GEOMETRY AND ARITHMETIC OF INTEGRABLE HIERARCHIES OF KDV TYPE. I. INTEGRALITY

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Abstract. For each of the simple Lie algebras $\mathfrak{g} = A_l$, D_l or E_6, we show that the all-genera one-point FJRW invariants of $\mathfrak{g}$-type, after multiplication by suitable products of Pochhammer symbols, are the coefficients of an algebraic generating function and hence are integral. Moreover, we find that the all-genera invariants themselves coincide with the coefficients of the unique calibration of the Frobenius manifold of $\mathfrak{g}$-type evaluated at a special point. For the A_4 (5-spin) case we also find two other normalizations of the sequence that are again integral and of at most exponential growth, and hence conjecturally are the Taylor coefficients of some period functions.

Contents

1. Introduction and description of the results for the 5-spin case

In this paper, we will study certain intersection numbers $\tau_{\mathfrak{g}}(g)$ (the precise definition will be given in Section 2) on the moduli space of stable algebraic curves $\overline{\mathcal{M}}_{g,n}$ [32] associated to simple Lie algebras, the case of A_{r-1}-type simple Lie algebra being essentially one-point r-spin intersection numbers, introduced by Witten [114]. (What we call $\tau_{A_{r-1}}(g)$ would be $\langle\tau_{s,m}\rangle$ in Witten's notation, where $2(r+1)g = r(s+1)+m+2$ with $s \geq 0$, $0 \leq m \leq r-2$; our notation in later sections will be slightly different from Witten's.) In particular, we will give recursive, closed, and asymptotic formulas for these numbers. Using these formulas, we will show for $\mathfrak{g} = A_l$ $(l \geq 1)$, or D_l $(l \geq 4)$, or E_6 that by multiplying $\tau_{\mathfrak{g}}(g)$ by appropriate gamma factors (products of Pochhammer symbols) we obtain new numbers whose generating functions are *algebraic*. In particular, these renormalized numbers are integral and grow only exponentially in g. Moreover, for the case of A_4, we find that there are *different* normalizations of the $\tau_{\mathfrak{g}}(g)$, obtained by multiplying by other gamma factors, that are again integral and of exponential growth, so that each of the corresponding generating series is conjecturally a period function for

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some family of algebraic varieties (or equivalently, a solution of some Picard–Fuchs differential equation). This latter point is also of interest from the point of view of the general arithmetic theory of differential equations (see for instance [118], where the conjecture relating integrality and geometric origin is discussed on pp. 728–729 and the A_4 example on pp. 768–769) and will be discussed from this point of view in the later paper [116].

For most of this introduction, we will assume that $\mathfrak{g} = A_4$ and describe our results in detail only for that case, indicating briefly at the end of the introduction where the statements of the general results can be found in the paper. For convenience, we write $\tau_g = \tau_{A_4}(g)$ for $g > 0$, and also set $\tau_0 = 1$ and $\tau_g = 0$ for $g < 0$. The first values are

g	0	1	2	3	4	5	6	7	8	9	10
τ_g	1	$\frac{1}{6}$	$\frac{11}{2^4 3^2 5^2}$	0	$\frac{341}{2^9 3^4 5^4}$	$\frac{161}{2^{10} 3^5 5^5}$	$\frac{3397}{2^{13} 3^6 5^6}$	$\frac{3421}{2^{13} 3^8 5^7}$	0	$\frac{1670581}{2^{20} 3^{10} 5^9 7^1}$	$\frac{26605753}{2^{23} 3^{12} 5^{12}}$

One-point 5-spin intersection numbers

The following theorem, which will be proved in Section 5, gives three different integrality statements about the numbers τ_g. One of these statements (the integrality of the numbers a_g) will be generalized to all A_l, D_l and E_6 in Sections 5–7 below. The two others will be given in this paper for the A_4 case only, with a discussion of the integrality properties of $\tau_{\mathfrak{g}}(g)$ for other simple Lie algebras postponed to the later paper [116] mentioned above.

Theorem 1. *Each of the renormalized values*

$$
\begin{aligned}
a_{5n} &= -\tfrac{1}{5}\big(\tfrac{4}{5}\big)_{2n-1}\tau_{5n}, & b_{5n} &= \big(\tfrac{4}{5}\big)_n\big(\tfrac{1}{5}\big)_n\tau_{5n}, & c_{5n} &= \big(\tfrac{4}{5}\big)_n\big(\tfrac{3}{5}\big)_n\tau_{5n},\\
a_{5n-1} &= -\tfrac{1}{5}\big(\tfrac{2}{5}\big)_{2n-1}\tau_{5n-1}, & b_{5n-1} &= \big(\tfrac{2}{5}\big)_n\big(\tfrac{3}{5}\big)_n\tau_{5n-1}, & c_{5n-1} &= \big(\tfrac{2}{5}\big)_n\big(\tfrac{4}{5}\big)_n\tau_{5n-1},\\
a_{5n-3} &= \tfrac{1}{5}\big(\tfrac{3}{5}\big)_{2n-2}\tau_{5n-3}, & b_{5n-3} &= \big(\tfrac{3}{5}\big)_{n-1}\big(\tfrac{2}{5}\big)_n\tau_{5n-3}, & c_{5n-3} &= \big(\tfrac{3}{5}\big)_{n-1}\big(\tfrac{1}{5}\big)_n\tau_{5n-3},\\
a_{5n-4} &= \tfrac{1}{5}\big(\tfrac{1}{5}\big)_{2n-2}\tau_{5n-4}, & b_{5n-4} &= \big(\tfrac{1}{5}\big)_n\big(\tfrac{4}{5}\big)_{n-1}\tau_{5n-4}, & c_{5n-4} &= \big(\tfrac{1}{5}\big)_n\big(\tfrac{2}{5}\big)_{n-1}\tau_{5n-4}
\end{aligned}
$$

belongs to $\mathbb{Z}\big[\frac{1}{30}\big]$. *Here* $(x)_k := x(x+1)\cdots(x+k-1)$ *denotes the ascending Pochhammer symbol.*

The point here is that the numbers τ_g decay like $1/\Gamma(\frac{2g}{5})$, as we will see in a moment, and that therefore each of the numbers a_g, b_g and c_g, as well as being integral (away from the primes 2, 3 and 5), is of only exponential growth in g and hence is expected to be the gth Taylor coefficient of some period function. For a_g, Theorem 2 below includes the stronger statement that the generating function $\sum a_g x^g$ is not only a period function, but is in fact algebraic. For a long time we were unable to identify the other two generating functions $\sum b_g x^g$ and $\sum c_g x^g$ as period functions, but in the end it turned out that they were both not just period functions, but again algebraic, although much more complicated and of much higher degree than $\sum a_g x^g$. This will be discussed in a later publication [116].

Remark 1. The formulas for a_g in Theorem 1 can be written more uniformly as

$$a_g \;=\; \frac{(-1)^m}{5}\,(A)_m\,\tau_g\,, \tag{1}$$

where $m = \big[\frac{2g-1}{5}\big]$ and $A = \big\{\frac{2g-1}{5}\big\}$ denote the integral and fractional parts of $\frac{2g-1}{5}$, respectively (recall that the fractional part of a real number x is defined as $\{x\} := x - [x]$), and similarly the arguments of the Pochhammer symbols in the formulas for b_g and c_g are always A and B or A and C, respectively, where $B = \big\{\frac{3g+1}{5}\big\}$ and $C = \big\{\frac{4g+3}{5}\big\}$. Note also that $A = 0$ if $g \equiv 3 \,(\mathrm{mod}\, 5)$, and that both a_g (as defined for all g by (5) below) and τ_g vanish in this case.

In the following theorem we collect many further properties of τ_g. All of the statements of this theorem will be generalized in the main body of the paper to the A_l, D_l, and E_6 cases.

