

NU-INVARIANTS OF EXTRA-TWISTED CONNECTED SUMS

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ABSTRACT. We analyse the possible ways of gluing twisted products of circles with asymptotically cylindrical Calabi-Yau manifolds to produce manifolds with holonomy G_2 , thus generalising the twisted connected sum construction of Kovalev and Corti, Haskins, Nordström, Pacini. We then express the extended ν -invariant of Crowley, Goette, and Nordström in terms of fixpoint and gluing contributions, which include different types of (generalised) Dedekind sums. Surprisingly, the calculations involve some non-trivial number-theoretical arguments connected with special values of the Dedekind eta-function and the theory of complex multiplication. One consequence of our computations is that there exist compact G_2 -manifolds that are not G_2 -nullbordant.

Though compact Riemannian manifolds with holonomy G_2 , also known as compact manifolds with a torsion-free G_2 -structure, have been constructed more than 25 years ago, we still do not know very much about them. On one hand, only a few obstructions against G_2 -metrics on a given compact 7-manifold are known (see Joyce [25, §10.2]). On the other hand, our current supply of examples is much smaller than allowed by these obstructions. It is therefore still interesting to explore new invariants of G_2 -manifolds in the hope to discover new obstructions, and to find new examples on which these invariants can be tested.

The extended ν -invariant was introduced in [14] to exhibit 2-connected 7-manifolds with a disconnected moduli space of G_2 -metrics. In the present paper, we apply it to a larger class of examples to find the first examples of G_2 -manifolds whose G_2 -bordism class can be shown to be nontrivial. We compute the η -invariants that appear in the definition of the extended ν -invariant using gluing formulas as well as variation and adiabatic limit formulas for manifolds with boundary. The details may be of interest to index theorists because in contrast to [14], we cannot rely on spectral symmetry here.

Extra-twisted connected sums. There are currently two major sources of compact G_2 -manifolds, that is, compact Riemannian manifolds whose holonomy group is isomorphic to G_2 . The first is Joyce's Kummer construction [25], based on resolution of singularities in flat orbifolds. It has recently been generalised by Joyce and Karigiannis to more complicated spaces [26]. The second is the twisted sum construction pioneered by Kovalev [28] and systematically studied by Corti, Haskins, Nordström and Pacini [12, 13].

For the latter, one starts with two asymptotically cylindrical Calabi-Yau manifolds V_{\pm} . The cross-sections of their ends approach the products of a K3 surface Σ_{\pm} and a circle $S_{\zeta_{\pm}}^1$ that we wish to call the *interior* circle. In the classical set up, one takes the products of V_{\pm} and an *exterior* circle $S_{\xi_{\pm}}^1$ of length $\xi_{\pm} = \zeta_{\mp}$. Then one glues truncated copies M_{\pm} of $V_{\pm} \times S_{\xi_{\pm}}^1$ along their ends, in a way that swaps the roles of interior and exterior circles.

Here, we assume in addition that some finite cyclic groups $\Gamma_{\pm} \cong \mathbb{Z}/k_{\pm}$ act on V_{\pm} , preserving the Calabi-Yau structure. We also assume that the induced actions on the K3 factors Σ_{\pm} are trivial, and that the induced actions on the interior circles $S_{\zeta_{\pm}}^1$ are free. We consider twisted products $M_{\pm} \cong (V_{\pm} \times S_{\xi_{\pm}}^1)/\Gamma_{\pm}$, where Γ_{\pm} acts diagonally, and freely on the exterior

circle factors. The manifolds M_+ and M_- can now be glued with an “extra” twist. The cases where $k_{\pm} \leq 2$ have already been considered in [14, 31].

The known supply of asymptotically cylindrical Calabi-Yau manifolds with nontrivial symmetries that fix the K3 surface Σ is very limited. In the present paper, we use examples constructed from Fano threefolds of higher index, and from hypersurfaces in weighted projective spaces. We only consider examples of Picard rank 1 for simplicity. The possible groups obtained this way are $\Gamma \cong \mathbb{Z}/k$ with $k \leq 6$; see Table 1.

But even though we use only a few asymptotically cylindrical Calabi-Yau manifolds with nontrivial symmetry group, there are typically several different ways to glue two given twisted products M_{\pm} by changing the size of the exterior circles and the gluing angle between them. Some of the G_2 -manifolds we construct this way will not be simply connected—we obtain cyclic fundamental groups of order up to 21; see example 250 of Table 2—but the universal covers of these examples will again be extra-twisted connected sums; see Proposition 3.5.

Apart from the choice of V_+ , V_- and Γ_+ , Γ_- , an extra-twisted connected sum is described by two square matrices encoding the gluing of the tori and of the K3 surfaces, and two small integers that fix the actions of Γ_{\pm} . These data have to satisfy certain conditions, as described below. There are typically several ways to describe the same extra-twisted connected sum up to isometries and orientation reversal; see Proposition 3.7. In addition, passing to dual tori often leads to non-isometric extra-twisted connected sums built from the same blocks with the same K3-matching and the same gluing angle. We thus have a kind of partial t -duality for extra-twisted connected sums; see Proposition 3.9 and Example 3.11.

Our combinatorial description of extra-twisted connected sums allows us to find all possible combinations (among the asymptotically cylindrical Calabi-Yau 3-folds used in this paper) by a small computer program. Table 2 lists 255 examples of extra-twisted considered sums, 192 of which are not contained in [31]. Of all the examples, 125 are simply connected, representing at least 106 different G_2 -deformation types, that is, classes of G_2 -manifolds related by diffeomorphisms and deformations through torsion-free G_2 -structures.

The ν -invariant and its analytic refinement. The ν -invariant of G_2 -structures on closed 7-manifolds was introduced in [16]. It takes values in $\mathbb{Z}/48$, and its parity is determined by the Betti numbers of the underlying manifold. The examples in this paper show that all odd elements of $\mathbb{Z}/48$ appear as ν -invariants of torsion-free G_2 -structures on extra-twisted connected sums; see Table 2. To obtain more examples with even ν from our construction we would need to use blocks with Picard rank higher than 1. As we explain below, such blocks exist, and we expect that all even elements of $\mathbb{Z}/48$ can be realised as well. The total number of extra-twisted connected sums we can currently construct is much less than the number of twisted connected sums constructed in [13]. Nevertheless, the present method is more efficient at constructing different G_2 -deformation types that we can distinguish.

To date, only very few obstructions against the existence of a metric with full holonomy G_2 on a seven-manifold M are known. It is clear that M must be spin and have a finite fundamental group, and the de Rham class of the defining three-form φ must satisfy certain cohomological inequalities. When Crowley and Nordström discovered the ν -invariant in [16], it was hoped that the ν -invariant could be an obstruction against deforming a given topological G_2 -structure into a G_2 -holonomy metric. While computations in [14] showed that G_2 -manifolds can have non-zero ν -invariants, all examples considered there (and also in [31]) have a ν -invariant that is divisible by 3. In particular, those examples are all G_2 -nullbordant, as a consequence of the result of Schelling [35] that $3 \mid \nu(M)$ if and only if M is G_2 -nullbordant. However, we now find that G_2 -bordism does not present any obstruction to G_2 holonomy metrics.

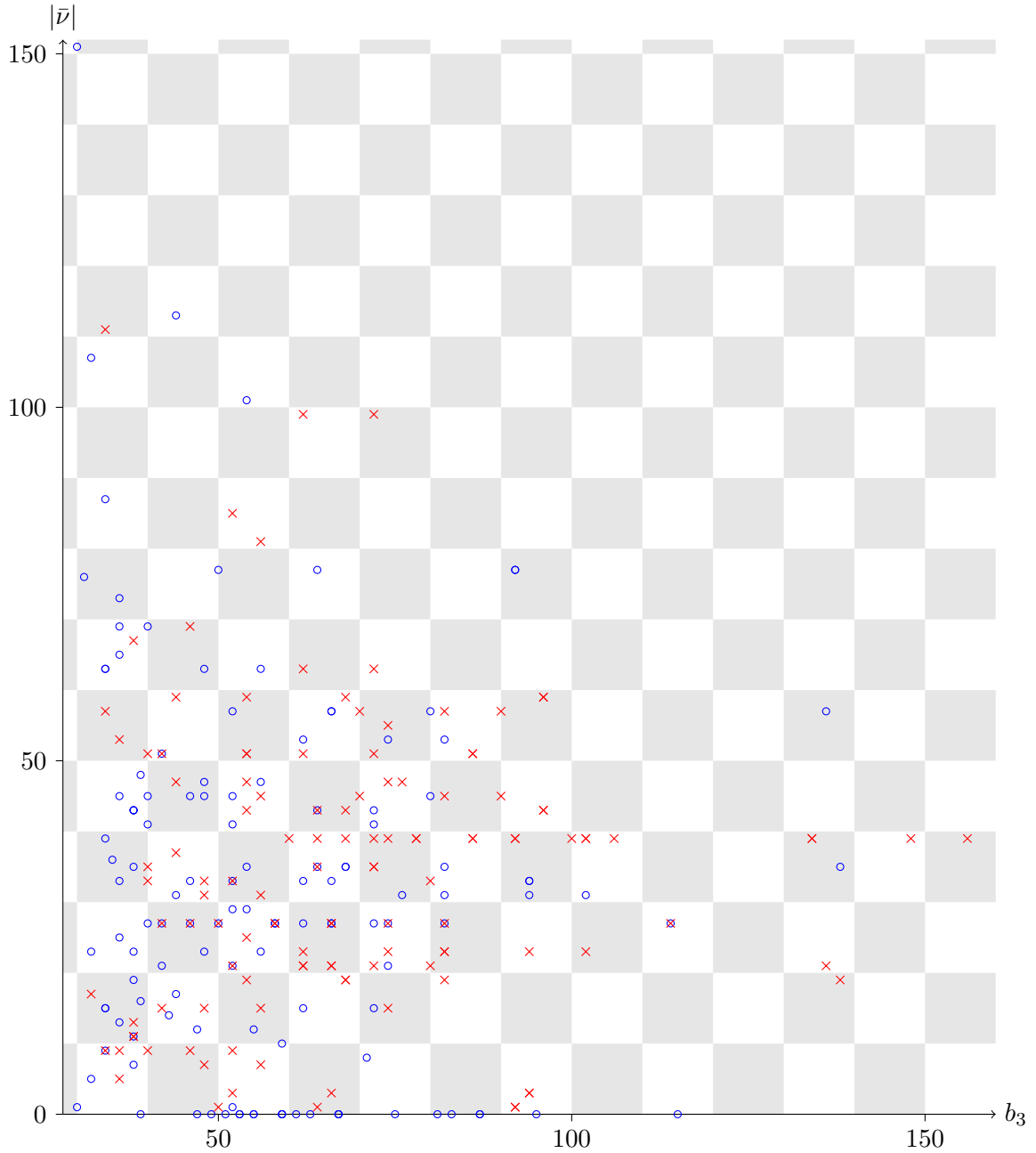


FIGURE 1. The landscape of examples in Table 2

Observation 1. *There exist compact G_2 -manifolds that are not G_2 -nullbordant.*

To compute the ν -invariant, we use the method described in [14]. That is, we consider an integer-valued invariant

$$\bar{\nu}(M) = 3\eta(B) - 24\eta(D) \quad \text{such that} \quad \nu(M) \equiv \bar{\nu}(M) - 24(1 + b_1(M)) \pmod{48}. \quad (0.1)$$

Here, B is the signature operator and D the spin Dirac operator on M ; see Section 2. In Example 2.15, we exhibit an extra-twisted connected sum M with $\bar{\nu}(M) = -11$; see also

Example 4.16. Hence, $\nu(M) \equiv 13 \pmod{48}$. It follows from [35] that M is not G_2 -nullbordant, and we have an example for Observation 1.

Our results are summarized in Table 2 and plotted in Figure 1, where crosses stand for simply-connected (in fact, two-connected) examples and circles for examples with nontrivial fundamental group. Because orientation reversal changes the sign of $\bar{\nu}$, we used the absolute value of $\bar{\nu}$ here. To get all values of $\bar{\nu}$, the whole picture should thus be extended by reflexion along the horizontal axis. In comparison with [14], we get some examples where the absolute value of $\bar{\nu}$ is much larger, for example $\bar{\nu} = -151$ in line 254 of Table 2 with fundamental group $\mathbb{Z}/3$, or example 240 with $\bar{\nu} = -111$, which is simply connected. All this seems to indicate that there could still be a wealth of unknown examples of G_2 -manifolds.

A gluing formula for $\bar{\nu}$. We still follow the route of computation for $\bar{\nu}$ that we laid out in [14]. That is, we first write $M = M_+ \cup M_-$ and use the gluing formula of Bunke [9] and Kirk, Lesch [27] to write $\bar{\nu}(M, g)$ as a sum of the contributions $\bar{\nu}(M_{\pm}, g)$ from the two pieces and a gluing term; see Theorem 2.1.

The gluing term contains two pieces of information. The first is an integer contribution $3m_{\rho}(L; N_+, N_-)$ that depends on the relative positions of the images of $H^2(V_{\pm})$ in $H^2(\Sigma)$, where Σ is the K3 surface that appears as a factor of the cross-section at infinity of M_{\pm} . If $\vartheta \in (0, \pi)$ denotes the oriented angle between the exterior circles, which we call the *gluing angle*, then the second contribution is $-72\frac{\rho}{\pi}$, where $\rho = \pi - 2\vartheta$. If the gluing angle is an irrational multiple of π , then this term will be irrational, too.

In [14], restricting attention to the case when both $\#\Gamma_{\pm} \leq 2$ ensured that both pieces have spectral symmetry, so that $\bar{\nu}(M_{\pm}, g) = 0$. Moreover, the gluing angle was forced to be a rational multiple of π simply by the combinatorics of isometries of tori (see §3.1). However, if M_{\pm} is a twisted product $(V_{\pm} \times S^1)/\Gamma_{\pm}$ and $\#\Gamma_{\pm} \geq 3$, then the spectra of the relevant Dirac operators on M_{\pm} with respect to the appropriate boundary conditions are no longer necessarily symmetric.

To compute the contribution $\bar{\nu}(M_{\pm}, g)$, we consider an adiabatic limit by scaling the exterior circle factor to 0. By an adiabatic limit formula generalising both Dai's theorem for manifolds with boundary [19] and the formula for adiabatic limits of Seifert fibrations [22], we see that the isolated fixpoints of the Γ_{\pm} -action on V_{\pm} contribute by generalised Dedekind sums $D_{\gamma}(V) \in \mathbb{Q}$ introduced in Definition 2.3 (Note that involutions on Calabi-Yau 3-folds cannot have isolated fixed points, consistent with the claim that the η -invariants vanish when $\#\Gamma_{\pm} \leq 2$).

When passing to the adiabatic limit, we also modify the metric on the boundary of M_{\pm} . Hence, we also have to consider the variational formula for η -invariants on manifolds with boundary due to Cheeger [11], Bismut, Cheeger [4] and Dai, Freed [20]. The relevant boundary contribution can be computed from a universal η -form $\tilde{\eta}(\mathbb{A})$ for families of two-dimensional tori; see Proposition 2.8. In the appendix, Don Zagier explains how to rewrite the relevant η -form integrals in terms of the logarithm \mathcal{L} of the Dedekind η -function. In the following and throughout the paper, we represent the inverse of ε_{\pm} in \mathbb{Z}/k_{\pm} by $\varepsilon_{\pm}^* \in \mathbb{Z}$. Also, the ratio of the lengths of the exterior and the interior circle is denoted $s_{\pm} = \frac{\xi_{\pm}}{\zeta_{\pm}}$.

Theorem 2. *For all extra-twisted connected sums M , the extended ν -invariant is given by*

$$\bar{\nu}(M) = \bar{\nu}(M_+, g) + \bar{\nu}(M_-, g) - 72\frac{\rho}{\pi} + 3m_{\rho}(L; N_+, N_-), \quad (0.2a)$$

$$\text{where} \quad \bar{\nu}(M_{\pm}, g) = D_{\gamma_{\pm}}(V_{\pm}) - \frac{288}{\pi} \operatorname{Im} \mathcal{L} \left(\frac{s_{\pm}^{-1}i - \varepsilon_{\pm}^*}{k_{\pm}} \right) - 24 \frac{\varepsilon_{\pm}^*}{k_{\pm}}. \quad (0.2b)$$

This is proved in Section 2.5 using Proposition A.1 from the appendix. The occurrence of the Dedekind η -function can also be motivated by regarding $\tilde{\eta}(\mathbb{A})$ as a connection form of

the Chern connection on a holomorphic determinant line bundle; see Remark 2.14. From the theory of complex multiplication one knows that the values of $\text{Im } \mathcal{L}$ in (0.2b) can be expressed in terms of logarithms of algebraic numbers; for the values used in this paper, this is done explicitly in A.2. The linear combinations that appear in (0.2a) can be figured out from the functional equation (A.1) of \mathcal{L} (see Proposition A.3), giving one proof of Theorem 3 below.

Computing $\bar{\nu}$ via elementary hyperbolic geometry. In Section 4, we follow a different path to rewrite the right hand side of (0.2a) in terms of Dedekind sums. The universal η -form $\tilde{\eta}(\mathbb{A})$ for bundles of flat tori can be understood as a 1-form on the upper half plane, whose exterior derivative turns out to be a constant multiple of the hyperbolic area form. The relevant path of integration consists of two sides of some ideal hyperbolic polygon P , depending on the gluing data. The remaining sides can be chosen such that $\tilde{\eta}(\mathbb{A})$ vanishes along them. To apply Stokes' Theorem, it remains to determine the contribution from the cusps. At this point, we use a strict version of an adiabatic limit formula for η -forms that was proved by Bunke, Ma [10] modulo exact forms; see also Liu [30].

Following a suggestion by Zagier, we determine the remaining corners of the polygon P using continued fractions. Here, we also need the entries of the gluing matrix $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ that encodes the matching of the tori as described in (1.6). At this point, one can already finish the computation of $\bar{\nu}(M)$ for any particular example by hand. However, one can simplify these computations by observing that the contributions from the cusps and the hyperbolic area formula add up to a classical Dedekind sum $S(k, n)$ given for integers $n > 0$ and k by

$$S(k, n) = \sum_{j=1}^{n-1} \left(\left(\frac{j}{n} \right) \right) \left(\left(\frac{jk}{n} \right) \right) \in \frac{1}{6n} \mathbb{Z}, \quad \text{where } ((x)) = \begin{cases} 0 & \text{for } x \in \mathbb{Z} \\ x - [x] - \frac{1}{2} & \text{for } x \notin \mathbb{Z} \end{cases} \quad (0.3)$$

denotes a sawtooth function. For more background on the Dedekind η -function, Dedekind sums, and their appearance in topology we refer the reader to [24]. We may use Proposition 3.7 to make sure that $m \geq 0$ and $n > 0$.

Theorem 3. *Assume that $n > 0$. Then $A = \frac{m - \varepsilon_+^* n}{k_+}$ is an integer and*

$$\bar{\nu}(M) = D_{\gamma_+}(V_+) + D_{\gamma_-}(V_-) + 3m_\rho(L; N_+, N_-) + 24 \left(\frac{q}{k_- n} - \frac{m}{k_+ n} + 12S(A, n) \right). \quad (0.4)$$

The proof outlined above can be found in Section 4.5. We see that the possibly irrational term $-72 \frac{\rho}{\pi}$ from Theorem 2 has been subsumed into the hyperbolic area formula. Hence, this presentation makes it easier to check that $\bar{\nu}(M)$ is an integer; see also Remark 4.15.

All in all, our way to a tractable formula for the ν -invariant consists of many small steps, but in each of our examples, the sum of the various contributions is an integer. This, together with the fact that the two completely different approaches described above give the same expression, could be seen as a sanity check for our results presented here.

Scope. To keep this article within reasonable size, we had to leave out some aspects of the construction.

- (i) We only consider examples built from blocks of Picard rank 1. These examples automatically have matchings of pure angle in the sense of Remark 1.16. On the other hand, by (5.2) most examples obtained this way have even $b_3(M)$, and hence odd $\bar{\nu}(M)$.

There are building blocks of Picard rank 2 with automorphism group \mathbb{Z}/k for $k = 1, \dots, 6$. Thus we expect that there are examples of extra-twisted connected sums that realise all 24 even values of $\nu(M) \in \mathbb{Z}/48$ as well.

- (ii) All examples constructed in Section 5 are either two-connected or have two connected universal cover. We distinguish them only by their extended ν -invariants and their third Betti number. See Figure 1 for a plot of all possible pairs of these invariants. We do not attempt to compute the torsion part of their fourth cohomology or the divisibility of their first Pontryagin classes. That would be needed in order to apply diffeomorphism classification results [17] to exhibit examples of 7-manifolds where the moduli space of G_2 -metrics is disconnected, but such examples have already been seen in [14].
- (iii) We might consider asymptotic Calabi-Yau manifolds with arbitrary automorphisms, so Σ is invariant under Γ only as a set. But first of all, it looks more difficult to construct matchings in this situation. And worse, these examples would never be simply connected. Instead, their universal covers would again be extra-twisted connected sums of the type considered here. This has been explained in [15, Remark 1.12].
- (iv) It is not clear how to define a ν -invariant for non-compact or singular G_2 -spaces. Theorem 2.1 contains a possible definition for G_2 -manifolds with an asymptotically cylindrical end. However, it is also not clear to us how to interpret the resulting numbers in (0.2b). From Dai and Freed's point of view in [20], the invariant $e^{2\pi i \frac{\nu(M_{\pm}; g)}{6}}$ should take values in a certain determinant line associated to the cross-section at infinity. The invariant $\bar{\nu}(M)$ would therefore take values in a "logarithm" of this determinant line.

Organisation. In Section 1, we recall the extra-twisted connected sum construction. We discuss the matching problem for tori in Subsection 1.3, and for K3 surfaces in Subsection 1.4. Theorem 2 is proved in Section 2. The fixpoint contributions are computed in Subsection 2.3, the variational formula is discussed in 2.4, and a direct computation of the η -form integrals employing Zagier's approach can be found in 2.5. We discuss the combinatorics of torus matchings in Section 3. Theorem 3 is proved by elementary hyperbolic geometry in Section 4. Adiabatic deformations of tori are identified with hyperbolic geodesics in Subsection 4.2, and the contribution from the cusps is explained in 4.4. In Subsection 4.5, following another of Zagier's suggestions, we use continued fractions to construct ideal polygons, and in 4.6, we rewrite the cusp contributions as a Dedekind sum. Section 5 contains more details about the construction of examples, in particular we describe some building blocks with group actions in Subsection 5.2, and possible K3 matchings in Subsection 5.3. Section 6 contains the proofs of two adiabatic limit theorems used in Subsections 2.3 and 4.4. The appendix by Don Zagier contains the evaluation of η -form integrals in terms of the Dedekind η -function, as well as explicit formulas for the values that we use in this paper.

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1. EXTRA-TWISTED CONNECTED SUMS

We generalise the twisted connected sum construction of [28, 13] by allowing twisted products of asymptotically cylindrical Calabi-Yau manifolds with circles. This approach was already employed in [14], where we considered products twisted by an involution (on one or both sides). Here, we allow twists by more general finite cyclic groups.

1.1. The gluing construction. Let V_{\pm} be asymptotically cylindrical Calabi-Yau manifolds of complex dimension 3, and assume that their ends are asymptotic to $\Sigma_{\pm} \times S^1_{\zeta_{\pm}} \times (0, \infty)$, where Σ_{\pm} are K3 surfaces, and $S^1_{\zeta_{\pm}} = \mathbb{R}/\zeta_{\pm}\mathbb{Z}$. The Calabi-Yau structure on V_{\pm} can be described in terms of a pair $(\Omega_{\pm}, \omega_{\pm})$, where Ω_{\pm} is a complex 3-form (holomorphic with respect to the complex structure) and ω_{\pm} is a Kähler form, normalised so that $8\omega^3 = 6\Omega \wedge \bar{\Omega}$. Along the cylindrical end $\Sigma_{\pm} \times S^1_{\zeta_{\pm}} \times (0, \infty)$, the asymptotic limits of Ω and ω are of the form

$$\Omega := (du - idt) \wedge (\omega_{\pm}^J + i\omega_{\pm}^K), \quad \omega := dt \wedge du + \omega_{\pm}^I$$

respectively, where t is the coordinate on the $(0, \infty)$ factor, u_{\pm} is the coordinate on $S^1_{\zeta_{\pm}} = \mathbb{R}/\zeta_{\pm}\mathbb{Z}$, and the triple $(\omega_{\pm}^I, \omega_{\pm}^J, \omega_{\pm}^K)$ defines a hyper-Kähler structure on Σ_{\pm} . Such a Calabi-Yau structure induces a metric $g_{V_{\pm}}$ of holonomy $SU(3)$ whose asymptotic limit is of the form $dt^2 + du_{\pm}^2 + g_{\Sigma_{\pm}}$, where $g_{\Sigma_{\pm}}$ is a metric of holonomy $SU(2)$ induced by the hyper-Kähler structure. Note in particular that the circumference of the circle factor in the asymptotic cylinder is ζ_{\pm} .

Remark 1.1. The condition on the cylindrical ends forces V_{\pm} to be simply-connected by [23, Theorem A].

Letting $S^1_{\xi_{\pm}} = \mathbb{R}/\xi_{\pm}\mathbb{Z}$ and denoting its coordinate by v_{\pm} , we can define a product torsion-free G_2 -structure on $V_{\pm} \times S^1_{\xi_{\pm}}$ by

$$\varphi_{\pm} = \operatorname{Re} \Omega_{\pm} + dv_{\pm} \wedge \omega_{\pm}.$$

Then φ_{\pm} defines the metric $dv_{\pm}^2 + g_{V_{\pm}}$, with holonomy contained in G_2 . Note that the circle factor has circumference ξ_{\pm} . The asymptotic limit of φ_{\pm} has the form

$$dv_{\pm} \wedge dt \wedge du_{\pm} + dv_{\pm} \wedge \omega_{\pm}^I + du_{\pm} \wedge \omega_{\pm}^J + dt \wedge \omega_{\pm}^K. \quad (1.1)$$

Now assume further that two groups $\Gamma_{\pm} \cong \mathbb{Z}/k_{\pm}$ act isometrically on V_{\pm} , preserving the Calabi-Yau structures, such that the actions on the end are products of trivial actions on $\Sigma_{\pm} \times (0, \infty)$ and free actions on $S^1_{\zeta_{\pm}}$. We extend the Γ_{\pm} -action diagonally to $\tilde{M}_{\pm} = V_{\pm} \times S^1_{\xi_{\pm}}$, such that Γ_{\pm} acts freely on $S^1_{\xi_{\pm}}$. Then the torsion-free G_2 -structure φ_{\pm} descends to the quotient $M_{\pm} = \tilde{M}_{\pm}/\Gamma_{\pm}$, which we can thus regard as an asymptotically cylindrical G_2 -manifold. The cross-section of the asymptotic cylinder is the product X_{\pm} of Σ_{\pm} with a torus $(S^1_{\zeta_{\pm}} \times S^1_{\xi_{\pm}})/\Gamma_{\pm}$.

We now suppose that we have a suitable isometry between the cross-sections. This isometry will necessarily be a product of isometries

$$\mathfrak{t} : (S^1_{\zeta_+} \times S^1_{\xi_+})/\Gamma_+ \rightarrow (S^1_{\zeta_-} \times S^1_{\xi_-})/\Gamma_-$$

and

$$\mathfrak{r} : \Sigma_+ \rightarrow \Sigma_-.$$

We require that the isometry

$$\begin{aligned} \Sigma_+ \times (S^1_{\zeta_+} \times S^1_{\xi_+})/\Gamma_+ \times \mathbb{R} &\rightarrow \Sigma_- \times (S^1_{\zeta_-} \times S^1_{\xi_-})/\Gamma_- \times \mathbb{R} \\ (x, z, t) &\mapsto (\mathfrak{r}(x), \mathfrak{t}(z), -t) \end{aligned} \quad (1.2)$$

identifies the asymptotic limits (1.1). In particular, exactly one of \mathfrak{t} and \mathfrak{r} is orientation-preserving. Our convention is to require \mathfrak{t} to be orientation-reversing. We will refer to \mathfrak{t} as a *torus matching* and to \mathfrak{r} as a *hyper-Kähler rotation*.

A key feature of the construction is how the torus matching aligns the external circle directions. As above, we denote by

$$u_{\pm} \in \mathbb{R}/\zeta_{\pm}\mathbb{Z}, \quad v_{\pm} \in \mathbb{R}/\xi_{\pm}\mathbb{Z}$$

the coordinates in the direction of the interior and exterior circles respectively. In [14, (10)], the *gluing angle* ϑ was introduced as the directed angle between the exterior circles under \mathfrak{t} , so

$$\begin{aligned} \partial_{v_-} &= \cos \vartheta \partial_{v_+} + \sin \vartheta \partial_{u_+}, \\ \partial_{u_-} &= \sin \vartheta \partial_{v_+} - \cos \vartheta \partial_{u_+}. \end{aligned} \tag{1.3}$$

The condition that (1.2) preserves the asymptotic limits of the G_2 -structures is now equivalent to the following condition; see [31, §1.2].

Definition 1.2. Let Σ_{\pm} be K3 surfaces with hyper-Kähler structures $(\omega_{\pm}^I, \omega_{\pm}^J, \omega_{\pm}^K)$. Call a diffeomorphism $\mathfrak{r} : \Sigma_+ \rightarrow \Sigma_-$ a *hyper-Kähler rotation* with angle ϑ if

$$\begin{aligned} \mathfrak{r}^* \omega_-^K &= -\omega_+^K \\ \mathfrak{r}^*(\omega_-^I + i\omega_-^J) &= e^{i\vartheta}(\omega_+^I - i\omega_+^J). \end{aligned} \tag{1.4}$$

Let $V_{\pm, \ell} = V_{\pm} \setminus ((\ell + 2, \infty) \times S_{\zeta}^1 \times \Sigma)$, $\tilde{M}_{\pm, \ell} = V_{\pm, \ell} \times S_{\xi_{\pm}}^1$ and $M_{\pm, \ell} = \tilde{M}_{\pm, \ell}/\Gamma_{\pm}$ denote truncations of the manifolds above, and let $\tilde{X}_{\pm} \cong \Sigma_{\pm} \times S_{\zeta_{\pm}}^1 \times S_{\xi_{\pm}}^1$ and $X_{\pm} = \tilde{X}_{\pm}/\Gamma_{\pm}$ denote their boundaries. For sufficiently large ℓ , it is possible to obtain a new closed G_2 -manifold M_{ℓ} by gluing $M_{+, \ell}$ and $M_{-, \ell}$ along a diffeomorphism $X_+ \cong X_-$. This procedure is described in detail in [31] following the ideas in [28, 13]. Let us summarise.

Theorem 1.3. *Given a pair of ACyl Calabi-Yau 3-folds V_{\pm} with asymptotic cross-sections $\Sigma_{\pm} \times S_{\zeta_{\pm}}^1$ and automorphisms Γ_{\pm} , a torus matching $\mathfrak{t} : (S_{\zeta_+}^1 \times S_{\xi_+}^1)/\Gamma_+ \rightarrow (S_{\zeta_-}^1 \times S_{\xi_-}^1)/\Gamma_-$ and a hyper-Kähler rotation $\mathfrak{r} : \Sigma_+ \rightarrow \Sigma_-$ with angle ϑ equal to the gluing angle of \mathfrak{t} , the manifold M_{ℓ} above admits torsion-free G_2 -structures.*

On \tilde{M}_{\pm} and M_{\pm} , let $t_{\pm} \in (0, \infty)$, be the coordinate in the cylindrical direction. On M_{ℓ} , we will later consider the cylindrical coordinate $t = t_- - \ell - 2$ that increases going from M_- to M_+ ; see [14, (14)].

Note that after gluing, M_+ and M_- induce opposite orientations on the 2-torus. Note that ϑ is invariant under swapping the roles of M_+ and M_- . The angle between the interior circles is $\pi - \vartheta$; see Figure 2.

For further discussion of how extra-twisted connected sums using different data can be essentially the same see Remark 3.8.

Remark 1.4. The G_2 -structure on M defines a unique spin structure on M that we need for the analytic description of the ν -invariant. Its restriction to M_{\pm} is the spin structure induced by the $SU(3)$ -structure, and hence by the Calabi-Yau structure on V_{\pm} .

Because Γ_{\pm} preserves the Calabi-Yau structure on V_{\pm} , it acts canonically on the associated spinor bundle $SV_{\pm} \cong \Lambda^{0, \bullet} T^*V_{\pm}$. The spinor bundle on M_{\pm} that is induced by the G_2 -structure then satisfies $SM_{\pm} \cong p^*SV_{\pm}/\Gamma_{\pm}$. On the cylinder $\Sigma_{\pm} \times (0, \infty) \times (S_{\zeta_{\pm}}^1 \times S_{\xi_{\pm}}^1)/\Gamma_{\pm}$, it is isomorphic to the pullback of the direct sum of two copies of $S\Sigma_{\pm}$.

1.2. Setting up the matching problem. Understanding the possible torus matchings $t : (S_{\xi_+}^1 \times S_{\xi_+}^1)/\Gamma_+ \rightarrow (S_{\xi_-}^1 \times S_{\xi_-}^1)/\Gamma_-$ for given values of $k_{\pm} = |\Gamma_{\pm}|$ —and in particular the possible gluing angles ϑ —is essentially a combinatorial problem, which will be discussed in the next subsection and in Subsection 3.1. Given a torus matching, Theorem 1.3 raises the question of how to find pairs of ACyl Calabi-Yau 3-folds with automorphisms and a hyper-Kähler rotation of the correct angle ϑ between the K3 surfaces in the asymptotic cross-section. We now explain how this question can be reduced to complex algebraic geometry, as in [13, §6] and [31, §6].

Definition 1.5. Let Z be a non-singular algebraic 3-fold and $\Sigma \subset Z$ a non-singular K3 surface. Let N be the image of $H^2(Z) \rightarrow H^2(\Sigma)$. We call (Z, Σ) a *building block* if

- (i) the class in $H^2(Z)$ of the anticanonical line bundle $-K_Z$ is indivisible,
- (ii) $\Sigma \in |-K_Z|$ (i.e. Σ is an anticanonical divisor), and there is a projective morphism $f : Z \rightarrow \mathbb{P}^1$ with $\Sigma = f^*(\infty)$,
- (iii) The inclusion $N \hookrightarrow H^2(\Sigma)$ is primitive, that is, $H^2(\Sigma)/N$ is torsion-free.
- (iv) The group $H^3(Z)$ —and thus also $H^4(Z)$ —is torsion-free.

We call N , equipped with the restriction of the intersection form on $H^2(\Sigma)$, the *polarising lattice* of the block.

If Γ is a group acting faithfully on Z by biholomorphisms that fix Σ pointwise then we call (Z, Σ, Γ) a *building block with automorphisms*. (Γ is then necessarily cyclic.)

Given such a (Z, Σ) , [23, Theorem D] gives the existence of ACyl Calabi-Yau structures on $V := Z \setminus \Sigma$, and it is easy to see that Γ restricts to isomorphisms of these structures.

Rather than to look for a hyper-Kähler rotation given a pair of ACyl Calabi-Yau structures, it is easier to first choose the diffeomorphism $r : \Sigma_+ \rightarrow \Sigma_-$ satisfying obvious necessary conditions in terms of cohomology classes and then find Calabi-Yau structures that make r a hyper-Kähler rotation. Recall that the *period* of a complex K3 surface Σ is the positive-definite 2-plane $\Pi \subset H^2(\Sigma; \mathbb{R})$ spanned by the real and imaginary parts of elements of $H^{2,0}(\Sigma; \mathbb{C})$.

Definition 1.6. Let (Z_{\pm}, Σ_{\pm}) be a pair of building blocks, and let $\Pi_{\pm} \subset H^2(\Sigma_{\pm}; \mathbb{R})$ be the periods. Call a diffeomorphism $r : \Sigma_+ \rightarrow \Sigma_-$ a *K3 matching* with angle ϑ if there are Kähler classes $k_{\pm} \in H^2(Z_{\pm}; \mathbb{R})$ such that the angle between $r^*(k_-)$ and Π_+ is ϑ , the angle between $(r^{-1})^*(k_+)$ and Π_- is ϑ , and $\Pi_+ \cap r^*\Pi_-$ is non-trivial.

It is easy to see that the ACyl Calabi-Yau structures of [23, Theorem D] can be chosen to ensure that a given K3 matching is a hyper-Kähler rotation; see [31, Theorem 1.1 and Lemma 6.2].

Theorem 1.7. *Given $\zeta_{\pm} > 0$, blocks (Z_{\pm}, Σ_{\pm}) and a K3 matching $r : \Sigma_+ \rightarrow \Sigma_-$ with angle ϑ , there exist ACyl Calabi-Yau structures on $V_{\pm} := Z_{\pm} \setminus \Sigma_{\pm}$ with asymptotic limit*

$$(du_{\pm} - idt_{\pm}) \wedge (\omega_{\pm}^J + i\omega_{\pm}^K), dt_{\pm} \wedge du_{\pm} + \omega_{\pm}^I,$$

such that r is an angle ϑ hyper-Kähler rotation of the hyper-Kähler structures $(\omega_{\pm}^I, \omega_{\pm}^J, \omega_{\pm}^K)$.

