{\kappa}(w) \in \mathbb{Q}[[w]]$  by

$$\begin{pmatrix} G_0(w) \\ G_1(w) \\ G_2(w) \\ G_3(w) \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ w & 1 & -1 \\ 1 & -1 & -1 \\ w & -3 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{4(1-w^2)} \\ \frac{1}{8\sqrt{1-\frac{2}{3}w+w^2}} \\ \frac{1}{8\sqrt{1+\frac{2}{3}w+w^2}} \end{pmatrix} = \begin{pmatrix} 1 + \frac{7}{27}w^4 + \frac{32}{81}w^6 + \dots \\ \frac{1}{3}w + \frac{4}{27}w^3 + \frac{1}{3}w^5 + \dots \\ \frac{1}{3}w^2 + \frac{20}{81}w^4 + \frac{49}{243}w^6 + \dots \\ \frac{5}{9}w^3 + \frac{91}{243}w^7 + \frac{1760}{6561}w^9 + \dots \end{pmatrix} \quad (3)$$

and denote by  $\Lambda_{\kappa} \subset \mathbb{Z}[\frac{1}{3}]$  the multiset of coefficients of  $G_{\kappa}(w)$ , counted with multiplicity. Since  $[w^n](G_{\kappa}) = \frac{1}{4} + O(n^{-1/2})$  for  $n \equiv \kappa \pmod{2}$  (see Figure 1) and vanishes otherwise,

<sup>1</sup>The  $\sqrt{12}$  normalization factor has been included here for aesthetic reasons only.

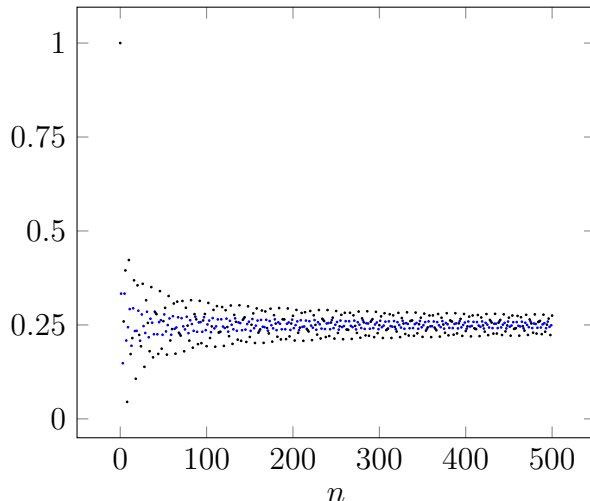


FIGURE 1. First 501 nonzero power series coefficients of  $G_0$  (black) and  $G_1$  (blue).

the only non-zero limit point of  $\Lambda_\kappa$  is  $\frac{1}{4}$  and this is also the only value that can have infinite multiplicity.<sup>2</sup> Our main result is the following.

**Theorem 1.** *For each  $\kappa = 0, 1, 2, 3$  there is an orthogonal decomposition*

$$L_\kappa^2(\mathbb{R}^3) = \bigoplus_{\lambda \in \Lambda_\kappa} V_\kappa(\lambda),$$

that diagonalizes  $Q$  such that  $Q(f) = \lambda \|f\|^2$  for  $f \in V_\kappa(\lambda)$ . In particular, the largest eigenvalue  $\alpha_\kappa = \max(\Lambda_\kappa)$  of  $Q$  on  $L_\kappa^2$  is given explicitly by  $\alpha_0 = 1$ ,  $\alpha_1 = \alpha_2 = \frac{1}{3}$ , and  $\alpha_3 = \frac{5}{9}$ . The eigenspaces  $V_\kappa(\alpha_\kappa)$  corresponding to these largest eigenvalues are given explicitly by

$$\begin{aligned} V_0(0) &= \{f \in L^2(\mathbb{R}^3) : f(x) = g(|x|)\}, \\ V_1(\frac{1}{3}) &= \{f \in L^2(\mathbb{R}^3) : f(x) = x_1 g(|x|) + x_1 P(x_1, x_2, x_3) h(|x|)\}, \\ V_2(\frac{1}{3}) &= \{f \in L^2(\mathbb{R}^3) : f(x) = x_1 x_2 g(|x|)\}, \\ V_3(\frac{5}{9}) &= \{f \in L^2(\mathbb{R}^3) : f(x) = x_1 x_2 x_3 g(|x|)\}, \end{aligned}$$

where  $P(x_1, x_2, x_3) = x_1^4 - 5(x_2^2 + x_3^2)x_1^2 + 15(x_2^4 - 5x_3^2x_2^2 + x_3^4)$ .

The proof will actually identify the set of eigenvalues of  $Q$  on  $L_\kappa^2$  with  $\Lambda_\kappa$  as a multiset, in the sense that each  $V_\kappa(\lambda)$  is a module over the set  $V_0(0)$  of radial  $L^2$ -functions of rank equal to the multiplicity of  $\lambda$  in  $\Lambda_\kappa$  (and hence finite for  $\lambda$  different from 0 and possibly  $\frac{1}{4}$ ). The fact that  $V_1(\frac{1}{3})$  looks “larger” than the other three  $V_\kappa(\alpha_\kappa)$  is then simply a reflection of the fact that  $\frac{1}{3}$  appears twice as a coefficient in the power series  $G_1(w)$ .

Theorem 1 gives not only the largest eigenvalue, but the whole spectrum for each  $\kappa$ . In particular, one can check by a numerical computation (Lemma 5 below) that the second largest eigenvalue  $\beta_\kappa$  of  $Q$  on  $L_\kappa^2$  is given by

$$(\beta_0, \beta_1, \beta_2, \beta_3) = \left( \frac{8320}{3^9}, \frac{469136}{3^{13}}, \frac{232}{3^6}, \frac{221312}{3^{12}} \right).$$

From Theorem 1 we then get the following sharp trilinear inequality.