Theorem 2. *The numbers $\tau_g = \tau_{A_4}(g)$ have the following properties:*
(i) [recursion] *The numbers τ_g satisfy the recursion relation*

$$\begin{aligned} &2^8\,3^4\,5^{17}\,31\,g\,(g-1)\,(g-2)\,(g-4)\,\tau_g \\ &\quad -\,5^{11}\left(2^8\,3^4\,g^4 - 2^{13}\,3^4\,g^3 + 2^4\,3^2\,54331\,g^2 - 2^4\,3^2\,43^1\,6329\,g + 5^1\,7^1\,2013229\right)\tau_{g-5} \\ &\quad +\,2^2\,5^6\left(2^2\,3^2\,5\,g^2 - 2^2\,3^3\,5^1\,7\,g + 19739\right)\tau_{g-10} \,-\, \tau_{g-15} \;=\; 0\,, \quad g\in\mathbb{Z} \end{aligned} \tag{2}$$

with the initial conditions given by the above table.
(ii) [dual topological ODE] *The generating series*

$$\varphi(X) \;:=\; \sum_{g\ge 0} \tau_g\, X^{\frac{g}{5}} \tag{3}$$

satisfies the fourth-order linear differential equation $Q\varphi = 0$, where

$$\begin{aligned} Q \;=\;& 2^8\,3^4\,5^{15}\,X^3\left(X - 5^6\,31\right)\frac{d^4}{dX^4} \,+\, 2^8\,3^4\,5^{14}\,X^2\left(2^1\,3^2\,X - 5^6\,23^1\,31\right)\frac{d^3}{dX^3} \\ &-\,2^4\,3^2\,5^9\,X\left(X^2 - 5^4\,6091\,X + 2^6\,3^3\,5^{10}\,7^1\,31\right)\frac{d^2}{dX^2} \\ &-\,2^5\,3^2\,5^8\left(2\,X^2 - 5^4\,3209\,X + 2^5\,3^3\,5^{10}\,31\right)\frac{d}{dX} \\ &+\left(X^2 + 2^2\,5^6\,61\,X + 5^{13}\,7^1\,23^1\,31\right). \end{aligned} \tag{4}$$

(iii) [algebraicity] *The generating function of the numbers a_g defined in Theorem 1 is algebraic. More precisely, we have*

$$y \;:=\; \sum_{g\ge 0} a_g\, z^g \quad\Rightarrow\quad y^5 \,-\, \frac{z}{6}\,y^3 \,+\, \frac{z^2}{400}\,y \;=\; 1\,. \tag{5}$$

(iv) [closed formula] *Denote $m = [(2g-1)/5]$ as above, and define $c_{p,j} \in \mathbb{Q}$ $(0 \le p \le j)$ by*

$$c_{p,j} \;:=\; \text{coefficient of } x^j \text{ in } \frac{1}{p!}\left(\frac{(1+x)^6 - 1 - 6\,x}{6\,x}\right)^p. \tag{6}$$

Then for all $g \ge 0$ with $g \not\equiv 3 \pmod 5$, we have

$$\tau_g \;=\; \Gamma\big(\{\tfrac{2g-1}{5}\}\big)\,\frac{(-1)^{g+m-1}}{5^g}\sum_{p=0}^{2g}\frac{c_{p,2g}}{\Gamma\big(\frac{2g-1}{5} - p + 1\big)}\,. \tag{7}$$

(v) [product formula] *Let*

$$w(u) \;=\; 1 + \sum_{n\ge 0} C_n\, u^{n+1} \;=\; 1 \,+\, u \,-\, \frac{2}{3}\,u^2 \,+\, \frac{11}{18}u^3 - \dots \tag{8}$$

be the unique power-series solution in $1 + u + u^2\mathbb{Q}[[u]]$ to the sextic equation

$$\frac{w^6}{30} \,-\, \frac{w}{5} \,+\, \frac{1}{6} \;=\; \frac{u^2}{2}\,, \tag{9}$$

and let $f(T) := \sum_{k\geq 0}(2k-1)!!\, C_{2k}\,(-T)^k$. *Then*

$$f(T)\,f(-T) \;=\; \sum_{g\geq 0}\Bigl(1+\frac{2g-1}{5}\Bigr)_{2g}\,(-1)^{g-1}\,5^{3g}\,a_g\,T^{2g} \tag{10}$$

with a_g *as in part* (iii).

(vi) [terminating hypergeometric sum] *For all* $g\in\mathbb{Z}$ *with* $g\not\equiv 3\,(\mathrm{mod}\,5)$, *we have*

$$\tau_g \;=\; \frac{6^{-g}}{\Gamma\bigl(1-\{\frac{2g-1}{5}\}\bigr)}\sum_{0\leq s\leq g/2}(-1)^s\Bigl(\frac{3}{10}\Bigr)^{2s}\frac{\Gamma\bigl(\frac{1+3g-5s}{5}\bigr)}{s!\,(g-2s)!}\,. \tag{11}$$

(vii) [asymptotics] *For* $g\not\equiv 3\,(\mathrm{mod}\,5)$ *and as* $g\to\infty$, τ_g *is given asymptotically by*

$$\tau_g \;\sim\; \frac{5}{\sqrt{6}^{\frac{3}{5}}\sqrt{\pi g^3}}\,\frac{\sin\bigl(\{\frac{2g-1}{5}\}\pi\bigr)}{\bigl(\{\frac{2g-1}{5}\}\bigr)_m}\,\Bigl(6^{\frac{2}{5}}\,20\,\sin^2\bigl(\frac{\pi}{5}\bigr)\Bigr)^{-g}, \tag{12}$$

where $m=[(2g-1)/5]$ *is again as above.*

The proof of this theorem is given in Sections 2 and 5.

Remark 2. Of course, parts (i) and (ii) are equivalent, by the usual bijection between power series satisfying a linear differential equation with polynomial coefficients and sequences of numbers satisfying a recursion with polynomial coefficients. More explicitly, since the recursion (2) involves only τ_g, τ_{g-5}, τ_{g-10} and τ_{g-15}, it is equivalent to five separate recursions for τ_g with $g\equiv s\ (\mathrm{mod}\,5)$, $0\leq s\leq 4$, the values for $s=3$ being uninteresting because $\tau_g=0$. Similarly, by the Frobenius method, the differential equation in part (ii) has four fundamental solutions in $x^\lambda\mathbb{Q}[[x]]$ with $\lambda=0,\frac{1}{5},\frac{2}{5},\frac{4}{5}$, and the generating function (3) is simply a certain linear combination of these with rational coefficients.

Remark 3. The integrality (away from 2, 3, and 5) of a_g in Theorem 1 follows immediately from part (iii) of Theorem 2, since the Taylor coefficients of any algebraic function are integral after some fixed rescaling. In fact, $\sum_{g\neq 3}(g-3)^{-1}a_g x^g$ is also algebraic, as we will see later at the end of Section 5, and correspondingly we have $(g-3)^{-1}a_g\in\mathbb{Z}[\frac{1}{30}]$ for $g\geq 3$. The integrality of b_g and c_g is harder and will follow from formula (11), as a consequence of the stronger statement, also proved at the end of Section 5, that each summand on the right-hand side of (11) becomes p-adically integral for every $p>5$ after multiplication with the two Pochhammer symbols given in Theorem 1.

Remark 4. We say something here about the origin of the problem and about some of the various approaches that can be used to solve it. The way that we are approaching intersection numbers is to use integrable systems. In 1990, Witten [112] proposed his famous conjecture that the partition function of ψ-class intersection numbers is a tau-function for the Korteweg-de Vries (KdV) integrable hierarchy. This conjecture was later proved by Kontsevich [83]. The r-spin Witten conjecture [114], stating that the partition function of the r-spin intersection numbers gives a tau-function for the Gelfand–Dickey integrable hierarchy [34], was proved by Faber–Shadrin–Zvonkine [63]. (The $r=2$ case corresponds to Witten's conjecture in 1990.) More generally, the ADE Witten conjecture [114], given its precise form by Fan–Jarvis–Ruan [66] (cf. Givental–Milanov [73]), states that the partition function of the FJRW invariants associated to a certain simple singularity [3] gives a tau-function for the Drinfeld–Sokolov (DS) integrable hierarchy [13, 25, 31, 38, 49]. The ADE Witten conjecture was proved by Fan–Jarvis–Ruan [66] (cf. [69, 73, 90]) with the D_4 case confirmed in [64]. For all these cases, the main mathematical object of the study is the so-called *topological solution* u^{top} [48, 56] to the corresponding integrable system together with its tau-function. Perhaps, the simplest way to describe this particular solution is to use its initial value (observe that each member of the DS hierarchy is an evolutionary PDE), which in terms of the normal coordinates [56] $r_1, \dots, r_l$ (where l is the rank) reads:

$$r_\alpha^{\rm top}|_{\text{higher times}=0} \;=\; \eta_{\alpha 1}\, t^{1,0}\,, \tag{13}$$

where $\alpha = 1, \dots, l$ and $\eta_{\alpha 1}$ are constants. One could then apply methods in integrable systems to compute $u^{\rm top}$ and more importantly its *tau-function*. These methods include the wave-function approach [5, 8, 9, 14, 20, 25, 34, 39, 53, 78, 108, 109], the ΨDOs [29, 34, 80, 87], the Sato Grassmannian approach [14, 30, 34, 108, 109], the Dubrovin–Zhang approach [24, 43, 48, 50, 54, 56], the Givental quantization [24, 70, 71, 72, 73], the topological recursion of Chekhov–Eynard–Orantin type (or of Bouchard–Eynard type) (cf. [15, 58, 119, 121] and the references therein), the matrix-resolvent method [11, 12, 13, 52, 53] (see also [21, 88, 89, 98]), etc. One of the keys for several of these methods is the *Lax pair* (see [5, 34, 38, 39]). In the second paper [117] of the series, the mathematical object will be a different solution to the DS hierarchy, whose tau-function has a different topological meaning; more general cases are considered in joint papers currently in preparation with Daniele Valeri. Taking the dispersionless limit of the DS hierarchy, one can also obtain an interesting geometric structure, suitable for studying *arbitrary* solutions, i.e., the Frobenius structure [41, 42, 43, 48, 56], which plays one of the central roles in understanding the above-mentioned methods. Besides the methods from integrable systems, there are other important methods for computing the intersection numbers from other theories, including the theory of matrix models [1, 2, 16, 17, 18, 33, 74, 83, 96], vertex algebras [6, 35, 69, 79, 92, 120, 123], emergent geometry [121, 122, 124], etc.