Thus given a K3 matching of two building blocks with automorphism and a torus matching with the corresponding k_+ , k_- and gluing angle ϑ , we can find ACyl Calabi-Yau structures so that Theorem 1.3 can be applied to build a G_2 -manifold.

Given a pair of blocks, there is no reason to expect to be able to find any K3 matchings at all. However, if we consider the sets of complex deformations of a pair of blocks, one can in many cases guarantee that there exist some elements of each of the two sets that admit a K3 matching, and moreover control the topology of the resulting G_2 -manifold. This will be discussed further in §1.4.

Represent $T_- = \tilde{T}_-/\Gamma_-$ similarly, and define $\varepsilon_- \in \mathbb{Z}/k_-$ with $\gcd(\varepsilon_-, k_-) = 1$ analogously. The isometry $\tilde{T}_+/\Gamma_+ \rightarrow \tilde{T}_-/\Gamma_-$ determines a sublattice $\tilde{\Lambda}_- \subset \Lambda$ such that $\tilde{T}_- \cong \mathbb{C}/\tilde{\Lambda}_-$. Let (μ_-, λ_-) denote a basis of $\tilde{\Lambda}_-$, where λ_- and μ_- correspond to the interior and exterior circle as above. We represent this basis as

$$(\mu_-, \lambda_-) = \frac{1}{k_+} \cdot (\mu_+, \lambda_+) \cdot \begin{pmatrix} m & p \\ n & q \end{pmatrix} \quad (1.6)$$

where $\begin{pmatrix} m & p \\ n & q \end{pmatrix} \in M_2(\mathbb{Z})$, and call $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ the *gluing matrix*. In Figure 2, we have $k_- = 3$, $\varepsilon_- = 1$, and the gluing matrix is $\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$; see entry 209 in Table 2.

In summary, we can associate to a torus matching the following data that is clearly invariant under our notion of equivalence

- k_+ and k_-
- the gluing matrix
- $\varepsilon_+ \in \mathbb{Z}/k_+$ and $\varepsilon_- \in \mathbb{Z}/k_-$ with $\gcd(\varepsilon_+, k_+) = \gcd(\varepsilon_-, k_-) = 1$.

For the construction, we also need the more geometric data

- the angle ϑ between the exterior circle directions
- the ratios $\frac{\xi_+}{\xi_-}$ and $s_{\pm} = \frac{\xi_{\pm}}{\zeta_{\pm}}$

which are not obviously invariant. However, among the (selection of) compatibility conditions that we now show, we see that the angle ϑ is in fact also determined by the equivalence class of the torus matching, and that if $\vartheta \notin \frac{\pi}{2}\mathbb{Z}$, then the ratios $\frac{\xi_{\pm}}{\zeta_{\pm}}$ and $\frac{\xi_+}{\xi_-}$ are as well.

Proposition 1.8. (i) *The data of a torus matching satisfies the following relations.*

$$\det \begin{pmatrix} m & p \\ n & q \end{pmatrix} = -k_-k_+, \quad (1.7)$$

$$\varepsilon_+m - n \equiv \varepsilon_+p - q \equiv 0 \pmod{k_+} \quad (1.8a)$$

$$\varepsilon_-p + m \equiv \varepsilon_-q + n \equiv 0 \pmod{k_-} \quad (1.8b)$$

$$\gcd\left(\frac{n - \varepsilon_+m}{k_+}, m\right) = \gcd\left(\frac{q - \varepsilon_+p}{k_+}, p\right) = \gcd(\varepsilon_+, k_+) = 1, \quad (1.9a)$$

$$\gcd\left(\frac{m + \varepsilon_-p}{k_-}, p\right) = \gcd\left(\frac{n + \varepsilon_-q}{k_-}, q\right) = \gcd(\varepsilon_-, k_-) = 1, \quad (1.9b)$$

(ii) *Either $m = q = 0$, or $n = p = 0$, or $\frac{nq}{mp} < 0$ and $s_+ = \frac{\xi_+}{\zeta_+} = \sqrt{-\frac{nq}{mp}}$. In the latter case,*

$$\text{we also have } \zeta_- = \sqrt{-\frac{qk_-}{mk_+}} \zeta_+, \xi_- = \sqrt{\frac{nk_-}{pk_+}} \zeta_+, \text{ and } s_- = \frac{\xi_-}{\zeta_-} = \sqrt{-\frac{mn}{pq}}.$$

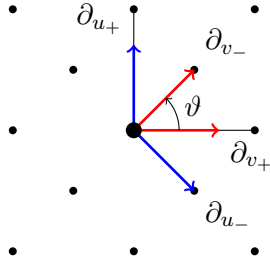
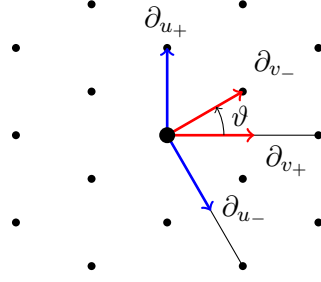
(iii) *The gluing angle ϑ is given as*

$$\vartheta = \arg(ms_+ + in) \in (-\pi, \pi].$$

In particular, $\vartheta \in (0, \pi)$ if and only if $n > 0$, and $\cos \vartheta = \text{sign}(m) \sqrt{-\frac{qm}{k_+k_-}}$.

Proof. For (1.7), we note that the bases (μ_-, λ_-) and (μ_+, λ_+) induce opposite orientations, and compute

$$-\frac{1}{k_+^2} \det \begin{pmatrix} m & p \\ n & q \end{pmatrix} = \frac{\text{vol}(\tilde{T}_-)}{\text{vol}(\tilde{T}_+)} = \frac{k_- \text{vol}(T)}{k_+ \text{vol}(T)} = \frac{k_-}{k_+}.$$

FIGURE 3. $G = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\vartheta = \frac{\pi}{4}$ FIGURE 4. $G = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$, $\vartheta = \frac{\pi}{6}$

We represent λ_- and μ_- in the basis (1.5) of Λ . By (1.6), we get

$$\begin{aligned} (\mu_-, \lambda_-) &= \frac{1}{k_+} \cdot (\mu_+, \lambda_+) \cdot \begin{pmatrix} \frac{1}{k_+} & 0 \\ \frac{\varepsilon_+}{k_+} & 1 \end{pmatrix} \cdot \begin{pmatrix} k_+ & 0 \\ -\varepsilon_+ & 1 \end{pmatrix} \cdot \begin{pmatrix} m & p \\ n & q \end{pmatrix} \\ &= (\nu_+, \lambda_+) \cdot \begin{pmatrix} m & p \\ \frac{n-\varepsilon_+m}{k_+} & \frac{q-\varepsilon_+p}{k_+} \end{pmatrix}. \end{aligned} \quad (1.10)$$

Because $\tilde{\Lambda}_- \subset \Lambda$, the coefficients are integers, and (1.8a) follows.

The group $\Gamma_- \cong \Lambda/\tilde{\Lambda}_-$ again acts freely by rotations on both the interior and the exterior circle of \tilde{T}_- . Equivalently, the elements λ_- and μ_- corresponding to the factors $S_{\zeta_-}^1$ and $S_{\xi_-}^1$ are primitive in Λ , which gives the first two gcd conditions in 1.9a. The last condition in 1.9a holds if and only if Γ_+ acts freely on \tilde{T}_+ .

Using (1.7), we can invert the gluing matrix. Then equation (1.6) is equivalent to

$$(\mu_+, \lambda_+) = \frac{1}{k_-} \cdot (\mu_-, \lambda_-) \cdot \begin{pmatrix} -q & p \\ n & -m \end{pmatrix}. \quad (1.11)$$

Now, the same arguments as above give (1.8b) and 1.9b.

The vectors λ_- , μ_- in (1.6) are perpendicular with respect to the standard metric on $\mathbb{C} \cong \mathbb{R}^2$ if and only if

$$0 = \frac{mp\xi_+^2 + nq\zeta_+^2}{k_+^2} = (mps_+^2 + nq) \cdot \frac{\zeta_+^2}{k_+^2},$$

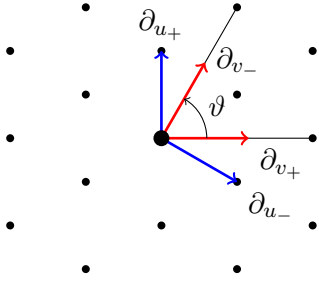
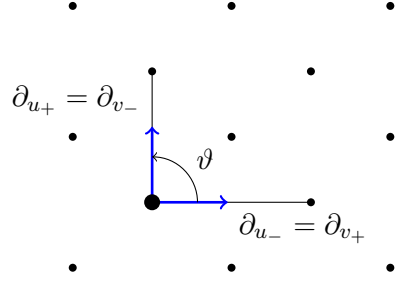
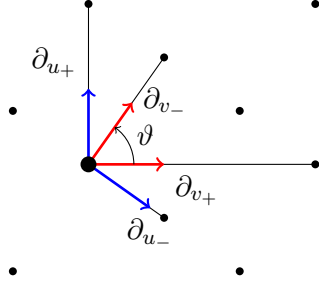
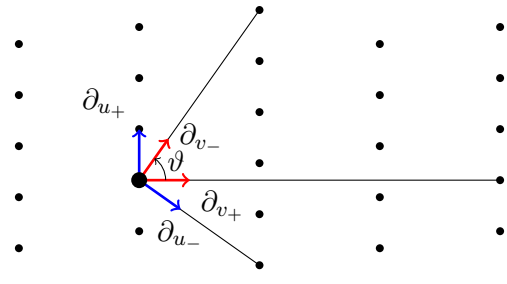
and the condition on $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ and s_+ follows. The remaining claims in (ii) follow because

$$\begin{aligned} \zeta_- = |\lambda_-| &= \frac{|q + ips_+| \zeta_+}{k_+} = \frac{\sqrt{q^2 + p^2 s_+^2}}{k_+} \zeta_+ = \sqrt{\frac{q(mq - np)}{m}} \frac{\zeta_+}{k_+} = \sqrt{-\frac{qk_-}{mk_+}} \zeta_+, \\ \xi_- = |\mu_-| &= \frac{\sqrt{n^2 + m^2 s_+^2}}{k_+} \zeta_+ = \sqrt{\frac{n(np - mq)}{p}} \frac{\zeta_+}{k_+} = \sqrt{\frac{nk_-}{pk_+}} \zeta_+. \end{aligned}$$

In [14], the gluing angle $\vartheta \in (-\pi, \pi]$ has been defined as the directed angle between μ_- and μ_+ , see also (1.3). We have $\vartheta \in (0, \pi)$ if and only if the scalar product $\langle \mu_-, \lambda_+ \rangle = \frac{n}{k_+} |\zeta_+|^2$ is positive. Hence, we get (iii) by

$$\vartheta = \arg \frac{\mu_+}{\mu_-} = \arg \frac{ik_+ \xi_+}{n\zeta_+ + im\xi_+} = \arg \frac{k_+(m\xi_+^2 + in\xi_+\zeta_+)}{n^2\zeta_+^2 + m^2\xi_+^2} = \arg(ms_+ + in). \quad \square$$

When $k_{\pm} \leq 2$, the only possibilities (up to swapping M_+ and M_-) with $m, n, q \geq 0$ and $p = 1$ are the ones already studied in [31, 14], illustrated in Figures 3–5. If we allow $p \geq 1$, there are


 FIGURE 5. $G = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}, \vartheta = \frac{\pi}{3}$

 FIGURE 6. $G = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \vartheta = \frac{\pi}{2}$

 FIGURE 7. $G = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \vartheta = \arccos \frac{1}{\sqrt{3}}$

 FIGURE 8. $G = \begin{pmatrix} 1 & 1 \\ 10 & -5 \end{pmatrix}, \vartheta = \arccos \frac{1}{\sqrt{3}}$

two more with gluing matrices $\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$; the latter is depicted in Figure 6. Notice that $\vartheta = \frac{\pi}{2}$ in this example, so the radii $\xi_+ = \zeta_-$ and $\xi_- = \zeta_+$ can be chosen independently.

Once we allow k_+ or k_- to be larger than 2, there are many more possibilities. Figure 7 illustrates a torus matching with $k_+ = 1$ and $k_- = 3$, where $s_+ = \sqrt{2}$ and $s_- = \frac{1}{\sqrt{2}}$ (so the tori have proportions of A4 paper). We consider this further in Sections 3. Let us for now give a single more complicated example that we will refer to in the course of our calculations.

Example 1.9. For $k_+ = 3$ and $k_- = 5$, one valid gluing matrix is

$$\begin{pmatrix} 1 & 1 \\ 10 & -5 \end{pmatrix}$$

with $\varepsilon_+ = 1$ and $\varepsilon_- = -1$. The torus matching is illustrated in Figure 8. The aspect ratios are $s_+ = 5\sqrt{2}$ and $s_- = \sqrt{2}$, and the gluing angle is $\vartheta = \arg(1 + \sqrt{2}i) = \arccos \frac{1}{\sqrt{3}}$. One example with this gluing matrix may be found in Table 2, no. 228.

Generalising the computations from [31, §1.3], the gluing matrix also determines the fundamental group of the extra-twisted connected sum.

Proposition 1.10. *An extra-twisted connected sum M with gluing matrix $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ has fundamental group isomorphic to \mathbb{Z}/p .*

Proof. Let $\iota_{\pm}: T^2 \rightarrow M_{\pm}$ denote the inclusion map and note that $\pi_1(T^2) \cong \pi_1(X) \cong \mathbb{Z}^2$. Since $\pi_1 V_{\pm} = 1$ by Remark 1.1, we also have $\pi_1(M_{\pm}) \cong \mathbb{Z}$, and the interior circle $S_{\zeta_{\pm}}^1$ is null-homotopic in M_{\pm} , and we have a short exact sequence

$$0 \longrightarrow \pi_1(S_{\zeta_{\pm}}^1) \longrightarrow \pi_1(T^2) \xrightarrow{\iota_{\pm*}} \pi_1(M_{\pm}) \longrightarrow 0.$$

Because $\iota_{\pm*}$ is surjective, it follows from the Seifert-van Kampen theorem that

$$\pi_1(M) \cong \pi_1(T^2) / (\ker(\iota_{+*}) + \ker(\iota_{-*})).$$

As basis of $\Lambda = \pi_1(T^2)$, we choose the vectors $\nu_+ = \frac{\mu_+ + \varepsilon_+ \lambda_+}{k_+}$ and λ_+ as in (1.5). Dividing out $\pi_1(S_{\zeta_+}^1) = \ker(\iota_+)$, we are left with a cyclic group generated by ν_+ . Modulo $\pi_1(S_{\zeta_+}^1)$, the group $\pi_1(S_{\zeta_-}^1) = \ker(\iota_-)$ is generated by $p\nu_+$, so $\pi_1(M) \cong \mathbb{Z}/p$. \square

We will discuss covering spaces in Proposition 3.5. Some examples of non-simply connected extra-twisted connected sums will be given in Examples 3.1 (i) and 3.11.

1.4. Matchings and polarising lattices. In Theorem 1.3 we set up our gluing construction, using a torus matching \mathfrak{t} and a hyper-Kähler rotation \mathfrak{r} . We studied the torus matchings in §1.3, while Theorem 1.7 reduced the problem of finding hyper-Kähler rotations to the less metric problem of finding K3 matchings. For the final piece of the machine, we review from [31, §6] how to find K3 matchings between building blocks.

The properties of the G_2 -manifolds produced by Theorem 1.3 can clearly depend not just on the choices of building blocks and torus matching but also on the choice of hyper-Kähler rotation. However, the topological properties that we care about depend on the hyper-Kähler rotation only via what we term its associated “configuration” of the polarising lattices of the building blocks.

Recall from Definition 1.5 that the polarising lattice of a block (Z, Σ) refers to the image N of $H^2(Z)$ in $H^2(\Sigma)$, equipped with the intersection form. We use L to denote a fixed even unimodular lattice with signature $(3, 19)$, so that $H^2(\Sigma)$ is isometric to L for any K3 surface Σ .

Definition 1.11 ([31, Definition 6.3]). A *configuration* of polarising lattices N_+, N_- is a pair of primitive embeddings $N_{\pm} \hookrightarrow L$. Two configurations are equivalent if they are related by the action of the isometry group $O(L)$.

Given the claim that the topology depends mainly on the blocks and the configuration, it is natural to phrase the matching problem as follows.

Question 1.12. Given $\vartheta \in \mathbb{R}$ and a pair of sets of building blocks (Z_{\pm}, Σ_{\pm}) (each family with fixed topology and in particular fixed polarising lattice N_{\pm}), which configurations of N_+ and N_- are realised by a ϑ -hyper-Kähler rotation of some elements of the families?

Given a configuration, let $\pi_{\pm} : L \rightarrow N_{\pm}$ denote the orthogonal projection, and let N_{\pm}^{μ} denote the $(\cos \mu)^2$ -eigenspace of the self-adjoint endomorphism $\pi_{\pm} \pi_{\mp} : N_{\pm} \rightarrow N_{\pm}$, and let $N_{\pm}^{\neq \vartheta}$ denote the orthogonal complement to N_{\pm}^{ϑ} (i.e. the direct sum of the eigenspaces with eigenvalue other than ϑ).

The condition (1.4) implies that if there exists a ϑ -hyper-Kähler rotation compatible with a given configuration then there are positive classes $[\omega_{\pm}^I], [\omega_{\pm}^J] \in N_{\pm}$ and $[\omega_{\pm}^K] \in N_{\pm}^{\perp}$ such that

$$\begin{aligned} [\omega_-^K] &= -[\omega_+^K] \\ ([\omega_-^I] + i[\omega_-^J]) &= e^{i\vartheta}([\omega_+^I] - i[\omega_+^J]). \end{aligned}$$

From this we can deduce the following necessary conditions for realising a given configuration (see [31, §6.3] for explanation) by a ϑ -hyper-Kähler rotation of some (Z_+, Σ_-) and (Z_-, Σ_-) .

- (i) $N_+ + N_-$ is non-degenerate of signature $(2, \text{rk} - 2)$.
- (ii) N_{\pm}^{ϑ} contains the restriction of some Kähler class from Z_{\pm} ; in particular N_{\pm}^{ϑ} is non-trivial.
- (iii) If we let Λ_{\pm} be a primitive overlattice of $N_{\pm} + N_{\mp}^{\neq \vartheta}$ in L , then Σ_{\pm} is Λ_{\pm} -polarised, i.e. $\text{Pic } \Sigma_{\pm}$ contains Λ_{\pm} .

On the other hand, it turns out to be possible to express a sufficient condition for being able to match some elements from a pair of families in terms of those families containing suitably generic Λ_{\pm} -polarised K3 surfaces. Recall that marked K3 surfaces whose Picard lattice contains

a fixed lattice $\Lambda \subset L$ can be parametrised by their periods, which belong to the Griffiths domain

$$D_\Lambda = \{\text{positive-definite planes } \Pi \subset \Lambda^\perp(\mathbb{R})\} \cong \{\Pi \in \mathbb{P}(\Lambda^\perp(\mathbb{C})) : \Pi^2 = 0, \Pi \bar{\Pi} > 0\},$$

where the second description gives rise to a complex analytic structure.

Definition 1.13 ([31, Definition 2.12]). Let $N \subset L$ be a primitive sublattice, $\Lambda \subset L$ a primitive overlattice of N , and $\text{Amp}_{\mathcal{Z}}$ an open subcone of the positive cone in $N_{\mathbb{R}}$. We say that a set of building blocks \mathcal{Z} with polarising lattice N is $(\Lambda, \text{Amp}_{\mathcal{Z}})$ -generic if there is a subset $U_{\mathcal{Z}}$ of the Griffiths domain D_Λ with complement a countable union of complex analytic submanifolds of positive codimension with the property that: for any $\Pi \in U_{\mathcal{Z}}$ and $k \in \text{Amp}_{\mathcal{Z}}$ there is a building block $(Z, \Sigma) \in \mathcal{Z}$ and a marking $h : L \rightarrow H^2(\Sigma; \mathbb{Z})$ such that $h(\Pi) = H^{2,0}(\Sigma)$, and $h(k)$ is the image of the restriction to Σ of a Kähler class on Z .

Proposition 1.14 ([31, Theorem 6.8]). Let \mathcal{Z}_\pm be a pair of sets of building blocks with polarising lattices N_\pm , and $\vartheta \in \mathbb{R}$. Let $N_\pm \hookrightarrow L$ be a configuration of the polarising lattices, and define Λ_\pm as in (iii) above. Suppose that the set \mathcal{Z}_\pm is $(\Lambda_\pm, \text{Amp}_{\mathcal{Z}_\pm})$ -generic. If

$$\cos \vartheta \neq 0 \text{ and } (\text{sign } \cos \vartheta) \pi_-(N_+(\mathbb{R})^\vartheta \cap \text{Amp}_{\mathcal{Z}_+}) \cap \text{Amp}_{\mathcal{Z}_-} \neq \emptyset. \quad (1.12)$$

or

$$\cos \vartheta = 0 \text{ and } N_+(\mathbb{R})^{\frac{\pi}{2}} \cap \text{Amp}_{\mathcal{Z}_+} \neq \emptyset \text{ and } N_-(\mathbb{R})^{\frac{\pi}{2}} \cap \text{Amp}_{\mathcal{Z}_-} \neq \emptyset \quad (1.13)$$

then there exist $(Z_\pm, \Sigma_\pm) \in \mathcal{Z}_\pm$ with an angle ϑ K3 matching $r : \Sigma_+ \rightarrow \Sigma_-$ with the prescribed configuration.

In [14] we found that the following property of a configuration plays a key role in the calculation of ν (see Theorem 2.1).

Definition 1.15. Given a configuration $N_+, N_- \subset L$, let $A_\pm : L_{\mathbb{R}} \rightarrow L_{\mathbb{R}}$ denote the reflection of $L_{\mathbb{R}} := L \otimes \mathbb{R}$ in N_\pm (with respect to the intersection form of $L_{\mathbb{R}}$; this is well-defined since N_\pm is non-degenerate). Suppose $A_+ \circ A_-$ preserves some decomposition $L_{\mathbb{R}} = L^+ \oplus L^-$ as a sum of positive and negative-definite subspaces. Then the *configuration angles* are the arguments $\alpha_1^+, \alpha_2^+, \alpha_3^+$ and $\alpha_1^-, \dots, \alpha_{19}^-$ of the eigenvalues of the restrictions $A_+ \circ A_- : L^+ \rightarrow L^+$ and $A_+ \circ A_- : L^- \rightarrow L^-$ respectively.

Remark 1.16. Since Λ_\pm is always at least as big as N_\pm , the genericity results required to apply Proposition 1.14 are the weakest possible when $\Lambda_\pm = N_\pm$. This happens in particular if $N_\pm^\vartheta = N_\pm$, that is, if $\pi_\pm \circ \pi_\mp|_{N_\pm} = \cos^2 \vartheta \text{id}_{N_\pm}$. In that case we will say that N_+ and N_- meet at pure angle ϑ .

Unless $\vartheta = \pm \frac{\pi}{2}$, meeting at pure angle ϑ implies that $\text{rk } N_+ = \text{rk } N_- = \text{multiplicity of } \pm 2\vartheta$ as configuration angles, while the remaining configuration angles are all 0.

Even with Proposition 1.14 in hand, the genericity hypothesis required makes it hard in general to completely answer Question 1.12 concerning which configurations can be realised by matching. However, all examples we know of building blocks do in fact have the property that they come in families that are (N, Amp) generic, for N the polarising lattice, and Amp some open cone in $N_{\mathbb{R}}$ (in particular, Proposition 5.11 asserts this for the examples in this paper). Therefore, finding all matchings of a pair of blocks where the configurations are at pure angle ϑ is only a lattice-arithmetic problem. That can certainly be solved by a brute force algorithm, though not very easily by hand if the ranks of the polarising lattices are greater than 1.

In this paper, we will restrict attention to blocks where the polarising lattices have rank 1, which makes it possible to answer Question 1.12 decisively. Condition (ii) then automatically requires the configurations to have pure angle. If the generators of the polarising lattices have

square-norms n_+ and n_- , then the bilinear form on $N_+ + N_-$ imposed by the configuration will be defined by a matrix

$$\begin{pmatrix} n_+ & h \\ h & n_- \end{pmatrix},$$

and the gluing angle is determined by

$$(\cos \vartheta)^2 = \frac{h^2}{n_+ n_-}. \quad (1.14)$$

Thus there exists a matching of the blocks with gluing angle ϑ if and only if $\cos \vartheta \sqrt{n_+ n_-}$ is an integer.

We give here one example that we will refer to while developing the calculations in Sections 2 and 4; see Subsection 5.3 for further examples of matchings.

Example 1.17. Consider two building blocks Z_+ , Z_- of rank 1 with polarising lattices (6) and (2), respectively. We consider the configuration

$$\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix},$$

which has pure angle $\vartheta = \arccos \frac{1}{\sqrt{3}}$. We will combine this configuration with the gluing data of Example 1.9, using a $\mathbb{Z}/3$ -block from Example 5.5 as Z_+ and a $\mathbb{Z}/5$ -block from Example 5.10 as Z_- , see Table 2, no. 228. The configuration angles are

$$\alpha_1^+ = -\alpha_2^+ = 2 \arccos \frac{1}{\sqrt{3}} \quad \text{and} \quad \alpha_3^+ = \alpha_1^- = \dots = \alpha_{19}^- = 0.$$

2. COMPUTING THE ν -INVARIANT

Using various results on η -invariants, including the gluing formula for η -invariants [9, 27], the variational formula for η -invariants on manifolds with boundary [4, 11, 20], and a combination of the adiabatic limit formula for manifolds with boundary in [19] with the one for Seifert fibrations in [22], we rewrite the ν -invariant of an extra-twisted connected sum in more explicit terms; see Theorem 2.13. For one of the contributions, we will give another description in the next chapter.

2.1. A modification of the spin Dirac operator. The extended ν -invariant of a G_2 -manifold is defined in (0.1) using the η -invariant of the signature operator B and the spin Dirac operator D . For computations, it is much more comfortable to work with a Riemannian metric that is of product type in the gluing region and sufficiently close to some G_2 -metric. However, the η -invariant of the spin Dirac operator of such a gluing metric typically differs from the one in the G_2 -case both by a small local contribution and by a \mathbb{Z} -valued spectral flow. To avoid the latter, we modified the spin Dirac operator in [14]. Because all our following considerations rely on the modified metric and the modified Dirac operator, we take the time to introduce them now.

2.1.1. We first recall some properties of the family of gluing metrics g_ℓ on M for $\ell \gg 1$ from [14, Rem 4.1]. The Riemannian manifold (M_ℓ, g_ℓ) contains a cylindrical piece diffeomorphic to $\Sigma \times T^2 \times (-\ell, \ell)$. Let $t: M \rightarrow \mathbb{R}$ be a smooth function that agrees with the cylindrical coordinate on $\Sigma \times T^2 \times (-\ell, \ell)$ and takes values outside $(-\ell, \ell)$ otherwise.

- (i) For $\pm t \geq -1$, the Riemannian manifold (M, g_ℓ) is isometric to a twisted product $(V_\pm \times S_{\xi_\pm}^1)/\Gamma_\pm$ of $(V_\pm, g_\ell^{V_\pm})$ and a circle $S_{\xi_\pm}^1$ of length ξ_\pm .

- (ii) For $\pm t \geq 2$, the metric $g_\ell^{V_\pm}$ is isometric to the original asymptotically cylindrical Calabi-Yau metric g^{V_\pm} .
- (iii) The manifold $(\Sigma \times T^2 \times (-1, 1))$ is the Riemannian product of the K3 surface Σ , the torus T^2 and the interval $(-1, 1)$ of length 2.
- (iv) There exists $c > 0$ such that $\|g_\ell|_{X \times (\pm[1, 2])} - g^X \oplus dt^2\|_{C^k} = O(e^{-c\ell})$ for all k .
- (v) Among the G_2 -metrics provided by Theorem 1.3, there is one—called \bar{g}_ℓ below—such that for some $c > 0$ and for each k , we have an estimate of the form

$$\|g_\ell - \bar{g}_\ell\|_{C^k} = O(e^{-c\ell}).$$

It follows from (ii) and (iii) that g_ℓ has local holonomy in G_2 except over the set $\Sigma \times T^2 \times ([-2, -1] \cup [1, 2])$, which is controlled by (i) and (iv).

Let $s_\pm = \xi_\pm / \zeta_\pm$ as in Proposition 1.8 (ii). We now consider the two halves M_\pm separately and put

$$M_{\pm, \ell, a} = (V_\pm \times S_{a\zeta_\pm}^1) / \Gamma_\pm, \quad (2.1)$$

where $S_{a\zeta_\pm}^1$ denotes an exterior circle of length $\xi_\pm = a\zeta_\pm$, and where V_\pm carries the metric $g_\ell^{V_\pm}$ considered in (i) above. Then the new metric $g_{\pm, \ell, a}$ on $M_{\pm, \ell, a}$ satisfies properties analogous to (i)–(v) above. For $a = s_\pm$, we recover the restriction of the metric g_ℓ . We will consider the odd signature operator $B_{M_{\pm, a, \ell}}$, and we write $B_{M_\pm, \ell}$ if $a = s_\pm$.

2.1.2. By Remark 1.4 and property (i) of the metrics $g_{\pm, \ell, a}$, we may describe the spinor bundle on $M_{\pm, \ell, a}$ with Hermitian metric and Clifford connection $\nabla^{SM} = \nabla^{SM_{\pm, \ell, a}}$ as

$$SM_{\pm, \ell, a} = p^* SV_{\pm, \ell} / \Gamma_\pm, \quad (2.2)$$

where $p: V_{\pm, \ell} \times S_{a\zeta_\pm}^1 \rightarrow V_{\pm, \ell}$ is the projection. The glued manifold M_ℓ with G_2 -metric \bar{g}_ℓ carries a parallel spinor s . Recall that ∂_{v_\pm} is the unit tangent vector to the exterior circle factor in the twisted Riemannian product $M_{\pm, \ell, a}$. We identify the spinor bundles for the metrics $\bar{g}_\ell|_{M_{\pm, \ell}}$ and $g_{\pm, \ell, a}$ in such a way that

- (i) the spinor s on $M_{\pm, \ell, a}$ is pulled back from a Γ_\pm -invariant unit spinor on $V_{\pm, \ell}$,
- (ii) its derivative $\nabla^{SM} s$ is supported on $X \times (\pm[1, 2])$,
- (iii) there exists $c > 0$ such that $\|\nabla^{SM} s\| = O(e^{-c\ell})$,
- (iv) we have $\nabla_{\partial_{v_\pm}}^{SM} s = 0$.

2.1.3. Let $D'_{M_{\pm, \ell, a}}$ denote the geometric spin Dirac operator of $M_{\pm, \ell, a}$, and let c_{v_\pm} denote Clifford multiplication by ∂_{v_\pm} . Decomposing $D'_{M_{\pm, \ell, a}} s$ using (2.2) and the properties of $g_{\pm, \ell, a}$ and s above, we find functions f_\pm, h_\pm on V_\pm and a spinor $r_\pm \in \Gamma(SM)$ that is pulled back from a Γ_\pm -invariant spinor on $V_{\pm, \ell}$, all independent of a , such that

$$D'_{M_{\pm, \ell, a}} s = f_\pm \cdot s + h_\pm \cdot c_{v_\pm} s + r_\pm, \quad (2.3)$$

and such that r_\pm is perpendicular to s and $c_{v_\pm} s$ everywhere. As in [14, (31), (32)], put

$$\begin{aligned} D_{M_{\pm, a, \ell}} &= D'_{M_{\pm, \ell, a}} - \langle \cdot, s \rangle (f_\pm s + h_\pm c_{v_\pm} s + r_\pm) - \langle \cdot, r_\pm \rangle s \\ &\quad - \langle \cdot, c_{v_\pm} s \rangle (h_\pm s - f_\pm c_{v_\pm} s - c_{v_\pm} r_\pm) + \langle \cdot, c_{v_\pm} r_\pm \rangle c_{v_\pm} s. \end{aligned} \quad (2.4)$$

Then $\ker D_{M_{\pm, a, \ell}}$ contains the parallel spinors s and $c_{v_\pm} s$ for all ℓ and all $a > 0$.

If we consider the special case $a = s_\pm$, the operators above combine to an operator $D_{M, \ell}$.

- (i) On $M_\ell \setminus (X \times ([-2, -1] \cup [1, 2]))$, the operator $D_{M, \ell}$ agrees with the geometric spin Dirac operator of the gluing metric g_ℓ described above.
- (ii) By [14, Prop 5.7], the kernel of $D_{M, \ell}$ is spanned by a nowhere vanishing section s .

- (iii) We have $D_{M,\ell}|_{M_{\pm,\ell}}(c_{v_{\pm}}s) = 0$ by [14, (34)].
- (iv) Again by [14, Prop 5.7], there is a constant $c > 0$ such that for ℓ sufficiently large, there exists a G_2 -metric \bar{g}_ℓ on M with geometric spin Dirac operator $D_{(M,\bar{g}_\ell)}$ such that

$$\eta(D_{(M,\bar{g}_\ell)}) = \eta(D_{M,\ell}) + O(e^{-c\ell}).$$

In particular, there is no spectral flow if we deform $D_{(M,\bar{g}_\ell)}$ into $D_{M,\ell}$.

From 2.1.1 (v) and 2.1.3 (iv), we conclude that

$$\bar{\nu}(M) = \lim_{\ell \rightarrow \infty} (3\eta(B_{M,\ell}) - 24\eta(D_{M,\ell})). \quad (2.5)$$

2.2. The gluing formula. In the following, we cut (M, g_ℓ) into two halves $M_{\pm,\ell,s_{\pm}}$ with common boundary $\Sigma \times T^2 \times \{0\}$. We identify $\Lambda^\bullet T^*(\Sigma \times T^2)$ with the restriction of $\Lambda^{\text{ev}} T^*M$, let the boundary operator A of the odd signature operator B act on $\Omega^\bullet(X)$ as in [14, (25)], and put

$$L_{B_{\pm}} = \text{im}(H^\bullet(M_{\pm}) \rightarrow H^\bullet(X)) \subset H^\bullet(X), \quad (2.6)$$

then $L_{B_{\pm}}$ are Lagrangian subspaces of $H^\bullet(X) \cong \ker A$. As in [14, sec 4.2], let $\eta(B_{M_{\pm,a,\ell}}; L_{B_{\pm}})$ denote the η -invariant of $B_{M_{\pm,a,\ell}}$ with respect to APS boundary conditions modified by $L_{B_{\pm}}$; in particular, the forms in the domain of $B_{M_{\pm,a,\ell}}$ orthogonally project to 0 on the Lagrangian in $\Omega^\bullet(X)$ given as the direct sum of $L_{B_{\pm}}$ with the sum of all eigenspaces A of eigenvalues of sign \pm .

For the operator $D_{M_{\pm,a,\ell}}$, we define similar boundary conditions as in [14, (37)]. We identify the spinor bundle of $\Sigma \times T^2$ with the restriction of the spinor bundle of M and write

$$D_{M_{\pm,a,\ell}}|_{X \times (-1,1)} = \gamma \left(\frac{\partial}{\partial t} + A_{\pm,a} \right),$$

where $\gamma = c_t$ denotes Clifford multiplication with $\frac{\partial}{\partial t}$ and $A_{\pm,a}$ now denotes the boundary operator of the modified spin Dirac operator $D_{M_{\pm,a,\ell}}$. We may write A as a shorthand for $A_{+,s_+} = A_{-,s_-}$. Note that $\ker A_{\pm,a} \cong H^{0,\bullet}(X) \cong \mathbb{C}^4$ independent of a .

Together with the L^2 -metric on spinors, γ introduces a symplectic structure on $\ker A_{\pm,a}$. Let s span $\ker(D_{M,\ell})$ as in 2.1.3 (ii) above, then by 2.1.3 (iii),

$$L_{D_-} = \text{span}\{s, c_{v_-}s\} \quad \text{and} \quad L_{D_+} = \text{span}\{s, c_{v_+}s\} \quad (2.7)$$

are Lagrangian subspaces of $\ker A_{\pm,a}$, namely, the space of $A_{\pm,a}$ -harmonic spinors on X that extend to $D_{M_{\pm,a,\ell}}$ -harmonic spinors on $M_{\pm,\ell,a}$ for each a . Define $\eta(D_{M_{\pm,a,\ell}}; L_{D_{\pm}})$ as above. In particular, the spinors in the domain of $D_{M_{\pm,a,\ell}}$ project orthogonally to 0 on the Lagrangian in $\Gamma(SX)$ given as the direct sum of $L_{D_{\pm}}$ with the sum of all eigenspaces of $A_{\pm,a}$ of eigenvalues of sign \pm .