<sup>2</sup>This could happen only if  $\frac{1}{4}$  occurred infinitely often as a Taylor coefficient of  $G_\kappa$ . In fact, numerical computations up to  $n = 10^5$  suggest that it never occurs at all, but we could not prove this.

**Corollary 1.** *Let  $g_1, g_2, g_3 \in L^2(\mathbb{R})$  be a collection of functions with the first  $\kappa$  being odd and the others even. Then*

$$\begin{aligned} & \sqrt{12} \int_{\mathbb{R}} \int_{\mathbb{R}} |u_{g_1}(y, t) u_{g_2}(y, t) u_{g_3}(y, t)|^2 dy dt \\ & \leq \alpha_{\kappa} \prod_{j=1}^3 \int_{\mathbb{R}} |g_j(x)|^2 dx - \beta_{\kappa} \text{dist}(g_1 \otimes g_2 \otimes g_3, V_{\kappa}(\alpha_{\kappa}))^2 \end{aligned}$$

with  $\alpha_{\kappa}$ ,  $\beta_{\kappa}$  and  $V_{\kappa}(\alpha_{\kappa})$  as above. Equality holds if and only if  $g_j(x) = b_j x e^{-ax^2}$  for  $j \leq \kappa$  and  $g_j(x) = b_j e^{-ax^2}$  for  $j > \kappa$  for some  $a, b_j \in \mathbb{C}$  with  $\text{Re } a > 0$ .

Here we have used the fact that the function  $g_1 \otimes g_2 \otimes g_3$  belongs to  $V_{\kappa}(\alpha_{\kappa})$  if and only if each  $g_j(x)$  or  $g_j(x)/x$ , depending whether  $g_j$  is even or odd, is a multiple of the same Gaussian for  $j = 1, 2, 3$ . (We leave the details to the reader).

By setting  $g_1 = g_2 = g_3$  in Corollary 1 for an odd function  $g_1$  (and removing the distance term) we obtain the following sharp inequality for odd initial data.

**Corollary 2.** *Let  $f \in L^2(\mathbb{R})$  be an odd function and let  $u_f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$  solve the Schrödinger equation (2) with initial data  $f$ . Then*

$$\|u_f\|_{L^6(\mathbb{R} \times \mathbb{R})} \leq 12^{-1/12} (5/9)^{1/6} \|f\|_{L^2(\mathbb{R})}, \quad (4)$$

with inequality if and only if  $f(x) = b x e^{-ax^2}$  with  $a, b \in \mathbb{C}$  and  $\text{Re } a > 0$ .

Foschi [3] proved inequality (4) with no parity condition on  $g$  and optimal constant  $12^{-\frac{1}{12}}$ , which he showed is attained by Gaussian initial data  $b e^{-ax^2}$ . Our result is a sharpening of his inequality when the initial data is odd.

## 2. PROOFS

In this section we collect the necessary facts to prove our main result. Since in most part our techniques are general, we will work with a higher dimensional analogue of  $Q$  and later on specialize to dimension 3.

For a given  $f \in L^2(\mathbb{R}^k)$  we denote by  $u_f(x, t)$  the solution of the Schrödinger equation (2) with initial data  $f$ . The Strichartz inequality (see [9, Theorem 2.3]) states that there exists  $A > 0$  such that

$$\|u_f\|_{L^p(\mathbb{R}^k \times \mathbb{R})} \leq A \|f\|_{L^2(\mathbb{R}^k)}, \quad (5)$$

where  $p = 2 + 4/k$ . It is conjectured that Gaussians are the only maximizers of this inequality, that is, letting  $A$  be the best possible constant, the inequality is attained if and only if  $f(x) = c \exp(-a|x|^2 + bx)$ , where  $c, a \in \mathbb{C}$ ,  $b \in \mathbb{C}^k$  and  $\text{Re } a > 0$ . In lower dimensions  $k = 1, 2$  this conjecture was solved by Foschi [3] and subsequent alternative proofs were given in [1, 2, 5, 6, 8]. The results of [2] are of special interest to this paper. By embedding the problem into a higher dimensional version and using the delta calculus tool, Carneiro [2, Lemma 6] proved the following useful representation<sup>3</sup>.

<sup>3</sup>Generalizing the work of [8].

**Proposition 1.** *If  $u_f : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$  solves Schrödinger's equation with initial data  $f \in L^2(\mathbb{R}^d)$  then*

$$c_d \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u_f((y, \dots, y), t)|^2 dy dt = \int_{\mathbb{R}^d} |P_E(w_d \widehat{f})(x)|^2 dx, \quad (6)$$

where  $E \subset L^2(\mathbb{R}^d)$  is the subspace of functions invariant under rotations fixing the direction  $\mathbf{1} = \frac{1}{\sqrt{d}}(1, \dots, 1)$ ,  $P_E$  is the orthogonal projection onto  $E$ ,

$$w_d(x) = (|x|^2 - (x \cdot \mathbf{1})^2)^{\frac{d-3}{4}},$$

$\widehat{f}$  is the Fourier transform of  $f$  defined by

$$\widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} dx,$$

and

$$c_d = \sqrt{d} 2^{d-2} \pi^{(d-3)/2} \Gamma((d-1)/2).$$

Notice when  $d = 3$  and  $f(x) = g(x_1)g(x_2)g(x_3)$  for some function  $g \in L^2(\mathbb{R})$  we have  $u_f((y, y, y), t) = u_g(y, t)^3$ ,  $w = 1$  and  $c_d = \sqrt{12}$ , and thus we obtain

$$\sqrt{12} \int_{\mathbb{R}} \int_{\mathbb{R}^3} |u_g(y, t)|^6 dy dt = \int_{\mathbb{R}^3} |P_E(\widehat{f})(x)|^2 dx \leq \int_{\mathbb{R}^3} |\widehat{f}(x)|^2 dx = \left( \int_{\mathbb{R}^3} |g(x)|^2 dx \right)^3.$$

This is the sharp Strichartz inequality (5) for  $k = 1$  proven by Foschi. Note that equality is attained if and only if  $\widehat{f} \in E$ , which in turn is equivalent to  $g$  being a Gaussian<sup>4</sup>.