Our next result (see Theorem 8 and its corollary) gives an explicit relationship between the all-genera FJRW intersection numbers of $\mathfrak{g}$-type (with $g = A_l$, D_l or E_6) and genus zero, which again will be stated in the introduction only for the 5-spin case. Let B be the Frobenius manifold associated to the A_4 Coxeter group [43, 45, 106, 125], and let $(\theta_{\alpha,m})_{\alpha=1,\dots,4,m\ge 0}$ be the unique *calibration* [50, 56] on B. The reader who is not familiar with the theory of Frobenius manifolds could simply identify $\theta_{\alpha,m}$ with the following formal power series of infinitely many variables:

$$\theta_{\alpha,m} \;=\; \frac{\partial^2 \mathcal{F}_0(\mathbf{t})}{\partial t^{\alpha,m}\partial t^{1,0}}\,, \tag{14}$$

where $\mathbf{t} = (t^{\alpha,q})_{\alpha=1,\dots,4,\,q\ge 0}$ and the definition of $\mathcal{F}_0$ can be found in (33). Denote $v_\alpha = \theta_{\alpha,0}$, $\alpha = 1, \dots, 4$, and denote $v = (v_1, \dots, v_4)$. Then from the theory of Frobenius manifolds (see Appendix B) we know that all the $\theta_{\alpha,m}$ are *polynomials in* v, with the first values being

$$\begin{aligned}
\theta_{\alpha,0} &= v_\alpha\,,\\
\theta_{1,1} &= v_1 v_4 + v_2 v_3\,,\\
\theta_{2,1} &= \frac{v_3^2}{2} + \frac{1}{10} v_1^2 v_3 + v_4 v_2 + \frac{1}{10} v_2^2 v_1\,,\\
\theta_{3,1} &= \frac{v_2^3}{15} + \frac{1}{75} v_1^3 v_2 + \frac{1}{5} v_3 v_1 v_2 + v_4 v_3\,,\\
\theta_{4,1} &= \frac{v_1^5}{2500} + \frac{1}{50} v_2^2 v_1^2 + \frac{1}{10} v_3^2 v_1 + \frac{v_4^2}{2} + \frac{1}{10} v_3 v_2^2\,,\\
\theta_{1,2} &= \frac{v_1^6}{3750} + \frac{1}{50} v_2^2 v_1^3 + \frac{1}{10} v_3^2 v_1^2 + \frac{1}{2} v_4^2 v_1 + \frac{1}{5} v_3 v_2^2 v_3 + \frac{v_2^4}{30} + \frac{v_3^3}{6} + v_4 v_3 v_2\,.
\end{aligned}$$

Theorem 3. *Define*

$$v_1^* = \frac{1}{6}\,, \quad v_2^* = 0\,, \quad v_3^* = \frac{11}{3600}\,, \quad v_4^* = 0\,.$$

Then for all $g \geq 1$ with $g \not\equiv 3 \,(\mathrm{mod}\, 5)$, we have

$$\tau_g \;=\; \theta_{\alpha,m}(v^*)\,, \tag{15}$$

where $\alpha \in \{1,2,3,4\}$ and $m \geq 0$ are such that $2g-1 = \alpha + 5m$.

Organization of the paper. In Section 2, we recall the definition of $\tau_{\mathfrak{g}}(g)$ for all $\mathfrak{g}$. In Section 3 we generalize parts (i), (ii) of Theorem 2 from A_4 to an arbitrary $\mathfrak{g}$. In Section 4, we provide several technical preparations for the subsequent sections. In Section 5 we prove the generalization of the rest of Theorem 2 for the A series, as well as proving Theorem 1. The analogues of Theorem 2 for the D_l and E_6 cases are given in Sections 6 and 7, respectively, while Theorem 3 is generalized to the A_l, D_l and E_6 cases in Section 8. The necessary material on the wave-function-pair approach for computing residues of pseudodifferential operators and a brief review of the theory of Frobenius manifolds are provided in Appendices A and B.

Note on authorship. In answer to questions that we have been asked, we would like to make the following statement to clarify the role of the first author, since for various reasons (several other joint projects, the time spent to settle the E_6 case, and our efforts to resolve the questions about period functions posed in Section 1) this paper was only finalized nearly two years after his death. The project started in Trieste when B.D. and D.Y. were members of SISSA and D.Z. was a senior visiting scientific member of ICTP. It originated from a question raised by D.Z. during a talk given by B.D. at ICTP on topological ODEs [12] in November 2015, concerning the integrality of the invariants in the 5-spin case after multiplication by suitable Pochhammer symbols. This question was answered affirmatively within the next few months in two completely different ways, by D.Y. and D.Z. by proving the integrality (away from 30) of the numbers denoted c_g in Section 1, and then by B.D. and D.Y. by proving the algebraicity of the generating function of the differently normalized numbers a_g. Both results were reported by D.Z. in [118], but it was only the second discovery, on the algebraicity of the generating series $\sum a_g x^g$, that generalized to all ADE cases (except possibly E_7 and E_8) and became the central result of the current paper. During 2015–2018 all three authors communicated frequently by email and also met in person several times in Trieste to work on the project. The general algebraicity conjecture for all ADE cases was made in 2017, and the precise form of the current Theorem 8 for all ADE cases was stated as a conjecture in an early draft of the paper (March 2018) signed by all three of us. These conjectures were then proved for the A cases using the method sketched in Remark 7 in Section 5, and also for D_4, D_5, D_6 using topological ODEs. Then in May, B.D. suggested the idea, motivated by his explicit computation for the D_4 case, of proving the conjectures by using wave functions rather than topological ODEs, and in the subsequent two months D.Y. and D.Z. worked out the details of this proof for all A- and D-cases and sent it by email to Boris. In July, D.Y. again visited B.D. in Trieste and together they checked all details of this complete proof. A draft of the paper was written out at the end of July 2018 and contained all parts, including most of the appendices, of the current paper except for the details of the proof of Theorem 4, the precise formula and proof of the asymptotics (parts (vii)) in Theorems 5, 6, and the proof of algebraicity for the E_6 case, which was found (now again using topological ODEs) by D.Y. and D.Z. only after Boris passed away. The second and third authors also wrote a new introduction and edited and polished many parts of the remaining text. Thus all of the present manuscript except for the proofs of Theorem 4 and of the E_6 case and the asymptotics statements in Theorems 5 and 6 was joint work with B.D. and was seen and approved by him. The arXiv submission of the paper [53], made while Boris was still alive, cited both the present paper and its planned second part, but only a small part of that paper was finished and it is now planned as a publication of only the two other authors [117], with a statement there specifying his role.

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2. Cohomological field theories and FJRW invariants of $\mathfrak{g}$-type

Let $\mathfrak{g}$ be a simply-laced simple Lie algebra. In this section, we review the definitions of the enumerative invariants $\tau_{\mathfrak{g}}(g)$, called the one-point FJRW (Fan–Jarvis–Ruan, Witten) invariants, that are studied in this paper.

We start with the definitions for the A case. For $g, n \geq 0$, let $1 \leq \alpha_1, \ldots, \alpha_n \leq r$ be integers satisfying the divisibility condition $r \,\big|\, \sum_{i=1}^n \alpha_i - n - (2g-2)$. For an algebraic curve C of genus g with n distinct marked points $x_1, \ldots, x_n$, there exists a line bundle $\mathcal{T}$ on C such that

$$\mathcal{T}^{\otimes r} \;=\; K_C \otimes \mathcal{O}\big((1-\alpha_1)\, x_1\big) \otimes \cdots \otimes \mathcal{O}\big((1-\alpha_n)\, x_n\big)\,, \tag{16}$$

where K_C is the canonical class of C. For C smooth there are r^{2g} such line bundles. A choice of such an "r-th root" of the bundle (16) is called an r-spin structure, and it defines a point in a covering of $\overline{\mathcal{M}}_{g,n}$. After a suitable compactification, this covering is denoted by

$$p : \overline{\mathcal{M}}^{1/r}_{g,n}(\alpha_1, \ldots, \alpha_n) \to \overline{\mathcal{M}}_{g,n}\,.$$