Theorem 2.1 ([14, Thm 1]). *Let M be an extra-twisted connected sum with gluing angle $\vartheta \in (0, \pi)$, and let $\rho = \pi - 2\vartheta$. Let $A_{N_{\pm}}$ denote the reflection in the subspace $N_{\pm} = \text{Im}(H^2(V_{\pm}) \rightarrow H^2(\Sigma)) \subset L = H^2(\Sigma)$, and assume that the orthogonal automorphism $A_{N_+}A_{N_-}$ of $H^{2,\pm}(\Sigma) \otimes \mathbb{C}$ has eigenvalues $e^{i\alpha_j^{\pm}}$ with $\alpha_1^+, \alpha_2^+, \alpha_3^+, \alpha_1^-, \dots, \alpha_{19}^- \in (-\pi, \pi]$ (i.e. α_i^{\pm} are configuration angles in the sense of Definition 1.15). Put*

$$\begin{aligned} \bar{\nu}(M_{\pm,a}) &= \lim_{\ell \rightarrow \infty} (3\eta(B_{M_{\pm,a,\ell}}; L_{B_{\pm}}) - 24\eta(D_{M_{\pm,a,\ell}}; L_{D_{\pm}})) \\ \text{and} \quad m_\rho(L; N_+, N_-) &= \text{sign } \rho (\#\{j \mid \alpha_j^- \in \{\pi - |\rho|, \pi\}\} - 1) \\ &\quad + 2 \text{sign } \rho \#\{j \mid \alpha_j^- \in (\pi - |\rho|, \pi)\} \in \mathbb{Z}, \end{aligned}$$

then the extended ν -invariant of M is given by

$$\bar{\nu}(M, g) = \bar{\nu}(M_{+,s_+}) + \bar{\nu}(M_{-,s_-}) - 72 \frac{\rho}{\pi} + 3m_\rho(L; N_+, N_-). \quad (2.8)$$

Note that in the examples in [14], we had $\Gamma_\pm = \mathbb{Z}/k_\pm$ with $k_\pm \in \{1, 2\}$. In these cases, one could find orientation reversing isometries of $M_{\pm, \ell, a}$ that anticommute with $B_{M_{\pm, \ell, a}}$, $D_{M_{\pm, \ell, a}}$ and preserve the boundary conditions, leading to $\eta(B_{M_{\pm, \ell, a}}; L_{B_\pm}) = \eta(D_{M_{\pm, \ell, a}}; L_{D_\pm}) = 0$ independent of a . Here, we want to deal with examples where this is no longer the case. We have examples where $\vartheta \notin \mathbb{Q}\pi$, so that $\frac{\rho}{\pi} \notin \mathbb{Q}$. In these cases at least some of the η -invariants above must have irrational limits. In particular, they can no longer vanish.

Example 2.2. With the configuration angles of Example 1.17, we get $\rho = \pi - 2 \arccos \frac{1}{\sqrt{3}} > 0$ and hence

$$-72 \frac{\rho}{\pi} + 3m_\rho(L; N_+, N_-) = -72 + \frac{144}{\pi} \arccos \frac{1}{\sqrt{3}} - 3.$$

2.3. The adiabatic limit of η -invariants. We still work on the manifolds $M_{\pm, \ell, a}$, which are twisted Riemannian products by property 2.1.1 (i) above. We also still consider the modification $D_{M_{\pm, \ell, a}}$ of the spin Dirac operator considered in (2.4).

We write

$$\bar{\nu}(M_\pm) = \bar{\nu}(M_{\pm, s_\pm}) = \lim_{a \rightarrow 0} \bar{\nu}(M_{\pm, a}) + \int_0^{s_\pm} \frac{d}{da} \bar{\nu}(M_{\pm, a}) da. \quad (2.9)$$

We consider $W_\pm = V_\pm/\Gamma_\pm$ as an orbifold with boundary, where the boundary itself is a manifold by assumption. Let ΛW_\pm denote its inertia orbifold. The orbifold \hat{A} -class on ΛW is defined in [22, (1.6)]. We will also need the \hat{L} -class; see [22, Cor 1.10]. Let \mathbb{A} denote the Bismut superconnection of the fibrewise spin Dirac operator for the map $p: M_{\pm, \ell, a} \rightarrow W_\pm$ with respect to the fibrewise trivial spin structure; see Remark 1.4. Let $\eta(\mathbb{A}) \in \Omega^\bullet(\Lambda W)$ denote the orbifold η -form as in [22, Def 1.7].

Then by Theorem 6.1, we find

$$\lim_{a \rightarrow 0} \bar{\nu}(M_{\pm, a}) = \int_{\Lambda W_\pm \setminus W_\pm} \left(3\hat{L}_{\Lambda W_\pm}(TW_\pm, \nabla^{TW_\pm}) - 24\hat{A}_{\Lambda W_\pm}(TW_\pm, \nabla^{TW_\pm}) \right) 2\eta_{\Lambda W_\pm}(\mathbb{A}). \quad (2.10)$$

Because W_\pm is even-dimensional, there is no contribution from η -invariants on W_\pm . Moreover, there are no very small eigenvalues in our situation. We remark that the circle orbibundle $M_\pm \rightarrow W_\pm$ is flat by construction, so the integral above localises at the orbifold singularities of W_\pm , and there is no contribution from the principal stratum. We are in a local product situation, so the orbifold η -forms all reduce to equivariant η -invariants.

The action of $\Gamma = \mathbb{Z}/k\mathbb{Z}$ on V is faithful because it is free on ∂V . At each fixpoint $p \in V^\gamma$ of $\gamma \in \Gamma$, the tangent space $T_p V$ splits as a sum of complex eigenspaces of the differential of γ with eigenvalues $e^{i\alpha_\ell}$, with $\alpha_1, \alpha_2, \alpha_3 \in \frac{2\pi}{k}\mathbb{Z}$. Because γ preserves the holomorphic volume form, the angles α_ℓ add up to a multiple of 2π . Hence, the complex codimension of the fixpoint set has to be at least 2. If the fixpoints are not isolated, then $V^\gamma \subset V$ is totally geodesic, and the eigenspaces locally form bundles over V^γ . The tangent bundle TV^γ corresponds to $\alpha_\ell = 0$. Let $\nu_\gamma \rightarrow V^\gamma$ denote the normal bundle.

We assume that the coordinate $v \in \mathbb{R}/\xi\mathbb{Z}$ on the exterior circle has been chosen such that inserting ∂_v into the G_2 -form φ gives the Kähler form on V ; see [14, (8)]. Then let $\gamma \in \Gamma$ be the generator that acts on the exterior circle by sending v to $v + \frac{\xi}{k}$. We start by defining a *generalised Dedekind sum* as in [22]. Note that it depends on the particular choice of generator γ .

Definition 2.3. Let $\gamma \in \Gamma \cong \mathbb{Z}/k\mathbb{Z}$ be a generator. For $0 < j < k$, let $V^{0,j}$ denote the set of isolated fixpoints of γ^j , and for each $p \in V^{0,j}$, let $\alpha_{j,1}(p)$, $\alpha_{j,2}(p)$, $\alpha_{j,3}(p)$ denote the angles of the action of γ^j on the complex vector space $(T_p V)$, chosen such that $\alpha_{j,1}(p) + \alpha_{j,2}(p) + \alpha_{j,3}(p) \in 4\pi\mathbb{Z}$. Then define

$$D_\gamma(V) = \frac{3}{k} \sum_{j=1}^{k-1} \cot \frac{\pi j}{k} \sum_{p \in V^{0,j}} \frac{\cos \frac{\alpha_{j,1}(p)}{2} \cos \frac{\alpha_{j,2}(p)}{2} \cos \frac{\alpha_{j,3}(p)}{2} - 1}{\sin \frac{\alpha_{j,1}(p)}{2} \sin \frac{\alpha_{j,2}(p)}{2} \sin \frac{\alpha_{j,3}(p)}{2}},$$

Theorem 2.4. Let $\gamma_\pm \in \Gamma_\pm \cong \mathbb{Z}/k_\pm\mathbb{Z}$ be the generator that acts on the exterior circle $\mathbb{R}/\xi_\pm\mathbb{Z}$ by sending v_\pm to $v_\pm + \frac{\xi_\pm}{k}$. Define $D_{\gamma_\pm}(V_\pm)$ as above, then

$$\lim_{a \rightarrow 0} \bar{\nu}(M_{\pm,a}) = D_{\gamma_\pm}(V_\pm).$$

Note that the theorem entails that non-isolated fixpoints do not contribute to the adiabatic limit of the extended ν -invariant. We have shown in [14] that $\bar{\nu}(M_\pm) = 0$ if $k_\pm = 1$ or $k_\pm = 2$. This fits with our considerations above because involutions of odd-dimensional Calabi-Yau manifolds cannot have isolated fixpoints.

Proof. As Calabi-Yau manifold, V has a preferred spin structure with spinor bundle $\Lambda^{0,\bullet} T^*V$. The Kähler metric identifies $\Lambda^{0,1} T^*V$ with TV with its natural complex structure. Let $\gamma \in \Gamma$. Because V^γ is at most one-dimensional, we can split $T_p V$ into one-dimensional eigenspaces that are also invariant under the curvature tensor $F \in \Lambda^{1,1} \text{End}(T_p V)$. This allows us to decompose the action of $\gamma e^{-\frac{F}{2\pi i}}$ on the spinor space $\Lambda^{0,\bullet} T^*V|_{V^\gamma}$ as

$$\gamma e^{-\frac{F}{2\pi i}}|_{\Lambda^{0,\bullet} T_p^* V} \cong \bigotimes_{j=1}^3 \left(1 \quad e^{i\alpha_\ell(1 + \beta_\ell)} \right),$$

where $\beta_\ell \in \Lambda^{1,1} T^*V^\gamma$ are real differential forms that represent the Chern roots of the subbundle of $TV|_{V^\gamma}$ corresponding to the eigenvalue $e^{i\alpha_\ell}$. We assume that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Because V is Ricci-flat, we know that $\beta_1 + \beta_2 + \beta_3 = 0$. This allows us to twist each tensor factor above with a line L_ℓ on which $\gamma e^{-\frac{F}{2\pi i}}$ acts as $e^{-i\frac{\alpha_\ell}{2}}(1 - \frac{\beta_\ell}{2})$. This gives us the decomposition

$$\gamma e^{-\frac{F}{2\pi i}}|_{\Lambda^{0,\bullet} T_p^* V} \cong \bigotimes_{\ell=1}^3 \left(e^{-i\frac{\alpha_\ell}{2}} \left(1 - \frac{\beta_\ell}{2} \right) \quad e^{i\frac{\alpha_\ell}{2}} \left(1 + \frac{\beta_\ell}{2} \right) \right). \quad (2.11)$$

Finally, we note that the Seifert fibration $M_\pm \rightarrow V_\pm/\Gamma_\pm$ is locally of product geometry. Therefore, the equivariant η -form $\tilde{\eta}_{\gamma^j}(\mathbb{A})$ reduces to half the equivariant η -invariant. If $\gamma \in \Gamma$ denotes the preferred generator, then

$$\eta_{\gamma^j}(D_{S^1}) = \eta_{\gamma^j}(B_{S^1}) = -i \cot \frac{\pi j}{k} \in \Omega^0(V) \quad (2.12)$$

with respect to the preferred orientations.

Complex one-dimensional fixpoint sets. Assume that $C \subset V^{\gamma^j}$ is a connected component of the fixpoint set of γ^j with $\dim_{\mathbb{C}} C = 1$, and with normal bundle $\nu_C \rightarrow C$ in V . Along C , we have $\alpha_2 = -\alpha_1$ and $\alpha_3 = 0$. To compute the orbifold \hat{A} -class following [22, (1.6), (1.7)] and [3,

sect. 6.4], we need

$$\begin{aligned} \text{ch}(\gamma^j, \Lambda^{0,\text{ev}}\nu_C^* - \Lambda^{0,\text{odd}}\nu_C^*)|_{(C,\gamma^j)} &= \prod_{\ell=1}^2 \text{str} \begin{pmatrix} e^{-\frac{i\alpha_\ell}{2}} \left(1 - \frac{\beta_\ell}{2}\right) & \\ & e^{\frac{i\alpha_\ell}{2}} \left(1 + \frac{\beta_\ell}{2}\right) \end{pmatrix} \\ &= 4 \sin^2 \frac{\alpha_1}{2} - 2i(\beta_1 - \beta_2) \sin \frac{\alpha_1}{2} \cos \frac{\alpha_1}{2}, \end{aligned}$$

which follows from (2.11). For dimension reasons, $\hat{A}(TC) = 1$, so β_3 cannot contribute. Then by (2.12), the whole contribution of $(C, \gamma^j) \in \Lambda V$ to the untwisted η -invariant is

$$\begin{aligned} &\frac{(-1)^{\text{rk}_C \nu_C} \hat{A}(TC)}{k \text{ch}(\gamma^j, \Lambda^{0,\text{ev}}\nu_C^* - \Lambda^{0,\text{odd}}\nu_C^*)} [C] \cdot \eta_{\gamma^j}(D_{S^1}) \\ &= \frac{1}{4 \sin^2 \frac{\alpha_1}{2} - 2i(\beta_1 - \beta_2) \sin \frac{\alpha_1}{2} \cos \frac{\alpha_1}{2}} [C] \cdot \frac{\eta_{\gamma^j}(D_{S^1})}{k} \\ &= \frac{i \cos \frac{\alpha_1}{2}}{8k \sin^3 \frac{\alpha_1}{2}} \cdot (\beta_1 - \beta_2) [C] \cdot \eta_{\gamma^j}(D_{S^1}). \end{aligned} \quad (2.13)$$

For the signature η -invariant, we have to compute the equivariant twist Chern character following [3, Def 6.15]. The spinor bundle $\Lambda^{0,\bullet} T_p^* C$ of $T_p C \subset T_p V$ contributes only by its rank. By (2.11), we have

$$\begin{aligned} \text{ch}(\gamma^j, \Lambda^{0,\bullet} T_p^* V)|_{(C,\gamma^j)} &= 2 \prod_{\ell=1}^2 \text{tr} \begin{pmatrix} e^{-\frac{i\alpha_\ell}{2}} \left(1 - \frac{\beta_\ell}{2}\right) & \\ & e^{\frac{i\alpha_\ell}{2}} \left(1 + \frac{\beta_\ell}{2}\right) \end{pmatrix} \\ &= 8 \cos^2 \frac{\alpha_1}{2} + 4i(\beta_1 - \beta_2) \cos \frac{\alpha_1}{2} \sin \frac{\alpha_1}{2}. \end{aligned}$$

By (2.12) and the above, the whole contribution of V^{γ^j} to the signature η -invariant is

$$\begin{aligned} &\frac{(-1)^{\text{rk}_C \nu_C} \hat{A}(TC) \text{ch}(\gamma^j, \Lambda^{0,\bullet} T_p^* V)}{k \text{ch}(\gamma^j, \Lambda^{0,\text{ev}}\nu_C^* - \Lambda^{0,\text{odd}}\nu_C^*)} [C] \cdot \eta_{\gamma^j}(B_{S^1}) \\ &= \frac{8 \cos^2 \frac{\alpha_1}{2} + 4i(\beta_1 - \beta_2) \cos \frac{\alpha_1}{2} \sin \frac{\alpha_1}{2}}{4 \sin^2 \frac{\alpha_1}{2} - 2i(\beta_1 - \beta_2) \cos \frac{\alpha_1}{2} \sin \frac{\alpha_1}{2}} [C] \cdot \eta_{\gamma^j}(D_{S^1}) \\ &= \frac{i \cos \frac{\alpha_1}{2}}{k \sin^3 \frac{\alpha_1}{2}} \cdot (\beta_1 - \beta_2) [C] \cdot \eta_{\gamma^j}(D_{S^1}). \end{aligned} \quad (2.14)$$

From (2.10), (2.13) and (2.14), we see that (V^{γ^j}, γ^j) does not contribute to $\lim_{a \rightarrow 0} \bar{\nu}(M_a)$.

Isolated fixpoints. At an isolated fixpoint p of $\gamma = \gamma^j$, we have $\nu_p = T_p V$. The action of γ is determined by three nonzero angles $\alpha_\ell = \alpha_{j,\ell}(p)$ for $\ell = 1, 2, 3$ that sum up to 0. If necessary, we add a multiple of 2π to one of the angles.

The contribution to the orbifold \hat{A} -form is the number

$$\text{ch}(\gamma, \Lambda^{0,\text{ev}} T_p^* V - \Lambda^{0,\text{odd}} T_p^* V)|_{(p,\gamma)} = \prod_{\ell=1}^3 \text{str} \begin{pmatrix} e^{-\frac{i\alpha_\ell}{2}} & \\ & e^{\frac{i\alpha_\ell}{2}} \end{pmatrix} = 8i \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \sin \frac{\alpha_3}{2}.$$

By (2.12), the contribution to the untwisted η -invariant is

$$\frac{(-1)^{\text{rk}_C T_p V}}{k \text{ch}(\gamma, \Lambda^{0,\text{ev}} T_p^* V - \Lambda^{0,\text{odd}} T_p^* V)} [p] \cdot \eta_{\gamma^j}(D_{S^1}) = \frac{\cot \frac{\pi j}{k}}{8k \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \sin \frac{\alpha_3}{2}}. \quad (2.15)$$

For the signature η -invariant, we multiply the above with the equivariant Chern character

$$\mathrm{ch}(\gamma, \Lambda^{0, \bullet} T_p^* V) = \prod_{\ell=1}^3 \mathrm{tr} \begin{pmatrix} e^{-\frac{i\alpha_\ell}{2}} & \\ & e^{\frac{i\alpha_\ell}{2}} \end{pmatrix} = 8 \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} \cos \frac{\alpha_3}{2}$$

and obtain the contribution to the signature η -invariant

$$\frac{(-1)^{\mathrm{rk}_{\mathbb{C}} T_p V} \mathrm{ch}(\gamma, \Lambda^{0, \bullet} T_p^* V)}{k \mathrm{ch}(\gamma, \Lambda^{0, \mathrm{ev}} T_p^* V - \Lambda^{0, \mathrm{odd}} T_p^* V)} [p] \cdot \eta_{\gamma^j}(B_{S^1}) = \frac{1}{k} \cot \frac{\alpha_1}{2} \cot \frac{\alpha_2}{2} \cot \frac{\alpha_3}{2} \cot \frac{\pi j}{k}. \quad (2.16)$$

From (2.10), (2.15) and (2.16), we obtain the Theorem. \square

Remark 2.5. When considering examples, it is more convenient to fix the generator τ that acts on the interior circle $S^1_{\zeta} \cong \mathbb{R}/\zeta\mathbb{Z}$ by $u \mapsto u + \frac{\zeta}{k}$. If a unit $\varepsilon \in \mathbb{Z}/k$ is chosen as in equation (1.5), then the generator chosen above is $\gamma = \tau^\varepsilon$, and the contribution of the isolated fixpoints to the extended ν -invariant is given by $D_{\tau^\varepsilon}(V)$.

Now assume that Z is a building block with a Γ -action that fixes an anticanonical divisor Σ of Z pointwise, and that $V \cong Z \setminus \Sigma$. The orientation convention of [13, equation (3.2)] says that the complex structure rotates the outward cylindrical direction into the positive direction of the interior circle. If we identify the asymptotical cylinder with the normal bundle to Σ in Z , the outward cylindrical direction becomes the inward normal direction. This means that the interior circle through $v \in \nu_\Sigma$ is oriented by $-iv \in T_v \nu_\Sigma$. Hence $\tau \in \Gamma$ should act on ν_Σ by $e^{-\frac{2\pi i}{k}}$.

Example 2.6. In Example 5.10, we describe a building block with $\mathbb{Z}/5$ -symmetry. It has one isolated fixpoint p . Let $\zeta_5 = e^{\frac{2\pi i}{5}}$, then there is a generator τ of $\mathbb{Z}/5$ that acts on $T_p Z$ as $\mathrm{diag}(\zeta_5, \zeta_5, \zeta_5^{-2})$ and on ν_Σ by $\zeta_5^{-1} = e^{-\frac{2\pi i}{5}}$. Hence, the generator γ in Theorem 2.4 corresponds to τ^ε , for $\varepsilon \not\equiv 0 \pmod{5}$. Then γ acts on $T_p Z$ as $\mathrm{diag}(\zeta_5^\varepsilon, \zeta_5^\varepsilon, \zeta_5^{-2\varepsilon})$, so

$$\alpha_{j,1}(p) = \alpha_{j,2}(p) = \frac{2\varepsilon\pi}{5} \quad \text{and} \quad \alpha_{j,3}(p) = -\frac{4\varepsilon\pi}{5}.$$

If we represent $\varepsilon \in \mathbb{Z}/5 \setminus \{0\}$ by an element of $\{-2, -1, 1, 2\}$, we get

$$D_\gamma(V) = D_{\tau^\varepsilon}(V) = \lim_{r \rightarrow 0} \bar{\nu}((V \times S_r^1)/\Gamma) = \frac{24}{5\varepsilon}.$$

We can use this block as Z_- in Example 1.9, with $\varepsilon_- = -1$ and hence $D_{\gamma_-}(V^-) = -\frac{24}{5}$.

2.4. The variation of η -invariants. In this section, we apply a variation formula for η -invariants on manifolds with boundary by Dai and Freed [20, Thm 1.9]. Similar formulas in the case where the boundary operator is invertible have been established by Cheeger [11, Section 8] and Bismut-Cheeger [4, Thm 6.36]. Dai and Freed actually interpret the reduced η -invariant in \mathbb{R}/\mathbb{Z} with respect to a certain class of possible boundary conditions as a section of the dual of the determinant line bundle of the fibrewise boundary operators, equipped with the Quillen metric and the Bismut-Freed connection. Because we have fixed the boundary condition in Section 2.1, we recover the reduced η -invariant as an \mathbb{R}/\mathbb{Z} -valued function.

We now consider the family $\mathcal{M}_\pm = M_\pm \times (0, \infty) \rightarrow (0, \infty)$ with fibre $M_{\pm, a}$ over $a \in (0, \infty)$. We choose the trivial connection $T^H \mathcal{M}_\pm \subset T\mathcal{M}_\pm$, and the fibrewise metric is induced from the metric $g^{TV_\pm} \oplus a^2 g^{S^1}$ on \tilde{M}_\pm . Using these data, Bismut and Freed [6, (1.7)] construct a connection $\tilde{\nabla}^u$ on the infinite-dimensional vector bundle $\Omega^\bullet(\mathcal{M}_\pm/(0, \infty)) \rightarrow (0, \infty)$ of fibrewise exterior differential forms that is unitary with respect to the fibrewise L^2 -metrics. In our situation, it is not hard to see that the subbundle of fibrewise harmonic forms and its subbundle representing the subspaces L_{B_\pm} of (2.6) in each fibre are parallel with respect to $\tilde{\nabla}^u$. Dai and

Freed regard $L_{B_{\pm}}$ as graphs of isometries $H^+(X) \rightarrow H^-(X)$ whose determinants defines a section of unit length of the determinant line bundle $\det H^{\bullet}(X) = \Lambda^{\max} H^+(X)^* \otimes \Lambda^{\max} H^-(X)$. This section is again parallel with respect to the connection induced by $\tilde{\nabla}^u$ on $\det H^{\bullet}(X)$.

The variational formula is typically phrased in terms of the Bismut-Freed connection on the determinant line bundle over $(0, \infty)$ (which preserves the Quillen metric). However, if the kernels of the boundary operators form a bundle over the base, as in the case at hand, it is easier to work with the L^2 -metric above. A simple fibrewise rescaling of the determinant line bundle transforms one metric into the other, as in [20, Prop. 2.15]. It is shown in [20, (3.8)] that the Bismut-Freed connection becomes a unitary connection with respect to the L^2 -metric given by

$$\tilde{\nabla}^u - 2\pi i \tilde{\eta}(\mathbb{B}), \quad \text{where} \quad \tilde{\eta}(\mathbb{B}) = -\frac{1}{4\pi i} \int_0^{\infty} \text{str} \left(A_X [\tilde{\nabla}^u, A_X] e^{-tA_X^2} \right) dt \quad (2.17)$$

is the η -form of the family of boundary operators A , and $\mathbb{B}_t = \sqrt{t} A + \tilde{\nabla}^u$ is the corresponding Bismut superconnection. Note that we do not need to specify the degree 1 component of $\tilde{\eta}(\mathbb{B})$ explicitly because our base space is one-dimensional here.

The situation for the spin Dirac operators is completely analogous. Hence, fixing APS boundary conditions modified by the Lagrangian of (2.6), (2.7) as before, the variational formula in the version of [20, Thm 3.3] in our situation reads

$$\begin{aligned} d\eta(B_{M_{\pm,a}}) &= \int_{\mathcal{M}_{\pm}/(0,\infty)} 2\hat{L}(\nabla^{T(\mathcal{M}_{\pm}/(0,\infty))}) - 2\tilde{\eta}(\mathbb{B}) \in \Omega^1((0,\infty)), \\ d\eta(D_{M_{\pm,a}}) &= \int_{\mathcal{M}_{\pm}/(0,\infty)} 2\hat{A}(\nabla^{T(\mathcal{M}_{\pm}/(0,\infty))}) - 2\tilde{\eta}(\mathbb{D}) \in \Omega^1((0,\infty)), \end{aligned} \quad (2.18)$$

where $\int_{\mathcal{M}_{\pm}/(0,\infty)}$ denotes integration along the fibres. The first term is the usual local variation formula for η -invariants on closed manifolds. The second term is the boundary contribution. Here \mathbb{D} is the superconnection associated to the boundary operators $C_{X_{\pm,a}}$.

Proposition 2.7. *The local variation terms vanish, that is*

$$\int_{\mathcal{M}_{\pm}/(0,\infty)} \hat{L}(\nabla^{T(\mathcal{M}_{\pm}/(0,\infty))}) = \int_{\mathcal{M}_{\pm}/(0,\infty)} \hat{A}(\nabla^{T(\mathcal{M}_{\pm}/(0,\infty))}) = 0.$$

Proof. We split the vertical tangent bundle

$$T(\mathcal{M}_{\pm}/(0,\infty)) \cong p_{V_{\pm}}^* TV_{\pm} \oplus \underline{\mathbb{R}},$$

where $p_{V_{\pm}}: \mathcal{M}_{\pm} \rightarrow V_{\pm}$ denotes obvious projection. This splitting is parallel with respect to the Bismut connection on the vertical tangent bundle. Because the metric on V_{\pm} is unchanged, the Bismut connection on $p_{V_{\pm}}^* TV_{\pm}$ is pulled back from V_{\pm} . The connection on $\underline{\mathbb{R}}$ is Euclidean and therefore flat. We conclude that

$$\hat{A}(\nabla^{T\mathcal{M}_{\pm}/(0,\infty)}) = \hat{A}(\nabla^{p_{V_{\pm}}^* TV_{\pm}}) \cdot \hat{A}(\nabla^{\underline{\mathbb{R}}}) = p_{V_{\pm}}^* \hat{A}(\nabla^{TV_{\pm}}).$$

Because this expression is of horizontal degree 0 and the fibres are odd-dimensional, the integral in the proposition vanishes. The same holds for the \hat{L} -form integral above. \square

Proposition 2.8. *Let \mathbb{A} denote the superconnection associated to the fibrewise spin Dirac operator for the family over $(0, \infty)$ with fibre $(S^1 \times S_a^1)/\Gamma_{\pm}$ at $a \in (0, \infty)$, equipped with the trivial spin structure. Then*

$$d\bar{\nu}(\mathcal{M}_{\pm}/(0,\infty)) = 288\tilde{\eta}(\mathbb{A}). \quad (2.19)$$

Proof. By equation (2.18) and Proposition 2.7, we are left with

$$d\bar{\nu}(\mathcal{M}_{\pm}/(0, \infty)) = 48\tilde{\eta}(\mathbb{D}) - 6\tilde{\eta}(\mathbb{B}) .$$

By assumption $X_{\pm,a} \cong \Sigma_{\pm} \times (S^1 \times S_r^1)/\Gamma_{\pm}$. In particular, the geometry of Σ_{\pm} is independent of a . We may therefore split the index of the appropriate operator on Σ_{\pm} off from the η -forms above. For the spin Dirac operator, it remains to compute the η -form of a superconnection \mathbb{A} acting on the spinor bundle of a bundle $E \rightarrow (0, \infty)$ of flat tori $(S^1 \times S_a^1)/\Gamma_{\pm}$ as in the proposition, which we can see as the quotient bundle of a vector bundle $W \rightarrow (0, \infty)$ by a bundle of lattices in W ; see Sections 2.5 and 4.1 below.

The fibrewise spinor bundle is the pullback of the virtual complex spinor bundle $S^+ \oplus S^- \rightarrow (0, \infty)$ of W with connection $\nabla^S = \nabla^{S^+} \oplus \nabla^{S^-}$. Let $\tilde{\eta}(\mathbb{A})$ denote the spinorial η -form. By the following remark, the η -form associated with the fibrewise signature operator on $E \rightarrow (0, \infty)$ equals $2\tilde{\eta}(\mathbb{A})$, at least in degree 1. We know that $\text{ind}(D_{\Sigma_{\pm}}) = 2$ and $\text{sign}(\Sigma_{\pm}) = -16$, so

$$d\bar{\nu}(\mathcal{M}_{\pm}/(0, \infty)) = 48 \text{ind}(D_{\Sigma_{\pm}}) \tilde{\eta}(\mathbb{A}) - 6 \text{sign}(\Sigma_{\pm}) 2\tilde{\eta}(\mathbb{A}) = 288\tilde{\eta}(\mathbb{A}) . \quad \square$$

Remark 2.9. We want to relate the η -forms for the spin Dirac operator and the signature operator of a bundle $p: E \rightarrow (0, \infty)$ of flat 2-tori. Let TT^2 denote the vertical tangent bundle of this fibration. The signature operator acts on the spinor bundle twisted by the pullback of $S = S^+ \oplus S^-$. We therefore have

$$p_*\Lambda^{\bullet}T^*T^2 \cong p_*p^*S \otimes S ,$$

and the corresponding superconnection is now given by $\mathbb{A}_t \otimes \text{id} + \text{id} \otimes \nabla^S$. The two terms above supercommute, so we have

$$\text{str}_{p_*\Lambda^{\bullet}T^*T^2} \left(\frac{\partial(\mathbb{A}_t \otimes \text{id} + \text{id} \otimes \nabla^S)}{\partial t} e^{-(\mathbb{A}_t \otimes \text{id} + \text{id} \otimes \nabla^S)^2} \right) = \text{str}_{p_*p^*S} \left(\frac{\partial \mathbb{A}_t}{\partial t} e^{-\mathbb{A}_t^2} \right) \cdot \text{tr}_S \left(e^{-(\nabla^S)^2} \right) .$$

This implies that the signature η -form of the torus bundle is given as $\eta(\mathbb{A}) \text{ch}(\nabla^S)$. In degree 1, this equals twice the spinorial η -form because $\text{rk } S = 2$. Equivalently, the reader is invited to compare Bismut and Cheeger's results for the universal spinorial η -form and the signature η -form of bundles of flat tori in [5, Thms 2.22, 2.25].

2.5. A direct computation of the η -form integral. We rewrite the η -form integral directly in terms of the eigenvalues of the Dirac operator on the family of flat tori over \mathcal{H} . Bismut and Cheeger did similar computations in [5]. In the appendix, Don Zagier will exhibit a new way to compute the contribution from the variational formula to the ν -invariant in terms of logarithms of Dedekind η -functions.

We consider a family of tori $(S^1 \times S_a^1)/\Gamma$ for $a \in (0, \infty)$ as in Proposition 2.8. With ε relatively prime to k as in Section 1.3, we have $(S^1 \times S_a^1)/\Gamma \cong \mathbb{R}^2/\Lambda_a$, where

$$\Lambda_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} \varepsilon \\ a \end{pmatrix} \frac{1}{k} \mathbb{Z} \quad \subset \quad \mathbb{R}^2 .$$

Let us denote the total space of this family by E and the fibres by Z . Consider a flat connection on $\mathbb{R}^2 \rightarrow (0, \infty)$ given by

$$\nabla = d - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{da}{a} ,$$

then Λ is parallel with respect to ∇ . This connection induces a splitting $TE = TZ \oplus T^H E$. A horizontal lift of $V = \frac{\partial}{\partial a}$ at a point $(x, y, a) \in \mathbb{R}^2 \times (0, \infty)$ is given as

$$\bar{V}_{(x,y,a)} = \frac{y}{a} \frac{\partial}{\partial y} + \frac{\partial}{\partial a} .$$

We equip $\mathbb{R}^2 \rightarrow (0, \infty)$ and E with the fibrewise metric g^{TZ} induced from the standard metric on \mathbb{R}^2 . The Levi-Civita connection on E induces a Euclidean connection ∇^{TZ} on $TZ \cong \pi^*\mathbb{R}^2 \rightarrow E$ that coincides with the pullback of the trivial connection d . The mean curvature of the fibres is given as

$$h = -\frac{1}{2} \operatorname{tr}((g^{TZ})^{-1} \mathcal{L}_{\tilde{V}} g^{TZ}) da = -\frac{da}{a}.$$

We consider the fibrewise product spin structure, so $S^+ \cong S^- \cong \underline{\mathbb{C}} \rightarrow E$. Let c_1, c_2 denote Clifford multiplication with the standard orthonormal basis vectors e_1, e_2 on $S^+ \oplus S^-$, then the complex Clifford volume element ic_1c_2 acts by ± 1 on S^\pm . The Levi-Civita connection induces the trivial connection on S^\pm . For the Bismut superconnection, we have to consider a connection on π_*S^\pm of the form

$$\tilde{\nabla}_{\frac{\partial}{\partial a}}^u f = \left(\nabla^{S^\pm} - \frac{h}{2} \right)_{\tilde{V}} f = \left(\frac{y}{a} \frac{\partial}{\partial y} + \frac{\partial}{\partial a} + \frac{1}{2a} \right) f$$

under the natural identification $\Gamma(\pi_*S^\pm) \cong \Gamma(S^\pm)$. Then the Bismut superconnection is given by

$$\mathbb{A}_t = \sqrt{t} A_{TZ} + \tilde{\nabla}^u = \sqrt{t} \left(c_1 \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y} \right) + da \left(\frac{y}{a} \frac{\partial}{\partial y} + \frac{\partial}{\partial a} + \frac{1}{2a} \right). \quad (2.20)$$

Starting from (2.17), we compute

$$\tilde{\eta}(\mathbb{A}) = -\frac{1}{4\pi} \frac{da}{a} \int_0^\infty \operatorname{tr}_{\pi_*S} \left(\frac{\partial^2}{\partial x \partial y} e^{t(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})} \right) dt, \quad (2.21)$$

We have used the definition $\operatorname{str}(\cdot) = \operatorname{tr}(ic_1c_2 \cdot)$ of the supertrace. Also $\operatorname{str}(1) = \operatorname{tr}(ic_1c_2) = 0$.

With respect to the standard Euclidean metric, the lattice dual to Λ_a is given by

$$\Lambda_a^* = \left\{ \mu \in \mathbb{C} \mid \langle \lambda, \mu \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda \right\} = \left\{ \binom{n}{m/a} \mid \varepsilon n + m \equiv 0 \pmod{k} \right\}.$$

For m, n as above, we consider sections

$$\varphi_{m,n}^\pm(x, y, a) = \frac{1}{\sqrt{a}} e^{2\pi i(nx+my/a)} \in \Gamma(S^\pm) \cong C^\infty(E; \mathbb{C})$$

of L^2 -norm 1. They are parallel under $\tilde{\nabla}^u$, and they are eigensections of the fibrewise Laplacian for the eigenvalue $4\pi^2(n^2 + \frac{m^2}{a^2})$. Note that each admissible pair (m, n) appears twice (once for S^+ and once for S^-), hence (2.21) becomes

$$\begin{aligned} \tilde{\eta}(\mathbb{A}) &= -\frac{1}{2\pi} \frac{da}{a} \int_0^\infty \sum_{\substack{m+\varepsilon n \equiv 0 \\ \pmod{k}}} \binom{-4\pi^2 \frac{mn}{a}}{\pmod{k}} e^{-4\pi^2 t(n^2 + \frac{m^2}{a^2})} dt \\ &= \frac{da}{2\pi} \int_0^\infty \sum_{\substack{m+\varepsilon n \equiv 0 \\ \pmod{k}}} mn e^{-t(m^2 + a^2 n^2)} dt. \end{aligned}$$

In the definition below, we substitute $-m$ for m .

Definition 2.10. For each ε relatively prime to k , we define a function $F_{k,\varepsilon}: (0, \infty) \rightarrow \mathbb{R}$ by

$$F_{k,\varepsilon}(s) = \int_0^\infty \int_0^s \sum_{\substack{m \equiv \varepsilon n \\ \pmod{k}}} mn e^{-t(m^2 + n^2 a^2)} da dt. \quad (2.22)$$

Proposition 2.11. Consider the family $E \rightarrow (0, \infty)$ above. Then

$$\int_{[0,s]} \tilde{\eta}(\mathbb{A}) = -\frac{1}{2\pi} F_{k,\varepsilon}(s). \quad \square$$

Theorem 2.12. *The variation of $\bar{\nu}(M_{\pm,a})$ is given by*

$$\bar{\nu}(M_{\pm,s_{\pm}}) - \lim_{a \rightarrow 0} \bar{\nu}(M_{\pm,a}) = -\frac{144}{\pi} F_{k_{\pm}, \varepsilon_{\pm}}(s_{\pm}) .$$

Proof. This follows from Propositions 2.8 and 2.11. \square

We can now give a formula for the extended ν -invariant.

