Ramanujan's congruences for partitions modulo 5, 7 and 11

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Let $p(n)$ denote the number of partitions of n. Ramanujan proved that $p(n)$ is divisible by ℓ whenever $\ell \in \{5, 7, 11\}$ and $24n-1$ is divisible by ℓ [3,4]. Many proofs of these three congruences are now known. The one here is inspired by those given in a paper by M. Hirschhorn [1] and is substantially the same as his for $\ell = 5$ or $\ell = 7$, but is simpler when $\ell = 11$.

Theorem (Ramanujan). *If $\ell \in \{5, 7, 11\}$ and $24\,n \equiv 1 \pmod{\ell}$, then $p(n) \equiv 0 \pmod{\ell}$.*

Proof. For each of these primes (and, by a famous theorem of Serre [5], for no other prime $\ell > 3$) the $(\ell-1)$st power of the Dedekind eta-function can be represented as a binary theta series. Choosing one such representation in each case, we find

$$\ell = 5: \quad \eta(\tau)^4 \;=\; \sum_{\substack{a\equiv 1 \ (\mathrm{mod}\ 6)\\ b\equiv 1 \ (\mathrm{mod}\ 4)}} (-1)^{[a/6]}\, b\, Q^{a^2+3b^2} \;\equiv\; \sum Q^{(\mathrm{R\ or\ 0})\,+\,\mathrm{N}} \;=\; \sum Q^{\not\equiv 0}\,,$$

$$\ell = 7: \quad \eta(\tau)^6 \;=\; \sum_{\substack{a\equiv 1 \ (\mathrm{mod}\ 4)\\ b\equiv 1 \ (\mathrm{mod}\ 4)}} a\, b\, Q^{3a^2+3b^2} \;\equiv\; \sum Q^{\mathrm{N}\,+\,\mathrm{N}} \;=\; \sum Q^{\not\equiv 0}\,,$$

$$\ell = 11: \quad \eta(\tau)^{10} \;=\; \sum_{\substack{a\equiv 2 \ (\mathrm{mod}\ 6)\\ b\equiv 1 \ (\mathrm{mod}\ 6)}} \frac{ab(a^2-b^2)}{6}\, Q^{2a^2+2b^2} \;\equiv\; \sum Q^{\mathrm{N}\,+\,\mathrm{N}} \;=\; \sum Q^{\not\equiv 0}\,,$$

where $Q = e^{\pi i\tau/12}$ and where the symbols R, N, 0, and $\equiv$ denote respectively quadratic residues, quadratic non-residues, zero, and congruence modulo ℓ. Hence in all three cases we have

$$\sum_{n\ge 0} p(n)\, Q^{24n-1} \;=\; \frac{1}{\eta(\tau)} \;\equiv\; \frac{\eta(\tau)^{\ell-1}}{\eta(\ell\tau)} \;\equiv\; \frac{\sum Q^{\not\equiv 0 \ (\mathrm{mod}\ \ell)}}{\sum Q^{\equiv 0 \ (\mathrm{mod}\ \ell)}} \;=\; \sum Q^{\not\equiv 0 \ (\mathrm{mod}\ \ell)}$$

and thus $p(n) \equiv 0$ if $24n-1 \equiv 0$. □

We should say a few words about the formulas for $\eta^{\ell-1}$ given above. The formula for η^4 is just the product of the two famous identities

$$\eta(\tau) \;=\; \sum_{a\equiv 1 \ (\mathrm{mod}\ 6)} (-1)^{[a/6]}\, Q^{a^2} \qquad \text{and} \qquad \eta(\tau)^3 \;=\; \sum_{b\equiv 1 \ (\mathrm{mod}\ 4)} b\, Q^{3b^2}\,,$$

due to Euler and Jacobi, respectively, and the formula for η^6 is just the square of the latter of these. The formula for η^{10} is given, in a somewhat more complicated form, in the paper by Serre already quoted ([5], eq. (27)) and is quoted on p. 121 in the book [2], which is a huge compendium of formulas of this type. But actually one does not need to give references or proofs for any of these formulas, since any true identity of this type, once discovered, can be proved trivially by a finite computation. Specifically, the theory of theta series says that any function of the form $\sum_{\mathbf{a}\in\mathbb{Z}^r} \varepsilon(\mathbf{a})\, P(\mathbf{a})\, e^{2\pi i F(\mathbf{a})\tau}$, where F is a positive definite quadratic form in r variables with rational coefficients, ε a periodic function on $\mathbb{Z}^r$, and P a homogeneous polynomial of degree d that is spherical with respect to F, is a modular form of weight $d+r/2$ and calculable level. If $r = 2$ and $F(a,b)$, say, is a multiple of a^2+b^2, then "spherical with respect to F" simply means "harmonic," i.e. $\partial^2P/\partial a^2 + \partial^2P/\partial b^2 = 0$. This is certainly true for $P(a,b) = a^3b - ab^3$, so the two sides of the formula for η^{10} are both modular forms of weight 5 and small level and their equality follows simply by verifying that their first few Fourier coefficients agree. A typical example of the magic of modular forms!

Bibliography

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