In genus zero, for a point $\big(C, x_1, \ldots, x_n, \mathcal{T}\big)$ in the covering space $\overline{\mathcal{M}}^{1/r}_{0,n}(\alpha_1, \ldots, \alpha_n)$, denote $V = H^1(C, \mathcal{T})$. This gives a vector bundle $\mathcal{V} \to \overline{\mathcal{M}}^{1/r}_{0,n}(\alpha_1, \ldots, \alpha_n)$ as the space V has constant dimension thanks to the fact that $H^0(C, \mathcal{T})$ vanishes. Put

$$c_0(\alpha_1, \ldots, \alpha_n) \;:=\; p_*\, e\big(\mathcal{V}^\vee\big) \;\in\; H^{2(s-1)}\big(\overline{\mathcal{M}}_{0,n}\big)\,, \tag{17}$$

where $e(\mathcal{V}^\vee)$ is the Euler class of the dual vector bundle $\mathcal{V}^\vee$, and $s := \frac{\sum_{i=1}^n \alpha_i - n + 2}{r}$. The cohomology class $c_0(\alpha_1, \ldots, \alpha_n)$ is called the *Witten class*. For higher genus, $H^0(C, \mathcal{T})$ is only generically zero and the vector bundle can only be defined on a generic stratum. The Witten class $c_g(\alpha_1, \ldots, \alpha_n)$ could still be defined as a particular cohomology class in $H^{2(s+g-1)}(\overline{\mathcal{M}}_{g,n})$ with $s = \frac{\sum_{i=1}^n \alpha_i - n - (2g-2)}{r}$, but the construction is more involved (cf. [26, 27, 28, 65, 66, 76, 97, 102, 103, 114]). The genus g r-spin intersection numbers are defined as the following integrals:

$$\int_{\overline{\mathcal{M}}_{g,n}} c_g(\alpha_1, \ldots, \alpha_n)\, \psi_1^{q_1} \cdots \psi_n^{q_n} \;=:\; \langle \tau_{\alpha_1, q_1} \cdots \tau_{\alpha_n, q_n} \rangle_g\,, \qquad q_1, \ldots, q_n \geq 0\,, \tag{18}$$

where ψ_i $(1 \leq i \leq n)$ denotes the first Chern class of the ith tautological line bundle over $\overline{\mathcal{M}}_{g,n}$. These integrals vanish unless the degree and the dimension match:

$$s + g - 1 + q_1 + \cdots + q_n \;=\; 3g - 3 + n\,.$$

The so-called Vanishing Axiom, conjectured in [76] and proved in [102, 103], says that the Witten class $c_g(\alpha_1, \ldots, \alpha_n)$ vanishes if any of $\alpha_1, \ldots, \alpha_n$ reaches r. Therefore we assume that $\alpha_1, \ldots, \alpha_n$

are in $\{1,\dots,r-1\}$. The numbers $\tau_{A_{r-1}}(g)$ that we are looking at are the r-spin intersection numbers with $n=1$ (one-point). More precisely, they are defined by

$$\tau_{A_{r-1}}(g) \;:=\; \langle \tau_{\alpha,q}\rangle_g\,, \tag{19}$$

where $\alpha \in \{1,\dots,r-1\}$ and $q\geq 0$ are uniquely determined by

$$2\,(r+1)\,g \,-\, 1 \;=\; \alpha \,+\, r\,(q+1)\,. \tag{20}$$

The Witten class gives a particular *cohomological field theory (CohFT)* [84, 94] of rank $(r-1)$. Let us recall the definition of a general CohFT. A rank l CohFT is a quadruple $\big(V^l, \langle,\rangle, \mathbb{1}, \Omega_{g,n}\big)$, where V is a $\mathbb{C}$-vector space, $\langle,\rangle$ is a symmetric non-degenerate bilinear form, $\mathbb{1}$ is a particular element in V, called the unity, and $\{\Omega_{g,n}\}_{2g-2+n>0}$ is a collection of linear maps from $V^{\otimes n}$ to $H^{\rm even}\big(\overline{\mathcal{M}}_{g,n};\mathbb{C}\big)$, satisfying the following axioms:
C1. (total symmetry) Each $\Omega_{g,n}$ is S_n-invariant, where the action of S_n permutes both the marked points of $\overline{\mathcal{M}}_{g,n}$ and the tensor products $V^{\otimes n}$.
C2. (splitting) Let e_α be a basis of V. Denote $\eta_{\alpha\beta} := \langle e_\alpha, e_\beta\rangle$, $\eta = (\eta_{\alpha\beta})$, $\big(\eta^{\alpha\beta}\big) := \eta^{-1}$, and denote by $q_{g,n}$ and s_{g_1,n_1,g_2,n_2} the gluing maps

$$q_{g,n} : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}\,, \tag{21}$$

$$s_{g_1,n_1,g_2,n_2} : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g=g_1+g_2,n=n_1+n_2}\,. \tag{22}$$

Then it is required that $\forall\; x_1,\dots,x_n \in V$,

$$q_{g,n}^*\Omega_{g,n}(x_1,\dots,x_n) \;=\; \Omega_{g-1,n+2}(x_1,\dots,x_n,e_\alpha,e_\beta)\,\eta^{\alpha\beta}\,, \tag{23}$$

$$s_{g_1,n_1,g_2,n_2}^*\Omega_{g,n}(x_1,\dots,x_n) \;=\; \Omega_{g_1,n_1+1}(x_1,\dots,x_{n_1},e_\alpha)\,\eta^{\alpha\beta}\,\Omega_{g_2,n_2+1}(e_\beta,x_{n_1+1},\dots,x_n)\,. \tag{24}$$

Here and below, free Greek indices take the integer values from 1 to l, and the Einstein summation convention is used for repeated Greek indices with one up and one down, and the tensors $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$ will be used to raise and lower the Greek indices.
C3. (unity) Let $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. Then $\forall\; x_1,\dots,x_n \in V$,

$$\Omega_{g,n+1}(x_1,\dots,x_n,\mathbb{1}) \;=\; p^*\Omega_{g,n}(x_1,\dots,x_n)\,, \tag{25}$$

$$\Omega_{0,3}(x_1,x_2,\mathbb{1}) \;=\; \eta(x_1,x_2)\,. \tag{26}$$

Take e_α a basis of V with $e_1 = \mathbb{1}$. It is natural to view V as the complex coordinate space $\{(v^1,\dots,v^l) \mid v^\alpha \in \mathbb{C}\}$ and therefore as the complex manifold $\mathbb{C}^l$. Define a power series $F = F(v) \in \mathbb{C}[[v^1,\dots,v^l]]$ by

$$F \;:=\; \sum_{n\geq 3} \int_{\overline{\mathcal{M}}_{0,n}} \Omega_{0,n}(e_{\alpha_1},\dots,e_{\alpha_n})\,\frac{v^{\alpha_1}\cdots v^{\alpha_n}}{n!}\,. \tag{27}$$

We call F the *genus-zero primary potential* (cf. [43, 56, 84, 94, 99, 100, 111]). Denote by $B\subset\mathbb{C}^l$ the domain of convergence for F around $v=0$. Throughout the paper we assume that B contains an open ball centered at $v=0$.

Remark 5. For a projective variety X, the Gromov–Witten classes associated to X give rise to a CohFT. For this case, it is sometimes helpful or even necessary to replace the ring $H^{\rm even}\big(\overline{\mathcal{M}}_{g,n};\mathbb{C}\big)$ with $\mathcal{N}\otimes H^{\rm even}\big(\overline{\mathcal{M}}_{g,n};\mathbb{C}\big)$, where $\mathcal{N}$ is the Novikov ring [84, 94]. We did not write the axioms involving the Novikov ring because for the specific CohFTs that are considered in this paper the power series F is actually a polynomial and so $B=\mathbb{C}^l$. Nevertheless, it will be interesting to generalize the results of this paper to the situation when the Novikov ring is introduced.

A CohFT $\big(V,\langle,\rangle,\mathbb{1},\Omega_{g,n}\big)$ is called *homogeneous* of *charge* d if it satisfies the following axiom:
C4. There is a vector field E on B which for some choice of the basis e_α of V has the form

$$E \;=\; \big(1-\tfrac{d}{2}\big)\,v^\alpha\,\partial_\alpha \,+\, r^\alpha\,\partial_\alpha \,-\, \sum_{\alpha=1}^{l}\mu_\alpha\,v^\alpha\,\partial_\alpha\,, \tag{28}$$

and such that if we define the action of E on Ω by

$$\begin{aligned}(E\Omega)_{g,n}(e_{\alpha_1},\dots,e_{\alpha_n}) \;:=\; &\Big(\mathrm{gr}+\tfrac{2-d}{2}n-\sum_{i=1}^{n}\mu_{\alpha_i}\Big)\,\Omega_{g,n}(e_{\alpha_1},\dots,e_{\alpha_n})\\ &+\,p_*\,\Omega_{g,n+1}\big(e_{\alpha_1},\dots,e_{\alpha_n},\,r^\alpha e_\alpha\big)\,,\end{aligned} \tag{29}$$

where p denotes the forgetful map, and gr is the grading operator defined by

$$\mathrm{gr}\,\phi \;:=\; q\,\phi\,,\quad \text{if } \phi\in H^{2q}(\overline{\mathcal{M}}_{g,n};\mathbb{C})\,, \tag{30}$$

then

$$(E\Omega)_{g,n} \;=\; \big((g-1)\,d+n\big)\,\Omega_{g,n}\,,\quad \forall\; 2g-2+n>0\,. \tag{31}$$

The genus g correlators of the CohFT are defined by

$$\int_{\overline{\mathcal{M}}_{g,n}}\Omega_{g,n}(e_{\alpha_1},\dots,e_{\alpha_n})\,\psi_1^{q_1}\cdots\psi_n^{q_n} \;=:\; \langle\tau_{\alpha_1,q_1}\cdots\tau_{\alpha_n,q_n}\rangle_g^{\Omega}\,,\qquad q_1,\dots,q_n\ge 0\,. \tag{32}$$