Theorem 2.13. *The extended ν -invariant of an extra-twisted connected sum is given as*

$$\begin{aligned} \bar{\nu}(M) &= D_{\gamma_+}(V_+) + D_{\gamma_-}(V_-) \\ &\quad - \frac{144}{\pi} (F_{k_+, \varepsilon_+}(s_+) + F_{k_-, \varepsilon_-}(s_-)) - 72 \frac{\rho}{\pi} + 3m_{\rho}(L; N_+, N_-) . \end{aligned} \quad (2.23)$$

Proof. This follows from Theorems 2.1, 2.4 and 2.12. \square

Proof of Theorem 2. Combine Theorem 2.13 with Proposition A.1. \square

Remark 2.14. Using ideas and results of Atiyah [1], Bismut-Freed [6] and Ray-Singer [32], we can motivate the appearance of the Dedekind η -function. We consider the universal family $p: E \rightarrow \mathcal{H}$ of flat tori over the upper half plane that we will describe in more detail in Section 4. There exists a Kähler structure on E whose restriction to each fibre $p^{-1}(\tau)$ induces the flat Riemannian metric of volume 1 with the conformal structure induced by $\tau \in \mathcal{H}$. The fibrewise canonical bundle of p is holomorphically trivial, so we may regard the bundle of fibrewise antiholomorphic forms as a model for the fibrewise spinor bundle on E .

Following [6], the η -form $\tilde{\eta}(\mathbb{A})$ describes a natural connection on the determinant line bundle of the fibrewise Dirac operator. Atiyah explains that this connection agrees with the Chern connection on the determinant line bundle with respect to the Quillen metric. Using results of Ray and Singer [32, Theorem 4.1], he shows that the determinant line bundle admits a holomorphic section whose norm can be written in terms of the Dedekind η -function, see the discussion before [1, (5.19)]. This implies that the η -form itself can be described by the logarithmic derivative of $\eta(\tau)$.

Example 2.15. We consider the gluing data from Examples 1.9, then $s_- = \sqrt{2}$, $s_+ = 5\sqrt{2}$ and $\vartheta = \arccos \frac{1}{\sqrt{3}}$. From Theorem 2 and Examples 2.2, 2.6, we get

$$\begin{aligned} \bar{\nu}(M) &= -\frac{24}{5} + \frac{144}{\pi} \left(\arccos \frac{1}{\sqrt{3}} - \frac{1}{2} \right) - 3 \\ &\quad - \frac{144}{\pi} \left(2 \operatorname{Im} \mathcal{L} \left(\frac{\sqrt{2}i - 10}{30} \right) + \frac{\pi}{18} + 2 \operatorname{Im} \mathcal{L} \left(\frac{\sqrt{2}i + 2}{10} \right) - \frac{\pi}{30} \right) . \end{aligned}$$

The functional equation (A.1) for \mathcal{L} allows us to conclude that

$$\begin{aligned} 2 \operatorname{Im} \mathcal{L} \left(\frac{\sqrt{2}i - 10}{30} \right) + \frac{\pi}{18} + 2 \operatorname{Im} \mathcal{L} \left(\frac{\sqrt{2}i + 2}{10} \right) - \frac{\pi}{30} + \frac{1}{2} - \arccos \frac{1}{\sqrt{3}} \\ = \frac{\pi}{6} \left(\frac{1}{30} + \frac{1}{10} - 12 S(3, 10) \right) , \end{aligned}$$

see Proposition A.3. Because $3^2 \equiv -1 \pmod{10}$, we have $S(3, 10) = 0$, and hence we confirm entry 228 of Table 2 because

$$\bar{\nu}(M) = -\frac{24}{5} - 3 - 24 \left(\frac{1}{30} + \frac{1}{10} \right) = -11 .$$

This is of course exactly the formula we would get from Theorem 3.

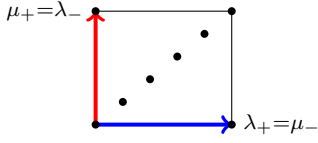


FIGURE 9.

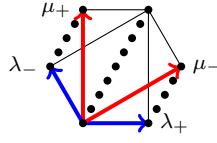


FIGURE 10.

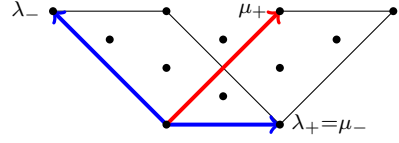


FIGURE 11.

3. TORUS MATCHINGS

In this section we collect some arguments concerning the combinatorics of gluing matrices and torus isometries that are tangential to the main narrative of the paper, but which may be relevant to a reader who wishes to understand in more detail how many different torus isometries there are. We first discuss in detail which gluing matrices can be realised, and then the various symmetries that relate torus isometries that are in some sense the same.

3.1. Combinatorics of torus isometries. We now pick up the thread from §1.3 to enumerate torus matchings systematically.

Example 3.1. Let us give some examples and counterexamples of torus matchings.

- (i) Let M be a twisted connected sum as in [13, 28]. Assume that the group $\Gamma \cong \mathbb{Z}/k$ acts by isomorphisms on the two ACyl Calabi-Yau manifolds V_{\pm} used in the construction of M such that the induced action on the cross-section acts trivially on the K3 factor and freely on the interior circle $S^1_{\zeta_{\pm}}$. Then $(\mathbb{Z}/k)^2$ acts on M , where each factor \mathbb{Z}/k acts on the ACyl Calabi-Yau manifold Y_{\pm} on one side and on the exterior circle on the other. For each $\varepsilon_+ \in \mathbb{Z}/k$ with $\gcd(\varepsilon_+, k) = 1$, we obtain a free \mathbb{Z}/k -action on M where a generator acts as $(1, \varepsilon_+) \in (\mathbb{Z}/k)^2$, see Figure 9 for $k = 5$ and $\varepsilon_+ = 1, 3$. The points of the lattice Λ are indicated by dots.

The corresponding torus matching has $k_+ = k_- = k$, gluing matrix $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$, and $\varepsilon_+ \varepsilon_- \equiv 1 \pmod{k}$. The gluing angle is $\vartheta = \pm \frac{\pi}{2}$, and we have $\xi_+ = \zeta_-$ and $\zeta_+ = \xi_-$, but the ratio $s_+ = s_-^{-1}$ can be chosen arbitrarily.

- (ii) There are also examples with gluing angle $\vartheta \notin \frac{\pi}{2}\mathbb{Z}$ where the gluing matrix together with the numbers k_+, k_- does not determine the torus matching completely. As an example, consider $k_+ = k_- = 8$ and the gluing matrix $\begin{pmatrix} 4 & 4 \\ 12 & -4 \end{pmatrix}$. This determines $s_+ = s_- = \sqrt{3}$ and $\vartheta = \arctan \sqrt{3}$. We can either pick $\varepsilon_+ = \varepsilon_- = 1$ or $\varepsilon_+ = \varepsilon_- = -3$, see Figure 10.
- (iii) If we want to construct a torus matching from a gluing matrix $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$, numbers k_{\pm} and ε_{\pm} , it is not quite enough to satisfy only the conditions listed in Proposition 1.8 (i). If we set $k_+ = k_- = 4$ and pick the gluing matrix $\begin{pmatrix} 0 & 4 \\ 4 & -8 \end{pmatrix}$, we may choose $\varepsilon_+ = -\varepsilon_- = \pm 1$. Then all conclusions in Proposition 1.8 (i) hold, but equation (3.2) is violated, which is part of the conclusions in Proposition 1.8 (ii). Hence we cannot have a matching of quotients of rectangular tori, see Figure 11.

To search systematically for torus matchings with fixed k_+ and k_- , it is helpful to first note the following formal consequences of (1.8) and Proposition 1.8 (ii).

$$\gcd(m, n) = \gcd(m, k_+) = \gcd(n, k_+) \text{ and } \gcd(p, q) = \gcd(p, k_+) = \gcd(q, k_+) \quad (3.1a)$$

$$\gcd(m, p) = \gcd(m, k_-) = \gcd(p, k_-) \text{ and } \gcd(n, q) = \gcd(n, k_-) = \gcd(q, k_-) \quad (3.1b)$$

$$np \cdot mq \leq 0, \quad \text{and if } np \cdot mq = 0, \text{ then either } n = p = 0 \text{ or } m = q = 0. \quad (3.2)$$

Proposition 3.2. *Let $k_+ > 0$, $\varepsilon_+ \in (\mathbb{Z}/k_+)^*$, and let $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ be a gluing matrix with $\det \begin{pmatrix} m & p \\ n & q \end{pmatrix}$ negative and divisible by k_+ . Suppose that (1.8a), (1.9a) and (3.2) are satisfied. Then there exists a unique torus matching with these data. If one chooses $a, b \in \mathbb{Z}$ such that*

$$1 = bp - a \frac{q - \varepsilon_+ p}{k_+}, \quad (3.3a)$$

$$\text{then } \varepsilon_- \equiv a \frac{n - \varepsilon_+ m}{k_+} - bm \pmod{k_-}. \quad (3.3b)$$

Proof. Let k_+ , $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ and ε_+ be given as above. Let $\tilde{\Lambda}_+ \subset \mathbb{C}$ be a lattice with basis (μ_+, λ_+) , then we can construct a sublattice $\Lambda \subset \frac{1}{k_+} \tilde{\Lambda}_+$ of index k_+ with basis (ν_+, λ_+) given by (1.5), and $\tilde{\Lambda}_+ \subset \Lambda$ is also a sublattice of index k_+ . By assumption (1.8a), the gluing matrix then determines a sublattice $\tilde{\Lambda}_- \subset \Lambda$ of index $k_- = -\det \begin{pmatrix} m & p \\ n & q \end{pmatrix} / k_+$ with basis (μ_-, λ_-) determined by (1.10).

We conclude that $\Lambda \subset \frac{1}{k_-} \tilde{\Lambda}_-$ is again a sublattice of index k_- , and let $\lambda(c, d) = \frac{c}{k_-} \mu_- + \frac{d}{k_-} \lambda_- \in \frac{1}{k_-} \tilde{\Lambda}_-$ with $c, d \in \{0, \dots, k_- - 1\}$. Assume that $\lambda(c, d) \in \Lambda$. The vectors λ_- and μ_- are primitive in Λ by (1.9a) and (1.10), so $c = 0$ if and only if $d = 0$. Similarly, for each c there can at most be one d in the given range such that $\lambda(c, d) \in \Lambda$ and vice versa. Because there are exactly k_- elements of Λ with coordinates c, d in the given range, for each c there is exactly one d such that $\lambda(c, d) \in \Lambda$, and vice versa. Specifying $c = 1$, we hence get a unique $\varepsilon_- = d \in \mathbb{Z}/k_-$. Moreover, $\gcd(\varepsilon_-, k_-) = 1$, and Λ is an extension of $\tilde{\Lambda}_-$ by a cyclic group $\Gamma_- \cong \mathbb{Z}/k_-$. This proves existence and uniqueness of the gluing data.

We set $k_- = -\frac{1}{k_+} \det \begin{pmatrix} m & p \\ n & q \end{pmatrix}$. To determine ε_- , we fix the basis $(\frac{\lambda_-}{k_-}, \frac{\mu_-}{k_-})$ of \mathbb{R}^2 . With respect to this basis, we have $\lambda_+ = \begin{pmatrix} -m \\ p \end{pmatrix}$ and $\mu_+ = \begin{pmatrix} n \\ -q \end{pmatrix}$ by (1.11). The lattice Λ is spanned by the vectors λ_+ and

$$\nu_+ = \frac{\mu_+ + \varepsilon_+ \lambda_+}{k_+} = \frac{1}{k_+} \begin{pmatrix} n - \varepsilon_+ m \\ \varepsilon_+ p - q \end{pmatrix},$$

which has integer coordinates by (1.8a). By (1.9a), we can find a and b satisfying (3.3a). Then the vector $\nu_- = a\nu_+ + b\lambda_+$ has second coordinate 1, hence together with λ_- , it also spans Λ . Its first coordinate is given by the right hand side of (3.3b), which therefore agrees with ε_- modulo k_- . \square

Remark 3.3. As a consequence of (3.3), we check that

$$n - \varepsilon_+ m + \varepsilon_- q - \varepsilon_+ \varepsilon_- p = n - \varepsilon_+ m + \left(a \frac{n - \varepsilon_+ m}{k_+} - bm \right) (q - \varepsilon_+ p) = bk_+ k_-, \quad (3.4)$$

in particular, the left hand side is always divisible by $k_+ k_-$. If ε_{\pm}^* is inverse to ε_{\pm} modulo k_{\pm} , we can deduce from the above that similarly

$$k_+ k_- \mid p - \varepsilon_+^* q + \varepsilon_-^* m - \varepsilon_+^* \varepsilon_-^* n. \quad (3.5)$$

We can now prove a claim used in Remark 4.14 below by computing

$$-\frac{m - \varepsilon_+^* n}{k_+} \cdot \frac{q + \varepsilon_-^*}{k_-} = \frac{np - mq - n(p - \varepsilon_+^* q + \varepsilon_-^* m - \varepsilon_+^* \varepsilon_-^* n)}{k_+ k_-} \equiv 1 \pmod{n}. \quad (3.6)$$

For given positive integers k_- and k_+ , there are only finitely many matrices $\begin{pmatrix} m & p \\ n & q \end{pmatrix} \in M_2(\mathbb{Z})$ that satisfy conditions (1.7) and (3.2). For

$$np - mq = k_-k_+ > 0$$

and (3.2) imply that

$$np \in \{0, \dots, k_-k_+\} \quad \text{and} \quad mq \in \{-k_-k_+, \dots, 0\}. \quad (3.7)$$

By Remark 3.8 below, we may assume in addition $\vartheta \in (0, \pi)$. Then $n > 0$ by Proposition 1.8 (iii), and therefore also $p > 0$. In other words, $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ can be chosen to be either off-diagonal with non-negative entries, or with exactly three positive entries and one negative entry, which can only be m or q .

For small k_+ and k_- it is now easy even by hand to enumerate all $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ that satisfy (3.1) in addition. Most of those have $\gcd\begin{pmatrix} m & p \\ n & q \end{pmatrix} = 1$ and thus give rise to a unique torus matching, while for the few remaining ones it is easy to enumerate any ε_+ that satisfy (1.8a) and (1.9a).

Remark 3.4. Assume that we are given gluing data $k_{\pm} \in \mathbb{Z}$, $\varepsilon_{\pm} \in \mathbb{Z}/k_{\pm}$ and $\begin{pmatrix} m & p \\ n & q \end{pmatrix} \in M_2(\mathbb{Z})$. Then the lattice Λ is a sublattice of the lattice $\tilde{\Lambda}_+ + \tilde{\Lambda}_-$ spanned by $\tilde{\Lambda}_+$ and $\tilde{\Lambda}_-$. Because we have assumed the groups $\Gamma_{\pm} = \mathbb{Z}/k_{\pm}$ act freely on the interior and on the exterior circles, we can compute the index of Λ in $\tilde{\Lambda}_+ + \tilde{\Lambda}_-$ in four different ways:

$$[\tilde{\Lambda}_+ + \tilde{\Lambda}_- : \Lambda] = \gcd(m, p, k_+) = \gcd(n, q, k_+) = \gcd(m, n, k_-) = \gcd(p, q, k_-).$$

For the first equation, we choose the vectors $\frac{\lambda_+}{k_+}$ and $\frac{\mu_+}{k_+}$ as a basis for \mathbb{R}^2 ; see Figure 2. Then the smallest positive second coordinate of an element of Λ is 1, because $\Gamma_+ \cong \mathbb{Z}/k_+$ acts freely on the exterior circle of \tilde{M}_+ . On the other hand, the smallest positive second coordinate of an element of $\tilde{\Lambda}_+ + \tilde{\Lambda}_-$ is $\gcd(m, p, k_+)$. The other equations follow similarly.

We note that we have not used the numbers ε_{\pm} in the argument above. Hence, if $[\tilde{\Lambda}_+ + \tilde{\Lambda}_- : \Lambda] = 1$, then Λ is uniquely determined by k_{\pm} and the gluing matrix, and so are ε_{\pm} . This is the case in most examples with $\vartheta \neq \frac{\pi}{2}$. In Table 2. If $\vartheta = \frac{\pi}{2}$ and $k_+ = k_- \geq 3$, then there are at least two possible choices for ε_+ . On the other hand, in examples 237, 238, 254 and 255 in Table 2, we have $[\tilde{\Lambda}_+ + \tilde{\Lambda}_- : \Lambda] > 1$, but nevertheless, Λ and ε_{\pm} are uniquely determined.

3.2. New extra-twisted connected sums from old. Having discussed how to enumerate torus matchings, we now move on to discuss relations between them. We will find several ways to describe isometric extra-twisted connected sums, but we also discuss covering spaces and a kind of “ t -duality”.

Let us start with coverings. By Proposition 1.10, an extra-twisted connected sum with gluing matrix $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ is simply-connected if and only if $p = 1$. Let us enumerate its connected covering spaces if $p > 1$.

Proposition 3.5. *Assume that M is an extra-twisted connected sum with gluing data given by k_{\pm} , $\varepsilon_{\pm} \in \mathbb{Z}/k_{\pm}$ and gluing matrix $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$. Assume that $p > 1$ and that $\ell \mid p$. Then there exists a unique connected ℓ -fold covering space \tilde{M} . It is an extra-twisted connected sum constructed from the same building blocks as M with the same gluing angle ϑ , and with gluing data*

$$\tilde{k}_{\pm} = \frac{k_{\pm}}{\gcd(\ell, k_{\pm})}, \quad \tilde{\varepsilon}_{\pm} = \frac{\ell \varepsilon_{\pm}}{\gcd(\ell, k_{\pm})}, \quad (3.8a)$$

$$\begin{pmatrix} \tilde{m} & \tilde{p} \\ \tilde{n} & \tilde{q} \end{pmatrix} = \begin{pmatrix} \frac{m}{\gcd(\ell, k_-)} & \frac{p}{\ell} \\ \frac{n\ell}{\gcd(\ell, k_+) \gcd(\ell, k_-)} & \frac{q}{\gcd(\ell, k_+)} \end{pmatrix} \in M_2(\mathbb{Z}). \quad (3.8b)$$

Proof. Because $\pi_1(M)$ is cyclic, there is a unique connected ℓ -fold covering $\pi: \tilde{M} \rightarrow M$. Let $\tilde{M}_\pm \rightarrow M_\pm$ denote its restriction to the two halves of M . Because $\pi_*: \pi_1(M_\pm) \rightarrow \pi_1(M)$ is surjective, we see that $\tilde{M}_\pm \rightarrow M_\pm$ are also connected ℓ -fold coverings, which are uniquely determined by ℓ up to isomorphism since $\pi_1(M_\pm) \cong \mathbb{Z}$. It follows that \tilde{M} is an extra-twisted sum, glued from \tilde{M}_+ and \tilde{M}_- .

Let $\tilde{X} \rightarrow X$ denote the restriction of π to $X = M_+ \cap M_-$. The corresponding sublattice $\tilde{\Lambda} \subset \Lambda = \pi_1(X)$ is spanned by the vectors λ_\pm corresponding to the interior circles, and by

$$\tilde{\nu}_\pm = \ell \nu_\pm = \frac{\ell}{k_\pm} (\mu_\pm + \varepsilon_\pm \lambda_\pm).$$

The smallest multiples of μ_\pm inside $\tilde{\Lambda}$ are

$$\tilde{\mu}_\pm = \frac{k_\pm}{\gcd(\ell, k_\pm)} \cdot \tilde{\nu}_\pm - \frac{\ell \varepsilon_\pm}{\gcd(\ell, k_\pm)} \lambda_\pm = \frac{\ell}{\gcd(\ell, k_\pm)} \mu_\pm,$$

and $\tilde{\Lambda}$ is an extension of the lattice spanned by λ_\pm and $\tilde{\mu}_\pm$ by a finite cyclic group $\tilde{\Gamma}_\pm \cong \mathbb{Z}/\tilde{k}_\pm$. With respect to the new bases $(\tilde{\mu}_\pm, \lambda_\pm)$, \tilde{k}_\pm and $\tilde{\varepsilon}_\pm$ are given by (3.8a), and the new gluing matrix takes the form of (3.8b), which has determinant $-\tilde{k}_+ \tilde{k}_-$. \square

Remark 3.6. If $\tilde{k}_\pm = k_\pm$, then the covering is constructed using the same finite groups Γ_\pm . On the other hand, it is possible that one has to pass to a proper subgroup of one or both these groups.

Conversely, an extra-twisted connected sum admits a quotient by \mathbb{Z}/ℓ if

- (i) There exist multiples \bar{k}_\pm of k_\pm such that $\bar{k}_\pm = k_\pm \gcd(\ell, \bar{k}_\pm)$,
- (ii) The action of $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ on V_\pm extends to an action by \mathbb{Z}/\bar{k}_\pm with respect to the embedding $\mathbb{Z}/k_\pm \ni [1] \mapsto [\bar{k}_\pm/k_\pm] \in \mathbb{Z}/\bar{k}_\pm$,
- (iii) The number $\frac{n}{\ell} \gcd(\ell, \bar{k}_+) \gcd(\ell, \bar{k}_-)$ is an integer.

Table 2 describes 255 deformation families of extra-twisted connected sums, 125 of which are simply connected. Among the remaining examples, there are 64 where taking the universal cover implies passing to subgroups of Γ_+ or Γ_- .

Among the examples in Table 2, the one with largest fundamental group $\pi_1(M) \cong \mathbb{Z}/21$ is entry 250, which has $k_+ = 4$ and $k_- = 6$. The universal cover has $\tilde{k}_+ = 4$ and $\tilde{k}_- = 2$. It can be found as entry 174 with M_+ and M_- swapped. Entries 175 and 248 are the two intermediate covering spaces.

Proposition 3.7. *Let M be an extra-twisted connected sum constructed from asymptotically cylindrical Calabi-Yau manifolds V_\pm with gluing data $k_\pm \in \mathbb{Z}$, $\varepsilon_\pm \in \mathbb{Z}/k_\pm$ and $\begin{pmatrix} m & p \\ n & q \end{pmatrix} \in M_2(\mathbb{Z})$, and with gluing angle ϑ . Then the following gluing data describe an isometric extra-twisted connected sum M' , possibly with opposite orientation:*

$$\begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} = \begin{pmatrix} -q & p \\ n & -m \end{pmatrix}, \quad k'_+ = k_-, \quad k'_- = k_+, \quad (3.9)$$

$$\varepsilon'_+ = \varepsilon_-, \quad \varepsilon'_- = \varepsilon_+, \quad \text{and} \quad \vartheta' = \vartheta;$$

$$\begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} = \begin{pmatrix} m & -p \\ -n & q \end{pmatrix}, \quad k'_+ = k_+, \quad k'_- = k_-, \quad (3.10)$$

$$\varepsilon'_+ = -\varepsilon_+, \quad \varepsilon'_- = -\varepsilon_-, \quad \text{and} \quad \vartheta' = -\vartheta;$$

$$\begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} = \begin{pmatrix} -m & -p \\ -n & -q \end{pmatrix}, \quad k'_+ = k_+, \quad k'_- = k_-, \quad (3.11)$$

$$\varepsilon'_+ = \varepsilon_+, \quad \varepsilon'_- = \varepsilon_-, \quad \text{and} \quad \vartheta' = \vartheta \pm \pi.$$

In (3.9), we have to swap the roles of V_+ and V_- . In (3.10), we change the orientation of M . In (3.11), we pass to the opposite Calabi-Yau structure on one side.

The three elements above generate a group $H \cong (\mathbb{Z}/2)^3$ that acts on the set of gluing data describing a given deformation family up to isomorphism.

Proof. We obtain (3.9) by exchanging the roles of V_+ and V_- , see equation (1.11). Because the definition of the gluing angle is symmetric, it does not change.

For (3.10), we change the orientation of M by swapping the orientations of the two exterior circles. This changes the sign of the gluing angle and of ε_{\pm} . The new gluing matrix arises by conjugating with $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

In (3.11), we rotate one of the two sides, say M_+ , by π , which leads to the new gluing angle. This has the effect of changing the orientations of both V_+ and the exterior circle, and hence the gluing matrix is multiplied by $-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If ω_+ and Ω_+ describe the old Calabi-Yau structure on V_+ , the new one carries the opposite complex structure and is given by $-\omega_+$ and $-\bar{\Omega}_+$. \square

Remark 3.8. The subgroup spanned by (3.10) and (3.11) is rich enough to make sure that we can always assume $m, n, p \geq 0$ and $q \leq 0$. Moreover, if we use the same building block and the same finite group $\Gamma \cong \mathbb{Z}/k$ for M_+ and M_- , we may apply (3.9) to get $m + q \leq 0$. In Table 2, we only list gluing data satisfying these conventions.

Recall that $\Sigma \times T \cong \Sigma_{\pm} \times T_{\pm}$ denote the isometric cross-sections at infinity of the asymptotically cylindrical G_2 -manifolds M_{\pm} used in Theorem 1.3 to construct M .

Proposition 3.9. *Let M be an extra-twisted connected sum glued along $\Sigma \times T$ with gluing data $k_{\pm} \in \mathbb{Z}$, $\varepsilon_{\pm} \in \mathbb{Z}/k_{\pm}$ and $\begin{pmatrix} m & p \\ n & q \end{pmatrix} \in M_2(\mathbb{Z})$ and with gluing angle ϑ . Then there exists a deformation family of extra-twisted connected sums M' with isomorphic asymptotically flat Calabi-Yau manifolds V_{\pm} , glued along $\Sigma \times T^*$ with gluing data*

$$\begin{aligned} \begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} &= \begin{pmatrix} -q & n \\ p & -m \end{pmatrix}, & k'_+ &= k_+, & k'_- &= k_-, \\ \varepsilon'_+ &= -\varepsilon_+^*, & \varepsilon'_- &= -\varepsilon_-^*, & \text{and} & \vartheta' &= \vartheta. \end{aligned} \quad (3.12)$$

Together with the group H , this transformation generates a group isomorphic to $(\mathbb{Z}/2)^4$ that acts on the possible gluing data compatible with a given K3 matching.

Thus, the new extra-twisted connected sum is in a certain sense “ t -dual” to the original one, but can in general not be deformed into the original one.

Proof. We recall the generators for the involved lattices in \mathbb{C} from Section 1.3:

$$\begin{aligned} \tilde{\Lambda}_+ &= \langle \zeta_+, i\xi_+ \rangle, \\ \Lambda &= \left\langle \zeta_+, \frac{\varepsilon_+ \zeta_+ + i\xi_+}{k_+} \right\rangle, \\ \text{and } \tilde{\Lambda}_- &= \left\langle \frac{q\zeta_+ + ip\xi_+}{k_+}, \frac{n\zeta_+ + im\xi_+}{k_+} \right\rangle. \end{aligned}$$

Because T is a k_{\pm} -fold quotient of T_{\pm} , the dual torus T^* is a k_{\pm} -fold covering of \tilde{T}_{\pm}^* , or equivalently, a k_{\pm} -fold quotient of $k_{\pm}\tilde{T}_{\pm}^*$. With respect to standard Euclidean metric on $\mathbb{C} \cong \mathbb{R}^2$,

we have generators for the dual lattices, where we take ε_+^* in \mathbb{Z}/k_+ as above:

$$\begin{aligned} k_+ \tilde{\Lambda}_+^* &= \left\langle \frac{k_+}{\zeta_+}, \frac{ik_+}{\xi_+} \right\rangle, \\ \Lambda^* &= \left\langle \frac{k_+}{\zeta_+}, -\frac{\varepsilon_+^*}{\zeta_+} + \frac{i}{\xi_+} \right\rangle, \\ \text{and } k_- \tilde{\Lambda}_-^* &= \left\langle -\frac{m}{\zeta_+} + \frac{in}{\xi_+}, \frac{p}{\zeta_+} - \frac{iq}{\xi_+} \right\rangle. \end{aligned}$$

The gluing data (3.12) can be read off from this description. And because the gluing angle does not change, the K3 matching used to construct M also works for M' . \square

Alternatively, one can rotate both tori \tilde{T}_\pm by a right angle, thus swapping the role of exterior and interior circles. This leads to the same gluing data as above.

Remark 3.10. For the sake of completeness, let us add the following observations.

- (i) Given gluing data k_\pm , ε_\pm and $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ as above with gluing angle $\vartheta \notin \frac{\pi}{2} + \pi\mathbb{Z}$. Then we can write valid gluing data of the form

$$\begin{aligned} \begin{pmatrix} m' & p' \\ n' & q' \end{pmatrix} &= \begin{pmatrix} p & m \\ -q & -n \end{pmatrix}, & k'_+ &= k_+, & k'_- &= k_-, \\ \varepsilon'_+ &= -\varepsilon_+, & \varepsilon'_- &= \varepsilon_-^*, & \text{and } \vartheta' &= \pm \frac{\pi}{2} - \vartheta. \end{aligned} \quad (3.13)$$

One can check that this transformation together with those of Propositions 3.7 and 3.9 generates a group isomorphic to $D_4 \times (\mathbb{Z}/2)^2$ that acts on the set of valid gluing data.

However, to construct an extra-twisted connected sum, we need a K3 matching that is compatible with the gluing angle. If both blocks have rank 1, the compatibility condition is given by (1.14). The new gluing data above will in general not be compatible with the K3 matching used for the original extra-twisted connected sum M . And it is not hard to find examples in Table 2 where the new gluing angle is not compatible with any possible K3 matching of rank 1 blocks.

- (ii) Let us now consider matchings with gluing angle $\vartheta = \pm \frac{\pi}{2}$ as in Example 3.1 (i). Then the transformation (3.13) above would lead to a gluing angle $\vartheta \in \{0, \pi\}$, and hence to a manifold with infinite fundamental group.

We also recall that $k_+ = k_- = k$, $\varepsilon_- = \varepsilon_+^* \in \mathbb{Z}/k$, and the gluing matrix takes the form $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$ by Example 3.1 (i). This implies that the transformation (3.12) of Proposition 3.9 acts exactly as the composition of the three elements (3.9)–(3.11) of H . Hence, we are reduced to an action of the group H in this special case. This is not surprising because (3.12) mainly affects the ratio of circle lengths, which is not specified by the gluing data if $\vartheta = \frac{\pi}{2}$, see the discussion before Proposition 1.8. Exploiting the action of H , we may assume in Table 2 that $p = n > 0$ and that $\varepsilon_+ > 0$.

One may note that in all our examples in Table 2 we have $\varepsilon_+^* = \pm \varepsilon_+$, so in none of these examples the group H acts effectively.

Example 3.11. Consider the Example 1.9 (228), where numbers in parentheses refer to Table 2, possibly up to the isometry (3.9). Applying the transformation (3.12), we get the gluing matrix

$$\begin{pmatrix} 5 & 10 \\ 1 & -1 \end{pmatrix} \quad \text{with} \quad \varepsilon_+ = 1, \quad \varepsilon_- = -1, \quad \bar{\nu} = -43 \quad (231).$$

By Proposition 1.10, the corresponding extra-twisted connected sum M' is not simply connected. By Proposition 3.5, its universal covering and some intermediate covering spaces have gluing

data and $\bar{\nu}$ -invariant

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad \text{with} \quad \begin{matrix} k_+ = 3, \\ k_- = 1, \end{matrix} \quad \begin{matrix} \varepsilon_- = -1, \\ \varepsilon_+ = 1, \end{matrix} \quad \bar{\nu} = -19 \quad (21),$$

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \quad \text{with} \quad \begin{matrix} k_+ = 3, \\ k_- = 1, \end{matrix} \quad \begin{matrix} \varepsilon_- = 1, \\ \varepsilon_+ = 2, \end{matrix} \quad \bar{\nu} = -35 \quad (23),$$

$$\begin{pmatrix} 5 & 5 \\ 2 & -1 \end{pmatrix} \quad \text{with} \quad \begin{matrix} k_+ = 3, \\ k_- = 5, \end{matrix} \quad \begin{matrix} \varepsilon_+ = 1, \\ \varepsilon_- = 2, \end{matrix} \quad \bar{\nu} = -23 \quad (230).$$

Note that the universal covering is different from the original manifold M from Example 1.9 (228). In particular, we have forgotten the $\mathbb{Z}/5$ -action on block 12 of Table 1, thus obtaining block 9. The first two lines above are again related by (3.12). Applying (3.12) to the last line gives an extra-twisted connected sum with fundamental group $\mathbb{Z}/2$, whose universal cover is the original manifold M , and which is described by

$$\begin{pmatrix} 1 & 2 \\ 5 & -5 \end{pmatrix} \quad \text{with} \quad \begin{matrix} k_+ = 3, \\ k_- = 5, \end{matrix} \quad \begin{matrix} \varepsilon_+ = -1, \\ \varepsilon_- = 2, \end{matrix} \quad \bar{\nu} = -7 \quad (229).$$

4. HYPERBOLIC GEOMETRY AND η -FORMS

Because we do not have a closed formula for the function \mathcal{L} in Theorem 2, we do not see immediately that the right hand side of equation (0.2a) is an integer (or even rational). Here, we will pursue a different approach to compute the integrals in Proposition 2.8. We will instead regard the η -form of the tautological family of flat tori over the upper half plane $\mathcal{H} \subset \mathbb{C}$ as a primitive of the hyperbolic area form. We will then use elementary hyperbolic geometry and an adiabatic limit formula for η -forms as in [10, 30] to complete the computation.

Throughout this section, we will assume that $m \geq 0$, $n, p > 0$, whereas $q \leq 0$. This is no loss of generality by Remark 3.8. As a consequence, we always have $\vartheta \in (0, \frac{\pi}{2}]$ and $\rho \geq 0$.

4.1. Hyperbolic area and the η -form. We extend the η -form $\tilde{\eta}(\mathbb{A})$ of (2.21) to $\mathcal{H} \times (0, \infty)$, which we regard as the moduli space of flat tori. We then want to apply the so-called transgression formula

$$d\tilde{\eta}(\mathbb{A}) = \int_{E/\mathcal{H}} \hat{A}(T(E/\mathcal{H})) - \text{ch}(\text{ind}(A_{T^2})), \quad (4.1)$$

see [3, Thm 10.32], where \mathbb{A} extends the Bismut superconnection (2.20). We will see that the integral of the fibrewise \hat{A} -form over the fibres vanishes. The index bundle of the fibrewise operators A_{T^2} consists of fibrewise parallel sections of the spinor bundle. This will allow us to give a simple formula for its Chern character form, and hence for $d\tilde{\eta}(\mathbb{A})$. We follow Bismut and Cheeger [5], but consider tori of varying area.

Consider $\mathbb{R}^2 \cong \mathbb{C}$ with the standard Euclidean metric. We represent flat tori T^2 by lattices in \mathbb{C} , for which we choose integral bases (z, w) . In the gluing situation, $w = 1$ will correspond to a generator of the fundamental group of the interior circle, and z will be the complementary generator $\frac{\varepsilon_+ + s_+}{k_+}$ of the fundamental group of T^2 . With respect to the time parameter $t = t_-$, the basis (z, w) will be negatively oriented.

Because the adiabatic limit will produce degenerate lattices, we allow all pairs $(z, w) \in \mathbb{C}^2 \setminus \{0\}$. We let $\mathcal{H} \subset \mathbb{C}$ denote the upper halfplane and $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ its closure in $\mathbb{C}P^1$.

We write $\partial_\infty \mathcal{H} = \overline{\mathcal{H}} \setminus \mathcal{H} = \mathbb{R} \cup \infty$. We get a map $\tau: \mathbb{C}^2 \setminus \{0\} \rightarrow \overline{\mathcal{H}}$ by setting

$$\tau \begin{pmatrix} z \\ w \end{pmatrix} = \begin{cases} \infty & \text{if } w = 0, \\ z/w & \text{if } \operatorname{Im}(z/w) \geq 0, \text{ and} \\ \bar{z}/\bar{w} & \text{if } \operatorname{Im}(z/w) < 0. \end{cases}$$

The group $GL(2, \mathbb{Z})$ acts on $\mathbb{C}^2 \setminus \{0\}$ by matrix multiplication. It also acts on $\overline{\mathcal{H}}$ by Möbius transformations, more precisely,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \text{ and} \\ \frac{a\bar{z}+b}{c\bar{z}+d} & \text{if } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -1. \end{cases} \quad (4.2)$$

It is easy to check that the map τ intertwines these two $GL(2, \mathbb{Z})$ -actions.

We construct a right inverse $\Phi: \mathcal{H} \rightarrow \mathbb{C}^2 \setminus \{0\}$ of τ by

$$\Phi: z = x + iy \mapsto \frac{1}{\sqrt{y}} \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

This way we obtain a tautological bundle $p: E \rightarrow \mathcal{H}$ of flat tori with fibres $p^{-1}(z) = \mathbb{C}/\operatorname{im} \Phi^t(z)$ of area 1, where we regard $\Phi^t(z)$ as a map $\mathbb{Z}^2 \rightarrow \mathbb{C}$. We fix the standard torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ to obtain a trivialisation $E \cong T^2 \times \mathcal{H} \rightarrow \mathcal{H}$ of p as follows: At $z = x + iy \in \mathcal{H}$, we map $\mathbb{R}^2/\mathbb{Z}^2$ to $\mathbb{C}/\operatorname{im} \Phi^t(z)$ by

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \Phi^t(z) \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \frac{zu + v}{\sqrt{y}}.$$

Then the fibre $\mathbb{R}^2/\mathbb{Z}^2$ is equipped with the pullback metric

$$g_z = \Phi^t(z)^* g_0 = \frac{1}{y} \begin{pmatrix} x^2 + y^2 & x \\ x & 1 \end{pmatrix}.$$

We consider the diagonal $GL(2, \mathbb{Z})$ -action on $E \cong T^2 \times \mathcal{H}$, with the usual action on T^2 and the action (4.2) by Möbius transformations on \mathcal{H} . Then $GL(2, \mathbb{Z})$ acts by fibrewise isometries with respect to the fibrewise metrics considered above.