Let  $\mathcal{H}_n$  denote the space of spherical harmonics of degree  $n$  on the  $(d-1)$ -dimensional unit sphere  $\mathbb{S}^{d-1}$  (i.e., the restrictions to  $\mathbb{S}^{d-1}$  of degree  $n$  homogeneous polynomials annihilated by  $\Delta$ ). Let also

$$\mathcal{H}_n \otimes \text{Radial} = \{f \in L^2(\mathbb{R}^d) : f(x) = Y_n(x/|x|) g(|x|) \text{ and } Y_n \in \mathcal{H}_n\}.$$

It is well known that

$$L^2(\mathbb{R}^d) = \bigoplus_{n \geq 0} \mathcal{H}_n \otimes \text{Radial},$$

where the sum is orthogonal, that is, any function  $f \in L^2(\mathbb{R}^d)$  can be expanded in the form

$$f(x) = \sum_{n \geq 0} Y_n(x/|x|) g_n(|x|) \quad (Y_n \in \mathcal{H}_n). \quad (7)$$

**Lemma 1.** *We have*

$$P_E f(x) = \sum_{n \geq 0} \frac{Y_n(\mathbf{1})}{C_n^{d/2-1}(1)} C_n^{d/2-1} \left( \frac{x}{|x|} \cdot \mathbf{1} \right) g_n(|x|), \quad (8)$$

if  $f \in L^2(\mathbb{R}^d)$  is expanded as in (7), where  $C_n^{d/2-1}(z)$  is the Gegenbauer polynomial.

*Proof.* Firstly, it is not hard to realize that

$$P_E f(x) = \int_G f(\rho x) dG(\rho)$$

<sup>4</sup>This is the proof method of [8] in a nutshell, which was later generalized by Carneiro [2]. Foschi's original proof is a bit more complicated.

where  $G$  is the subgroup of  $SO(d)$  that fixes the vector  $\mathbf{1}$  and  $dG$  its induced  $G$ -invariant Haar probability measure. Secondly, noting that the function

$$\omega \in \mathbb{S}^{d-1} \mapsto \int_G Y_n(\rho\omega) dG(\rho)$$

still is an spherical harmonic of degree  $n$  that depends only on  $\omega \cdot \mathbf{1}$ , and must therefore be a multiple of the zonal harmonic  $C_n^{d/2-1}(\omega \cdot \mathbf{1})$ . We conclude that

$$\int_G Y_n(\rho\omega) dG(\rho) = \frac{Y_n(\mathbf{1})}{C_n^{d/2-1}(1)} C_n^{d/2-1}(\omega \cdot \mathbf{1}) \quad (\omega \in \mathbb{S}^{d-1}),$$

which proves the lemma. ■

In what follows we let

$$L_\kappa^2(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : f \text{ is odd in } x_1, x_2, \dots, x_\kappa \text{ and even in } x_{\kappa+1}, \dots, x_d\} \quad (9)$$

for  $\kappa = 0, 1, \dots, d$ . We also let  $P_\kappa$  denote the orthogonal projection onto  $L_\kappa^2(\mathbb{R}^d)$  and set  $T_\kappa = P_\kappa P_E P_\kappa$ . Abusing notation, we use the same symbol  $T_\kappa$  to denote the restriction of  $T_\kappa$  to  $\mathcal{H}_n$ ; this is justified since  $T_\kappa = \tilde{T}_\kappa \otimes \text{Id}$  on  $\mathcal{H}_n \otimes \text{Radial}$ , and we identify  $\tilde{T}_\kappa$  with  $T_\kappa$ . Using the fact that

$$P_\kappa(Y)(\omega) = \frac{1}{2^d} \sum_{\varepsilon \in \{\pm 1\}^d} \varepsilon_1 \cdots \varepsilon_\kappa Y(\varepsilon_1 \omega_1 + \cdots + \varepsilon_d \omega_d),$$

we can apply (8) to obtain the following lemma.

**Lemma 2.** *We have*

$$T_\kappa(Y_n)(\omega) = \frac{Y_n(\mathbf{1})}{C_n^{d/2-1}(1)} P_{d,\kappa,n}(\omega)$$

for every  $Y_n \in \mathcal{H}_n$ , where, for all  $n \geq 0$ , we set

$$P_{d,\kappa,n}(\omega) = \frac{1}{2^d} \sum_{\varepsilon \in \{\pm 1\}^d} \varepsilon_1 \cdots \varepsilon_\kappa C_n^{d/2-1}\left(\frac{\omega \cdot \varepsilon}{\sqrt{d}}\right) \quad (\omega \in \mathbb{S}^{d-1}).$$

In particular,  $T_\kappa$  leaves each subspace  $\mathcal{H}_n$  invariant and  $\text{rank}(T_\kappa|_{\mathcal{H}_n}) \leq 1$ , with equality if and only if the eigenvalue

$$\lambda_{d,\kappa,n} = \frac{1}{2^d} \sum_{\varepsilon \in \{\pm 1\}^d} \varepsilon_1 \cdots \varepsilon_\kappa C_n^{d/2-1}\left(\frac{\varepsilon_1 + \cdots + \varepsilon_d}{d}\right)$$

is nonzero, in which case  $P_{d,\kappa,n}$  is the unique associated eigenvector of  $T_\kappa$  in  $\mathcal{H}_n$ .