Below, the label Ω will often be omitted. The genus g free energy of the CohFT is defined by

$$\mathcal{F}_g(\mathbf{t}) \;:=\; \sum_{n\ge0}\sum_{q_1,\dots,q_n\ge0}\frac{t^{\alpha_1,q_1}\cdots t^{\alpha_n,q_n}}{n!}\,\langle\tau_{\alpha_1,q_1}\cdots\tau_{\alpha_n,q_n}\rangle_g\,, \tag{33}$$

where $\mathbf{t}:=(t^{\alpha,q})_{\alpha=1,\dots,l,q\ge0}$ denotes the infinite vector of indeterminates. The exponential

$$e^{\sum_{g\ge0}\epsilon^{2g-2}\mathcal{F}_g(\mathbf{t})} \;=:\; Z \tag{34}$$

is called the *partition function* of the CohFT, where ϵ is an indeterminate. It satisfies the following string equation

$$\sum_{q\ge1}t^{\alpha,q}\frac{\partial Z}{\partial t^{\alpha,q-1}} \,+\, \frac{1}{2\epsilon^2}\eta_{\alpha\beta}t^{\alpha,0}t^{\beta,0}Z \;=\; \frac{\partial Z}{\partial t^{1,0}}\,. \tag{35}$$

An immediate consequence of the string equation is that

$$\langle\tau_{\alpha,q-1}\rangle_g \;=\; \langle\tau_{\alpha,q}\tau_{1,0}\rangle_g\,. \tag{36}$$

The CohFT $\Omega_{g,n}$ given by the Witten class $c_g(\alpha_1,\dots,\alpha_n)$ is also recognized as the FJRW CohFT of A_{r-1}-type. This is the reason for the notation used in (19). For each simply-laced simple Lie algebra $\mathfrak{g}$ of rank l, Fan, Jarvis and Ruan [65, 66] constructed a rank l homogeneous CohFT from a certain simple singularity such that the corresponding Frobenius manifold is isomorphic to the Frobenius manifold associated to the Weyl group of $\mathfrak{g}$ [45, 125]. This CohFT will be referred to as the FJRW CohFT of $\mathfrak{g}$-type, whose partition function is shown to be a particular tau-function for the DS hierarchy of $\mathfrak{g}$-type [38, 66, 112, 114] (cf. [49, 64, 69, 72, 73, 79, 83, 90, 115]. In [90] Liu, Ruan and Zhang introduced the notion of a *partial CohFT*, and constructed certain partial CohFTs associated to simple singularities, whose partition functions are proved *ibid.* to be tau-functions for the DS hierarchies of BCFG-type. Correlators of these CohFTs or partial CohFTs are referred to as the FJRW–LRZ invariants. The main focus of this paper will be on the ADE cases, leaving the more detailed studies of the BCFG cases to future publications. It should be noted that the terminology "DS hierarchy of $\mathfrak{g}$-type" refers to the DS hierarchy, under the choice of a principal nilpotent element, associated to the untwisted affine Kac–Moody algebra $\hat{\mathfrak{g}}^{(1)}$ (see page 1402 of [12] for more details and see [13]).

Now let $\mathfrak{g}$ be a *simply-laced* simple Lie algebra of rank l with the normalized Cartan–Killing form $(\cdot|\cdot)$. Denote by r the Coxeter number of $\mathfrak{g}$, and by $m_1, \dots, m_l$ the exponents of $\mathfrak{g}$. Here we order $m_1, \dots, m_l$ such that $1 = m_1 \leq m_2 \leq \cdots \leq m_l = r-1$, except for the D_l case, where we set $m_\alpha = 2\alpha - 1$ for $\alpha = 1, \dots, l-1$ and $m_l = l-1$. (Observe the difference with [12, 13], where for the D_l case the exponents are numbered by $1 = m_1 \leq m_2 \leq \cdots \leq m_l = r-1$.) Let $\big(V^l, \langle,\rangle, \mathbb{1}, \Omega_{g,n}\big)$ denote the FJRW CohFT of $\mathfrak{g}$-type. Choose a basis e_α of V satisfying

$$\Omega_{g,n}(e_{\alpha_1}, \dots, e_{\alpha_n}) \,\in\, H^{2(s+g-1)}\big(\overline{\mathcal{M}}_{g,n}\big)\,, \quad s \,:=\, \frac{\sum_{i=1}^{l} m_{\alpha_i} - n - (2g-2)}{r}\,. \tag{37}$$

The existence of such a choice can be verified case by case from the data provided in [66], and this implies that the CohFT $\Omega_{g,n}$ is *homogeneous* of charge $d = \frac{r-2}{r}$. The associated correlators (32) are called *genus g FJRW invariants of $\mathfrak{g}$-type.* The degree-dimension matching implies that $\langle \tau_{\alpha_1,q_1} \cdots \tau_{\alpha_n,q_n} \rangle_g$ vanishes unless

$$\frac{\sum_{i=1}^{n} m_{\alpha_i} - n - (2g-2)}{r} \,+\, g - 1 \,+\, \sum_{i=1}^{n} q_i \;=\; 3g \,-\, 3 \,+\, n\,. \tag{38}$$

For given (α_i, q_i), $i = 1, \cdots, n$, for simplicity we sometimes omit the subindex g in $\langle \cdots \rangle_g$, because for all possibly non-zero invariants this subindex can be reconstructed by (38); in such an omission, if the reconstructed g is not an integer, $\langle \tau_{\alpha_1,q_1} \cdots \tau_{\alpha_n,q_n} \rangle$ is defined as 0. It is also convenient to take $\epsilon = 1$ in (34) for the definition of the partition Z, i.e., we have

$$Z \;=\; Z(\mathbf{t}) \;:=\; e^{\sum_{g\geq 0} \mathcal{F}_g(\mathbf{t})}\,, \tag{39}$$

and the string equation reads

$$\sum_{q\geq 1} t^{\alpha,q}\, \frac{\partial Z(\mathbf{t})}{\partial t^{\alpha,q-1}} \,+\, \frac{1}{2}\, \eta_{\alpha\beta}\, t^{\alpha,0}\, t^{\beta,0}\, Z(\mathbf{t}) \;=\; \frac{\partial Z(\mathbf{t})}{\partial t^{1,0}}\,. \tag{40}$$

The numbers $\tau_{\mathfrak{g}}(g)$ that we are studying in this paper are defined by

$$\tau_{\mathfrak{g}}(g) \;:=\; \langle \tau_{\alpha,q} \rangle\,, \quad g \geq 0\,, \tag{41}$$

where

$$2\,(r+1)\,g \,-\, 1 \;=\; m_\alpha \,+\, r\,(q+1)\,. \tag{42}$$

It should be noticed that for the case $\mathfrak{g} = D_l$ with l being an even number, there are two equal exponents $m_{l/2}$ and m_l, so for this case, the different $(\alpha = l/2, q)$ and $(\alpha = l, q)$ correspond to the same g. However, with an appropriate choice of the basis, the numbers $\langle \tau_{l,q} \rangle$ vanish, and we use $\tau_{\mathfrak{g}}(g)$ to denote $\langle \tau_{l/2,q} \rangle$.

3. Differential equation

The topological and dual topological ODEs of $\mathfrak{g}$-type are introduced in [12] for computing the FJRW–LRZ invariants, which are obtained as a result of the theorems of Fan–Jarvis–Ruan and Liu–Ruan–Zhang together with an application of the matrix-resolvent method [11, 12, 13] for the Drinfeld–Sokolov hierarchy of $\mathfrak{g}$-type [38]. Differential equations similar to the topological ODE of $\mathfrak{g}$-type were also considered from other perspectives; see for examples [67, 68, 95]. In this section, via reducing the dual topological ODE to a scalar differential equation, we generalize parts (i) and (ii) of Theorem 2 of the Introduction from A_4 to an arbitrary simple Lie algebra $\mathfrak{g}$.

Fix $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and let $\triangle$ be the root system. Choose a set of simple roots Π, and let $E_1, \dots, E_l, F_1, \dots, F_l$ be the Weyl generators. Denote by θ the highest root with respect to Π, and by $E_{-\theta}$ a root vector associated to $-\theta$. The semisimple element $\Lambda = \Lambda(\lambda)$ (aka the Kostant element) is defined by $\Lambda(\lambda) = \sum_{\alpha=1}^{l} E_\alpha + \lambda\, E_{-\theta}$, $\lambda \in \mathbb{C}$. Kostant [85] shows that for any $\lambda \neq 0$, $\mathfrak{g}$ has the orthogonal decomposition with respect to $(\cdot|\cdot)$, i.e. $\mathfrak{g} = \operatorname{Ker}\operatorname{ad}_{\Lambda(\lambda)} \oplus \operatorname{Im}\operatorname{ad}_{\Lambda(\lambda)}$, $\operatorname{Ker}\operatorname{ad}_{\Lambda(\lambda)} \perp \operatorname{Im}\operatorname{ad}_{\Lambda(\lambda)}$. Denote by $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}\big[\lambda, \lambda^{-1}\big]$ the loop algebra, and by $\rho^\vee$ the Weyl co-vector (which is the unique element in $\mathfrak{h}$ satisfying $[\rho^\vee, E_\alpha] = E_\alpha$). Introduce a grading operator on $L(\mathfrak{g})$

$$\mathrm{gr} \;:=\; \mathrm{ad}_{\rho^\vee} \,+\, r\,\lambda\,\frac{d}{d\lambda}\,. \tag{43}$$

An element q in $L(\mathfrak{g})$ is called homogeneous of principal degree k if $\mathrm{gr}\, q = k\, q$. It was proven by Kac [77] that the kernel of $\operatorname{ad}_{\Lambda(\lambda)}$ in $L(\mathfrak{g})$ has the form $\bigoplus_{j\in E} \mathbb{C}\Lambda_j(\lambda)$. Here, E is the exponent set of $L(\mathfrak{g})$, and $\Lambda_j(\lambda) \in L(\mathfrak{g})$ are homogeneous of principal degree j, normalized by $\Lambda_1 = \Lambda$, $\Lambda_{m_\alpha + \ell h}(\lambda) = \lambda_{m_\alpha}(\lambda)\,\lambda^\ell$, $\ell \geq 0$, and $\big(\Lambda_{m_\alpha}|\Lambda_{m_\beta}\big) = r\,\lambda\,\eta_{\alpha\beta}$ for some non-degenerate symmetric constant matrix η.