We extend E trivially to a bundle over $\mathcal{H} \times (0, \infty)$ and let r be a coordinate on $(0, \infty)$. Let $GL(2, \mathbb{Z})$ act trivially on $(0, \infty)$. We extend the family of metrics above to $g_{z,r} = r^2 g_z$. Let $W = \mathbb{R}^2 \times \mathcal{H} \times (0, \infty) \rightarrow \mathcal{H} \times (0, \infty)$ be the fibrewise universal covering of E , regarded as an oriented Euclidean vector bundle with fibrewise metrics $g_{z,r}$. The tangent bundle of $E = T^2 \times \mathcal{H} \times (0, \infty)$ splits as $TT^2 \oplus T\mathcal{H} \oplus T(0, \infty)$, and $GL(2, \mathbb{Z})$ preserves this splitting. We use this splitting to define a $GL(2, \mathbb{Z})$ -invariant connection $T^H E = \{0\} \oplus T\mathcal{H} \oplus T(0, \infty)$ on p . Together with $g_{z,r}$, it induces a metric connection

$${}^0\nabla = d + \frac{1}{2} g^{-1} dg = d + \frac{1}{2y^2} \begin{pmatrix} x dx + y dy & dx \\ (y^2 - x^2) dx - 2xy dy & -x dx - y dy \end{pmatrix} + \frac{dr}{r} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

on W . As in [5, Prop 2.1], the vertical tangent bundle $T(E/\mathcal{H})$ of $E \rightarrow \mathcal{H}$ together with its natural connection is just the pullback of $(W, {}^0\nabla)$. In particular the integral of the form $\hat{A}(T(E/\mathcal{H}))$ over the fibres in (4.1) vanishes.

Because W is trivial, there is an associated spinor bundle $S = S^+ \oplus S^- \rightarrow \mathcal{H} \times (0, \infty)$. Let $\widetilde{GL}(2, \mathbb{R})$ denote the double cover of $GL(2, \mathbb{R})$ that is nontrivial over both connected components, and let $\widetilde{GL}(2, \mathbb{Z}) \subset \widetilde{GL}(2, \mathbb{Z})$ denote the preimage of $GL(2, \mathbb{Z})$. Then the induced action on E and W lifts to S in a way that is compatible with Clifford multiplication. Let $\widetilde{SL}(2, \mathbb{Z})$ denote the preimage of $SL(2, \mathbb{Z})$ in $\widetilde{GL}(2, \mathbb{Z})$, then elements of $\widetilde{GL}(2, \mathbb{Z}) \setminus \widetilde{SL}(2, \mathbb{Z})$ swap the bundles S^+ and S^- . Because the vertical tangent bundle is isomorphic to p^*W , the

bundle p^*S becomes a fibrewise spinor bundle on E . Moreover, the kernel of the fibrewise Dirac operator consists of fibrewise parallel spinors. Therefore, the index bundle $\text{ind}(A_{T^2})$ in (4.1) is isomorphic to S , and the L^2 -unitary connection $\tilde{\nabla}^u$ and the connection ${}^0\nabla$ induce the same connection on the index bundle.

The basis $e_1 = \frac{\sqrt{y}}{r} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $e_2 = \frac{1}{r\sqrt{y}} \begin{pmatrix} 1 \\ -x \end{pmatrix}$ of \mathbb{R}^2 is oriented and g_{x+iy} -orthonormal. In this basis, we have

$${}^0\nabla = d + \frac{dx}{2y} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad {}^0\nabla^2 = \frac{dx dy}{2y^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In particular, the additional coordinate r no longer appears. We can now compute

$$\text{Pf}\left(\frac{{}^0\nabla^2}{2\pi}\right) = \frac{dx dy}{4\pi y^2} = \frac{1}{4\pi} dA_{\text{hyp}}.$$

The data considered above suffice to define the Bismut superconnection \mathbb{A} for the spinor bundle S on $E \rightarrow \mathcal{H} \times (0, \infty)$, which extends the superconnection \mathbb{A} introduced in Proposition 2.8; see also (2.20). Following Bismut and Cheeger, we get an explicit expression for the right hand side of (4.1) even if the area of the fibres is not constant.

Theorem 4.1 ([5, Thm 2.22]). *The spinorial η -form of the tautological family $E \rightarrow \mathcal{H} \times (0, \infty)$ has the exterior derivative*

$$d\tilde{\eta}(\mathbb{A}) = -(-1)^{\frac{\text{rk}W}{2}} \text{Pf}\left(\frac{{}^0\nabla^2}{2\pi}\right) \hat{A}^{-1}\left(\frac{{}^0\nabla^2}{2\pi}\right).$$

Hence, in our setting, we have

$$d\tilde{\eta}(\mathbb{A}) = \frac{1}{4\pi} dA_{\text{hyp}}. \quad (4.3)$$

Note that the η -form itself might still see the additional coordinate r , but only through an additional term of the form $f(r) dr$. Because we will integrate the η -form over closed contours below, this term will not affect any of the following computations. In particular, we can from now on ignore constant rescalings of the fibrewise metric and forget about the factor $(0, \infty)$ entirely.

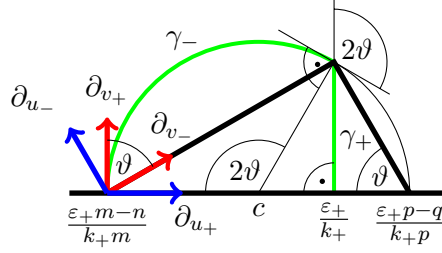
Remark 4.2. The η -form is not exact on \mathcal{H} . This does not contradict Proposition 2.8. If we were to leave the path in \mathcal{H} given by the adiabatic limit construction in section 2.4, the vertical tangent bundle of the family E_{\pm} would no longer split as in the proof of Proposition 2.7, so the local variation terms in (2.18) would no longer vanish and contribute to $d\bar{\nu}$ as well.

4.2. Adiabatic limits and hyperbolic geodesics. We represent the isometric tori $T_{\pm} = \tilde{T}_{\pm}/\Gamma_{\pm}$ by points in the upper halfplane \mathcal{H} . In particular, we make use of the fact that we may rescale all tori to area 1 without changing the contribution to the ν -invariant. When we consider adiabatic limits of M_{\pm} , the points corresponding to $X_{\pm,r}$ trace out geodesic arcs in \mathcal{H} . These arcs will be used to compute the sum of integrals of the η -form $\tilde{\eta}(\mathbb{A})$ of Proposition 2.8.

We represent $T = \tilde{T}_+/\Gamma_+$ by the basis (1.5). In equation (2.1) we have considered families of metrics on M_{\pm} . These induce two families of metrics on T_+ . We write $X_{+,a} = \partial M_{+,a} = \Sigma_+ \times T_{+,a}$. For the second one, we consider the isomorphism $T \cong \tilde{T}_-/\Gamma_-$ and write $X_{-,a} = \partial M_{-,a} = \Sigma_- \times T_{-,a}$. The map τ above represents both families by curves in \mathcal{H} .

Lemma 4.3. *Consider the families of flat tori $T_{+,a}$ and $T_{-,a}$ as above, with basis (1.5) for $a = s_{\pm}$.*

- (i) *The family $T_{+,a}$ is represented in \mathcal{H} by a vertical line γ_+ with real part $\frac{\varepsilon_+}{k_+}$. The adiabatic limit $a \rightarrow 0$ corresponds to $\frac{\varepsilon_+}{k_+} \in \partial_{\infty} \mathcal{H}$.*

FIGURE 12. The hyperbolic angle between γ_+ and γ_- .

- (ii) The family $T_{-,a}$ is represented in \mathcal{H} by a hyperbolic geodesic γ_- between $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+ m}$ and $\frac{\varepsilon_+}{k_+} - \frac{q}{k_+ p} \in \partial_\infty \mathcal{H}$. The adiabatic limit $a \rightarrow 0$ corresponds to $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+ m}$.
- (iii) The geodesics γ_+ and γ_- intersect at $\frac{\varepsilon_+ + is_+}{k_+}$ with unoriented angle

$$\left| \angle_{\frac{\varepsilon_+ + is_+}{k_+}} \left(\frac{\varepsilon_+}{k_+}, \frac{\varepsilon_+}{k_+} - \frac{n}{k_+ m} \right) \right| = 2\vartheta .$$

In the case of Example 3.1 (i), we have $2\vartheta = \pi$, so both γ_- and γ_+ agree with the vertical line with real part $\frac{\varepsilon_+}{k_+}$.

Proof. We write $T_{+,a} = \mathbb{C}/\Lambda_{+,a}$, where the lattice $\Lambda_{+,a}$ is generated by $\frac{\varepsilon_+ + ia}{k_+}$ and $1 \in \mathbb{C}$, so its complex structure is represented by the point $\frac{\varepsilon_+ + ia}{k_+} \in \mathcal{H}$ on the hyperbolic geodesic from $\frac{\varepsilon_+}{k_+}$ to ∞ .

Analogously, the torus $T_{-,a} = \mathbb{C}/\Lambda_{-,a}$ can be represented by $\frac{\varepsilon_- + ia}{k_-}$ on the hyperbolic geodesic from $\frac{\varepsilon_-}{k_-}$ to ∞ . We now consider the matrix

$$\begin{pmatrix} \frac{\varepsilon_+ p - q}{k_+} & \frac{n - \varepsilon_+ m + \varepsilon_- q - \varepsilon_- \varepsilon_+ p}{k_- k_+} \\ p & -\frac{\varepsilon_- p + m}{k_-} \end{pmatrix} \in GL(2, \mathbb{Z}) \setminus SL(2, \mathbb{Z}) ,$$

where the integrality of the upper right corner follows from (3.4). Assertion (ii) follows because under the action (4.2) by Möbius transformations, the matrix above maps

$$\frac{\varepsilon_-}{k_-} \mapsto \frac{\varepsilon_+}{k_+} - \frac{n}{k_+ m}, \quad \infty \mapsto \frac{\varepsilon_+}{k_+} - \frac{q}{k_+ p}, \quad \text{and} \quad \frac{\varepsilon_- + is_-}{k_-} \mapsto \frac{\varepsilon_+ + is_+}{k_+} .$$

We compute the angle in (iii) using Figure 12. We note that the hyperbolic upper half plane is conformal to the Euclidean half plane, so we may compute the angle using Euclidean geometry. Let c denote the Euclidean center of the circle through the points $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+ m}$, $\frac{\varepsilon_+}{k_+} - \frac{q}{k_+ p}$ and $\frac{\varepsilon_+ + is_+}{k_+}$. The angle between the two hyperbolic geodesic arcs from $\frac{\varepsilon_+}{k_+}$ and $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+ m}$ to $\frac{\varepsilon_+ + is_+}{k_+}$ equals the central angle subtending the arc from $\frac{\varepsilon_+}{k_+} - \frac{n}{k_+ m}$ to $\frac{\varepsilon_+ + is_+}{k_+}$. It is therefore twice the inscribed angle at $\frac{\varepsilon_+}{k_+} - \frac{q}{k_+ p}$, which we recognise as the gluing angle $\vartheta \in (0, \frac{\pi}{2}]$ as in Figure 12. Here we have used our assumption that $m, n \geq 0$. \square

4.3. Some elementary hyperbolic geometry. Some elementary properties of $\tilde{\eta}(\mathbb{A})$ are linked to hyperbolic geometry and the group $PGL(2, \mathbb{Z})$.

Lemma 4.4. *The spinorial η -form $\tilde{\eta}(\mathbb{A})$ is $PGL(2, \mathbb{Z})$ -equivariant, more precisely, for $g \in PGL(2, \mathbb{Z})$ acting on \mathcal{H} by Möbius transformations, we have*

$$g^* \tilde{\eta}(\mathbb{A}) = \det g \cdot \tilde{\eta}(\mathbb{A}).$$

Proof. The η -form is invariant under orientation preserving spin isometries. We know that each $g \in PGL(2, \mathbb{Z})$ has two possible lifts to $GL(2, \mathbb{Z})$ that act fibrewise on E over the given action on \mathcal{H} . Each of these lifts has two lifts to $\widetilde{GL}(2, \mathbb{Z})$ that also act on the spinor bundle $S \rightarrow \mathcal{H}$ and therefore also on the fibrewise spinor bundle p^*S over E .

If $g \in PSL(2, \mathbb{Z})$, all four lifts preserve the superconnection \mathbb{A} and the subbundles $S^\pm \subset S \rightarrow \mathcal{H}$. Therefore $g^* \tilde{\eta}(\mathbb{A}) = \tilde{\eta}(\mathbb{A})$ in this case.

If $g \in PGL(2, \mathbb{Z}) \setminus PSL(2, \mathbb{Z})$, then all four lifts of g to $\widetilde{GL}(2, \mathbb{Z})$ preserve the superconnection \mathbb{A} , but swap the bundles S^+ and S^- . This reverses the sign of the supertrace in the definition of the η -form, so we now have $g^* \tilde{\eta}(\mathbb{A}) = -\tilde{\eta}(\mathbb{A})$. \square

Remark 4.5. Each orientation reversing isometry $g \in PGL(2, \mathbb{Z}) \setminus PSL(2, \mathbb{Z})$ is a hyperbolic glide reflection along a hyperbolic geodesic γ_g . If g is a reflection, then the restriction of $\tilde{\eta}(\mathbb{A})$ to γ_g vanishes. If $g \in PGL(2, \mathbb{Z}) \setminus PSL(2, \mathbb{Z})$ is a reflection about γ_g , then hgh^{-1} is a reflection about $h(\gamma_g)$.

Elements g as above can be represented by matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}) \setminus SL(2, \mathbb{Z})$ as in (4.2). They represent reflections if and only if the trace $a + d$ vanishes. The line of reflection is vertical if and only if $c = 0$, so the corresponding reflections in $PSL(2, \mathbb{Z})$ are of the form $\begin{pmatrix} -1 & k \\ 0 & 1 \end{pmatrix}$, and the fixed line has real part $\frac{k}{2}$.

More generally, quotients of the standard rectangular lattice by a $\mathbb{Z}/k\mathbb{Z}$ -action are represented by points with real part in $\frac{1}{k}\mathbb{Z}$. The following Lemma helps us to recognise images of the geodesics associated with their adiabatic limits.

Lemma 4.6. *For any $\frac{e}{f} \in \mathbb{Q}$, there exists $\tau \in PSL(2, \mathbb{Z})$ such that $\tau(\frac{e}{f}) = \infty$.*

If $\frac{e}{f} \neq \frac{g}{h} \in \mathbb{Q}$ are represented by reduced fractions and $k := eh - fg > 0$, then $\tau(\frac{g}{h}) \equiv \frac{\varepsilon}{k} \pmod{\mathbb{Z}}$, where ε is the unique elements of $(\mathbb{Z}/k)^\times$ such that $g - \varepsilon e$ and $h - \varepsilon f$ are both divisible by k .

Proof. The first assertion is clear. For matrices $\begin{pmatrix} e & g \\ f & h \end{pmatrix}$ where the entries in each column are coprime, the determinant k and the $\varepsilon \in (\mathbb{Z}/k)^\times$ such that $(g, h) - \varepsilon(e, f)$ is divisible by k are clearly invariant under the action of $SL(2, \mathbb{Z})$. Hence if $\tau(e, f) = (1, 0)$ then $\tau(g, h) = (\varepsilon, k)$. \square

Corollary 4.7. *The geodesic between two reduced fractions $\frac{e}{f}$ and $\frac{g}{h} \in \mathbb{Q}$ in \mathcal{H} is fixed by a reflection in $PGL(2, \mathbb{Z})$ if and only if $eh - fg \in \{\pm 1, \pm 2\}$.*

4.4. The contribution from the cusps. Thanks to Theorem 4.1, we can in principle evaluate integrals of η -forms over closed contours using hyperbolic area formulas. However, we have to choose contours that touch the infinite boundary $\partial_\infty \mathcal{H} = \mathbb{R} \cup \{\infty\}$ because in (2.10), we consider adiabatic limits of tori. Moreover, Remark 4.5 and Corollary 4.7 only give a rather limited supply of geodesics on which the η -form $\tilde{\eta}(\mathbb{A})$ vanishes, and these connect rational points on the boundary $\partial_\infty \mathcal{H}$. In this section, we therefore compute the limit of the integral of the η -form over certain horocyclic arcs.

Adiabatic limit formulas for η -forms have been proved by Bunke, Ma [10] and Liu [30]. However, their results typically hold only up to exact forms. Here, we have to integrate the η -form over an interval, so we need an adiabatic limit formula that holds “on the nose”. We will prove such a formula in Section 6.2, but only for the simple special case at hand.

Recall that cusps are fixed by orientation preserving parabolic elements of $PSL(2, \mathbb{Z})$. They can be represented by matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ of trace $a + d = 2$. It follows that the set of cusp points is exactly $\mathbb{Q} \cup \{\infty\} \subset \partial_\infty \mathcal{H}$.

We want to define a distance between two hyperbolic geodesics ending in a cusp point $\frac{e}{f}$. To move $\frac{e}{f}$ to $\infty = \frac{1}{0}$, assume that the fraction $\frac{e}{f}$ is reduced and find a and $b \in \mathbb{Z}$ such that

$$ae + bf = 1. \quad (4.4)$$

Then $\begin{pmatrix} a & b \\ -f & e \end{pmatrix} \in SL(2, \mathbb{Z})$, and the Möbius transformation $z \mapsto \frac{az+b}{-fz+e}$ rotates the cusp $\frac{e}{f}$ into ∞ . Now consider hyperbolic geodesics starting at $x, y \in \mathbb{R} \cup \{\infty\}$ and ending in $\frac{e}{f}$. They get rotated by the matrix above to vertical lines with real parts $\frac{ax+b}{-fx+e}$ and $\frac{ay+b}{-fy+e}$. Because of (4.4), the difference is

$$\frac{ax+b}{-fx+e} - \frac{ay+b}{-fy+e} = \frac{x-y}{(fy-e)(fx-e)}.$$

Definition 4.8. The *cuspidal angle* between two hyperbolic geodesics going from $x, y \in \mathbb{R} \cup \{\infty\} = \partial_\infty \mathcal{H}$ to a cusp point represented by a reduced fraction $\frac{e}{f} \in \mathbb{Q} \cup \{\infty\}$, with $x \neq \frac{e}{f} \neq y$, is defined as

$$\angle_{\frac{e}{f}}(x, y) = \frac{x-y}{(fx-e)(fy-e)} \in \mathbb{R}$$

if $x, y \in \mathbb{R}$, and by the obvious extension of this formula if one of the points is ∞ .

Note that $\angle_{\frac{e}{f}}(x, y)$ is a geometric notion for the covering map $\mathcal{H} \rightarrow \mathcal{H}/SL(2, \mathbb{Z})$. It measures how often a line joining the two geodesics above in the universal covering space \mathcal{H} winds around the cusp in $\mathcal{H}/SL(2, \mathbb{Z})$. The sign is chosen such that oriented ideal triangles have positive cuspidal angles.

Proposition 4.9. Let $\frac{e}{f} \in \mathbb{Q} \cup \{\infty\} \subset \partial_\infty \mathcal{H}$ be a cusp point, and let $x, y \in \partial_\infty \mathcal{H}$. Assume that α_r is a family of horocyclic arcs centered at $\frac{e}{f}$ from the geodesic from $\frac{e}{f}$ to x to the geodesic from $\frac{e}{f}$ to y that converges to $\frac{e}{f}$ as $r \rightarrow \infty$. Then

$$\lim_{r \rightarrow \infty} \int_{\alpha_r} \tilde{\eta}(\mathbb{A}) = \frac{\angle_{e/f}(x, y)}{12}.$$

Proof. By the considerations above, we can rotate the cusp point $\frac{e}{f}$ to ∞ . Hence we consider the horocycle $x \mapsto x + iy$ for large fixed y and prove that

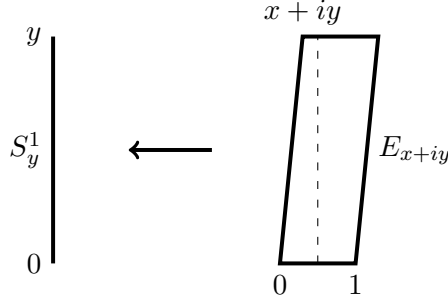
$$\lim_{y \rightarrow \infty} \tilde{\eta}(\mathbb{A})|_{\mathbb{R}+iy} = -\frac{dx}{12}. \quad (4.5)$$

Because η -forms are scale invariant, we can for the moment discard the Bismut-Cheeger convention that all tori have area 1. We choose fibrewise metrics induced by

$$\tilde{g}_{x+iy} = \begin{pmatrix} x^2 + y^2 & x \\ x & 1 \end{pmatrix}$$

We consider a family of fibered manifolds

$$\begin{array}{ccccc} S_1^1 & \longrightarrow & E_{x+iy} & \longrightarrow & E \\ & & \downarrow & & \downarrow \\ & & S_y^1 & \longrightarrow & F \\ & & & & \downarrow \\ & & & & \mathbb{R} \ni x. \end{array} \quad (4.6)$$


 FIGURE 13. The torus E_{x+iy} as a bundle.

Here, the interior circles form the fibres of E_{x+iy} of length 1, and the base can be identified with the exterior circles in \tilde{T}_+ . In particular, the orientation of E_{x+iy} agrees with the one in [10, 30]. The metric on E_{x+iy} defines a connection $T^H E_{x+iy}$ on $E_{x+iy} \rightarrow S_y^1$ with holonomy $-x \in \mathbb{R}/\mathbb{Z}$. In Figure 13, elements of $T^H E_{x+iy}$ correspond to vertical vectors. The dashed line is horizontal with respect to $T^H E_{x+iy}$.

Let $L \rightarrow F$ denote a Hermitian line bundle with a fibrewise flat connection that contains $E \rightarrow F$ as a circle bundle. Then over $F_x \cong S_y^1$, the bundle L has holonomy $e^{-2\pi i x}$. The bundle carries a Hermitian connection ∇^L that we can describe as

$$\nabla^L = d + \frac{2\pi i x}{y} d\varphi$$

in a trivialisation over a neighbourhood of F_x in F . Here, φ is a coordinate on $S_y^1 \cong \mathbb{R}/y\mathbb{Z}$. We compute curvature and first Chern form of (L, ∇^L) on F as

$$(\nabla^L)^2 = \frac{2\pi i}{y} dx d\varphi \quad \text{and} \quad c_1(\nabla^L) = -\frac{1}{y} dx d\varphi .$$

By Proposition 4.11 below, we have

$$\lim_{y \rightarrow \infty} \tilde{\eta}(\mathbb{A})|_{\mathbb{R}+iy} = \int_{F/\mathbb{R}} \tilde{\eta}(\mathbb{A}') , \quad (4.7)$$

where now \mathbb{A}' is the superconnection of the fibrewise spinor bundle over the circle bundle $E \rightarrow F$. The corresponding η -form of $E \rightarrow F$ has been computed by Zhang in [37, Thm 1.7]; see also [21, Rem 3.5]. Here, it only has a component of degree 2, given by

$$\tilde{\eta}(\mathbb{A}') = \frac{1}{12} c_1(\nabla^L) = -\frac{1}{12y} dx d\varphi .$$

Integration over the fibre of $F \rightarrow \mathbb{R}$ of length y proves (4.5). \square

Example 4.10. Consider the ideal triangle Δ with corners $0, 1$ and $\infty \in \partial_\infty \mathcal{H}$. By Remark 4.5, the form $\tilde{\eta}(\mathbb{A})$ vanishes on its sides. The cusp angles are $\angle_0(\infty, 1) = \angle_1(0, \infty) = \angle_\infty(1, 0) = 1$. Let Δ_r denote truncations of Δ by horocyclic arcs centered at the corners that converge to the full triangle Δ as $r \rightarrow \infty$; see Figure 14. Then by Theorem 4.1 and Proposition 4.9, we check that

$$\lim_{r \rightarrow \infty} \int_{\partial \Delta_r} \tilde{\eta}(\mathbb{A}) = \frac{\angle_0(\infty, 1) + \angle_1(0, \infty) + \angle_\infty(1, 0)}{12} = \frac{1}{4} = \frac{A_{\text{hyp}}(\Delta)}{4\pi} = \int_{\Delta} d\tilde{\eta}(\mathbb{A}) .$$

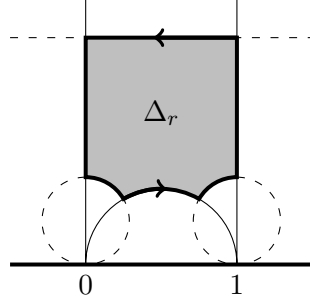


FIGURE 14. An ideal triangle, truncated by horocyclic arcs.

The following Proposition is inspired by a result [10, Thm 5.11] of Bunke and Ma; see also Liu in [30, Thm 1.3], where the following equality is proved up to exact forms. Moreover, Liu assumes that the fibrewise Dirac operator of the fibration $F \rightarrow \mathbb{R}$ is invertible [30, Ass 3.1], which is not the case here.

Proposition 4.11. *In the situation above, let \mathbb{A}' denote the superconnection associated with the fibrewise spin Dirac operator on the bundle $E \rightarrow F$. Then*

$$\lim_{y \rightarrow \infty} \tilde{\eta}(\mathbb{A})|_{\mathbb{R}+iy} = \int_{F/\mathbb{R}} \tilde{\eta}(\mathbb{A}').$$

We postpone the proof to Section 6.2.

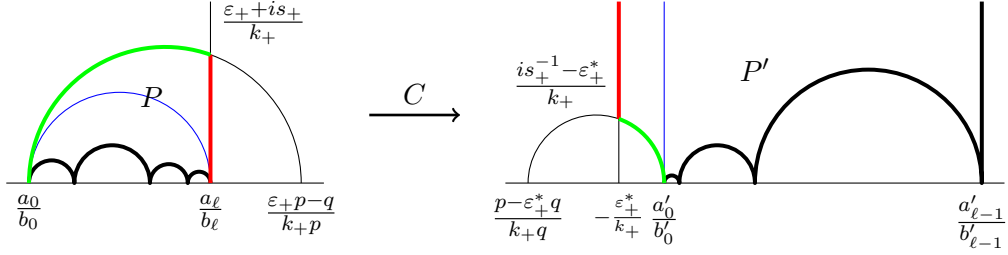
4.5. Continued Fractions and Hyperbolic Polygons. We can finally compute the variation of the $\bar{\nu}(M_{\pm,r})$ in the adiabatic limit $r \rightarrow 0$. We use the results of the last sections and elementary hyperbolic geometry.

Let $M_{\pm} = \tilde{M}_{\pm}/\Gamma_{\pm}$ be \mathbb{Z}/k_{\pm} -blocks with boundary $X_{\pm} \cong \Sigma_{\pm} \times T_{\pm}$ and $T_{\pm} = \tilde{T}_{\pm}/\Gamma_{\pm}$ as before. Define ε_{+} , s_{\pm} as in Section 1.3, and let γ_{\pm} be the hyperbolic geodesics of Lemma 4.3. For simplicity, we assume $n > 0$ and $m \geq 0$. To evaluate the variational formula (2.19) for M_{+} , we integrate $\tilde{\eta}(\mathbb{A})$ along the vertical line γ_{+} from $\frac{\varepsilon_{+}}{k_{+}}$ to $\frac{\varepsilon_{+}+is_{+}}{k_{+}}$. For M_{-} , we note that the orientation of the boundary was reversed during gluing. Hence, we integrate $\tilde{\eta}(\mathbb{A})$ along the geodesic ray γ_{-} from $\frac{\varepsilon_{+}+is_{+}}{k_{+}}$ to $\frac{\varepsilon_{+}}{k_{+}} - \frac{n}{k_{+}m} \in \partial_{\infty}\mathcal{H}$. Note that this last point is ∞ in case $m = 0$ and $\vartheta = \frac{\pi}{2}$.

We will now complete the two geodesic rays above to an ideal hyperbolic polygon P with one finite corner $\frac{\varepsilon_{+}+is_{+}}{k_{+}}$ and further corners represented by reduced fractions $\frac{\varepsilon_{+}}{k_{+}} - \frac{n}{k_{+}m} = \frac{a_0}{b_0}$, \dots , $\frac{a_{\ell}}{b_{\ell}} = \frac{\varepsilon_{+}}{k_{+}} \in \mathbb{Q} \cup \{\infty\} \subset \partial_{\infty}\mathcal{H}$ such that $a_j b_{j-1} - b_j a_{j-1} = 1$ for $j = 1, \dots, \ell$. Then $\tilde{\eta}(\mathbb{A})$ vanishes along the geodesics joining $\frac{a_{j-1}}{b_{j-1}}$ and $\frac{a_j}{b_j}$ by Lemma 4.4 and Corollary 4.7.

It turns out to be easier to construct the image P' of P under C , where $C = \begin{pmatrix} \varepsilon_{+}^{*} & -r \\ -k_{+} & \varepsilon_{+} \end{pmatrix} \in SL(2, \mathbb{Z})$ act as a Möbius transformation, with $\varepsilon_{+}\varepsilon_{+}^{*} = k_{+}r + 1$. Because the η -form $\tilde{\eta}(\mathbb{A})$, the hyperbolic area and the cusp angle are $SL(2, \mathbb{Z})$ -invariant, all computations can be performed on P' . We note that

$$C\left(\frac{\varepsilon_{+}}{k_{+}}\right) = \infty, \quad C\left(\frac{\varepsilon_{+}m - n}{k_{+}m}\right) = \frac{m - \varepsilon_{+}^{*}n}{k_{+}n}, \quad \text{and} \quad C\left(\frac{\varepsilon_{+}p - q}{k_{+}p}\right) = \frac{p - \varepsilon_{+}^{*}q}{k_{+}q}. \quad (4.8)$$


 FIGURE 15. The hyperbolic polygons P and P'

We follow Zagier's suggestion and use continued fractions. We let $a'_0 = \frac{m - \varepsilon_+^* n}{k_+}$, $b'_0 = n \in \mathbb{Z}$ and represent $\frac{a'_0}{b'_0}$ as a continued fraction with minus signs,

$$\frac{a'_0}{b'_0} = \frac{m - \varepsilon_+^* n}{k_+ n} = c_1 - \frac{1}{c_2 - \cdots - \frac{1}{c_\ell}}.$$

This way, we obtain a sequence of integers c_1, \dots, c_ℓ with $c_2, \dots, c_\ell \geq 2$ and reduced fractions

$$\frac{a'_0}{b'_0} < \frac{a'_1}{b'_1} = c_1 - \frac{1}{c_2 - \cdots - \frac{1}{c_{\ell-1}}} < \cdots < \frac{a'_{\ell-1}}{b'_{\ell-1}} = \frac{c_1}{1} \quad \text{and} \quad \frac{a'_\ell}{b'_\ell} = \frac{1}{0}.$$

As explained in [36, §V], the numbers a'_j, b'_j and c_j are related by the formula

$$\begin{pmatrix} a'_j & -a'_{j+1} \\ b'_j & -b'_{j+1} \end{pmatrix} = \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{\ell-j} & -1 \\ 1 & 0 \end{pmatrix} \in SL(2\mathbb{Z}) \quad (4.9)$$

for $j = 0, \dots, \ell - 1$, which also shows that $a'_{j+1}b'_j - a'_j b'_{j+1} = 1$.

Remark 4.12. For later use, we note that

- (i) because $a'_1 b'_0 - a'_0 b'_1 = 1$, the number $-b'_1$ is inverse to $a'_0 = \frac{m - \varepsilon_+^* n}{k_+}$ modulo $b'_0 = n$,
- (ii) because $-\frac{q + \varepsilon_-^* n}{k_-}$ is also inverse to a'_0 by equation (3.6), we have

$$b'_1 \equiv \frac{q + \varepsilon_-^* n}{k_-} \pmod{n}.$$

Let $\frac{a'_j}{b'_j}$ denote the preimage of $\frac{a'_j}{b'_j}$ under the Möbius transformation C for $0 < j < \ell$, the cases $j = 0, \ell$ being settled by (4.8). Then $a_j b_{j-1} - a_{j-1} b_j = 1$. The finite corner at $\frac{\varepsilon_+ + i s_+}{k_+}$ gets mapped to $\frac{i s_+^{-1} - \varepsilon_+^*}{k_+}$. Thus we have completed the construction of P and P' ; see Figure 15.

From (4.9), we see that $\begin{pmatrix} a'_j \\ b'_j \end{pmatrix} = C_j \begin{pmatrix} c_{\ell-j} \\ 1 \end{pmatrix}$, $\begin{pmatrix} a'_{j+1} \\ b'_{j+1} \end{pmatrix} = C_j \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} a'_{j+2} \\ b'_{j+2} \end{pmatrix} = C_j \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, with $C_j = \begin{pmatrix} a'_{j+1} & -a'_{j+2} \\ b'_{j+1} & -b'_{j+2} \end{pmatrix} \in SL(2, \mathbb{Z})$. Because cusp angles are $SL(2, \mathbb{Z})$ -invariant, we can now

compute the cusp angles of P . For $j = 0, \dots, \ell - 2$, we obtain

$$\angle_{\frac{a_{j+1}}{b_{j+1}}} \left(\frac{a_j}{b_j}, \frac{a_{j+2}}{b_{j+2}} \right) = \angle_{\frac{a'_{j+1}}{b'_{j+1}}} \left(\frac{a'_j}{b'_j}, \frac{a'_{j+2}}{b'_{j+2}} \right) = \angle_{\infty}(c_{\ell-j}, 0) = c_{\ell-j}, \quad (4.10a)$$

$$\angle_{\frac{a_0}{b_0}} \left(\frac{\varepsilon_+ + is_+}{k_+}, \frac{a_1}{b_1} \right) = \angle_{\frac{a'_0}{b'_0}} \left(\frac{p - \varepsilon_+^* q}{k_+ p}, \infty \right) + \angle_{\frac{a'_0}{b'_0}} \left(\infty, \frac{a'_1}{b'_1} \right) = \frac{-q}{k_- n} + \frac{b'_1}{b'_0}, \quad (4.10b)$$

$$\angle_{\frac{a_{\ell}}{b_{\ell}}} \left(\frac{a_{\ell-1}}{b_{\ell-1}}, \frac{\varepsilon_+ + is_+}{k_+} \right) = \angle_{\infty} \left(c_1, -\frac{\varepsilon_+^*}{k_+} \right) = c_1 + \frac{\varepsilon_+^*}{k_+}. \quad (4.10c)$$

For (4.10b), we have used Lemma 4.3 (ii) and equation (4.8).

4.6. Evaluation of the η -Form Integrals. With all preliminaries understood, we can now prove Theorem 3. We start by integrating the η -form from Proposition 2.8 along the geodesic rays γ_+ and γ_- .

Theorem 4.13. *Assume that $m \geq 0$, $n > 0$. Then we have*

$$\begin{aligned} \bar{\nu}(M_{+,s_+}) + \bar{\nu}(M_{-,s_-}) - \lim_{r \rightarrow 0} (\bar{\nu}(M_{+,r}) + \bar{\nu}(M_{-,r})) \\ = 72 \frac{\rho}{\pi} + 24 \left(\frac{q}{k_- n} - \frac{m}{k_+ n} + 12 S \left(\frac{m - \varepsilon_+^* n}{k_+}, n \right) \right). \end{aligned} \quad (4.11)$$

It will follow from the proof below that the first three terms on the right hand side stem from the area of the triangle spanned by γ_+ and γ_- . The Dedekind sum comes from the polygon we get by omitting the finite corner. Both areas are separated by the blue geodesic in Figure 15.