The next lemma gives part of the structure of the eigenvalues  $\lambda_{d,\kappa,n}$ .

**Lemma 3.** *We have  $\lambda_{d,\kappa,n} = 0$  if  $0 \leq n < \kappa$  or if  $n - \kappa$  is odd. Moreover,*

$$\lambda_{d,\kappa,\kappa} = \frac{2^\kappa \kappa! (d/2 - 1)_\kappa}{d^\kappa (d - 2)_\kappa}.$$

*Proof.* Since  $\sum_{\varepsilon \in \{\pm 1\}^d} \varepsilon_1 \cdots \varepsilon_\kappa$  vanishes for  $1 \leq \kappa \leq d$ , it follows that

$$\frac{1}{2^d} \sum_{\varepsilon \in \{\pm 1\}^d} \varepsilon_1 \cdots \varepsilon_\kappa (\varepsilon_1 + \cdots + \varepsilon_d)^m = \frac{1}{2^d} \sum_{\substack{m_1 + \cdots + m_d = m \\ m_1, \dots, m_d \geq 0 \\ m_1, \dots, m_\kappa \text{ odd} \\ m_{\kappa+1}, \dots, m_d \text{ even}}} \frac{m!}{m_1! \cdots m_d!}, \quad (10)$$

which implies that (10) vanishes if  $0 \leq m < \kappa$  or if  $m - \kappa$  is odd. Since  $C_n^{d/2-1}(z)$  is an even or odd polynomial depending whether  $n$  is even or odd, we conclude that  $\lambda_{d,\kappa,n} = 0$  if  $0 \leq n < \kappa$  or if  $n - \kappa$  is odd. Finally, if  $m = n = \kappa$  then (10) equals  $\kappa!$ , so we obtain

$$\lambda_{d,\kappa,\kappa} = \frac{\kappa! \times \text{leading coefficient of } C_\kappa^{d/2-1}(z)}{C_\kappa^{d/2-1}(1)} = \frac{2^\kappa \kappa! (d/2 - 1)_\kappa}{d^\kappa (d - 2)_\kappa}.$$

■

We believe that  $\lambda_{d,\kappa,\kappa} := \max\{\lambda_{d,\kappa,n}\}_{n \geq 0}$  for all  $d \geq 3$  and  $\kappa = 0, 1, 2, 3$ , but we could not prove that.

We now restrict our attention to the 3-dimensional case.

**Lemma 4.** *For  $0 \leq \kappa \leq 3$  the series  $G_\kappa(w) = \sum_{n \geq 0} \lambda_{3,\kappa,n} w^n$  is given by (3).*

*Proof.* The generating function for the Gegenbauer polynomials<sup>5</sup> is

$$(1 - 2zw + w^2)^{1-d/2} = \sum_{n \geq 0} w^n C_n^{d/2-1}(z).$$

Hence

$$\begin{aligned} G_\kappa(w) &= \frac{1}{8} \sum_{\varepsilon \in \{\pm 1\}^3} \varepsilon_1 \cdots \varepsilon_\kappa \sum_{n \geq 0} w^n C_n^{3/2-1} \left( \frac{\omega \cdot \varepsilon}{\sqrt{d}} \right) \\ &= \frac{1}{8} \sum_{\varepsilon \in \{\pm 1\}^3} \frac{\varepsilon_1 \cdots \varepsilon_\kappa}{\sqrt{1 - \frac{2}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)w + w^2}}, \end{aligned}$$

which is easily seen to be equivalent with (3). ■

**Lemma 5.** *Let  $\alpha_\kappa$  and  $\beta_\kappa$  be the two largest eigenvalues in  $\Lambda_\kappa = \bigcup_{n \geq 0} \{\lambda_{3,\kappa,n}\}$ . Then*

$$\begin{aligned} \alpha_0 = \lambda_{3,0,0} = 1 &> \beta_0 = \lambda_{3,0,10} = \frac{8320}{19683} = 0.4226997 \cdots, \\ \alpha_1 = \lambda_{3,1,1} = \lambda_{3,1,5} = \frac{1}{3} &> \beta_1 = \lambda_{3,1,15} = \frac{469136}{1594323} = 0.2942540 \cdots, \\ \alpha_2 = \lambda_{3,2,2} = \frac{1}{3} &> \beta_2 = \lambda_{3,2,8} = \frac{232}{729} = 0.3182441 \cdots, \\ \alpha_3 = \lambda_{3,3,3} = \frac{5}{9} &> \beta_3 = \lambda_{3,3,13} = \frac{221312}{531441} = 0.4164375 \cdots. \end{aligned}$$

*Proof.* One can check that the coefficient of  $w^n$  in  $(1 - \frac{2}{3}w + w^2)^{-1/2}$  is bounded in absolute value by  $(3/\pi n \sqrt{2})^{1/2} = 0.82172 \cdots n^{-1/2}$ , and from this it follows that  $\lambda_{3,\kappa,n} < \beta_\kappa$  for all  $n > 1000$  with  $\beta_\kappa$  as in the lemma. Then by explicitly computing the first 1000 eigenvalues  $\lambda_{3,\kappa,n}$  we can find the first and second biggest eigenvalues and their multiplicities exactly. ■