Write

$$\rho^\vee \;=\; \sum_{i=1}^{n} x_i\, H_i,\; x_i \in \mathbb{C}\,,$$

and define $I_- = 2\sum_{i=1}^{l} x_i\, F_i$. Then $I_+, I_-, \rho^\vee$ form an $sl_2(\mathbb{C})$ Lie algebra:

$$[\rho^\vee, I_+] = I_+\,, \quad [\rho^\vee, I_-] = -I_-\,, \quad [I_+, I_-] = 2\,\rho^\vee\,. \tag{44}$$

According to [7, 85], there exist elements $\gamma^1, \dots, \gamma^l \in \mathfrak{g}$ such that

$$\operatorname{Ker}\operatorname{ad}_{I_-} = \operatorname{Span}_{\mathbb{C}}\big\{\gamma^1, \dots, \gamma^l\big\}\,, \qquad [\rho^\vee, \gamma^i] \;=\; -m_i\,\gamma^i\,. \tag{45}$$

Fix $\big\{\gamma^1, \dots, \gamma^l\big\}$, then the lowest weight decomposition of $\mathfrak{g}$ has the form

$$\mathfrak{g} \;=\; \bigoplus_{i=1}^{l} \mathcal{L}^i\,, \quad \mathcal{L}^i \;=\; \operatorname{Span}_{\mathbb{C}}\big\{\gamma^i, \operatorname{ad}_{I_+}\gamma^i, \dots, \operatorname{ad}_{I_+}^{2m_i}\gamma^i\big\}\,. \tag{46}$$

Here each $\mathcal{L}^i$ is an $sl_2(\mathbb{C})$-module. Any $\mathfrak{g}$-valued function $M(\lambda)$ can be uniquely represented as

$$M(\lambda) = \sum_{i=1}^{l} S_i(\lambda)\, \operatorname{ad}_{I_+}^{2m_i}\gamma^i + \sum_{i=1}^{l}\sum_{m=0}^{2m_i-1} K_{im}(\lambda)\, \operatorname{ad}_{I_+}^{m}\gamma^i\,, \tag{47}$$

where $S_i(\lambda)$, $K_{im}(\lambda)$ are certain complex-valued functions. Note that $\operatorname{ad}_{I_+}^{2m_i}\gamma^i$ is the highest weight vector of $\mathcal{L}^i$, $i = 1, \dots, l$.

The dual topological ODE of $\mathfrak{g}$-type [12] is an ODE for a $\mathfrak{g}$-valued function G defined by

$$\left[\frac{dG}{dx}, E_{-\theta}\right] \,+\, \big[G, I_+\big] \,+\, x\,G \;=\; 0\,. \tag{48}$$

Write

$$G(x) \;=\; \sum_{\alpha=1}^{l} \phi_\alpha(x)\, \operatorname{ad}_{I_+}^{2m_\alpha}\gamma^\alpha \,+\, \sum_{\alpha=1}^{l}\sum_{m=0}^{2m_\alpha-1} \widetilde{K}_{\alpha m}(x)\, \operatorname{ad}_{I_+}^{m}\gamma^\alpha\,. \tag{49}$$

It is shown in [12] that the ODE (48) is equivalent to an ODE for $\phi = (\phi_1, \dots, \phi_l)^T$

$$\frac{d\phi}{dx} \;=\; \sum_{i=-1}^{2r-2} x^i\, V_i\, \phi\,, \tag{50}$$

where V_i $(i=-1,\dots,2r-2)$ are constant $l\times l$ matrices and $V_{-1}=\operatorname{diag}(-m_{l+1-\alpha}/r)_{\alpha=1,\dots,l}$. (Equivalence between systems of ODEs means that their solutions have a one-to-one correspondence.) Here we note that $x=0$ is a Fuchsian singular point of (50) and that the matrices V_{-1}, $\dots$, V_{2h-2} are determined by the lowest-weight-structure-constants of $\mathfrak{g}$. It is obvious from (50) that the dimension of the space of solutions to (48) is equal to the rank of $\mathfrak{g}$.

Denote by $(G_\alpha)_{\alpha=1,\dots,l}$ a solution basis of (48), and by $\phi_{\alpha;1},\dots,\phi_{\alpha;l}$ the ϕ-coefficients of G_α. The matrix Φ defined by $\Phi_{\beta\alpha}:=\phi_{\alpha;\beta}$ is called the fundamental solution matrix, which can be normalized by using the following initial condition near $x=0$:

$$\Phi \;=\; D\Big(I+\sum_{m\geq 1}\Phi_m\,x^m\Big)\,,\qquad D\;=\;\operatorname{diag}\Big(x^{-\frac{m_{l+1-\alpha}}{r}}\Big)_{\alpha=1,\dots,l}\,. \tag{51}$$

For each fixed α we allow G_α to have a multiplicative non-zero constant and keep using the notations G_α and $\phi_{\alpha;\beta}$. The following proposition is proved in [12], and we review its proof here.

Proposition 1. *The series $\phi_{\alpha;l}$ can be expressed in terms of the FJRW invariants of $\mathfrak{g}$-type by*

$$\phi_{\alpha;l}(x)\;=\;\frac{x^{-\frac{1}{r}}}{\Gamma(\frac{m_\alpha}{r})}\,\delta_{\alpha,l}\;+\;\frac{x^{\frac{m_\alpha}{r}}}{\Gamma(\frac{m_\alpha}{r})}\sum_{w\geq 0}c_{\alpha,w}\,x^{m_\alpha+(r+1)w+1}\,, \tag{52}$$

where

$$c_{\alpha,w}\;=\;(-1)^{m_\alpha+(r+1)w}\,\big\langle\tau_{\alpha,m_\alpha+(r+1)w}\big\rangle\,(-r)^{\frac{m_\alpha+1+rw}{2}}\,,\quad w\geq 0\,. \tag{53}$$

Here we note that when $\mathfrak{g}$ is a non-simply-laced simple Lie algebra, the $\langle\tau_{\alpha,m_\alpha+(r+1)w}\rangle$ in the right-hand side of (53) should be considered as the one-point correlators of the corresponding Liu–Ruan–Zhang partial CohFT (cf. [12, 13, 90]).

Proof of Proposition 1. Recall that the topological ODE of $\mathfrak{g}$-type [12] is the $\mathfrak{g}$-valued ODE

$$\frac{dM(\lambda)}{d\lambda}\;=\;\big[M(\lambda),\Lambda(\lambda)\big]\,. \tag{54}$$

About this ODE, the following statements are proved in [12, 13]:
(a) The dimension of the formal Puiseux series solutions to (54) is equal to the rank of $\mathfrak{g}$.
(b) There exists a unique basis $M_1,\dots,M_l$ of the formal solutions to (54) such that

$$M_\alpha(\lambda)\;=\;\lambda^{-\frac{m_\alpha}{r}}\Big[\Lambda_{m_\alpha}(\lambda)\;+\;\sum_{k\geq 1}M_{\alpha,k}(\lambda)\Big]\,, \tag{55}$$

$$M_{\alpha,k}(\lambda)\;\in\;L(\mathfrak{g})\,,\quad \operatorname{gr}M_{\alpha,k}(\lambda)\;=\;\big[m_\alpha-(r+1)k\big]\,M_{\alpha,k}(\lambda)\,. \tag{56}$$

(c) Denote $\kappa=(\sqrt{-r})^{-r}$. The following identity for one-point FJRW invariants of $\mathfrak{g}$-type is true:

$$\kappa^{\frac{2}{r+1}}\sqrt{-r}\sum_{g,q\geq 0}(-1)^q\,\frac{(\frac{m_\alpha}{r})_{q+2}}{(\kappa^{\frac{1}{r+1}}\lambda)^{\frac{m_\alpha}{r}+q+2}}\,\langle\tau_{\alpha,q}\rangle_g\;=\;\big(E_{-\theta}|M_\alpha(\lambda)\big)\;-\;\lambda^{-\frac{r-1}{r}}\,\delta_{\alpha,l}\,. \tag{57}$$