Proof. By Propositions 2.8, 4.9 and the discussion at the beginning of subsection 4.5, we have

$$\begin{aligned} \bar{\nu}(M_{+,s_+}) + \bar{\nu}(M_{-,s_-}) - \lim_{r \rightarrow 0} (\bar{\nu}(M_{+,r}) + \bar{\nu}(M_{-,r})) &= 288 \int_{\gamma_+ \cup \gamma_-} \tilde{\eta}(\mathbb{A}) = 288 \int_P d\tilde{\eta}(\mathbb{A}) \\ &- 24 \angle_{\frac{a_0}{b_0}} \left(\frac{\varepsilon_+ + is_+}{k_+}, \frac{a_1}{b_1} \right) - 24 \angle_{\frac{a_{\ell}}{b_{\ell}}} \left(\frac{a_{\ell-1}}{b_{\ell-1}}, \frac{\varepsilon_+ + is_+}{k_+} \right) - 24 \sum_{j=0}^{\ell-2} \angle_{\frac{a_{j+1}}{b_{j+1}}} \left(\frac{a_j}{b_j}, \frac{a_{j+2}}{b_{j+2}} \right). \end{aligned} \quad (4.12)$$

From (4.3), Lemma 4.3 (iii) and the hyperbolic area formula, we get

$$288 \int_P d\tilde{\eta}(\mathbb{A}) = \frac{72}{\pi} A_{\text{hyp}}(P) = 72\ell - 144 \frac{\vartheta}{\pi} = 72(\ell - 1) + 72 \frac{\rho}{\pi}. \quad (4.13)$$

From Definition 4.8 and equations (4.10), we get

$$\begin{aligned} \angle_{\frac{a_0}{b_0}} \left(\frac{\varepsilon_+ + is_+}{k_+}, \frac{a_1}{b_1} \right) + \angle_{\frac{a_{\ell}}{b_{\ell}}} \left(\frac{a_{\ell-1}}{b_{\ell-1}}, \frac{\varepsilon_+ + is_+}{k_+} \right) + \sum_{j=0}^{\ell-2} \angle_{\frac{a_{j+1}}{b_{j+1}}} \left(\frac{a_j}{b_j}, \frac{a_{j+2}}{b_{j+2}} \right) \\ = \left(\frac{-q}{k_- n} + \frac{b'_1}{b'_0} \right) + \left(\frac{\varepsilon_+^*}{k_+} + c_1 \right) + \sum_{j=2}^{\ell} c_j. \end{aligned} \quad (4.14)$$

This number can be interpreted along the lines of [36, §V]. The product of the matrices on the right hand side of (4.9) for $j = 0$ is given by

$$A = \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{\ell} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a'_0 & -a'_1 \\ b'_0 & -b'_1 \end{pmatrix}.$$

Now it follows from $a'_0 = \frac{m - \varepsilon_+^* n}{k_+}$, $b'_0 = n$ and [36, equations (6), (25), (26)] that

$$\begin{aligned}
 \left(\frac{-q}{k_- n} + \frac{b'_1}{b'_0} \right) + \left(\frac{\varepsilon_+^*}{k_+} + c_1 \right) + \sum_{j=2}^{\ell} c_j \\
 &= \frac{-q}{k_- n} + \frac{b'_1}{b'_0} + \frac{\varepsilon_+^*}{k_+} + 3(\ell - 1) + N(A) \\
 &= \frac{-q}{k_- n} + \frac{b'_1}{b'_0} + \frac{\varepsilon_+^*}{k_+} + 3(\ell - 1) + \frac{m - \varepsilon_+^* n}{k_+ n} - \frac{b'_1}{b'_0} - 12S(-b'_1, b'_0) \quad (4.15) \\
 &= 3(\ell - 1) + \frac{m}{k_+ n} - \frac{q}{k_- n} - 12S(-b'_1, n),
 \end{aligned}$$

where the Dedekind sum $S(-b'_1, n)$ is defined in (0.3), and $N: SL(2, \mathbb{Z}) \rightarrow \mathbb{Z}$ is introduced in (A.1), see also [36, (3)].

The Dedekind sum $S(k, n)$ is odd and n -periodic in k , and it does not change if k is replaced by its inverse modulo n . Our claim (4.11) follows from Remark 4.12 (i) and (4.12)–(4.15). \square

Proof of Theorem 3. We may assume that $n > 0$. If $m \geq 0$, the theorem follows from Theorems 2.1, 2.4 and 4.13. If $m < 0$, we additionally use Proposition 3.7, combining (3.10) and (3.11) to reduce to the case $m > 0$, $n > 0$. \square

Remark 4.14. The formula in Theorem 3 is indeed symmetric in the two halves of the twisted connected sum. Swapping the two halves amounts to exchanging m and $-q$, and ε_{\pm} and k_{\pm} ; see Proposition 3.7, in particular (3.9). By equation (3.6), the number $\frac{m - \varepsilon_+^* n}{k_+}$ is inverse to $-\frac{q + \varepsilon_-^* n}{k_-}$ modulo n , so the Dedekind sum above is the same in both cases.

Remark 4.15. We can evaluate $3n(P) - \ell(P) \bmod \mathbb{Z}$ in a slightly different way. By Remark 4.12 (ii), we have $\frac{-q}{k_- n} + \frac{b'_1}{b'_0} \equiv \frac{\varepsilon_-^*}{k_-} \bmod \mathbb{Z}$, so instead of (4.15), we get

$$\left(\frac{-q}{k_- n} + \frac{b'_1}{b'_0} \right) + \left(\frac{\varepsilon_+^*}{k_+} + c_1 \right) + \sum_{j=2}^{\ell} c_j \equiv \frac{\varepsilon_+^*}{k_+} + \frac{\varepsilon_-^*}{k_-} \bmod \mathbb{Z}.$$

Following the proofs of Theorems 4.13 and 3 above, we see that

$$\nu(M, g) \equiv D_{\gamma_+}(V_+) + D_{\gamma_-}(V_-) + 3m_{\rho}(L; N_+, N_-) - 24 \left(\frac{\varepsilon_+^*}{k_+} + \frac{\varepsilon_-^*}{k_-} \right) \bmod 24\mathbb{Z}. \quad (4.16)$$

The terms $\frac{\varepsilon_{\pm}}{k_{\pm}}$ and $D_{\gamma_{\pm}}(V_{\pm})$ depend on the Γ_{\pm} -action on \tilde{M}_{\pm} only, and one can check that in all our examples

$$D_{\gamma_{\pm}}(V_{\pm}) - 24 \frac{\varepsilon_{\pm}^*}{k_{\pm}} \in \mathbb{Z}, \quad (4.17)$$

as one would expect from the formula above. In particular for $k_{\pm} = 5$, we only found examples where the Γ_{\pm} -action on V_{\pm} has isolated fixpoints, and (4.17) holds for all choices of ε_{\pm} .

Example 4.16. Let us illustrate the remark above using our standard example. We start with the $\mathbb{Z}/5$ -block from Example 5.10, whose fixpoint contribution was computed in Example 2.6. We check that modulo 24,

$$\frac{24}{5\varepsilon} - 24 \frac{\varepsilon^*}{5} \equiv \begin{cases} 0 & \text{if } \varepsilon \equiv \pm 1, \text{ and} \\ 12 & \text{if } \varepsilon \equiv \pm 2. \end{cases}$$

The $\mathbb{Z}/3$ -block from Example 5.5 has no isolated fixpoints, so the contribution to $\nu \bmod 24$ is simply $-24 \frac{\varepsilon_3^*}{3} = -8\varepsilon$. Together with $m_\rho(L; N_+, N_-) = -1$ from Example 2.2, we find that $\nu(M) \equiv -11 \bmod 24$, which confirms our computations in Example 2.15.

Remark 4.17. One can check that we recover the formula for $\bar{\nu}(M)$ in [14] in the case where $k_+, k_- \in \{1, 2\}$. Involutive isomorphisms of Calabi-Yau manifolds cannot have isolated fixpoints; see Section 2.3, so the first two terms on the right side of (0.4) vanish.

We start with a rectangular twisted connected sum as in [28, 13], so $k_+ = k_- = 1$. Because $m = q = 0 = \rho = A$, we have $\bar{\nu}(M) = 0$ by (0.4).

Next, consider examples with $k_- = 1, k_+ = 2$ and gluing matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, so $\vartheta = \frac{\pi}{4}$. By (0.4), we get

$$\bar{\nu}(M) = -3 + 24 \left(-1 - \frac{1}{2} + 12 S(0, 1) \right) = -39 ,$$

see entries 1–18 in Table 2.

In [14], we only consider simply connected examples with $k_- = k_+ = 2$, so either $\begin{pmatrix} m & p \\ n & q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ and $\vartheta = \frac{\pi}{3}$, or $\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$ and $\vartheta = \frac{\pi}{6}$. In the first case,

$$\bar{\nu}(M) = -3 + 24 \left(-\frac{1}{6} - \frac{1}{6} + 12 S(-1, 3) \right) = -27 ,$$

see entries 105, 113, 120–123, 135, 142, 146 in Table 2. In the second case,

$$\bar{\nu}(M) = -3 + 24 \left(-\frac{3}{2} - \frac{1}{2} + 12 S(0, 1) \right) = -51 ,$$

see entries 110, 111, 138, 139 in Table 2.

5. EXAMPLES

In this section, we generate examples of extra-twisted connected sums and compute their $\bar{\nu}$ -invariants. To do this we will define some building blocks whose polarising lattice has rank 1. We will also describe the topology of those blocks, and compute some parts of the topology of the resulting extra-twisted connected sums.

5.1. The cohomology of an extra-twisted connected sum. We have previously explained how to compute the fundamental group of an extra-twisted connected sum M from the gluing matrix, in Proposition 1.10. We now compute some other basic topological features. In particular we show that all our examples have $H_2(M) = 0$ (so those that have $\pi_1 M = 0$ are in fact 2-connected), and give a formula (5.2) for $b_3(M)$.

Remark 5.1. The most important topological properties that we do not compute are the torsion in $H^4(M)$, and the Pontryagin class $p_1(M) \in H^4(M)$. In general, the torsion in $H^4(M)$ can have contributions both from the action of Γ_\pm on the two halves, as well as from the gluing. In [31], attention was focussed on blocks with involution such that the former contribution vanishes, and on certain matchings (namely ones with gluing matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$) where the torsion in $H^4(M)$ can then be determined in a simple way from the configuration $N_+ + N_-$. We do not attempt here to generalise those arguments.

Topology of V_\pm . Let us first recall from [12, §5] some relevant facts about the topology of the ACyl Calabi-Yau manifold $V := Z \setminus \Sigma$ constructed from a building block (Z, Σ) , and make some observations about the action on cohomology of an automorphism group Γ . Let us assume that the kernel of the restriction map $H^2(Z) \rightarrow H^2(\Sigma)$ is generated by $[\Sigma]$. Note that this implies that the action of Γ on $H^2(Z)$ is trivial.

We identify $\Sigma \subset Z$ with a standard K3 surface and denote the image of $H^2(Z)$ in $L = H^2(\Sigma)$ by N . Then $N \subset L$ is primitive, but typically not unimodular. Let $T = N^\perp \subset L$, and let N^* be the dual of N , such that we have a short exact sequence

$$0 \longrightarrow T \longrightarrow L \longrightarrow N^* \longrightarrow 0 .$$

By [12, Lemma 5.2], Z and V are simply connected. Using excision and suspension, we have

$$H^k(Z, V) \cong H^k(\Sigma \times D^2, \Sigma \times S^1) \cong H^{k-2}(\Sigma) .$$

The long exact sequence of the pair (Z, V) becomes

$$\dots \longrightarrow H^{k-2}(\Sigma) \xrightarrow{\iota} H^k(Z) \xrightarrow{j^*} H^k(V) \xrightarrow{\delta} H^{k-1}(\Sigma) \longrightarrow \dots . \quad (5.1)$$

According to [12, Lemma 5.4], we have $H^1(V) = 0 = H^5(V)$, and the exact sequence (5.1) gives rise to split short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathbb{Z} \longrightarrow H^2(Z) \longrightarrow H^2(V) \longrightarrow 0 , \\ 0 \longrightarrow H^3(Z) \longrightarrow H^3(V) \longrightarrow T \longrightarrow 0 , \\ 0 \longrightarrow N^* \longrightarrow H^4(Z) \longrightarrow H^4(V) \longrightarrow 0 . \end{aligned}$$

It follows that $H^\bullet(V)$ is torsion free if $H^\bullet(Z)$ is torsion free. The inclusion $\mathbb{Z} \hookrightarrow H^2(Z)$ maps 1 to the cohomology class $\iota_! 1$ induced by $\iota: \Sigma \rightarrow Z$. It is easy to see that the sequences above are Γ -equivariant. In particular, Γ acts trivially on $H^2(V)$, while the Γ -invariant part of $H^3(V)$ is the direct sum of T and $H^3(Z)^\Gamma$.

The map δ in (5.1) involves restriction to $\Sigma \times S_\zeta^1$ followed by integration over S_ζ^1 . Write

$$H^k(\Sigma \times S_\zeta^1) = H^k(\Sigma) \oplus H^{k-1}(\Sigma)\mathbf{u} ,$$

where $\mathbf{u} \in H^1(S_\zeta^1)$ is the generator. The restriction map $\iota^*: H^\bullet(V) \rightarrow H^\bullet(\Sigma \times S_\zeta^1)$ is described in [12, Cor 5.5]. We have in particular that $H^2(V)$ maps isomorphically to $N \subset H^2(\Sigma)$, while the image of $H^3(V)$ is $T\mathbf{u} \subset H^3(\Sigma)\mathbf{u}$.

Topology of M_\pm . We may regard $M_\pm = (V_\pm \times S_{\xi_\pm}^1)/\Gamma_\pm$ as the mapping torus of a generator $\gamma_0 \in \Gamma_\pm \cong \mathbb{Z}/k_\pm$. Generalising the discussion of blocks with involution from [31, §2.2], we can use excision and the Thom isomorphism to see that

$$H^\bullet(M_\pm, V_\pm) \cong H^{\bullet-1}(V_\pm) .$$

From the long exact sequence of the pair (M_\pm, V_\pm) , we get

$$\dots \longrightarrow H^{\ell-1}(V_\pm) \longrightarrow H^\ell(M_\pm) \xrightarrow{\iota_V^*} H^\ell(V_\pm) \xrightarrow{\gamma_0^* - \text{id}} H^\ell(V_\pm) \longrightarrow \dots ,$$

where $\iota_V: V_\pm \rightarrow M_\pm$ is the inclusion of V as fibre of the obvious projection $M_\pm \rightarrow S_{\xi_\pm/k_\pm}^1$.

Since $H^1(V) = 0$ while $H^2(V)$ is Γ -invariant, it is immediate that $H^2(M_\pm) \cong H^2(V_\pm)$. Since $H^3(V_\pm)^\Gamma$ is torsion-free, we also have a splitting $H^3(M_\pm) \cong H^2(V_\pm) \oplus H^3(V_\pm)^\Gamma$, and $H^3(M_\pm)$ is torsion-free too. While the splitting is not natural with \mathbb{Z} coefficients, it is natural with \mathbb{Q} coefficients.

We can treat the cross-section $\Sigma \times T^2$ similarly. Now $H^2(\Sigma \times T^2) = H^2(\Sigma) \oplus H^2(T^2)$, and $H^2(M_\pm) \rightarrow H^2(\Sigma \times T^2)$ maps isomorphically to N_\pm .

Meanwhile, the pull-back $H^1(T^2) \rightarrow H^1(S_\zeta^1 \times S_\zeta^1)$ of the quotient map is injective (with image of index k_\pm). We abuse notation slightly to use $\mathbf{u}_\pm, \mathbf{v}_\pm$ to denote not only the generators of $H^1(S_{\zeta_\pm}^1 \times S_{\zeta_\pm}^1)$ obtained by pulling back the generators of the two factors, but also their unique pre-images in $H^1(T^2; \mathbb{Q})$. (In terms of de Rham cohomology, these classes are represented by

the 1-forms $\frac{1}{\zeta_+} du_+$ and $\frac{1}{\zeta_{\pm}} dv_{\pm}$.) Switching to rational coefficients, we then have the splitting $H^3(\Sigma \times T^2; \mathbb{Q}) = H^2(\Sigma; \mathbb{Q})\mathbf{u}_{\pm} \oplus H^2(\Sigma; \mathbb{Q})\mathbf{v}_{\pm}$. In the splitting

$$H^3(M_{\pm}; \mathbb{Q}) = H^2(V_{\pm})\mathbf{v}_{\pm} \oplus H^3(V_{\pm}; \mathbb{Q})^{\Gamma},$$

the first term has image exactly $N_{\pm}\mathbf{v}_{\pm}$, the second term has image exactly $T_{\pm}\mathbf{u}_{\pm}$, and the kernel of $H^3(M_{\pm}; \mathbb{Q}) \rightarrow H^3(\Sigma \times T^2; \mathbb{Q})$ is the $H^3(Z_{\pm}; \mathbb{Q})^{\Gamma}$ component in $H^3(V_{\pm}; \mathbb{Q})^{\Gamma}$.

The topology of M . Generalising the discussion from [31, §7.1] of extra-twisted connected sums that involve only involutions, we can now apply the Mayer-Vietoris sequence to compute some basic features of the topology of an extra-twisted connected sum.

Proposition 5.2. *Let M be an extra-twisted connected sum of building blocks (Z, Σ) such that $H^2(Z) \rightarrow H^2(\Sigma)$ is generated by $[\Sigma]$, with configuration of polarising lattices $N_+, N_- \hookrightarrow L$. Let ρ_{\pm} be the ranks of the polarising lattices (so $\rho_{\pm} = b_2(Z_{\pm}) - 1$). If $\cos \vartheta \neq 0$ let d_{ϑ} be the rank of $N_+^{\vartheta} \cong N_-^{\vartheta}$ defined in §1.4, otherwise let $d_{\vartheta} = \text{rk } N_+^{\frac{\pi}{2}} + \text{rk } N_-^{\frac{\pi}{2}}$.*

- (i) *The free part of $H^2(M)$ is isomorphic to $N_+ \cap N_- \subset L$,*
- (ii) *The torsion in $H^3(M)$ is isomorphic to the cotorsion of $N_+ + N_-$ in L .*
- (iii) *$b_3(M) = b_2(M) + 23 - \rho_+ - \rho_- + d_{\vartheta} + b_3(Z_+)^{\Gamma_+} + b_3(Z_-)^{\Gamma_-}$.*

The examples considered in this paper use configurations where N_+ is transverse to N_- , and $N_+ + N_-$ is embedded primitively in L . Thus the proposition implies that our examples have $H_2(M) = 0$, and those examples that are simply-connected are in fact 2-connected. Moreover, all our examples have $\rho_+ = \rho_- = 1$, and hence $\text{rk } N_+^{\vartheta} = \text{rk } N_-^{\vartheta} = 1$. Thus if $\vartheta \notin \frac{\pi}{2}\mathbb{Z}$ we have $d_{\vartheta} = 1$ and

$$b_3(M) = 22 + b_3^{\Gamma}(Z_+) + b_3^{\Gamma}(Z_-), \quad (5.2)$$

while if $\vartheta \in \frac{\pi}{2}\mathbb{Z} \setminus \pi\mathbb{Z}$ then $d_{\vartheta} = 2$ and

$$b_3(M) = 23 + b_3^{\Gamma}(Z_+) + b_3^{\Gamma}(Z_-). \quad (5.3)$$

Proof. We have a Mayer-Vietoris sequence

$$\dots \longrightarrow H^k(M) \longrightarrow H^k(M_+) \oplus H^k(M_-) \longrightarrow H^k(\Sigma \times T^2) \longrightarrow H^{k+1}(M) \longrightarrow \dots$$

The image of $H^1(M_+) \oplus H^1(M_-)$ in $H^1(\Sigma \times T^2)$ has finite index, and indeed it is dual to the fundamental group computed in Proposition 1.10. Since $H^2(M_{\pm})$ maps isomorphically to $N_{\pm} \subset H^2(\Sigma \times T^2)$, the image of $H^2(M)$ in $H^2(M_+) \oplus H^2(M_-)$ is isomorphic to $N_+ \cap N_- \subset L$ (as determined by the configuration).

For $H^3(M)$ we get a short exact sequence

$$0 \longrightarrow L/(N_+ + N_-) \oplus \mathbb{Z} \longrightarrow H^3(M) \longrightarrow \ker(H^3(M_+) \oplus H^3(M_-) \rightarrow H^3(\Sigma \times T^2)) \longrightarrow 0$$

Since the last term is torsion-free, the sequence splits, and the torsion of $H^3(M)$ equals the torsion of $L/(N_+ + N_-)$.

Finally we want to determine $b_3(M)$. The contribution from $L/(N_+ + N_-) \oplus \mathbb{Z}$ equals $23 - \rho_+ - \rho_- + b_2(M)$. The other term we describe as the sum of the kernels of $H^3(M_{\pm}) \rightarrow H^3(\Sigma \times T^2)$, which by the above have rank $b_3(Z_{\pm})^{\Gamma}$, and the intersection of the images in $H^3(\Sigma \times T^2)$. Points in the intersection of the images are those that can be written as both $n_+\mathbf{v}_+ + t_+\mathbf{u}_+$ and $n_-\mathbf{v}_- + t_-\mathbf{u}_-$, with $n_{\pm} \in N_{\pm}$ and $t_{\pm} \in T_{\pm}$. Now, the gluing identifies the tori in such a way that

$$\zeta_+\mathbf{v}_+ = \cos \vartheta \zeta_-\mathbf{v}_- + \sin \vartheta \zeta_-\mathbf{u}_-, \quad \zeta_+\mathbf{u}_+ = \sin \vartheta \zeta_-\mathbf{v}_- - \cos \vartheta \zeta_-\mathbf{u}_-.$$

If $\cos \vartheta \neq 0$ then n_+ and n_- determine each other, because the orthogonal projection of $\frac{1}{\zeta_-}n_-$ to N_+ must be $\frac{\cos \vartheta}{\zeta_+}n_+$ and vice versa. Thus, in the notation of §1.4, in fact $n_{\pm} \in N_{\pm}^{\vartheta}$, so the

intersection of the images is isomorphic to $N_+^\vartheta \cong N_-^\vartheta$. On the other hand, if $\cos \vartheta = 0$ then we can simply take each n_\pm freely in $N_\pm^{\frac{\pi}{2}}$ (i.e. in the orthogonal complement to N_\mp in N_\pm).

Either way, the contribution to $b_3(M)$ of the intersection of the images in $H^3(\Sigma \times T^2)$ is what we denoted as d_ϑ . Adding that to the other contributions proves (iii). \square

5.2. Examples of building blocks. We now give examples of building blocks. For simplicity, we restrict attention to blocks whose polarising lattice N has rank 1. We list the relevant data for the examples in Table 1.

Each family of blocks Z is obtained by blowing up Fano 3-folds Y of Picard rank 1. We list the index r of Y , the anticanonical degree $-K_Y^3$, the norm-square of the generator of the Picard lattice N of Y (which is isometric to the polarising lattice of Z), the third Betti number $b_3(Y)$, and the result of evaluating $c_2(Z)$ on the pull-back $H \in H^2(Z)$ of the generator $-\frac{1}{r}K_Y \in H^2(Y)$ (the latter number is needed to compute the Pontryagin class of the extra-twisted connected sums built from the block, although we do not do that in this paper).

Recall that the Picard lattice of a Fano 3-fold Y is $H^2(Y)$ equipped with the bilinear form $(A, B) \mapsto A \cdot B \cdot (-K_Y)$. For rank 1 Fanos, the norm-square of the generator $-\frac{1}{r}K_Y$ is thus simply computed as $\frac{1}{r^2}(-K_Y)^3$.

Example 5.3. If we ignore the desire for automorphisms, then we can simply take the list of rank 1 blocks from [12, Table 1]. These are obtained by blowing up a Fano 3-fold Y of Picard rank 1 along the transverse intersection C of two smooth anticanonical divisors. As explained in [12, §5.2], the resulting building block Z has $b_3(Z) = b_3(Y) + b_1(C) = b_3(Y) + (-K_Y)^3 + 2$. For the final piece of data we wish to include in Table 1, [18, (4.4)] gives $c_2(Z)H = \frac{24 - K_Y^3}{r}$.

All other examples we consider will in fact be subfamilies of the families from Example 5.3 that admit automorphisms. For each suitable automorphism that we have found on some elements of the family, we list in Table 1 its order k and the rank $b_3^\Gamma(Z)$ of the invariant part of $H^3(Z)$ (so the number against $k = 1$ is $b_3(Z)$), and the number of isolated fixed points (among all elements of Γ). The formula (5.2) for the third Betti number of an extra-twisted connected sum involves $b_3^\Gamma(Z)$, while the computation of the $\bar{\nu}$ -invariant in Theorem 2.4 relies on some further details about the fixed points that is not included in the table, but only in the descriptions of the individual examples.

The pattern is that we consider special elements Y of the given family of Fano 3-folds that admit a group of automorphisms Γ , whose fixed set is a union of a K3 divisor Σ and (possibly) some isolated fixed points. After blowing up a curve $C \subset \Sigma$ like in Example 5.3, one obtains a building block Z with automorphisms whose fixed sets are the proper transforms of the union of the fixed set of the corresponding automorphism on Y and a copy $\tilde{C} \subset V \subset Z$ of C ; it is a section of the exceptional set, which is a trivial \mathbb{P}^1 -bundle over C .

In all but one of our examples (Example 5.9), the fixed set Z^γ is the same for all non-identity elements $\gamma \in \Gamma$. Because the cohomology of Z is Γ -invariant except in degree 3 (so that e.g. $b_2(Z/\Gamma) = b_2(Z) = 2$) we can in those cases easily compute $b_3^\Gamma(Z)$ from

$$\chi(Z/\Gamma) = \frac{1}{k}\chi(Z) + \frac{k-1}{k}\chi(Z^\gamma).$$

Example 5.4. Blocks with involutions were already considered in [31]. In some sense, the simplest way to obtain examples is to start from a Fano 3-fold X with even anti-canonical class $-K_X$ (i.e. \mathbb{P}^3 or a del Pezzo 3-fold), and consider a double cover Y of X branched over an anticanonical divisor; see [31, Example 3.22]. Blowing up the double cover Y along a

	Y	r	$-K_Y^3$	N	$b_3(Y)$	$c_2(Z)H$	Ex	k	$b_3^\Gamma(Z)$	$\#fix$
1	\mathbb{P}^3	4	64	4	0	22	5.3	1	66	
2	Q	3	54	6	0	26	5.3	1	56	
3	V_1	2	8	2	42	16	5.3	1	52	
4	V_2	2	16	4	20	20	5.3	1	38	
5							5.4	2	18	
6	V_3	2	24	6	10	24	5.3	1	36	
7	V_4	2	32	8	4	28	5.3	1	38	
8	V_5	2	40	10	0	32	5.3	1	42	
9		1	2	2	104	26	5.3	1	108	
10							5.4	2	46	
11							5.9	3	24	2
12							5.10	5	8	1
13							5.9	6	4	2
14		1	4	4	60	28	5.3	1	66	
15							5.4	2	26	
16							5.8	3	12	1
17							5.6	4	6	
18		1	6	6	40	30	5.3	1	48	
19							5.4	2	18	
20							5.5	3	8	
21		1	8	8	28	32	5.3	1	38	
22							5.4	2	14	
23		1	10	10	20	34	5.3	1	32	
24							5.4	2	12	
25		1	12	12	14	36	5.3	1	28	
26		1	14	14	10	38	5.3	1	26	
27		1	16	16	6	40	5.3	1	24	
28		1	18	18	4	42	5.3	1	24	
29		1	22	22	0	46	5.3	1	24	

TABLE 1. Rank 1 building blocks

transverse intersection C of the ramification divisor and another anticanonical divisor yields a block Z with involution.

Similarly to Example 5.4, we could take X to be one of the two Fano 3-folds with index $r > 2$. Then the r -fold branched cover Y of X branched over an anticanonical divisor can be blown up along the intersection of the ramification locus with another anticanonical divisor of Y to give a building block Z with an automorphism of order r .

Example 5.5. For $X = Q \subset \mathbb{P}^4$ the quadric 3-fold (which has $r = 3$), Y is isomorphic to a complete intersection of a quadric and a cubic in \mathbb{P}^5 of the forms $X_1^2 + \dots + X_5^2$ and $X_0^3 + F(X_1, \dots, X_5)$, and the branch switching automorphisms correspond to multiplying the homogeneous coordinate X_0 by cube roots of unity. The fixed set is the anticanonical divisor $\{X_0 = 0\}$, which is smooth for a generic F . Blowing up a transverse intersection C with another anticanonical divisor yields a block Z with an automorphism group of order 3

(number 20 in Table 1); these building blocks are then special elements of the family 18 in the table obtained in Example 5.3.

If we let $\tau \in \Gamma$ be the generator that multiplies X_0 by $\zeta^{-1} = e^{-\frac{2\pi i}{3}}$, then the fixed set $Z^\tau \subset Z$ consists of the proper transform Σ of the ramification locus and a section \tilde{C} of the exceptional set. Clearly τ acts on the normal bundle of Σ as multiplication by ζ^{-1} .

Example 5.6. For $X = \mathbb{P}^3$ (which has $r = 4$), Y is isomorphic to a quartic in \mathbb{P}^4 with defining equation of the form $X_0^4 + F(X_1, X_2, X_3, X_4)$, giving entry 17 in Table 1. These are special elements of the family 14 of blocks obtained in Example 5.3.

Further, we can of course also consider this as a family of blocks with involution; then we recover a subfamily of family 15, which already came up in Example 5.4.

Before producing some examples with isolated fixpoints, let us recall that we need to find a generator τ of Γ that acts on the normal bundle ν_Σ by $\zeta^{-1} = e^{-\frac{2\pi i}{k}}$. Then by Remark 2.5, the contribution of the fixpoint set to the extended ν -invariant is given by $D_{\tau^\varepsilon}(Z)$ with ε as in equation (1.5), where for brevity, we write $D_\gamma(Z)$ instead of $D_\gamma(V)$ as in Definition 2.3.

Remark 5.7. While the action of Γ on the normal bundle of the fixed curve \tilde{C} does not affect the $\bar{\nu}$ -invariant by Theorem 2.4, it is easy to describe in a uniform way in all our examples.

The exceptional divisor E in Z is biholomorphic to $C \times \mathbb{P}^1$. We can choose the identification so that $C \times \{(1 : 0)\}$ is the intersection $E \cap \Sigma$, while $C \times \{(0 : 1)\}$ is the 1-dimensional component \tilde{C} of the fixed set of Γ in Z . The action of $\tau \in \Gamma$ on E is trivial on the C factor, and can be identified with $(Y_0 : Y_1) \mapsto (\zeta Y_0 : Y_1)$ on the \mathbb{P}^1 factor, for ζ such that τ acts on ν_Σ by ζ^{-1} .

Then τ acts on the normal bundle of \tilde{C} in E (which is trivial) as multiplication by ζ . If we write the normal bundle of \tilde{C} in Z as a direct sum of this trivial bundle and another line bundle, then (because \tilde{C} is contained in V which has a Calabi-Yau structure preserved by Γ) the second summand must be isomorphic to $T^*\tilde{C}$, and τ must act on it as multiplication by ζ^{-1} .

Example 5.8. Consider a smooth quartic of the form $X_0^3 X_1 + F(X_1, X_2, X_3, X_4)$. Multiplying X_0 by a primitive third root of unity, say $\zeta^{-1} = e^{-\frac{2\pi i}{3}}$, defines an automorphism τ of order 3. Its fixed set is the union of the K3 surface $\Sigma = \{X_0 = 0\}$ and the isolated point $(1 : 0 : 0 : 0)$.

Blowing up Y along the intersection C (a quartic plane curve) of Σ with another anticanonical divisor yields a building block Z with automorphism group $\Gamma \cong \mathbb{Z}/3$, number 16 in Table 1. It is a different subfamily of family 14 than the one considered in Example 5.6.

The block Z has $\chi(Z) = -60$. The fixed set Z^τ of τ is the union of the proper transform of Σ , a section \tilde{C} of the exceptional divisor and the isolated fixed point, so its Euler characteristic is 21. Thus $\chi(Z/\Gamma) = -\frac{1}{3}60 + \frac{2}{3}21 = -6$, and hence $b_3^\Gamma(Z) = 12$.

Clearly, τ acts on the normal bundle of Σ as multiplication by ζ^{-1} . Meanwhile, in the affine chart $(z_1, \dots, z_4) \mapsto (1 : z_1 : \dots : z_4)$, the action of τ is represented by multiplication with ζ . Hence, the action of τ on the tangent space at $(1 : 0 : 0 : 0)$ is diagonal with eigenvalue ζ . We can now compute $D_{\tau^\varepsilon}(Z) = 2\varepsilon$ for $\varepsilon = \pm 1$.

Example 5.9. Consider a smooth sextic Y in the weighted projective space $\mathbb{P}^4(1^4, 3)$ of the form $X_0^6 + F(X_1, X_2, X_3) + X_4^2$. Multiplying X_0 by a primitive 6th root of unity, say $\zeta^{-1} = e^{-\frac{\pi i}{3}}$, defines an automorphism τ of order 6. Its fixed set is the K3 surface $\Sigma = \{X_0 = 0\}$. In addition, τ^2 has two isolated fixed points, at $x_\pm = (1 : 0 : 0 : 0 : \pm i)$, which are swapped by τ .

Blowing up Y along the intersection C of Σ with another anticanonical divisor that is stable under τ as a set yields a building block Z with automorphism group $\Gamma \cong \mathbb{Z}/6$, line 13 in

Table 1. It can be considered as a more special subfamily of family 9 appearing in Example 5.3 or of family 10 of involution blocks from Example 5.4. But if we consider it as a block with automorphism group of order 3 (number 11 in the table), then that is distinct from the previous examples.

Clearly, τ acts on the normal bundle of Σ as multiplication by ζ^{-1} . Meanwhile, in the affine chart $(z_1, \dots, z_4) \mapsto (1 : z_1 : \dots : z_4)$, the action of τ is represented by $(z_1, \dots, z_4) \mapsto (\zeta z_1, \zeta z_2, \zeta z_3, \zeta^3 z_4)$. The isolated fixed points correspond to $(0, 0, 0, \pm i)$, and have tangent space $z_4 = 0$. Thus the action of τ^2 on the tangent spaces is diagonal with eigenvalue ζ^2 . Again, we find $D_{\tau^\varepsilon}(Z) = 2\varepsilon$ for the automorphism group $\mathbb{Z}/6$, and $D_{\tau^\varepsilon} = 4\varepsilon$ if we restrict to the automorphism group $\mathbb{Z}/3$.

We have $\chi(\widetilde{Z}) = -102$, while the fixed set Z^{τ^2} of τ^2 is the union of the proper transform of Σ , a copy \widetilde{C} of C and the two isolated fixed points, so has Euler characteristic 24. In the case where we consider the automorphism group $\Gamma' \cong \mathbb{Z}/3$ generated by τ^2 , we thus find $\chi(Z/\Gamma') = -\frac{1}{3}102 + \frac{2}{3}24 = -18$, so $b_3^{\Gamma'}(Z) = 24$.

In turn, we can consider Z/Γ as a $\mathbb{Z}/2$ quotient of Z/Γ' with fixed set of Euler characteristic $\chi(\Sigma) + \chi(\widetilde{C}) = 22$. Thus $\chi(Z/\Gamma) = -\frac{1}{2}18 + \frac{1}{2}22 = 2$, and $b_3^\Gamma(Z) = 4$.

Example 5.10. Consider again a smooth sextic Y in the weighted projective space $\mathbb{P}^4(1^4, 3)$, but now of the form $X_0^5 X_1 + F(X_1, \dots, X_3) + X_4^2$. Multiplying X_0 by a primitive 5th root of unity, say $\zeta^{-1} = e^{-\frac{2\pi i}{5}}$, defines an automorphism τ of order 5. Its fixed set is the union of the K3 surface $\Sigma = \{X_0 = 0\}$ and the isolated point $(1 : 0 : 0 : 0 : 0)$.

Blowing up Y along the intersection C of Σ with another anticanonical divisor yields a building block Z with automorphism group $\Gamma \cong \mathbb{Z}/5$. It can be considered as another more special subfamily (entry 12 in Table 1) of the family 9 that we already considered in Example 5.9.

Clearly, τ acts on the normal bundle of Σ as multiplication by ζ^{-1} . Meanwhile, in the affine chart $(z_1, \dots, z_4) \mapsto (1 : z_1 : \dots : z_4)$, the action of τ is represented by $(z_1, \dots, z_4) \mapsto (\zeta z_1, \zeta z_2, \zeta z_3, \zeta^3 z_4)$. The tangent space at the fixed point is $z_1 = 0$, so the eigenvalues of τ on the tangent space are ζ, ζ and ζ^3 . In Example 2.6, we have computed $D_{\tau^\varepsilon}(Z) = -\frac{24}{5\varepsilon}$ for $\varepsilon \in \{\pm 1, \pm 2\}$.

We have $\chi(Z) = -102$, while the fixed set Z^τ of τ is the union of the proper transform of Σ , a copy \widetilde{C} of C , and the isolated fixed point so has Euler characteristic 23. We thus find $\chi(Z/\Gamma) = -\frac{1}{5}102 + \frac{4}{5}23 = -2$, so $b_3^\Gamma(Z) = 8$.

In order to construct extra-twisted connected sums from our examples of blocks, we need to note that they have a genericity property described in Definition 1.13.

Proposition 5.11. *Each family \mathcal{Z} of blocks above is (N, Amp) -polarised, where N is the polarising lattice of the family, and $\text{Amp} \subset N_{\mathbb{R}}$ is one of the two open half-lines.*

Proof. For the families of blocks without automorphism in Example 5.3, this is just an instance of [12, Proposition 6.9], which is a consequence of the results of Beauville [2] on anticanonical divisors in Fano 3-folds. The same argument applies to the families of blocks with involution that are obtained in Examples 5.4, 5.5 or 5.6 from a cover of a Fano 3-fold X branched over an anticanonical divisor Σ , since Σ can be any smooth anticanonical divisor in X .