**Proof of Theorem 1.** Noting that if  $f \in L_\kappa^3(\mathbb{R}^3)$  then  $\widehat{f} \in L_\kappa^2(\mathbb{R}^3)$ , we then use (6) to obtain

$$Q(f) = \int_{\mathbb{R}^3} |P_E(\widehat{f})(x)|^2 dx = \int_{\mathbb{R}^3} T_\kappa(\widehat{f})(x) \overline{\widehat{f}(x)} dx.$$

We can now apply Lemmas 1, 2 and 4 and Plancherel's theorem to obtain the desired orthogonal decomposition. To finish, Lemmas 5 and 2 tell us what the largest eigenvalues are and how to compute their associated eigenspaces, and direct computation then gives

$$P_{3,0,0}(x) = 1, \quad P_{3,1,1}(x) = \frac{1}{\sqrt{3}} x_1,$$

<sup>5</sup>Note that if  $d = 3$  then  $C_n^{1/2}(z)$  is the Legendre polynomial and  $C_n^{1/2}(1) = 1$

$$\begin{aligned} P_{3,1,5}(x) &= \frac{1}{6\sqrt{3}}x_1 \left( -x_1^4 + (5x_2^2 + 5x_3^2)x_1^2 + (-15x_2^4 + 75x_3^2x_2^2 - 15x_3^4) \right), \\ P_{3,2,2}(x) &= x_1x_2, \quad P_{3,3,3}(x) = \frac{5}{\sqrt{3}}x_3x_2x_1. \end{aligned}$$

This finishes the proof. ■

### 3. A MORE GENERAL EIGENVALUE PROBLEM

There is another approach to our problem, though rather more technical than the one presented above, by using Laguerre polynomials as the basis of  $L^2_\kappa(\mathbb{R}^3)$  (as in [6]) to reduce Theorem 1 to a statement about a specific sequence of matrices of increasing size, with entries defined by certain power series and with the property that the eigenvalues of the larger matrices contain the eigenvalues of the smaller ones. In this section we give a more general matrix problem and its rather simple solution, since this can potentially be used for other problems. The application to our original problem is described at the end.

**General problem.** To any power series  $F(w_1, w_2, w_3) \in \mathbb{C}[[w_1, w_2, w_3]]$  we associate an endomorphism  $\Phi_F$  of the vector space  $V = \mathbb{C}[[u, v]]$ , defined on monomials by

$$\Phi_F \left[ \binom{S}{a} u^a v^{S-a} \right] = \sum_{b=0}^S \binom{S}{b} R_S(a, b) u^b v^{S-b} \quad (0 \leq a \leq S), \quad (11)$$

where

$$R_S(a, b) = [w_1^a w_2^{S-a} w_3^b](F) \quad (0 \leq a, b \leq S). \quad (12)$$

Thus  $\Phi_F$  preserves the space  $V_S$  of homogeneous polynomials of degree  $S$  for every integer  $S \geq 0$  and  $R_S = (R_S(a, b))_{0 \leq a, b \leq S}$  is the matrix representation of the restriction of  $\Phi_F$  to  $V_S$  with respect to the basis  $\left\{ \binom{S}{a} u^a v^{S-a} \right\}_{0 \leq a \leq S}$ . We will call  $F$  *special* if the map  $\Phi_F$  commutes with multiplication by  $(u + v)$ . If this holds, then by considering the action of  $\Phi_F$  with respect to the basis  $\left\{ u^i (u + v)^{n-i} \right\}_{0 \leq i \leq n}$  of  $V_S$  we see that the matrix  $R_S$  has the eigenvalues  $(e_0, e_1, \dots, e_S)$ , where  $e_n$  is independent of  $S$  and is given by

$$e_n = \Phi_F[u^n](1, -1) = \sum_{b=0}^n (-1)^{n-b} \binom{n}{b} R_n(n, b) = [w_1^n w_3^n]((1 - w_3)^n F(w_1, 0, w_3)). \quad (13)$$

We remark that any endomorphism  $\Phi$  of  $\mathbb{C}[[u, v]]$  that commutes with the Euler operator  $u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}$  preserves each  $V_S$  and hence gives rise via (11) to a sequence of  $(S + 1) \times (S + 1)$  matrices  $R_S$  and via (12) to a power series  $F(w_1, w_2, w_3) = \sum_{a, b, c \geq 0} R_{a+c}(a, b) w_1^a w_2^b w_3^c$  with  $\Phi_F = \Phi$ . (For example, if  $\Phi$  is the identity then  $F$  is  $(1 - w_2)^{-1} (1 - w_1 w_3)^{-1}$ .) Equation (13) gives a very simple formula for the eigenvalues of the matrix  $R_S$  if the power series  $F$  is special. The next proposition tells us how to recognize when this is the case.

**Proposition 2.** *Let  $F \in \mathbb{C}[[w_1, w_2, w_3]]$ . Then the following are equivalent.*

- (i)  $F$  is special, i.e.,  $\Phi_F$  commutes with multiplication by  $(u + v)$ .
- (ii)  $F$  satisfies the first order linear differential equation

$$(1 - w_1) \frac{\partial F}{\partial w_1} + (1 - w_2) \frac{\partial F}{\partial w_2} + w_3(1 - w_3) \frac{\partial F}{\partial w_3} = (1 + w_3) F. \quad (14)$$

- (iii)  $F$  has the form

$$F(w_1, w_2, w_3) = \frac{1}{(1 - w_2)(1 - w_3)} H\left(\frac{w_1 - w_2}{1 - w_2}, w_3 \frac{1 - w_2}{1 - w_3}\right) \quad (15)$$

for some power series  $H(x, y) \in \mathbb{C}[[x, y]]$ .