Write

$$\big(E_{-\theta}|M_\alpha(\lambda)\big)\;=\;\lambda^{-\frac{m_\alpha}{r}}\sum_{q\geq 0}S_{\alpha,q}\,\lambda^{-q}\;=:\;S_\alpha(\lambda)\,,\qquad S_{\alpha,q}\in\mathbb{Q}\,. \tag{58}$$

We have for $w\in\mathbb{Z}$ and $(1+m_\alpha+rw)/2\in\mathbb{Z}_{\geq 0}$,

$$\big\langle\tau_{\alpha,m_\alpha+(r+1)w}\big\rangle\;=\;(-1)^{m_\alpha+(r+1)w}\,\frac{S_{\alpha,m_\alpha+(r+1)w+2}}{(-r)^{\frac{m_\alpha+1+rw}{2}}\,\big(\frac{m_\alpha}{r}\big)_{m_\alpha+(r+1)w+2}}\,. \tag{59}$$

According to the definition given in [12], solutions to topological and dual topological ODEs are related via the Laplace transform $M(\lambda)=\int G(x)\,e^{-\lambda x}\,dx$. In particular, the series $\phi_{\alpha;l}$ are related to S_α by

$$S_\alpha(\lambda) \;=\; \int_C \phi_{\alpha;l}(x)\,e^{-\lambda x}\,dx\,, \tag{60}$$

where C is a carefully chosen contour (which can depend on α) on the x-plane. We have

$$\phi_{\alpha;l}(x) \;=\; \sum_{q\geq 0} S_{\alpha,q}\,\frac{x^{q+\frac{m_\alpha}{r}-1}}{\Gamma(q+\frac{m_\alpha}{r})} \;=\; \frac{x^{\frac{m_\alpha}{r}-1}}{\Gamma(\frac{m_\alpha}{r})}\sum_{q\geq 0} S_{\alpha,q}\,\frac{x^q}{\left(\frac{m_\alpha}{r}\right)_q}\,. \tag{61}$$

The proposition is proved. □

An immediate consequence of Proposition 1 is that the intersections numbers $\tau_{\mathfrak{g}}(g)$ grow at most exponentially. Let us introduce the series $a(x)$ by

$$a(x) \;:=\; (-r)^{-\frac{1}{2(r+1)}}\sum_{\alpha=1}^{l}\Gamma\Big(\frac{m_\alpha}{r}\Big)\,(-1)^{\frac{m_\alpha-r}{r}}\,\phi_{\alpha;l}(x)\,, \tag{62}$$

where $\phi_{\alpha;l}(x)$ are given by (52). More explicitly,

$$\begin{aligned} a(x) \;&=\; (-r)^{-\frac{1}{2(r+1)}}\,(-x)^{-\frac{1}{r}} \;+\; \sum_{\alpha=1}^{l}\sum_{w\geq 0}(-1)^{m_\alpha+(r+1)w+\frac{m_\alpha-r}{r}} \\ &\qquad\qquad \langle\tau_{\alpha,m_\alpha+(r+1)w}\rangle\,(-r)^{\frac{m_\alpha+1+rw}{2}}\,x^{\frac{(r+1)m_\alpha+r(r+1)w+r}{r}} \\ &=\; \Big(-(-r)^{\frac{r}{2(r+1)}}\,x\Big)^{-\frac{1}{r}} \;+\; \Big(-(-r)^{\frac{r}{2(r+1)}}\,x\Big)^{-\frac{1}{r}}\sum_{g\geq 0}\tau_{\mathfrak{g}}(g)\,\Big(-(-r)^{\frac{r}{2(r+1)}}\,x\Big)^{2g+\frac{2g}{r}}\,. \end{aligned}$$

Denote $t=-(-r)^{\frac{r}{2(r+1)}}x$ and

$$b(t) \;=\; t^{-\frac{1}{r}} \;+\; t^{-\frac{1}{r}}\sum_{g\geq 1}\tau_{\mathfrak{g}}(g)\,t^{2g+\frac{2g}{r}} \;=\; \sum_{g\geq 0}\tau_{\mathfrak{g}}(g)\,t^{2g\frac{r+1}{r}-\frac{1}{r}}\,.$$

We have $a(x)=b(t)$. We are ready to generalize part (i) of Theorem 2 by showing that $b(t)$ satisfies an ODE with polynomial coefficients.

Theorem 4. *Let $\mathfrak{g}$ be an arbitrary simple Lie algebra of rank l. The component ϕ_l satisfies a linear ODE with polynomial coefficients of order at most l. In other words, for every fixed α, the series $\phi_{\alpha;l}$ satisfies this ODE. In particular, $b(t)$ satisfies an ODE of order at most l.*

Proof. [1] For $\mathfrak{g}$ with $l=1$, the statement is trivial. Below we consider $l\geq 2$. As we have mentioned, the dual topological ODE is equivalent to system (50). This system consists of l linear equations for $\phi_1,\dots,\phi_l$. Denote these equations by $e_1,\dots,e_l$. Let N be a positive integer. Now consider the following linear combination of equations

$$e \;:=\; \sum_{k=0}^{N} c_k^l\,\frac{d^k e_l}{dx^k} \;+\; \sum_{\alpha=1}^{l-1}\sum_{k=1}^{N} c_k^\alpha\,\frac{d^{k-1}e_\alpha}{dx^{k-1}}\,. \tag{63}$$

Requiring that the coefficients of $\phi_1,\dots,\phi_{l-1}$, $\frac{d\phi_1}{dx},\dots,\frac{d\phi_{l-1}}{dx},\,\dots,\,\frac{d^{N-1}\phi_1}{dx^{N-1}},\dots,\frac{d^N\phi_{l-1}}{dx^N}$ vanish gives rise to a system of linear equations for c_0^l and c_k^α, $k=1,\dots,N$. The number of equations is

[1] The proof of Theorem 4 was written after the first author B.D. passed away.

equal to $(l-1)(N+1)$, while the number of unknowns is $1+Nl$. Take $N=l-1$, we have

$$1+Nl\,-\,(l-1)(N+1)\;=\;2\,+\,N\,-\,l\;=\;1\,. \tag{64}$$

According to the Gauss elimination we know that this linear system over the field $\mathbb{C}(x)$ has a non-zero solution c_0^l and c_k^α, $k=1,\ldots,N$. The theorem is proved by further noticing that $\frac{d^k e_l}{dx^k}$ $(k=0,\ldots,N)$ and $\frac{d^{k-1}e_\alpha}{dx^{k-1}}$ $(\alpha=1,\ldots,l-1,\, k=1,\ldots,N)$ together are linearly independent. □

It is clear that part (i) of Theorem 2 is a special case of Theorem 4 with the particular form of the linear ODE (4) computed from the standard $sl_5(\mathbb{C})$ realization with the $\Lambda(\lambda)$ given by

$$\Lambda(\lambda)\;=\;\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\\lambda&0&0&0&0\end{pmatrix}\,, \tag{65}$$

and from the change of independent variable $t\mapsto X=t^{\frac{2(r+1)}{r}}$ and $\varphi(X)=b(t)\,t^{\frac{1}{r}}$.

4. Computing residues

In this section, we carry out several technical preparations for the later sections. It is very convenient here to permit the variable r, which originally came from r-spin classes and hence was a positive integer, to take on arbitrary complex values, since essentially all of the identities we need are polynomial in r (at least at the level of the individual coefficients of the generating functions appearing).

Let r be a complex number and let L be the pseudodifferential operator

$$L\;=\;\partial^r\,+\,C\,x\,, \tag{66}$$

where $\partial=d/dx$ and C is an arbitrarily given constant. We are to compute the residues $z_k(x)$ of the pseudodifferential operators $L^{k/r}$, i.e.

$$z_k(x)\;:=\;\operatorname{res}L^{\frac{k}{r}}\,. \tag{67}$$

We note that most of the time we consider k as a nonnegative integer, although sometimes we give results valid for any $k\in\mathbb{C}$. For the definition of a pseudodifferential operator and its residue we refer to [29, 34, 80].

4.1. Closed formula.

Let us first introduce some notations. Define polynomials $c_{p,j}(r)\in\mathbb{Q}[r]$ $(0\le p\le j)$ by the generating function

$$\frac{1}{p!}\left(\frac{(1+x)^{r+1}-1-(r+1)\,x}{(r+1)\,x}\right)^p\;=\;\sum_{j=p}^\infty c_{p,j}(r)\,x^j \tag{68}$$

(then $c_{p,j}(r)$ has degree j and is divisible by r^p), the first few values being

	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$j=0$	1					
$j=1$	0	$r/2$				
$j=2$	0	$\frac{1}{3}\binom{r}{2}$	$\frac{(r/2)^2}{2!}$			
$j=3$	0	$\frac{1}{4}\binom{r}{3}$	$\frac{r^2(r-1)}{12}$	$\frac{(r/2)^3}{3!}$		
$j=4$	0	$\frac{1}{5}\binom{r}{4}$	$\frac{r(5r-8)}{72}\binom{r}{2}$	$\frac{r^3(r-1)}{48}$	$\frac{(r/2)^4}{4!}$	
$j=5$	0	$\frac{1}{6}\binom{r}{5}$	$\frac{r(4r-7)}{60}\binom{r}{3}$	$\frac{r^3(r-1)(7r-10)}{576}$	$\frac{r^4(r-1)}{288}$	$\frac{(r/2)^5}{5!}$

The polynomials $c_{p,j}(r)$ $(0 \le p \le j \le 5)$

The result of this subsection is given by the following proposition.