In Example 5.8, the K3 divisor Σ is a hypersurface in \mathbb{P}^3 defined by the quartic polynomial F . Clearly F can be chosen to be any smooth quartic, so a generic K3 surface with Picard lattice containing an ample primitive class of norm-square 4 will appear this way. Indeed, in particular any K3 surface with Picard lattice exactly $\langle 4 \rangle$ can be embedded as a quartic in \mathbb{P}^3 (see Saint-Donat [34, Theorem 6.1]).

Similarly, we see directly that any K3 that is a double cover of \mathbb{P}^2 branched over a smooth sextic curve can appear as the K3 divisor in blocks of the classes from Examples 5.9 and 5.10, and a generic K3 surfaces whose Picard lattice contains an ample class of norm-square 2 can be presented that way (see Reid [33, Theorem 3.8(d)]). \square

5.3. Examples of matchings. We now study the matchings that can be produced from the blocks in the previous section. Table 2 lists all extra-twisted connected sums that can be made from the blocks in Table 1, except those where both blocks have trivial automorphism group, which were studied in [13, 28]. Note that some examples with $k_{\pm} \leq 2$ were already considered in [13, 14, 31], in particular tables 4 and 5 in [31] contain some of the examples with $k_+ \leq 2$, $k_- = 2$ with additional information on $p_1(TM)$ and the torsion in $H^4(M)$. Table 2 contains 192 examples where $k_- \geq 3$; these are genuinely new.

We explained in §3.1 one way to find all gluing matrices for a given pair of orders (k_+, k_-) of automorphism groups. The gluing matrix determines the gluing angle ϑ , and for each pair of rank 1 blocks one can then determine whether there is a corresponding configuration as explained in §1.4. However, we find it convenient here to do these steps in the opposite order, and first enumerate all possible configurations involving the blocks from Table 1.

Given k_+ and k_- and a configuration, there may be several different gluing matrices G that have the right gluing angle and $\det G = -k_+k_-$, and several different choices of blocks Z_+, Z_- with the right polarising lattices and automorphism groups of order k_{\pm} . Each such choice yields a family of extra-twisted connected sums M by application of Proposition 1.14. The fundamental group $\pi_1(M)$ depends only on the gluing matrix, while $b_3(M)$ depends on the choices of Z_+ and Z_- . The invariant $\bar{\nu}(M)$ depends on the gluing matrix together with data for the isolated fixed points of the automorphisms on Z_{\pm} . It turns out that for those pairs of configuration and gluing matrix where there is more than one choice of (Z_+, Z_-) , there is never any ambiguity in the isolated fixed point data, so in practice, $\bar{\nu}(M)$ only depends on the gluing matrix.

We therefore organise Table 2 with the data about the extra-twisted connected sums from blocks with polarising lattices of rank 1 as follows. For each extra-twisted connected sum we first list the orders k_{\pm} of Γ_{\pm} , the even lattice describing the configuration of the K3 matching and $\cos^2 \vartheta$ of the gluing angle ϑ as determined by (1.14). Then follow the building blocks Z_+ and Z_- (the numbers referring to the entries in Table 1) and the third Betti number of the extra-twisted connected sum, the gluing matrix $G = \begin{pmatrix} m & p \\ n & q \end{pmatrix}$, and the parameters ε_{\pm} (see Proposition 1.8). By Remark 3.8, we always assume that $m, n, p > 0$ and $q < 0$. Moreover, if $k_+ = k_-$ we may swap the blocks if necessary to make sure that $m + q \leq 0$. Finally, we list the value of the $\bar{\nu}$ -invariant. Where there are several different choices of Z_+ with the same k_+, k_- and configuration they are separated by commas, as are the corresponding values of $b_3(M)$, while the different choices of the gluing matrix G (and the corresponding $\varepsilon_+, \varepsilon_-$ and $\bar{\nu}(M)$) are listed on separate rows. The number at the very left is the running number of the first example in the line, for example, the third line contains examples 5, 6 and 7.

Remark 5.12. If a non simply-connected extra-twisted connected sum has a nontrivial covering constructed with the same groups Γ_+, Γ_- , see Remark 3.6, then it is listed a few lines above, *e.g.* entry 20 is the universal cover of entry 22. If one needs to pass to a subgroup of at least one of the groups Γ_+, Γ_- , then one should determine \tilde{k}_+, \tilde{k}_- and the gluing matrix by (3.8b) and find the covering in a different section of the table, possibly with the roles of Z_+ and Z_- swapped, except if $\vartheta = \frac{\pi}{2}$. In the latter case, the universal cover is an ordinary twisted sum of the type discussed in [13, 28], and therefore not listed here.

	k_+	k_-	N_++N_-	$\cos^2 \vartheta$	Z_+	Z_-	$b_3(M)$	G	ε_+	ε_-	$\bar{\nu}(M)$
1	1	2	$\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$	1/2	3, 9	5	92, 148	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$		1	-39
						15	100, 156				
5			$\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$	1/2	1, 4, 14	10	134, 106, 134	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$		1	-39
8			$\begin{pmatrix} 4 & 4 \\ 4 & 8 \end{pmatrix}$	1/2	1, 4, 14	22	102, 74, 102	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$		1	-39
11			$\begin{pmatrix} 8 & 4 \\ 4 & 4 \end{pmatrix}$	1/2	7, 21	5	78, 78	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$		1	-39
						15	86, 86				
15			$\begin{pmatrix} 12 & 6 \\ 6 & 6 \end{pmatrix}$	1/2	25	19	68	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$		1	-39
16			$\begin{pmatrix} 16 & 4 \\ 4 & 2 \end{pmatrix}$	1/2	27	10	92	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$		1	-39
17			$\begin{pmatrix} 16 & 8 \\ 8 & 8 \end{pmatrix}$	1/2	27	22	60	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$		1	-39
18			$\begin{pmatrix} 18 & 6 \\ 6 & 4 \end{pmatrix}$	1/2	28	5, 15	64, 72	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$		1	-39
20	1	3	$\begin{pmatrix} 2 & 2 \\ 2 & 6 \end{pmatrix}$	1/3	3, 9	20	82, 138	$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$		-1	-19
22								$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$		1	-35
24			$\begin{pmatrix} 4 & 4 \\ 4 & 6 \end{pmatrix}$	2/3	1, 4, 14	20	96, 68, 96	$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$		-1	-43
27								$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$		1	-59
30			$\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$	1/3	2, 6, 18	11	102, 82, 94	$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$		-1	-23
33								$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$		1	-31
36			$\begin{pmatrix} 6 & 4 \\ 4 & 4 \end{pmatrix}$	2/3	2, 6, 18	16	90, 70, 82	$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$		-1	-45
39								$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$		1	-57
42			$\begin{pmatrix} 8 & 4 \\ 4 & 6 \end{pmatrix}$	1/3	7, 21	20	68, 68	$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$		-1	-19
44								$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$		1	-35
46			$\begin{pmatrix} 12 & 4 \\ 4 & 2 \end{pmatrix}$	2/3	25	11	74	$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$		-1	-47
47								$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$		1	-55
48			$\begin{pmatrix} 12 & 4 \\ 4 & 4 \end{pmatrix}$	1/3	25	16	62	$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$		-1	-21
49								$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$		1	-33
50			$\begin{pmatrix} 16 & 8 \\ 8 & 6 \end{pmatrix}$	2/3	27	20	54	$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$		-1	-43
51								$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$		1	-59
52			$\begin{pmatrix} 18 & 6 \\ 6 & 6 \end{pmatrix}$	1/3	28	20	54	$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$		-1	-19
53								$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$		1	-35
54	1	4	$\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$	1/2	3, 9	17	80, 136	$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$		-1	-21
56								$\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$		1	-57
58			$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$	1/4	1, 4, 14	17	94, 66, 94	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$		-1	3
61								$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$		1	-33
64			$\begin{pmatrix} 8 & 4 \\ 4 & 4 \end{pmatrix}$	1/2	7, 21	17	66, 66	$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$		-1	-21
66								$\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$		1	-57
68			$\begin{pmatrix} 12 & 6 \\ 6 & 4 \end{pmatrix}$	3/4	25	17	56	$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$		-1	-45
69								$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$		1	-81
70			$\begin{pmatrix} 16 & 4 \\ 4 & 4 \end{pmatrix}$	1/4	27	17	52	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$		-1	3
71								$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$		1	-33
72			$\begin{pmatrix} 18 & 6 \\ 6 & 4 \end{pmatrix}$	1/2	28	17	52	$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$		-1	-21

Table 2: Examples of extra-twisted connected sums

k_+	k_-	$N_+ + N_-$	$\cos^2 \vartheta$	Z_+	Z_-	$b_3(M)$	G	ε_+	ε_-	$\bar{\nu}(M)$	
73							$\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$		1	-57	
74	1	5	$\begin{pmatrix} 10 & 2 \\ 2 & 2 \end{pmatrix}$	1/5	8, 23	12	72, 62	$\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$	-1	21	
76							$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$		2	-15	
78							$\begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix}$		1	-27	
80			$\begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix}$	4/5	8, 23	12	72, 62	$\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}$	-1	-51	
82							$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$		-2	-63	
84							$\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$		1	-99	
86	1	6	$\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$	1/2	1, 4, 14	13	92, 64, 92	$\begin{pmatrix} 1 & 1 \\ 3 & -3 \end{pmatrix}$	-1	-1	
89							$\begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix}$		1	-77	
92			$\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$	1/3	2, 6, 18	13	82, 62, 74	$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$	-1	23	
95							$\begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$		1	-53	
98			$\begin{pmatrix} 12 & 2 \\ 2 & 2 \end{pmatrix}$	1/6	25	13	54	$\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}$	-1	47	
99							$\begin{pmatrix} 1 & 5 \\ 1 & -1 \end{pmatrix}$		1	-29	
100			$\begin{pmatrix} 12 & 4 \\ 4 & 2 \end{pmatrix}$	2/3	25	13	54	$\begin{pmatrix} 1 & 1 \\ 2 & -4 \end{pmatrix}$	-1	-25	
101							$\begin{pmatrix} 4 & 2 \\ 1 & -1 \end{pmatrix}$		1	-101	
102			$\begin{pmatrix} 16 & 4 \\ 4 & 2 \end{pmatrix}$	1/2	27	13	50	$\begin{pmatrix} 1 & 1 \\ 3 & -3 \end{pmatrix}$	-1	-1	
103							$\begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix}$		1	-77	
104	2	2	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	0	10	10	115	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0
105			$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	1/4	10	10	114	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	1	1	-27
106							$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$		1	-27	
107			$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$	0	10	5, 15	87, 95	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0
109			$\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$	0	10	19	87	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0
110			$\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$	3/4	10	19	86	$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$	1	1	-51
111							$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$		1	-51	
112			$\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$	0	10	22	83	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0
113			$\begin{pmatrix} 2 & 2 \\ 2 & 8 \end{pmatrix}$	1/4	10	22	82	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	1	1	-27
114							$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$		1	-27	
115			$\begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}$	0	10	24	81	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0
116			$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	0	5, 15	5	59, 67 67, 75	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0
120			$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$	1/4	5, 15	5	58, 66 66, 74	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	1	1	-27
124							$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$		1	-27	
128			$\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$	0	5, 15	19	59, 67	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0
130			$\begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$	0	5, 15	22	55, 63	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0
132			$\begin{pmatrix} 4 & 0 \\ 0 & 10 \end{pmatrix}$	0	5, 15	24	53, 61	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0
134			$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$	0	19	19	59	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0
135			$\begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$	1/4	19	19	58	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	1	1	-27
136							$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$		1	-27	

Table 2: Examples of extra-twisted connected sums

k_+	k_-	$N_+ + N_-$	$\cos^2 \vartheta$	Z_+	Z_-	$b_3(M)$	G	ε_+	ε_-	$\bar{\nu}(M)$	
137		$\begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix}$	0	19	22	55	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0	
138		$\begin{pmatrix} 6 & 6 \\ 6 & 8 \end{pmatrix}$	3/4	19	22	54	$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$	1	1	-51	
139							$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$	1	1	-51	
140		$\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$	0	19	24	53	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0	
141		$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$	0	22	22	51	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0	
142		$\begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}$	1/4	22	22	50	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	1	1	-27	
143							$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$	1	1	-27	
144		$\begin{pmatrix} 8 & 0 \\ 0 & 10 \end{pmatrix}$	0	22	24	49	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0	
145		$\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$	0	24	24	47	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	1	1	0	
146		$\begin{pmatrix} 10 & 5 \\ 5 & 10 \end{pmatrix}$	1/4	24	24	46	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	1	1	-27	
147							$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$	1	1	-27	
148	2	3	$\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$	1/2	10	16	80	$\begin{pmatrix} 1 & 1 \\ 3 & -3 \end{pmatrix}$	1	-1	-33
149							$\begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix}$	1	1	-45	
150		$\begin{pmatrix} 2 & 2 \\ 2 & 6 \end{pmatrix}$	1/3	10	20	76	$\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$	1	1	-47	
151							$\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$	1	-1	-31	
152		$\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$	1/2	5, 15	11	64, 72	$\begin{pmatrix} 1 & 1 \\ 3 & -3 \end{pmatrix}$	1	-1	-35	
154							$\begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix}$	1	1	-43	
156		$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$	1/6	5, 15	20	48, 56	$\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}$	1	-1	-7	
158							$\begin{pmatrix} 1 & 5 \\ 1 & -1 \end{pmatrix}$	1	1	-23	
160		$\begin{pmatrix} 4 & 4 \\ 4 & 6 \end{pmatrix}$	2/3	5, 15	20	48, 56	$\begin{pmatrix} 4 & 1 \\ 2 & -1 \end{pmatrix}$	1	-1	-31	
162							$\begin{pmatrix} 1 & 2 \\ 1 & -4 \end{pmatrix}$	1	1	-47	
164		$\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$	1/3	19	11	64	$\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$	1	1	-43	
165							$\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$	1	-1	-35	
166		$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$	1/6	19	16	52	$\begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}$	1	-1	-9	
167							$\begin{pmatrix} 1 & 5 \\ 1 & -1 \end{pmatrix}$	1	1	-21	
168		$\begin{pmatrix} 6 & 4 \\ 4 & 4 \end{pmatrix}$	2/3	19	16	52	$\begin{pmatrix} 4 & 1 \\ 2 & -1 \end{pmatrix}$	1	-1	-33	
169							$\begin{pmatrix} 1 & 2 \\ 1 & -4 \end{pmatrix}$	1	1	-45	
170		$\begin{pmatrix} 8 & 4 \\ 4 & 4 \end{pmatrix}$	1/2	22	16	48	$\begin{pmatrix} 1 & 1 \\ 3 & -3 \end{pmatrix}$	1	-1	-33	
171							$\begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix}$	1	1	-45	
172		$\begin{pmatrix} 8 & 4 \\ 4 & 6 \end{pmatrix}$	1/3	22	20	44	$\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$	1	1	-47	
173							$\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$	1	-1	-31	
174	2	4	$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$	1/8	10	17	74	$\begin{pmatrix} 1 & 1 \\ 7 & -1 \end{pmatrix}$	1	-1	15
175							$\begin{pmatrix} 1 & 7 \\ 1 & -1 \end{pmatrix}$	1	1	-21	
176		$\begin{pmatrix} 6 & 3 \\ 3 & 4 \end{pmatrix}$	3/8	19	17	46	$\begin{pmatrix} 1 & 1 \\ 5 & -3 \end{pmatrix}$	1	-1	-9	
177							$\begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}$	1	1	-69	
178							$\begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}$	1	-1	-33	
179							$\begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$	1	1	-45	
180		$\begin{pmatrix} 8 & 2 \\ 2 & 4 \end{pmatrix}$	1/8	22	17	42	$\begin{pmatrix} 1 & 1 \\ 7 & -1 \end{pmatrix}$	1	-1	15	

Table 2: Examples of extra-twisted connected sums

	k_+	k_-	N_++N_-	$\cos^2 \vartheta$	Z_+	Z_-	$b_3(M)$	G	ε_+	ε_-	$\bar{\nu}(M)$
181								$\begin{pmatrix} 1 & 7 \\ 1 & -1 \end{pmatrix}$	1	1	-21
182			$\begin{pmatrix} 10 & 5 \\ 5 & 4 \end{pmatrix}$	5/8	24	17	40	$\begin{pmatrix} 1 & 1 \\ 3 & -5 \end{pmatrix}$	1	-1	-33
183								$\begin{pmatrix} 5 & 1 \\ 3 & -1 \end{pmatrix}$	1	-1	-9
184								$\begin{pmatrix} 1 & 3 \\ 1 & -5 \end{pmatrix}$	1	1	-45
185								$\begin{pmatrix} 5 & 3 \\ 1 & -1 \end{pmatrix}$	1	1	-69
186	2	5	$\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$	1/2	5, 15	12	48, 56	$\begin{pmatrix} 1 & 1 \\ 5 & -5 \end{pmatrix}$	1	-1	-15
188								$\begin{pmatrix} 5 & 5 \\ 1 & -1 \end{pmatrix}$	1	1	-63
190			$\begin{pmatrix} 10 & 2 \\ 2 & 2 \end{pmatrix}$	1/5	24	12	42	$\begin{pmatrix} 2 & 1 \\ 8 & -1 \end{pmatrix}$	1	-2	-27
191								$\begin{pmatrix} 1 & 8 \\ 1 & -2 \end{pmatrix}$	1	-2	-27
192			$\begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix}$	4/5	24	12	42	$\begin{pmatrix} 8 & 1 \\ 2 & -1 \end{pmatrix}$	1	2	-51
193								$\begin{pmatrix} 1 & 2 \\ 1 & -8 \end{pmatrix}$	1	2	-51
194	2	6	$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	1/4	10	13	72	$\begin{pmatrix} 1 & 1 \\ 9 & -3 \end{pmatrix}$	1	-1	35
195								$\begin{pmatrix} 3 & 9 \\ 1 & -1 \end{pmatrix}$	1	1	-41
196			$\begin{pmatrix} 6 & 1 \\ 1 & 2 \end{pmatrix}$	1/12	19	13	44	$\begin{pmatrix} 1 & 1 \\ 11 & -1 \end{pmatrix}$	1	-1	59
197								$\begin{pmatrix} 1 & 11 \\ 1 & -1 \end{pmatrix}$	1	1	-17
198			$\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$	3/4	19	13	44	$\begin{pmatrix} 1 & 1 \\ 3 & -9 \end{pmatrix}$	1	-1	-37
199								$\begin{pmatrix} 9 & 3 \\ 1 & -1 \end{pmatrix}$	1	1	-113
200			$\begin{pmatrix} 8 & 2 \\ 2 & 2 \end{pmatrix}$	1/4	22	13	40	$\begin{pmatrix} 1 & 1 \\ 9 & -3 \end{pmatrix}$	1	-1	35
201								$\begin{pmatrix} 3 & 9 \\ 1 & -1 \end{pmatrix}$	1	1	-41
202	3	3	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	0	11	11	71	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-8
203			$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$	0	11	16	59	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-10
204			$\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$	0	11	20	55	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-12
205			$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	0	16	16	47	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-12
206			$\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$	0	16	20	43	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-14
207			$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$	0	20	20	39	$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$	1	1	-16
208			$\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$	1/9	20	20	38	$\begin{pmatrix} 1 & 1 \\ 8 & -1 \end{pmatrix}$	-1	-1	13
209								$\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$	1	1	-43
210								$\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$	-1	-1	-11
211								$\begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix}$	1	1	-19
212			$\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$	4/9	20	20	38	$\begin{pmatrix} 1 & 1 \\ 5 & -4 \end{pmatrix}$	-1	-1	-11
213								$\begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}$	1	1	-67
214								$\begin{pmatrix} 1 & 5 \\ 1 & -4 \end{pmatrix}$	1	1	-43
215								$\begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}$	-1	-1	-35
216	3	4	$\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$	1/2	11	17	52	$\begin{pmatrix} 3 & 1 \\ 6 & -2 \end{pmatrix}$	1	1	-85
217								$\begin{pmatrix} 6 & 2 \\ 3 & -1 \end{pmatrix}$	1	-1	-1
218								$\begin{pmatrix} 1 & 3 \\ 2 & -6 \end{pmatrix}$	-1	1	-29
219								$\begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix}$	-1	-1	-41
220			$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$	1/4	16	17	40	$\begin{pmatrix} 3 & 1 \\ 9 & -1 \end{pmatrix}$	-1	1	-51

Table 2: Examples of extra-twisted connected sums

	k_+	k_-	N_++N_-	$\cos^2 \vartheta$	Z_+	Z_-	$b_3(M)$	G	ε_+	ε_-	$\bar{\nu}(M)$
221								$\begin{pmatrix} 1 & 9 \\ 1 & -3 \end{pmatrix}$	1	-1	-27
222			$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$	1/6	20	17	36	$\begin{pmatrix} 1 & 1 \\ 10 & -2 \end{pmatrix}$	1	-1	-5
223								$\begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$	1	1	-65
224								$\begin{pmatrix} 1 & 5 \\ 2 & -2 \end{pmatrix}$	-1	-1	-13
225								$\begin{pmatrix} 2 & 10 \\ 1 & -1 \end{pmatrix}$	-1	1	-25
226			$\begin{pmatrix} 6 & 4 \\ 4 & 4 \end{pmatrix}$	2/3	20	17	36	$\begin{pmatrix} 1 & 1 \\ 4 & -8 \end{pmatrix}$	1	-1	-53
227								$\begin{pmatrix} 8 & 4 \\ 1 & -1 \end{pmatrix}$	-1	1	-73
228	3	5	$\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$	1/3	20	12	38	$\begin{pmatrix} 1 & 1 \\ 10 & -5 \end{pmatrix}$	1	-1	-11
229								$\begin{pmatrix} 1 & 2 \\ 5 & -5 \end{pmatrix}$	-1	2	-7
230								$\begin{pmatrix} 5 & 5 \\ 2 & -1 \end{pmatrix}$	1	2	-23
231								$\begin{pmatrix} 5 & 10 \\ 1 & -1 \end{pmatrix}$	-1	1	-43
232	4	4	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	0	17	17	35	$\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$	1	1	-36
233			$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$	1/16	17	17	34	$\begin{pmatrix} 1 & 1 \\ 15 & -1 \end{pmatrix}$	-1	-1	57
234								$\begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix}$	1	1	-63
235								$\begin{pmatrix} 1 & 5 \\ 3 & -1 \end{pmatrix}$	-1	-1	9
236								$\begin{pmatrix} 1 & 15 \\ 1 & -1 \end{pmatrix}$	1	1	-15
237			$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$	1/4	17	17	34	$\begin{pmatrix} 2 & 2 \\ 6 & -2 \end{pmatrix}$	1	1	-87
238								$\begin{pmatrix} 2 & 6 \\ 2 & -2 \end{pmatrix}$	-1	-1	-15
239			$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$	9/16	17	17	34	$\begin{pmatrix} 1 & 1 \\ 7 & -9 \end{pmatrix}$	-1	-1	9
240								$\begin{pmatrix} 3 & 1 \\ 7 & -3 \end{pmatrix}$	1	1	-111
241								$\begin{pmatrix} 1 & 7 \\ 1 & -9 \end{pmatrix}$	1	1	-63
242								$\begin{pmatrix} 3 & 7 \\ 1 & -3 \end{pmatrix}$	-1	-1	-39
243	4	5	$\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$	1/2	17	12	36	$\begin{pmatrix} 2 & 1 \\ 10 & -5 \end{pmatrix}$	-1	-2	-9
244								$\begin{pmatrix} 1 & 2 \\ 5 & -10 \end{pmatrix}$	1	2	-69
245								$\begin{pmatrix} 10 & 5 \\ 2 & -1 \end{pmatrix}$	-1	2	-33
246								$\begin{pmatrix} 5 & 10 \\ 1 & -2 \end{pmatrix}$	1	-2	-45
247	4	6	$\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$	1/8	17	13	32	$\begin{pmatrix} 1 & 1 \\ 21 & -3 \end{pmatrix}$	1	-1	17
248								$\begin{pmatrix} 3 & 3 \\ 7 & -1 \end{pmatrix}$	1	1	-107
249								$\begin{pmatrix} 1 & 7 \\ 3 & -3 \end{pmatrix}$	-1	-1	5
250								$\begin{pmatrix} 3 & 21 \\ 1 & -1 \end{pmatrix}$	-1	1	-23
251	5	5	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	0	12	12	39	$\begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$	1	1	-48
252								$\begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$	2	-2	0
253	6	6	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	0	13	13	31	$\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$	1	1	-76
254			$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	1/4	13	13	30	$\begin{pmatrix} 3 & 3 \\ 9 & -3 \end{pmatrix}$	1	1	-151
255								$\begin{pmatrix} 3 & 9 \\ 3 & -3 \end{pmatrix}$	-1	-1	1

Table 2: Examples of extra-twisted connected sums

6. PROOFS OF THE ADIABATIC LIMIT THEOREMS

For completeness, we give short proofs of the claims (2.10) and (4.7). This section does not attempt to be self-contained. Instead, we will state the analogue of statements in existing proofs, and add explanations only where we deviate from those.

6.1. Adiabatic limits of twisted products. Let $\Gamma \cong \mathbb{Z}/k$ be a finite group that acts effectively and isometrically on an even-dimensional manifold V with boundary ∂V . We assume that V has product geometry near ∂V . We consider $W = V/\Gamma$ as an orbifold with inertia orbifold ΛW .

Consider $S^1 \cong \mathbb{R}/\mathbb{Z}$ and let a generator $\gamma_0 \in \Gamma$ act by sending $[v] \in \mathbb{R}/\xi\mathbb{Z}$ to $[v + \frac{\xi}{k}]$. Then we will consider the Seifert fibration

$$p: M = (V \times S^1)/\Gamma \longrightarrow W, \quad (6.1)$$

where Γ acts diagonally on $V \times S^1$. We split $TM = TW \oplus TS^1$ by abuse of notation and consider a family of metrics

$$g_\varepsilon^{TM} = \varepsilon^{-2} g^{TW} \oplus g^{TS^1}$$

for $\varepsilon > 0$. The example we have in mind is of course $M = M_{\pm, \ell}$ with metric $g_\varepsilon^{TM} = \frac{1}{\varepsilon_{\pm}^2} g_{\pm, \ell, \varepsilon}$; see paragraph 2.1.1.

By a *Dirac bundle* we mean a Hermitian vector bundle with Hermitian connection and a compatible Clifford multiplication; see [29, def II.5.2]. We assume that V is equipped with a fixed Dirac bundle $E_V \rightarrow V$, on which Γ acts, preserving its structure. On M , we consider the Dirac bundle

$$E = p^* E_V / \Gamma \longrightarrow M. \quad (6.2)$$

We let (e_1, \dots, e_m) denote a local orthonormal frame of TM for g_1^{TM} such that e_1 is vertical and e_2, \dots, e_m are horizontal. Clifford multiplication with e_i will be denoted c_i . Here, we assume that c_1 acts as $-i^{\frac{m+1}{2}} c_2 \cdots c_m$ on $p^* E_V$, so that the Clifford volume element $i^{\frac{m+1}{2}} c_1 \cdots c_m$ acts as 1. The examples we have in mind are the spinor bundle SV and the bundle of exterior forms $\Lambda^\bullet T^*V$ on V , leading to the spinor bundle and the bundle of even forms on M .

We consider a Dirac-type operator on E of the form

$$D_{M, \varepsilon} = D_{S^1} + \varepsilon D_W \quad (6.3)$$

as in [22, (2.3)], where $D_{S^1} = c_1 \nabla_{e_1}^E$ is the fibrewise Dirac operator. In the case of the odd signature operator on M , $D_W = B_W$ is the signature operator on the orbifold W .

In the case of the modified spin Dirac operator on M , we assume that SV admits a Γ -invariant spinor s such that ∇^{SV} is supported away from ∂V . If $D'_{M, \varepsilon}$ and D'_W denote the geometric Dirac operators on M and on the orbifold W with respect to the metrics above, equation (2.3) becomes

$$D'_{M, \varepsilon}(p^* s) = p^*(\varepsilon D'_W s) = \varepsilon p^*(f s + h c_1 s + r),$$

with f , h and r as before. We now consider the operator

$$\begin{aligned} D_W &= D'_W - \langle \cdot, s \rangle (f s + h c_1 s + r) - \langle \cdot, r \rangle s \\ &\quad - \langle \cdot, c_1 s \rangle (h s - f c_1 s - c_1 r) + \langle \cdot, c_1 r \rangle c_1 s. \end{aligned} \quad (6.4)$$

Then (6.3) and (6.4) are equivalent to (2.4). We also recall that $D_W - D'_W$ is supported away from ∂W by Property 2.1.3 (i).

The situation here is simpler than in [22] because D_W is independent of ε and

$$D_{S^1} D_W + D_W D_{S^1} = 0. \quad (6.5)$$

In the case of the modified spin Dirac operator, this follows from (6.4) because f , h , s and r all have vanishing vertical derivative.

Because we are in a local product situation, the space $L^2(E)$ splits into eigenspaces of D_{S^1} which we may regard as spaces of L^2 -sections of orbibundles over W . These spaces are invariant under D_W by (6.5). In particular, $H = \ker D_{S^1} \subset p_*E$ is isomorphic to the original E_V , and the connection ∇^E induces a unitary connection $\nabla^{p_*E} = \nabla^H \oplus \nabla^{H^\perp}$.

To avoid a clash of notation later, we write u for the inward normal coordinate on M and W near their respective boundary. Then let $e_2 = \frac{\partial}{\partial u}$ be the inward normal unit vector to ∂M with respect to g_1 , extended parallelly over the cylindrical neighbourhood $u \in [0, 1]$ of ∂M . The boundary operator $D_{\partial M, \varepsilon}$ splits in the same manner as $D_{M, \varepsilon}$, and in that cylindrical neighbourhood of ∂M , we have

$$D_{M, \varepsilon} = c_2 \left(\varepsilon \frac{\partial}{\partial u} + D_{S^1}^\partial + \varepsilon D_{\partial W} \right),$$

where $D_{S^1}^\partial = -c_2 D_{S^1}$ denotes the fibrewise boundary operator. By (6.5), the operators $D_{S^1}^\partial$ and $D_{\partial W}$ anticommute as well. Both respect the splitting of $(p|_{\partial M})_*E$ into $H|_{\partial W}$ and $H^\perp|_{\partial W}$.

Let $\Pi_{+, \varepsilon}$ denote the spectral projection onto the subspace of $L^2(\partial M; E)$ spanned by the eigenspinors of $D_{S^1}^\partial + \varepsilon D_{\partial W}$ with positive eigenvalues. Then $\Pi_{+, \varepsilon}$ respects the splitting into $H|_{\partial W}$ and its orthogonal complement, so

$$\Pi_{+, \varepsilon} = \Pi_+^H \oplus \Pi_{+, \varepsilon}^\perp.$$

Moreover, $\ker(D_{S^1}^\partial + \varepsilon D_{\partial W})$ in the sections of $H|_{\partial W}$ and equals the kernel of the restriction of $D_{\partial W}$ to $H|_{\partial W}$. The relevant symplectic structure on $\ker(D_{S^1}^\partial + \varepsilon D_{\partial W})$ is induced by c_2 .

We denote the restriction of D_W to H by $D_{W, 1}$ in analogy with [22]. It is a Dirac operator in the case of the odd signature operator, and a modified Dirac operator in the case of the modified spin Dirac operator. Clifford multiplication with the global vertical tangent vector field e_1 still acts on H and anticommutes with $D_{W, 1}$. Because c_1 and c_2 anticommute, c_1 commutes with the boundary operator $D_{\partial W}$, so the projection Π_+^H commutes with c_1 as well. The Lagrangian subspaces $L = L_D$ and $L = L_B$ of (2.6) and (2.7) are also invariant under c_1 . We immediately conclude that

$$\eta_{\text{APS}}(D_{W, 1}; L) = 0. \quad (6.6)$$

Recall that $\eta(\mathbb{A}) \in \Omega^\bullet(\Lambda W)$ denotes the orbifold η -form of the Bismut superconnection of the fibrewise spin Dirac operator with respect to the fibrewise trivial spin structure. We may regard the signature operator as a Dirac operator twisted by the pullback of the spinor bundle on the base; for this reason, the η -form $\eta(\mathbb{A})$ occurs in both formulas in the theorem below.

Theorem 6.1 (Compare Dai [19, Thm 1.1], see also [22, Thm 0.1]). *With the assumptions and notations above,*

$$\lim_{\varepsilon \rightarrow 0} \eta(D_{M, \varepsilon}; L_D) = \int_{\Lambda W \setminus W} \hat{A}_{\Lambda W}(TW, \nabla^{TW}) 2\eta_{\Lambda W}(\mathbb{A}), \quad (1)$$

$$\lim_{\varepsilon \rightarrow 0} \eta(B_{M, \varepsilon}; L_B) = \int_{\Lambda W \setminus W} \hat{L}_{\Lambda W}(TW, \nabla^{TW}) 2\eta_{\Lambda W}(\mathbb{A}). \quad (2)$$

This result is not covered by [19] because the fibrewise operator is allowed to have a kernel, and because $p: M \rightarrow V$ is a Seifert fibration. It is not covered by [22] because the base orbifold is allowed to have a boundary. But of course, the Seifert fibration is locally a twisted product, and hence the situation here is more specialised than in the two references above. A little extra complication comes from the construction (2.4) of the modified spin Dirac operator.

Our proof below relies crucially on the fact that the operators $D_{M,\varepsilon}$ and $D_{\partial M,\varepsilon}$ both respect the splitting of the bundle $p_*E \rightarrow B$ of fibrewise sections into the fibrewise harmonic spinors, which form a bundle $H \rightarrow B$ by assumption, and its orthogonal complement H^\perp . We believe that with a little extra work, this proof extends to totally geodesic Seifert fibrations. Probably Dai's proof also extends to totally geodesic fibre bundles because [19, Prop 5.2] then holds for sections of H^\perp .

Proof. We follow the proof in [22] as far as possible. We will view D_W as a differential operator on p_*E and D_{S^1} as an endomorphism of p_*E . We consider the restriction of $D_{M,\varepsilon}$ to H^\perp . In accordance with [22, sect 2.c], we denote it by $D_{M,\varepsilon,4} = D_{S^1} + \varepsilon D_{W,4}$, where $D_{W,4}$ describes the action of D_W on sections of p_*E . Note that in our setting, $D_{W,2} = D_{W,3} = 0$. Let $\langle \cdot, \cdot \rangle_{M/W}$ denote the fibrewise L^2 -product, let div_W denote the divergence of a vector field or a one-form on W , let Δ^{H^\perp} denote the horizontal Laplacian on $H^\perp \rightarrow W$, and let c^W denote Clifford multiplication by horizontal vectors. Then

$$\begin{aligned} \operatorname{div}_W \langle \nabla^{H^\perp} \sigma, \tau \rangle_{M/W} &= -\langle \Delta^{H^\perp} \sigma, \tau \rangle_{M/W} + \langle \nabla^{H^\perp} \sigma, \nabla^{H^\perp} \tau \rangle_{M/W}, \\ \operatorname{div}_W \langle c^W \sigma, \tau \rangle_{M/W} &= \langle D_W \sigma, \tau \rangle_{M/W} - \langle \sigma, D_W \tau \rangle_{M/W}. \end{aligned}$$

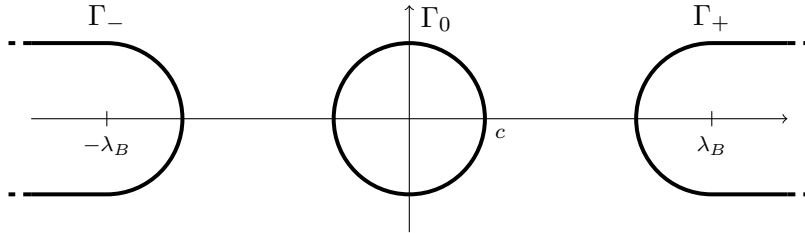
Because D_W has been modified to D_W by a self-adjoint operator of order 0 supported away from the boundary, and because $\frac{\partial}{\partial r}$ is the inward normal direction, we conclude that

$$\begin{aligned} &\| (i - \varepsilon^{-1} D_{M,\varepsilon,4}) \sigma \|_{L^2(W; H^\perp)}^2 - \| \nabla^{H^\perp} \sigma \|_{L^2(W; H^\perp)}^2 \\ &= \langle (1 + \varepsilon^{-2} D_{S^1}^2 + D_{W,4}^2 - \Delta^{H^\perp}) \sigma, \sigma \rangle_M - \langle i c_2 \sigma, \sigma \rangle_{\partial M} - \varepsilon^{-1} \langle D_{\partial M, \varepsilon, 4} \sigma, \sigma \rangle_{\partial M}. \end{aligned} \quad (6.7)$$

Let $H^1(W, H^\perp; \Pi_{+, \varepsilon}^\perp)$ denote the subspace of the first Sobolev space generated by sections that satisfy the APS boundary condition. If $\sigma \in H^1(W, H^\perp; \Pi_{+, \varepsilon}^\perp)$, then $\langle c_2 \sigma, \sigma \rangle = 0$ because c_2 anticommutes with $D_{\partial M, \varepsilon, 4}$, and $\langle D_{\partial M, \varepsilon, 4} \sigma, \sigma \rangle \leq 0$, so

$$\| (i - \varepsilon^{-1} D_{M,\varepsilon,4}) \sigma \|_{L^2(W; H^\perp)}^2 \geq \| \nabla^{H^\perp} \sigma \|_{L^2(W; H^\perp)}^2 + \langle (1 + \varepsilon^{-2} D_{S^1}^2 + D_{W,4}^2 - \Delta^{H^\perp}) \sigma, \sigma \rangle_M. \quad (6.8)$$

Let λ_B denote the smallest absolute value of a nonzero eigenvalue of the effective horizontal operator $D_{W,1}$ with respect to the given boundary conditions, and let $0 < c < \frac{\lambda_B}{2}$. Let $\Gamma = \Gamma_+ \dot{\cup} \Gamma_0 \dot{\cup} \Gamma_-$ denote a contour in \mathbb{C} , where Γ_\pm goes around $\pm[\lambda_B, +\infty]$ with distance c , and Γ_0 is a circle around 0 with radius c .