If these conditions hold, then the eigenvalues of  $\Phi_F$  are given by

$$e_n = [x^n y^n](H) \quad (n \geq 0).$$

*Example:* The  $H$  corresponding to  $\Phi = \text{Id}$  is  $(1 + y - xy)^{-1}$ , with  $e_n = 1$  for all  $n$ .

*Proof.* For  $a, b, c \geq 0$  set  $R(a, b, c) = [w_1^a w_2^b w_3^c](F) = R_{a+b}(a, c)$ . Then both conditions (i) and (ii) translate to the *same* recursion

$$(a+1)R(a+1, b, c) + (b+1)R(a, b+1, c) = cR(a, b, c-1) + (a+b-c+1)R(a, b, c),$$

as one checks by an easy direct calculation. This proves the equivalence of (i) and (ii). For that of (ii) and (iii) we use the inverse isomorphisms  $\mathbb{C}[[x, y, z]] \cong \mathbb{C}[[w_1, w_2, w_3]]$  given by

$$\begin{aligned} H(x, y, z) \mapsto F(w_1, w_2, w_3) &= \frac{1}{(1-w_2)(1-w_3)} H\left(\frac{w_1-w_2}{1-w_2}, w_3 \frac{1-w_2}{1-w_3}, w_2\right), \\ F(w_1, w_2, w_3) \mapsto H(x, y, z) &= \frac{(1-z)^2}{1+y-z} F\left(x+z-xz, z, \frac{y}{1+y-z}\right). \end{aligned}$$

Under this correspondence the differential equation (14) is sent to  $\frac{\partial H}{\partial z} = 0$ . Finally, if (15) holds with  $H(x, y) = \sum_{i,j \geq 0} h_{ij} x^i y^j$ , then

$$\begin{aligned} \Phi[u^n] &= \sum_{b=0}^n \binom{n}{b} [w_1^n w_3^b] \left( \sum_{i,j \geq 0} h_{ij} \frac{w_1^i w_3^j}{(1-w_3)^{j+1}} \right) u^b v^{n-b} \\ &= \sum_{0 \leq j \leq b \leq n} \binom{n}{b} \binom{b}{j} h_{nj} u^b v^{n-b} = \sum_{j=0}^n \binom{n}{j} h_{nj} u^j (u+v)^{n-j}, \end{aligned}$$

and therefore equation (13) gives  $e_n = h_{nn}$  as claimed. Note that this proof shows that the matrix  $R_S$  is conjugate to the triangular matrix  $R_S^*$  with entries  $R_S^*(a, b) = \binom{a}{b} h_{ab}$ .  $\blacksquare$

In the remainder of this section we will explain how Proposition 2 can be applied to the problem described in the introduction and whose solution was given in Theorem 1. For  $n \in \mathbb{Z}_{\geq 0}$  and  $\varepsilon \in \{\pm 1\}$  we define  $Q_n^\varepsilon \in L^2(\mathbb{R})$  by  $Q_n^\varepsilon(x) = x^r L_n^{(-\varepsilon/2)}(\pi x^2) e^{-\pi x^2/2}$ , where  $\pm 1 = (-1)^r$  with  $r \in \{0, 1\}$  and  $L_n^{(-\varepsilon/2)}(z)$  is the Laguerre polynomial of degree  $n$  and parameter  $-\varepsilon/2$ . These functions can be defined by the generating series identity

$$\mathcal{Q}^\varepsilon(x; t) := \sum_{n=0}^{\infty} Q_n^\varepsilon(x) t^n = \frac{x^r}{(1-t)^{r+1/2}} \exp\left(-\frac{\pi}{2} \frac{1+t}{1-t} x^2\right).$$

The easy calculation  $\int_{-\infty}^{\infty} \mathcal{Q}^\varepsilon(x; t_1) \mathcal{Q}^\varepsilon(x; t_2) dx = \frac{\pi^{-1/2} \Gamma(r+1/2)}{(1-t_1 t_2)^{r+1/2}}$  implies that the  $Q_n^\varepsilon$  are orthogonal with  $\|Q_n^\varepsilon\|^2 = \frac{\Gamma(n+r+1/2)}{\Gamma(1/2) n!}$ . In particular, these functions are orthogonal bases for  $L_\varepsilon^2(\mathbb{R})$ , and for each integer  $0 \leq \kappa \leq 3$  the multivariate functions

$$Q_{\mathbf{n}}(x) = Q_{\mathbf{n}}^{(\kappa)}(x) = \prod_{1 \leq j \leq \kappa} Q_{n_j}^-(x_j) \prod_{\kappa < j \leq 3} Q_{n_j}^+(x_j) \quad (\mathbf{n} \in \mathbb{Z}_{\geq 0})$$

form an orthogonal basis of the parity  $\kappa$  space  $L_\kappa^2(\mathbb{R}^3)$  as defined in (9). Instead of appealing to Proposition 1, we can then deal directly with the quadratic form (1) by applying



Lemma 11 of [5] to this orthonormal basis of  $L^2_\kappa(\mathbb{R}^3)$ . This leads after some computations to the problem of computing the eigenvalues of the sequence of  $\binom{S+2}{2} \times \binom{S+2}{2}$  matrices  $M_S = ((\tilde{Q}_m, \tilde{Q}_n)_{L^2(\mathbb{R})})_{|m|=|n|=S}$ , where  $\tilde{Q} \in L^2(\mathbb{R})$  is defined by  $\tilde{Q}(x) = Q(x\mathbf{1})$ , with  $\mathbf{1} = \frac{1}{\sqrt{3}}(1, 1, 1)$ . In the rest of this section we show how to solve this problem.

The first point is that  $M_S$  has the same non-zero eigenvalues (with multiplicities) as a much smaller matrix. To see this, we observe that  $M_S$  factors as  $A_S^* A_S$ , where  $A_S$  is the map defined by  $Q \mapsto \tilde{Q}$  from the  $\binom{S+2}{2}$ -dimensional subspace  $V_S = \langle Q_n \rangle_{|n|=S}$  of  $L^2(\mathbb{R}^3)$  to the  $(S+1)$ -dimensional space of  $L^2(\mathbb{R})$

$$\tilde{V}_S = \langle x^{\kappa+2n} e^{-\pi x^2/2} \rangle_{0 \leq n \leq S}.$$

Hence  $M_S$  has the same non-zero eigenvalues as the endomorphism  $\tilde{M}_S = A_S A_S^*$  of  $\tilde{V}_S$ . To compute them we will actually consider  $\tilde{M}_S$  acting in the larger subspace

$$\tilde{\tilde{V}}_S = \langle Q_n^\varepsilon \rangle_{0 \leq n \leq S + \kappa/2},$$

where  $\varepsilon = (-1)^\kappa$  (i.e., we consider  $A_S : V_S \rightarrow \tilde{V}_S$  and take its adjoint  $A_S^*$  in the larger subspace  $\tilde{\tilde{V}}_S$ ). Then we will show that the matrix representation of  $\tilde{M}_S$  is of the form (12) for some special power series  $F = F_\kappa$ , but with  $S$  replaced by  $S + [\kappa/2]$ .

To do this we use kernel functions. The two 3-variable generating series

$$\mathcal{P}^\varepsilon(x, y; w) := \frac{e^{-\pi \frac{1+w}{1-w} \frac{x^2+y^2}{2}}}{\sqrt{1-w}} \times \begin{cases} \cosh\left(\frac{2\pi xy\sqrt{w}}{1-w}\right) & \text{if } \varepsilon = +1, \\ \frac{1}{\sqrt{w}} \sinh\left(\frac{2\pi xy\sqrt{w}}{1-w}\right) & \text{if } \varepsilon = -1 \end{cases} \quad (16)$$

are the kernel functions for the multiplication maps on  $L^2_\varepsilon(\mathbb{R})$  sending  $Q_n^\varepsilon$  to  $w^n Q_n^\varepsilon$  for all  $n \geq 0$ . We can see this either by using formula 8.976-1 of [7] to write

$$\mathcal{P}^\varepsilon(x, y; w) = \sum_{n=0}^{\infty} \frac{Q_n^\varepsilon(x) Q_n^\varepsilon(y)}{\|Q_n^\varepsilon\|^2} w^n,$$

or else directly by using the algebraic identity

$$\frac{1+w}{1-w} \frac{x^2+y^2}{2} - \frac{2xy\sqrt{w}}{1-w} + \frac{1+t}{1-t} \frac{y^2}{2} = \frac{1+wt}{1-wt} \frac{x^2}{2} + \frac{1-wt}{(1-w)(1-t)} \left( y - \frac{x(1-t)\sqrt{w}}{1-wt} \right)^2$$

to get (setting  $p = [\kappa/2]$ ,  $r = \kappa - 2p$ ,  $\varepsilon = (-1)^r = (-1)^\kappa$ )

$$\begin{aligned} \int_{-\infty}^{\infty} \mathcal{P}^\varepsilon(x, y; w) \mathcal{Q}^\varepsilon(y; t) dy &= \frac{w^{-r/2} e^{-\pi \frac{1+wt}{1-wt} \frac{x^2}{2}}}{\sqrt{1-w} (1-t)^{r+1/2}} \int_{-\infty}^{\infty} y^r e^{-\pi \frac{1-wt}{(1-w)(1-t)} \left( y - \frac{x(1-t)\sqrt{w}}{1-wt} \right)^2} dy \\ &= \frac{w^{-r/2} e^{-\pi \frac{1+wt}{1-wt} \frac{x^2}{2}}}{\sqrt{1-w} (1-t)^{r+1/2}} \sqrt{\frac{(1-w)(1-t)}{1-wt}} \left( \frac{x(1-t)\sqrt{w}}{1-wt} \right)^r = \mathcal{Q}^\varepsilon(y; wt). \end{aligned} \quad (17)$$

(We have given this second proof because a more complicated form of the same calculation will be used in a moment.) It follows that the product

$$\mathcal{P}^{(\kappa)}(x, y; w) = \prod_{1 \leq j \leq \kappa} \mathcal{P}^-(x_j, y_j; w) \prod_{\kappa < j \leq 3} \mathcal{P}^+(x_j, y_j; w)$$

is the multivariate kernel function for the multiplication map  $Q_n^{(\kappa)} \mapsto w^{|n|} Q_n^{(\kappa)}$  on  $L_\kappa^2(\mathbb{R}^3)$  and hence that the coefficient of  $w^S$  in its diagonal restriction

$$\tilde{\mathcal{P}}^{(\kappa)}(x, y; w) = \mathcal{P}^{(\kappa)}(x \mathbf{1}, y \mathbf{1}; w) = \mathcal{P}^+\left(\frac{x}{\sqrt{3}}, \frac{y}{\sqrt{3}}; w\right)^{3-\kappa} \mathcal{P}^-\left(\frac{x}{\sqrt{3}}, \frac{y}{\sqrt{3}}; w\right)^\kappa$$

is the kernel function of the endomorphism  $\widetilde{M}_S$  of  $\widetilde{V}_S$ . From (16) we get the explicit formula