Assume that λ is not in the spectrum of $D_{M,\varepsilon,4}$ with APS boundary conditions given by $\Pi_{+, \varepsilon}^\perp$. Using parametrices on $\partial W \times [0, \infty)$ and on the double of W , one can construct a resolvent

$$R_\varepsilon(\lambda): L^2(W; H^\perp) \rightarrow H^1(W, H^\perp; \Pi_{+, \varepsilon}^\perp)$$

of $D_{M,\varepsilon,4}$. We define the family of Schatten norms of operators A acting on $L^2(M; E) \cong L^2(W; p_*E)$ by

$$\|A\|_p = \operatorname{tr} \left((A^* A)^{\frac{p}{2}} \right)$$

for $1 \leq p < \infty$, and let A_∞ denote the operator norm. Because $D_W^2 - \Delta^{p*E}$ is a bundle endomorphism on $E \rightarrow M$, we can use the inequality (6.8) above to prove the analogue of [22, Prop 2.7]. In particular, there exists a constant $\varepsilon_0 > 0$ such that for all $p > \dim M$, all $\varepsilon \in (0, \varepsilon_0)$ and all $\lambda \in \Gamma$, one has

$$\|R_\varepsilon(\lambda)\| = O(1, \varepsilon |\lambda|) \quad \text{and} \quad \|R_\varepsilon(\lambda)\| = O(|\lambda|). \quad (6.9)$$

Let $H^1(W, H; \Pi_{+, \Lambda}^H)$ denote the subspace of the first Sobolev space spanned by sections satisfying the Lagrangian APS boundary condition fixed above. Then we consider the resolvent

$$(\lambda - D_{W,1})^{-1}: L^2(W; H) \longrightarrow H^1(W, H; \Pi_{+, \Lambda}^H).$$

Obviously $(\lambda - D_{M,\varepsilon})^{-1} = (\lambda - D_{W,1})^{-1} \oplus R_\varepsilon(\lambda)$. Because $D_{W,1}$ is the effective horizontal operator, Proposition 2.8 in [22] reduces to

$$\|(\lambda - D_{W,1})^{-1}\|_\infty = O(1) \quad \text{and} \quad \|(\lambda - D_{W,1})^{-1}\|_p = O(|\lambda|) \quad (6.10)$$

for all $\lambda \in \Gamma$, which can be proved in the same way as (6.7). As an analogue of [22, Prop 2.9], we get

$$\|(\lambda - \varepsilon^{-1}D_{M,\varepsilon})^{-1} - (\lambda - D_{W,1})^{-1}\|_\infty = O(\varepsilon |\lambda|) \quad (6.11)$$

for all $\lambda \in \Gamma$.

Because $D_{M,\varepsilon} = \varepsilon D_{W,1} \oplus D_{M,\varepsilon,4}$, the spectral projection P_ε in [22, sect 2.f] coincides with the spectral projection onto $\ker(D_{W,1}) = \ker(D_{M,\varepsilon})$ independent of ε . Using (6.6), (6.9–6.11), we can adapt the proof of [22, Prop 2.10] to show that there exists a small $\alpha > 0$ such that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon^{\alpha-2}}^{\infty} \frac{1}{\sqrt{\pi t}} \operatorname{tr} \left(D_{M,\varepsilon} e^{-tD_{M,\varepsilon}^2} \right) dt \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon^{\alpha-2}}^{\infty} \frac{1}{\sqrt{\pi t}} \operatorname{tr} \left((1 - P_\varepsilon) \left(D_{M,\varepsilon} e^{-tD_{M,\varepsilon}^2} \right) (1 - P_\varepsilon) \right) dt = \eta(D_{W,1}) = 0. \end{aligned} \quad (6.12)$$

Note that the orbifold η -form $\eta_{\Lambda W}(\mathbb{A})$ vanishes on the principal stratum $W \subset \Lambda W$ because the Seifert fibration $M \rightarrow V$ is a twisted product and the fibrewise operator D_{S^1} has symmetric spectrum. The additional divergent terms in the heat asymptotics of the supertrace of e^{-tD_W} caused by the non-geometric terms introduced in (2.4) and (6.4) do not cause extra complications here because they are supported on the regular stratum (and away from the boundary). Because the singular stratum does not extend to the boundary ∂W , the right hand side of the expression in the theorem vanishes near the boundary.

We can now use finite propagation speed to obtain the analogue of [22, Prop 2.12], which says that

$$\lim_{\varepsilon \rightarrow 0} \int_0^{\varepsilon^{\alpha-2}} \frac{1}{\sqrt{\pi t}} \operatorname{tr} \left(D_{M,\varepsilon} e^{-tD_{M,\varepsilon}^2} \right) dt$$

equals the right hand side of the expression in the theorem. Together with (6.12), this finishes the proof. \square

6.2. Adiabatic limits of families of flat tori. We consider a family of fibred manifolds $E \rightarrow F \rightarrow \mathbb{R}$ as in Section 4.4, diagram (4.6). We will prove Proposition 4.11, which is a special case of the adiabatic limit formula for η -forms of Bunke, Ma [10] and Liu [30], but as an equation of forms, not as an equation of forms up to exact forms. To this end, we will simply compute both sides of the equation. We believe that under suitable conditions, the adiabatic limit formula for η -forms holds in this strict sense for much for general iterated fibre bundles.

We fix $y \in \mathbb{R}$; later we will consider the limit $y \rightarrow \infty$. For $x \in \mathbb{R}$, we identify

$$E_x = \mathbb{C}/(\mathbb{Z} + (x + iy)\mathbb{Z}) \quad \text{and} \quad F_x = \mathbb{R}/y\mathbb{Z}.$$

The fibration $E \rightarrow F$ is formed by taking the imaginary part. The standard Euclidean metric on \mathbb{C} induces a fibrewise metric on $E \rightarrow \mathbb{R}$. The group S^1 acts isometrically by translation in the real direction in \mathbb{C} .

On the total space of E , we consider the fibrewise orthonormal base induced by $e_1 = 1$ and $e_2 = i \in \mathbb{C}$. We choose a horizontal subspace $T^H E \subset TE$ for the fibration $E \rightarrow \mathbb{R}$ spanned by the vector field e_3 induced from the vector field

$$\mathbb{C} \times \mathbb{R} \longrightarrow \mathbb{C} \times \mathbb{R} \quad \text{with} \quad (u + iv, x) \longmapsto (v/y; 1),$$

which is invariant under the x -dependent action of \mathbb{Z}^2 on \mathbb{C} . Obviously,

$$[e_1, e_2] = [e_1, e_3] = 0 \quad \text{and} \quad [e_2, e_3] = \frac{1}{y} e_1. \quad (6.13)$$

Hence, the vertical connection $\nabla^{T(E/\mathbb{R})}$ is given by

$$\nabla^{T(E/\mathbb{R})} e_1 = \frac{1}{2y} e_2 dx \quad \text{and} \quad \nabla^{T(E/\mathbb{R})} e_2 = -\frac{1}{2y} e_1 dx.$$

Because $y \in \mathbb{R}$ is constant, this connection is flat.

We identify the fibrewise spinor bundles $S(E/\mathbb{R}) = S^+(E/\mathbb{R}) \oplus S^-(E/\mathbb{R}) \rightarrow E$ with $\underline{\mathbb{C}} \oplus \underline{\mathbb{C}}$. If ∇^0 denotes the trivial connection on the spinor bundle, then $\nabla^{T(E/\mathbb{R})}$ induces the connection

$$\nabla^{S(E/\mathbb{R})} = \nabla^0 + \frac{1}{4y} c_1 c_2 dx.$$

Let $W = p_* S(E/\mathbb{R})$ denote the infinite-dimensional vector bundle over \mathbb{R} with fibres the sections of $S(E/\mathbb{R})|_{E_x}$. We can identify sections of W with sections of $S(E/\mathbb{R})$. Because the fibres of p have vanishing mean curvature, the induced connection takes the form

$$\nabla^W s = \nabla_{e_3}^{S(E/\mathbb{R})} s dx.$$

Let D_x denote the fibrewise Dirac operator over $x \in \mathbb{R}$. Then the Bismut superconnection for the fibration $E \rightarrow \mathbb{R}$ takes the form

$$\mathbb{B}_t = \sqrt{t} D_x + \nabla^W.$$

Because

$$\begin{aligned} [\nabla^W, D_x] &= -c_1 [\nabla_{e_3}^0, \nabla_{e_1}^0] dx - c_2 [\nabla_{e_3}^0, \nabla_{e_2}^0] dx - [c_1 c_2, c_1] \nabla_{e_1}^0 \frac{dx}{4y} - [c_1 c_2, c_2] \nabla_{e_2}^0 \frac{dx}{4y} \\ &= c_2 \nabla_{e_1}^0 \frac{dx}{y} - c_2 \nabla_{e_1}^0 \frac{dx}{2y} + c_1 \nabla_{e_2}^0 \frac{dx}{2y} = c_2 \nabla_{e_1}^S \frac{dx}{2y} + c_1 \nabla_{e_2}^S \frac{dx}{2y}, \end{aligned}$$

the curvature of the Bismut superconnection is given by

$$\mathbb{B}_t^2 = t D_x^2 + \sqrt{t} [\nabla^W, D_x] = t D_x^2 + \frac{\sqrt{t}}{2y} (c_1 \nabla_{e_2}^S + c_2 \nabla_{e_1}^S) dx.$$

The η -form for bundles with even-dimensional fibres is given by

$$\begin{aligned} \tilde{\eta}(\mathbb{B}) &= \frac{1}{2\pi i} \int_0^\infty \text{str} \left(\frac{\partial \mathbb{B}_t}{\partial t} e^{-\mathbb{B}_t^2} \right) dt \\ &= -\frac{1}{8y\pi i} \int_0^\infty \text{tr} \left(i c_1 c_2 (c_1 \nabla_{e_1}^S + c_2 \nabla_{e_2}^S) (c_1 \nabla_{e_2}^S + c_2 \nabla_{e_1}^S) dx e^{-t D_x^2} \right) dt \\ &= \frac{1}{8y\pi} \int_0^\infty \text{tr} \left(((\nabla_{e_1}^S)^2 - (\nabla_{e_2}^S)^2) e^{-t D_x^2} \right) dt dx. \end{aligned} \quad (6.14)$$

The space of vertical sections is spanned by sections of the form $\varphi_{m,n}s_{\pm}$, where

$$\varphi_{m,n}(u, v) = e^{2\pi i \left(m \left(u - \frac{x}{y} v \right) + n \frac{v}{y} \right)}$$

for $m, n \in \mathbb{Z}$ and s_{\pm} is a fibrewise parallel section of $S^{\pm}(E/\mathbb{R})$. The vertical Laplacian takes the form $-\partial_u^2$, and its kernel is spanned by the functions $\varphi_{0,n}s_{\pm}$. Because $S(E/\mathbb{R})$ has rank 2, we can therefore rewrite the η -form as

$$\tilde{\eta}(\mathbb{B}) = \frac{dx}{4\pi y} \int_0^{\infty} \sum_{m,n \in \mathbb{Z}} 4\pi^2 \left(\left(\frac{n-mx}{y} \right)^2 - m^2 \right) e^{-4\pi^2 t \left(\left(\frac{n-mx}{y} \right)^2 + m^2 \right)} dt. \quad (6.15)$$

For fixed m , the sum over n describes the spectrum of a Dirac operator on a circle S_y^1 of length y with coefficients in a flat vector bundle. Approximating the heat kernel on S_y^1 by the Euclidean heat kernel gives

$$\begin{aligned} \sum_n 4\pi^2 \left(\left(\frac{n-mx}{y} \right)^2 - m^2 \right) e^{-4\pi^2 t \left(\frac{n-mx}{y} \right)^2} &= - \left(4\pi^2 m^2 + \frac{\partial}{\partial t} \right) \sum_{n \in \mathbb{Z}} e^{-4\pi^2 t \left(\frac{n-mx}{y} \right)^2} \\ &= - \left(4\pi^2 m^2 - \frac{1}{2t} \right) \frac{y}{\sqrt{4\pi t}} + O \left((1+m^2) e^{-\frac{y^2-c}{4t}} \right) \end{aligned}$$

for each small $c > 0$, uniformly in m . For $\alpha > 0$ small, we compute

$$\begin{aligned} \frac{dx}{4\pi y} \int_0^{y^{2-\alpha}} \sum_{m,n \in \mathbb{Z}} 4\pi^2 \left(\left(\frac{n-mx}{y} \right)^2 - m^2 \right) e^{-4\pi^2 t \left(\left(\frac{n-mx}{y} \right)^2 + m^2 \right)} dt \\ = - \frac{dx}{4\pi} \int_0^{y^{2-\alpha}} \sum_{m \in \mathbb{Z}} \left(4\pi^2 m^2 - \frac{1}{2t} \right) \frac{1}{\sqrt{4\pi t}} e^{-4\pi^2 m^2 t} dt \end{aligned} \quad (6.16)$$

For $t \geq y^{2-\alpha}$, we only need to study the contribution from $\ker D^{E/F}$, which can be written as

$$\begin{aligned} \frac{1}{2\pi i} \int_{y^{2-\alpha}}^{\infty} \text{str} \left(P^{\ker D^{E/F}} \frac{\partial \mathbb{B}_t}{\partial t} e^{-\mathbb{B}_t^2} \right) dt &= \frac{dx}{2y\pi} \int_{y^{2-\alpha}}^{\infty} \sum_n \frac{4\pi^2 n^2}{y^2} e^{-\frac{4\pi^2 n^2 t}{y^2}} dt \\ &= \frac{dx}{4y\pi} \sum_n e^{-4\pi^2 y^{-\alpha} n^2} dx. \end{aligned}$$

This sum converges to 0 as $y \rightarrow \infty$ for $\alpha < 1$ because

$$\frac{1}{y} \sum_n e^{-4\pi^2 y^{-\alpha} n^2} \leq \frac{2}{y} \sum_{n=0}^{\infty} e^{-4\pi^2 y^{-\alpha} n} = \frac{2}{y} \cdot \frac{1}{1 - e^{-4\pi^2 y^{-\alpha}}} = \frac{2}{4\pi^2 y^{1-\alpha}} + o(4\pi^2 y^{-\alpha}).$$

In general, one would expect here the η -form of the effective fibrewise operator on $F \rightarrow \mathbb{R}$, acting on sections of $\ker D^{E/F}$, and some extra terms in the case that there are very small eigenvalues. Because the kernel bundle is trivial here, it is not surprising that this form vanishes in our situation. Combining this with the computations above, we finally see that

$$\lim_{y \rightarrow \infty} \tilde{\eta}(\mathbb{B}) = \frac{dx}{4\pi} \int_0^{\infty} \sum_{m \in \mathbb{Z}} \left(\frac{1}{2t} - 4\pi^2 m^2 \right) e^{-4\pi^2 m^2 t} \frac{dt}{\sqrt{4\pi t}} \quad (6.17)$$

We now consider the fibration $E \rightarrow F$. We choose the horizontal bundle spanned by the vectors e_2 and e_3 above. We identify the spinor bundle $S(E/F) \rightarrow E$ with $\underline{\mathbb{C}}$. From

equation (6.13), we get the superconnection \mathbb{A}_t for the family $E \rightarrow F$ as

$$\mathbb{A}_t = \sqrt{t} c_1 \nabla_{e_1}^S + \nabla^W + \frac{1}{4y\sqrt{t}} c_1 dv dx .$$

Its curvature is given by

$$\mathbb{A}_t^2 = -t(\nabla_{e_1}^0)^2 + \frac{1}{2y} dv dx .$$

Assuming that the Clifford volume element ic_1 acts as 1, the η -form of the bundle $E \rightarrow F$ with odd-dimensional fibres takes the form

$$\begin{aligned} \tilde{\eta}(\mathbb{A}) &= (2\pi i)^{-\frac{NF}{2}} \int_0^\infty \text{tr} \left(\frac{\partial \mathbb{A}_t}{\partial t} e^{-\mathbb{A}_t^2} \right) \frac{dt}{\sqrt{\pi}} \\ &= \int_0^\infty \text{tr} \left(c_1 \left(\nabla_{e_1}^S - \frac{1}{8\pi i y t} dv dx \right) \left(1 - \frac{1}{4\pi i y} dv dx \nabla_{e_1}^S \right) e^{t(\nabla_{e_1}^S)^2} \right) \frac{dt}{\sqrt{4\pi t}} \\ &= \int_0^\infty \text{tr} \left(\left(-i \nabla_{e_1}^S + \frac{dv dx}{8\pi y t} (1 + 2t(\nabla_{e_1}^S)^2) \right) e^{t(\nabla_{e_1}^S)^2} \right) \frac{dt}{\sqrt{4\pi t}} \\ &= \int_0^\infty \text{tr} \left(\left(-i \nabla_{e_1}^S + \frac{dv dx}{8\pi y t} \left(1 + 2t \frac{\partial}{\partial t} \right) \right) e^{t(\nabla_{e_1}^S)^2} \right) \frac{dt}{\sqrt{4\pi t}} . \end{aligned} \tag{6.18}$$

The space of vertical sections is spanned by sections of the form φ_m for $m \in \mathbb{Z}$, where

$$\varphi_m(u) = e^{2\pi i m u} .$$

We can now compute the integral of $\tilde{\eta}(\mathbb{A})$ over the fibres of $F \rightarrow \mathbb{R}$ as

$$\begin{aligned} \int_{F/\mathbb{R}} \tilde{A}(\mathbb{A}) &= \frac{dx}{4\pi} \int_0^\infty \sum_{m \in \mathbb{Z}} \left(\frac{1}{2t} + \frac{\partial}{\partial t} \right) e^{-4\pi^2 m^2 t} \frac{dt}{\sqrt{4\pi t}} \\ &= \frac{dx}{4\pi} \int_0^\infty \sum_{m \in \mathbb{Z}} \left(\frac{1}{2t} - 4\pi^2 m^2 \right) e^{-4\pi^2 m^2 t} \frac{dt}{\sqrt{4\pi t}} . \end{aligned} \tag{6.19}$$

Proof of Proposition 4.11. The Proposition follows by comparing (6.17) and (6.19). \square

APPENDIX BY DON ZAGIER: ON THE VALUES OF THE FUNCTION $F_{k,\varepsilon}(s)$

In Section 2 of this paper (Proposition 2.11, Theorem 2.12, Theorem 2.13) it was shown that the ν -invariants of extra twisted connected sums can be computed in terms of values of the analytic function $F_{k,\varepsilon}: (0, \infty) \rightarrow \mathbb{R}$ defined for each $k \in \mathbb{N}$ and integer ε prime to k by

$$F_{k,\varepsilon}(s) = \int_0^\infty \int_0^s \sum_{m \equiv \varepsilon n \pmod{k}} mn e^{-t(m^2 + n^2 a^2)} da dt$$

(Definition 2.10). In this appendix we will give a closed formula for $F_{k,\varepsilon}(s)$ in terms of the Dedekind eta-function, show that it is equal to the arccosine (or arcsine, or arctangent) of a computable algebraic number whenever s^2 is rational, and show that the specific combinations of $F_{k,\varepsilon}$ -values occurring in Theorems 2 and 3 can be evaluated in terms of Dedekind sums.

A.1. Evaluation of $F_{k,\varepsilon}(s)$ in terms of the Dedekind eta-function. For τ in the upper half-plane \mathcal{H} we denote by $\eta(\tau)$ and $\mathcal{L}(\tau)$ the Dedekind η -function and the principal branch (real on the positive imaginary axis) of its logarithm, given explicitly by

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}), \quad \mathcal{L}(\tau) = \frac{\pi i \tau}{12} - \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} e^{2\pi i n \tau},$$

where $\sigma(n)$ denotes the sum of the positive divisors of n . The fact that $\eta(\tau)^{24}$ is a modular form of weight 12 on $SL_2(\mathbb{Z})$ implies that L satisfies the transformation equation

$$\mathcal{L}\left(\frac{a\tau + b}{c\tau + d}\right) = \mathcal{L}(\tau) + \frac{1}{4} \operatorname{Log}(-(c\tau + d)^2) + \frac{\pi i}{12} N(a, b, c, d), \quad (\text{A.1})$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, where Log denotes the principal branch (real on the positive real axis) of the logarithm on $\mathbb{C} \setminus (-\infty, 0]$ and $N(a, b, c, d)$ is an integer given by $N(a, b, c, d) = b/d$ ($= \pm b$) if $c = 0$ and by $N(a, b, c, d) = \frac{a+d}{c} - 12S(d, c)$ if $c \neq 0$, where the Dedekind sum $S(d, c)$ is defined in (0.3). Our first result is:

Proposition A.1. *The value of $F_{k,\varepsilon}(s)$ for any $k \in \mathbb{N}$, integer ε prime to k , and positive real number s is given by*

$$F_{k,\varepsilon}(s) = 2 \operatorname{Im} \mathcal{L}\left(\frac{-\varepsilon^* + is^{-1}}{k}\right) + \frac{\pi \varepsilon^*}{6k}, \quad (\text{A.2})$$

where $\varepsilon^* \in \mathbb{Z}$ is any solution of $\varepsilon \varepsilon^* \equiv 1 \pmod{k}$.

Proof. We first rewrite the definition of $F_{k,\varepsilon}$ as

$$s \frac{d}{ds} F_{k,\varepsilon}(s) = \frac{\pi}{k} \int_0^{\infty} \Theta_{k,\varepsilon}(s, t) dt \quad \text{and} \quad F_{k,\varepsilon}(0) = 0, \quad (\text{A.3})$$

where $\Theta_{k,\varepsilon}(s, t)$ is defined for $s, t > 0$ by

$$\Theta_{k,\varepsilon}(s, t) = \sum_{m \equiv \varepsilon n \pmod{k}} mn e^{-\pi t(m^2/s + n^2 s)/k}.$$

This theta series satisfies the functional equations

$$\Theta_{k,\varepsilon}(s, t) = -\Theta_{k,-\varepsilon}(s, t) = \Theta_{k,\varepsilon^*}(s^{-1}, t) = t^{-3} \Theta_{k,\varepsilon}(s, t^{-1}), \quad (\text{A.4})$$

(where $\varepsilon^* \equiv \varepsilon^{-1} \pmod{k}$ as above), as we see by changing the sign of m , interchanging m and n , or applying the Poisson summation formula with respect to both m and n . If instead we apply Poisson summation with respect to m only, we obtain the stronger identity

$$\Theta_{k,\varepsilon}(s, t) = \frac{(s/t)^{3/2}}{i\sqrt{k}} \sum_{m,n \in \mathbb{Z}} mn \zeta_k^{mn\varepsilon} e^{-\pi s(m^2/t + n^2 t)/k} \quad (\zeta_k := e^{2\pi i/k}), \quad (\text{A.5})$$

which also makes it clear that the integral in (A.3) converges, since it shows that $\Theta_{k,\varepsilon}(s, t)$ is exponentially small as t tends to either 0 or ∞ . Inserting (A.5) into (A.3) and applying the elementary formula

$$\int_0^{\infty} e^{-c_1 t - c_2/t} t^{-3/2} dt = \sqrt{\frac{\pi}{c_2}} e^{-2\sqrt{c_1 c_2}} \quad (c_1, c_2 > 0)$$

with $c_1 = \pi s n^2/k$, $c_2 = \pi s m^2/k$, we find

$$\frac{ik}{2\pi} F'_{k,\varepsilon}(s) = \sum_{m,n > 0} n (\zeta_k^{\varepsilon mn} - \zeta_k^{-\varepsilon mn}) e^{-2\pi mns/k} = \sum_{n=1}^{\infty} \sigma(n) (\zeta_k^{\varepsilon n} - \zeta_k^{-\varepsilon n}) e^{-2\pi ns/k},$$

and this can be integrated immediately using the definition of \mathcal{L} to give the formula

$$F_{k,\varepsilon}(s) = 2 \operatorname{Im} \mathcal{L}\left(\frac{\varepsilon + is}{k}\right) + c_{k,\varepsilon} \quad (\text{A.6})$$

for some constant $c_{k,\varepsilon}$ depending only on k and ε . We then use the modularity property (A.1) and the fact that $\frac{\pm\varepsilon+is}{k} = \gamma\left(\frac{\mp\varepsilon^*+is^{-1}}{k}\right)$ with $\gamma = \begin{pmatrix} \pm\varepsilon & * \\ k & \pm\varepsilon^* \end{pmatrix} \in SL_2(\mathbb{Z})$ to deduce (A.2) from (A.6) up to a constant whose value then follows immediately from the property $F_{k,\varepsilon}(0) = 0$, because $\mathcal{L}(\tau) = \pi i\tau/12 + o(1)$ for $\operatorname{Im}(\tau) \rightarrow \infty$. \square

Using the transformation law (A.1) again, we can evaluate the constant $c_{k,\varepsilon}$ in (A.6) to get

$$F_{k,\varepsilon}(s) = 2 \operatorname{Im} \mathcal{L}\left(\frac{\varepsilon + is}{k}\right) + 2\pi S(\varepsilon, k) - \frac{\pi\varepsilon}{6k}. \quad (\text{A.7})$$

giving an alternative formula for the function $F_{k,\varepsilon}(s)$. In some cases this same transformation law can be used to give a complete formula for $F_{k,\varepsilon}(s)$ in terms of Dedekind sums. This happens whenever one (and hence both) of the two $SL_2(\mathbb{Z})$ -equivalent numbers $\frac{\pm\varepsilon+is}{k}$ and $\frac{\mp\varepsilon^*+is}{k}$ is $SL_2(\mathbb{Z})$ -equivalent to its negative conjugate. An easy calculation shows that the equation $\gamma\tau = -\bar{\tau}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and τ in the upper half-plane holds if and only if $a = d$ and $|c\tau + a| = 1$, which in our situation says that $s^2 = \frac{1}{c^2} - \left(\frac{\varepsilon}{k} + \frac{a}{c}\right)^2$ for some integers a and c with $a^2 \equiv 1 \pmod{c}$. In all such cases, the number $F_{k,\varepsilon}(s)$ is the sum of a rational multiple of π and the arctangent of the square-root of a positive rational number. Concrete examples where this happens and where the Dedekind sum occurring can be evaluated in closed form are the special values

$$F_{k,1}\left(\frac{1}{\sqrt{k^2-1}}\right) = -F_{k,1}(\sqrt{k^2-1}) - \frac{(k-1)(k-2)}{6k}\pi = \arctan \sqrt{\frac{k+1}{k-1}} - \frac{3k+2}{12k}\pi$$

for integers $k > 1$ and

$$F_{k,1}\left(\sqrt{\frac{m}{n}}\right) = \arctan \sqrt{\frac{m}{n}} - \frac{km+2}{12k}\pi$$

for positive integers m and n with $m+n = 2k$. We omit the details.

A.2. Algebraic values. Except in the cases just mentioned, there is in general no simple closed formula for the values of $F_{k,\varepsilon}(s)$. However, if the square of the argument s is a rational number, as is the case for all of the special values needed in this paper, one has the following general result.

Proposition A.2. *If $s > 0$ is the square-root of a rational number, then the value of $F_{k,\varepsilon}(s)$ for any k and ε is i times the logarithm of a computable algebraic number.*

Proof. It is known from the theory of complex multiplication that the ratio of the values of the Dedekind eta-function at any two arguments belonging to the same imaginary quadratic field is a computable algebraic number. (More precisely, the value of $\eta(\tau)$ for τ belonging to any imaginary quadratic field is an algebraic multiple of a certain product of gamma-values, the so-called Chowla-Selberg number, that depends only on the field. For more details, see [8, Part 1, Section 6].) Since both $\frac{\varepsilon^*+i/s}{k}$ and $\frac{-\varepsilon^*+i/s}{k}$ belong to the imaginary quadratic field $\mathbb{Q}(is)$ when s^2 is rational, this proposition is an immediate corollary of Proposition A.1. \square

We do not describe here the algorithm for computing special eta-values at CM points, since it is standard in principle but is quite complicated. We limit ourselves instead to giving the

k	ε	s	$S(\varepsilon, k)$	b	σ	c
3	1	1	1/18	0	± 1	1
4	1	1	1/8	0	± 1	1
4	1	$\sqrt{3}$	1/8	1/12	± 1	1
5	1	1	1/5	0	± 1	1
6	1	1	5/18	0	± 1	1
6	1	$\sqrt{3}$	5/18	1/6	± 1	1
3	1	$\sqrt{2}$	1/18	-1/6	1	1/3
3	1	$\sqrt{5}$	1/18	-1/12	1	2/3
3	1	$2\sqrt{2}$	1/18	1/4	-1	1/3
4	1	$\sqrt{7}$	1/8	0	1	3/4
4	1	$\sqrt{15}$	1/8	-1/6	1	-1/4
4	1	$\sqrt{5/3}$	1/8	-1/6	1	1/4
5	1	2	1/5	0	1	3/5
5	2	1	0	1/10	-1	3/5
5	2	4	0	1/10	-1	4/5
6	1	$\sqrt{2}$	5/18	-1/12	1	1/3
6	1	$\sqrt{5}$	5/18	1/12	1	2/3
6	1	$\sqrt{11}$	5/18	1/6	1	5/6
3	1	2	1/18	1/6	-1	$\sqrt{3} - 1$
4	1	$\sqrt{2}$	1/8	-1/8	1	$\sqrt{2} - 1$
4	1	$\sqrt{5}$	1/8	-1/4	1	$\frac{1}{2}(1 - \sqrt{5})$
4	1	3	1/8	0	1	$\sqrt{3} - 1$
4	1	5	1/8	0	1	$3\sqrt{5} - 6$
5	2	2	0	1/10	-1	$3\sqrt{5} - 6$
6	1	$\sqrt{7}$	5/18	2/3	-1	$\frac{1}{4}(1 - \sqrt{21})$
3	1	$\sqrt{3}$	1/18	-1/6	1	$\sqrt[3]{2} - 1$
4	1	$3\sqrt{3}$	1/8	-1/12	1	$\sqrt[3]{2} - 1$
3	1	$2\sqrt{5}$	1/18	-1/6	1	$\frac{1}{3}(1 - \sqrt{5} + \sqrt{5(\sqrt{5} - 1)/2})$
3	1	$4\sqrt{2}$	1/18	-1/12	1	$\frac{1}{6}(6 - 5\sqrt{2} + (4\sqrt{2} + 2)\sqrt{\sqrt{2} - 1})$
3	1	$\sqrt{5}/2$	1/18	0	1	$\frac{1}{3}(\sqrt{5} - 1 + \sqrt{5(\sqrt{5} - 1)/2})$
4	1	$3\sqrt{7}$	1/8	0	1	$\frac{1}{16}(9 + \sqrt{21} - \sqrt{26\sqrt{21} - 114})$
4	1	$3/\sqrt{7}$	1/8	0	1	$\frac{1}{16}(9 + \sqrt{21} + \sqrt{26\sqrt{21} - 114})$
3	1	$5\sqrt{2}$	1/18	1/6	-1	$c = 0.766\dots, P(3c) = 0$
3	1	$5/\sqrt{2}$	1/18	0	1	$c = 0.940\dots, P(-3c) = 0$
5	1	$\sqrt{2}$	1/5	0	1	$c = 0.861\dots, Q(c) = 0$
5	2	$\sqrt{2}$	0	1/10	-1	$c = 0.634\dots, Q(-c) = 0$

TABLE 3. Data needed to compute $F_{k,\varepsilon}$

values of $F_{k,\varepsilon}(s)$ for the specific triples (k, ε, s) that are used in this paper. These values are given by

$$\begin{aligned} F_{k,\varepsilon}(s) &= \pi \left(S(\varepsilon, k) + b \right) + \frac{\sigma}{2} \arccos(c), \\ F_{k,\varepsilon^*}(1/s) &= \pi \left(S(\varepsilon, k) - b \right) - \frac{\sigma}{2} \arccos(c). \end{aligned}$$

with $b \in \mathbb{Q}$, $\sigma \in \{\pm 1\}$ and $c \in \overline{\mathbb{Q}}$ as in Table 3. The functions P and Q appearing in the last four lines of the table are the two sextic polynomials

$$\begin{aligned} P(X) &= 16 X^6 - 416 X^5 + 2440 X^4 + 4880 X^3 - 12615 X^2 - 1826 X - 32159, \\ Q(X) &= 16 X^6 - 32 X^5 + 200 X^4 + 560 X^3 + 105 X^2 - 402 X - 191. \end{aligned}$$

A.3. Evaluation of $A(k_+, \varepsilon_+; k_-, \varepsilon_-; G)$ in terms of the Dedekind sums. In this final subsection we place ourselves in the situation of Theorem 3. Specifically, this means that we have two pairs of coprime numbers $(k_{\pm}, \varepsilon_{\pm})$ with k_{\pm} positive and a 2×2 “gluing matrix” $G = \begin{pmatrix} m & p \\ n & q \end{pmatrix} \in M_2(\mathbb{Z})$ satisfying conditions (1.7)–(1.9). Equivalently, $\det G = -k_+k_-$ and

$$m - \varepsilon_+^* n = Ak_+, \quad p - \varepsilon_+^* q = Bk_+, \quad p + \varepsilon_-^* m = Ck_-, \quad q + \varepsilon_-^* n = Dk_- \quad (\text{A.8})$$

for some integers A, B, C, D with $(A, n) = (B, q) = (C, m) = (D, n) = 1$. We further assume that $n > 0$, $mnpq < 0$ and set $s_+ = \sqrt{-nq/mp}$, $s_- = \sqrt{-mn/pq}$, and $\rho = \pi - 2 \arg(ms_+ + in)$. Then the invariant we want to compute is the combination of $F_{k,\varepsilon}$ -values defined by

$$\mathcal{F}(k_+, \varepsilon_+; k_-, \varepsilon_-; G) := \frac{1}{\pi} \left(F_{k_+, \varepsilon_+}(s_+) + F_{k_-, \varepsilon_-}(s_-) + \frac{\rho}{2} \right).$$

Proposition A.3. *The number $\mathcal{F}(k_+, \varepsilon_+; k_-, \varepsilon_-; G)$ is always rational and is given by*

$$\mathcal{F}(k_+, \varepsilon_+; k_-, \varepsilon_-; G) = \frac{1}{6} \left(\frac{m}{k_+n} - \frac{q}{k_-n} - 12S(A, n) \right),$$

where $S(A, n)$ is the Dedekind sum as defined in (0.3).

Proof. Set $\lambda = \frac{\varepsilon_-^* A + B}{k_-} = \frac{C - \varepsilon_+^* D}{k_+}$, which is an integer by (3.5). The equations (A.8) can be rewritten as

$$\begin{pmatrix} q & p \\ -n & -m \end{pmatrix} = \begin{pmatrix} k_- & \varepsilon_-^* \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & \varepsilon_+^* \\ 0 & k_+ \end{pmatrix} \quad \text{with} \quad \gamma = \begin{pmatrix} D & \lambda \\ -n & -A \end{pmatrix} \in SL_2(\mathbb{Z}).$$

It is easily checked that γ maps $\tau_+ = \frac{\varepsilon_+^* + is_+^{-1}}{k_+}$ to $\tau_- = \frac{-\varepsilon_-^* + is_-^{-1}}{k_-}$. From the transformation law (A.1) of \mathcal{L} , we get

$$\mathcal{L}(\tau_-) - \mathcal{L}(\tau_+) = \frac{1}{4} \operatorname{Log} \left(- \left(\frac{m + ins_+^{-1}}{k_+} \right)^2 \right) + \frac{\pi i}{12} \left(\frac{A - D}{n} - 12S(A, n) \right).$$

Because $\mathcal{L}(-\bar{z}) = \overline{\mathcal{L}(z)}$, Proposition A.1 gives

$$\begin{aligned} \mathcal{F}(K_+, \varepsilon_+; k_-, \varepsilon_-; G) &= \frac{2}{\pi} (\operatorname{Im} \mathcal{L}(\tau_-) - \operatorname{Im} \mathcal{L}(\tau_+)) + \frac{\rho}{2\pi} + \frac{\varepsilon_+^*}{6k_+} + \frac{\varepsilon_-^*}{6k_-} \\ &= \frac{1}{6} \left(\frac{m}{k_+n} - \frac{q}{k_-n} - 12S(A, n) \right). \end{aligned} \quad \square$$

We observe that (0.2) and Proposition A.3 give an alternative proof of Theorem 3.

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