$$\tilde{\mathcal{P}}^{(\kappa)}(x, y; w) = \frac{e^{-\pi \frac{1+w}{1-w} \frac{x^2+y^2}{2}}}{w^{\kappa/2} (1-w)^{3/2}} \sum_{m \in \{\pm 3, \pm 1\}} C_m^{(\kappa)} e^{\frac{2\pi mxy\sqrt{w}}{3(1-w)}},$$

where  $C_m^{(\kappa)}$  are defined by  $\cosh^{3-\kappa} x \sinh^\kappa x = \sum C_m^{(\kappa)} e^{mx}$ , or explicitly by  $C_{\pm 3}^{(\kappa)} = (\pm 1)^\kappa / 8$  and  $C_{\pm 1}^{(\kappa)} = (\pm 1)^\kappa (3 - 2\kappa) / 8$ .

To compute the action of  $\widetilde{M}_S$  on  $\widetilde{V}_S$  we must compute the scalar product of  $\tilde{\mathcal{P}}^{(\kappa)}(x, \cdot)$  and  $Q_n^\varepsilon$ , which we again do by using generating functions and a Gaussian integral calculation. The calculation is exactly similar to the one given in (17): using the algebraic identity

$$\frac{1+w}{1-w} \frac{x^2+y^2}{2} - \frac{2mxy\sqrt{w}}{3(1-w)} + \frac{1+t}{1-t} \frac{y^2}{2} = \frac{1+z_m}{1-z_m} \frac{x^2}{2} + \frac{1-wt}{(1-w)(1-t)} \left(y - \frac{mx(1-t)\sqrt{w}}{3(1-wt)}\right)^2,$$

where

$$z_m = w \frac{1-wt - m^2(1-t)/9}{1-wt - m^2w(1-t)/9} = \begin{cases} wt & \text{if } m = \pm 3, \\ 1 - \frac{(1-w)(1-wt)}{1 - \frac{8}{9}wt - \frac{1}{9}w} & \text{if } m = \pm 1, \end{cases}$$

we find

$$\int_{-\infty}^{\infty} \mathcal{P}^{(\kappa)}(x, y; w) \mathcal{Q}^\varepsilon(y; t) dy = \sum_{m \in \{\pm 3, \pm 1\}} C_m^{(\kappa)} I_m^{(\kappa)}(x; w, t)$$

with

$$\begin{aligned} I_m^{(\kappa)}(x; w, t) &= \frac{w^{-\kappa/2} e^{-\pi \frac{1+z_m}{1-z_m} \frac{x^2}{2}}}{(1-w)(1-t)^{r+1/2}} \int_{-\infty}^{\infty} y^r e^{-\pi \frac{1-wt}{(1-w)(1-t)} \left(y - \frac{mx(1-t)\sqrt{w}}{3(1-wt)}\right)^2} du \\ &= \frac{w^{-p} e^{-\pi \frac{1+z_m}{1-z_m} \frac{x^2}{2}}}{1-w} \frac{(mx/3)^r}{(1-wt)^{r+1/2}} = \frac{w^{-p} (m/3)^r}{1-w} \left(\frac{1-z_m}{1-wt}\right)^{r+1/2} \mathcal{Q}^\varepsilon(x; z_m). \end{aligned}$$

Now replacing each  $Q_b^\varepsilon(y)$  by  $u^b$  to keep track of the matrix coefficients, inserting the values of  $C_m^{(\kappa)}$  and noticing that the resulting series is a power series in  $w, u$  and  $T = wt$ , we find

$$\widetilde{M}_S(a, b) = [T^a w^{S+p-a} u^b](F_\kappa)$$

for  $0 \leq a, b \leq S + p$ , where

$$F_\kappa(T, w, u) = \frac{1}{4} \left(\frac{1-z_3}{1-T}\right)^{r+1/2} \frac{1}{1-z_3u} + \frac{3-2\kappa}{4} \left(\frac{1-z_1}{1-T}\right)^{r+1/2} \frac{(1/3)^r}{1-z_1u}.$$

But this function is special because it has the form (15) with  $(w_1, w_2, w_3) = (T, w, u)$  and

$$H_\kappa(x, y) = \frac{1}{4} \frac{1}{1-y(x-1)} + \frac{3-2\kappa}{4 \cdot 3^r} \frac{(1 - \frac{8}{9}x)^{1/2-r}}{1 - \frac{8}{9}x - y(x-1)}.$$

Therefore Proposition 2 gives the eigenvalues of  $\widetilde{M}_S$  as

$$\begin{aligned}
e_n^{(\kappa)} &= [x^n y^n](H_\kappa) = [x^n] \left( \frac{(x-1)^n}{4} \left( 1 + \frac{3-2\kappa}{3^r(1-8x/9)^{n+r+1/2}} \right) \right) \\
&= \frac{1}{4} + \frac{1}{4} \frac{3-2\kappa}{3^r} [v^n] \left( \frac{1}{\sqrt{1+14v/9+v^2}} \left( \frac{1+v+\sqrt{1+14v/9+v^2}}{2} \right)^{1/2-r} \right),
\end{aligned}$$

where in the second line we have used the substitution  $v = x \frac{1-8x/9}{x-1}$  and the residue theorem, and it is easily checked that this equals the coefficient of  $w^{2n+r}$  in the series  $G_\kappa(w)$  defined in equation (3). It is also possible with this method to characterize the eigenspaces associated with the largest eigenvalues, but this requires a bit more work (especially in the case  $\kappa = 1$  where the space  $V_\kappa(\alpha_\kappa) = V_1(\frac{1}{3})$  has rank 2) and is not carried out here.

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