Proposition 2. *Let $L = \partial^r + Cx$. For any $r, \lambda \in \mathbb{C}$, we have*

$$L^\lambda \;=\; \sum_{j,s\ge 0} C^{j+s}\, d_{j,s}(\lambda, r)\, x^s\, \partial_x^{r(\lambda-s)-(r+1)j}\,, \tag{69}$$

where $d_{j,s}(\lambda, r)$ is a polynomial in λ and r with rational coefficients, given explicitly by

$$d_{j,s}(\lambda, r) \;=\; \frac{1}{s!}\sum_{p=0}^{j} c_{p,j}(r)\,(\lambda)^-_{s+p+j}\,. \tag{70}$$

Here $(\lambda)^-_n := \prod_{m=1}^n(\lambda - m + 1)$ denotes the descending Pochhammer symbol.

Proof. By the rules for manipulating pseudodifferential operators [29, 34, 80], it is clear that L^λ has the form (69) for some polynomials $d_{j,s}(\lambda, r)$. It then suffices to prove (70) for $\lambda \in \mathbb{Z}_{\ge 0}$, which we do by induction, the case $\lambda = 0$ being trivial (both sides of (70) then reduce to $\delta_{j,0}\,\delta_{s,0}$). Expanding the identity $L^{\lambda+1} = L^\lambda L$, we find that $d_{j,s}$ satisfies the recursive relation

$$d_{j,s}(\lambda+1, r) \,-\, d_{j,s-1}(\lambda, r) \,-\, d_{j,s}(\lambda, r) \;=\; \bigl(r(\lambda - s) - (r+1)(j-1)\bigr)\, d_{j-1,s}(\lambda, r)\,.$$

If we multiply both sides of this by $s!$ and replace $d_{j,s}$ everywhere by the expression given in (74), then the left-hand side equals

$$\sum_{p\ge 0} c_{p,j}\,\bigl[(\lambda+1)^-_{s+p+j} - s\,(\lambda)^-_{s+p+j-1} - (\lambda)^-_{s+p+j}\bigr] \;=\; \sum_{p\ge 0}(p+j)\,c_{p,j}\,(\lambda)^-_{s+p+j-1}$$

and the right-hand side equals

$$\sum_{p\ge 0} c_{p,j-1}\,\bigl[\,r\,(\lambda-s)-(r+1)\,(j-1)\,\bigr]\,(\lambda)^-_{s+p+j-1} \;=\; \sum_{p\ge 0} c_{p,j-1}\,\bigl[\,r\,(\lambda)^-_{s+p+j}+(rp-j+1)\,(\lambda)^-_{s+p+j-1}\,\bigr]\,.$$

(Define $c_{p,j} = 0$ if $p > j$.) Comparing the coefficients of $(\lambda)^-_n$ on both sides, we see that the claim follows from the recursion $(p+j)\,c_{p,j} = (rp - j + 1)\,c_{p,j-1} + r\,c_{p-1,j-1}$, which follows easily by differentiating (68).

We observe that there is a slightly different proof, which does not depend on the polynomiality, obtained by using the identity $L\,L^\lambda = L^\lambda\,L$ instead of $L^\lambda L = L^{\lambda+1}$. Expanding both sides and comparing coefficients, we find the identity

$$\bigl(\,r\,(\lambda-s+1)-(r+1)\,j\,\bigr)\,d_{j,s-1}(\lambda, r) - r\,s\,d_{j,s}(\lambda, r) \;=\; \sum_{m=1}^{j}\binom{r}{m+1}\,(s+m)^-_{m+1}\,d_{j-m,s+m}(\lambda, r)\,,$$

which determines the $d_{j,s}$ completely by a double induction (first on j, and then for a given j on s). The proposition then follows by verifying that the right-hand side of (70) satisfies the same identity, which is an elementary exercise using the generating function (68). □

Corollary 1. *For arbitrary positive integers r and k, the polynomial $z_k(x) = \operatorname{res} L^{\frac{k}{r}}$ is given by $z_k(0) = C^j d_{j,0}(k/r, r)$ if $k = -1 + (r+1)j$ with $j \geq 0$ and is 0 otherwise, and similarly for any $s \geq 0$, $z_k^{(s)}(0)$ is equal to $C^{s+j} d_{j,s}(k/r, r)$ if $k = -1 + (r+1)j$ with $j \geq 0$ and vanishes otherwise.*

4.2. **Product formula.** In this subsection we will derive another formula for $z_k(0)$ using the wave-function-pair approach (see Appendix A or [53]). Before doing this, we will first introduce some functions and prove several lemmas that are useful for the construction.

Definition 1. Define a power series (algebraic if r is rational)

$$w \;=\; 1 + u - \frac{r-1}{6}\,u^2 + \frac{(r-1)(2r+1)}{72}\,u^3 - \cdots \tag{71}$$

by

$$\frac{w^{r+1}}{r(r+1)} - \frac{w}{r} + \frac{1}{r+1} \;=\; \frac{u^2}{2} \tag{72}$$

and define coefficients $C_n(r,j) \in \mathbb{Q}[r,j]$, $n \geq 0$ by

$$\frac{w^{j+1}-1}{j+1} \;=\; \sum_{n\geq 0} C_n(r,j)\,u^{n+1} \quad \text{(replace LHS by } \log w \text{ if } j=-1\text{)}\,, \tag{73}$$

the first few values being

$$C_0(r,j) \;=\; 1\,, \qquad C_1(r,j) \;=\frac{j}{2} - \frac{r-1}{6}\,, \qquad C_2(r,j) \;=\; \frac{j(j-r)}{6} + \frac{(r-1)(2r+1)}{72}\,,$$
$$C_3(r,j) \;=\; \frac{j(j-r)(j-r-1)}{24} - \frac{(r-1)(r+2)(2r+1)}{540}\,.$$

Define $C_{-2}(r,j) = C_{-1}(r,j) = 0$.

Remark 6. Immediately from the definition we find a further property for the coefficients $C_n(r,j)$, that is, they have an S_3-symmetry generated by two involutions

$$C_n(r,j) \;=\; (-1)^n\, C_n\Big(-r-1, \frac{n-3}{2} - j\Big) \;=\; (r+1)^n\, C_n\Big(\frac{-r}{r+1}, \frac{j-r}{r+1}\Big)\,. \tag{74}$$

We hope to investigate this very intriguing property and its applications later.

We now define a power series $f_j(T) = f_{r,j}(T) \in \mathbb{Q}[r,j][[T]]$ (we usually omit r) by

$$f_j(T) \;:=\; \sum_{k\geq 0} (2k+1)!!\, C_{2k}(r,j)\,(-T)^k\,. \tag{75}$$

Note that this is defined for all r and j in $\mathbb{C}$, not just for non-negative integers, because the $C_n(r,j)$ are polynomials.

Lemma 1. *The power series $f_j(T)$ $(j \in \mathbb{C})$ satisfy the following two identities:*

$$f_{j+1}(T) \;=\; \Big(1 + \Big(\frac{r-1}{2} - j\Big)\,T + (r+1)\,T^2 \frac{d}{dT}\Big)\, f_j(T)\,, \tag{76}$$

$$f_{j+r}(T) \;=\; f_j(T) - r\,j\,T\,f_{j-1}(T)\,. \tag{77}$$

Proof. We have by definition

$$w^j \frac{dw}{du} = \sum_{n\geq 0} (n+1)\, C_n(r,j)\, u^n . \tag{78}$$

Applying d/du to (72) gives

$$\Big(\frac{w^r}{r} - \frac{1}{r}\Big) \frac{dw}{du} = u \quad \Rightarrow \quad w^{r+j} \frac{dw}{du} - w^j \frac{dw}{du} = r\, u\, w^j .$$

Substituting (78) and (73) into this identity, comparing the Taylor coefficients of u, and noticing that $C_0(r,j) = 1$, $C_1(r,j) = \frac{j}{2} - \frac{r-1}{6}$, we obtain that for all $n \geq 0$,

$$(n+1)\, C_n(r, r+j) = (n+1)\, C_n(r,j) + r\, j\, C_{n-2}(r, j-1) + r\, \delta_{n,1} ,$$

which proves (77). Similarly, we have

$$\begin{aligned}\big(w^{j+1} - w^j\big) \frac{dw}{du} &= w^{j+1}\Big(w^r \frac{dw}{du} - r\, u\Big) - w^j \frac{dw}{du} \\ &= w^j \Big(\frac{r(r+1)}{2} u^2 - r + (r+1) w\Big) \frac{dw}{du} - r\, u\, w^{j+1} - w^j \frac{dw}{du} .\end{aligned}$$

Namely,

$$\big(w^{j+1} - w^j\big) \frac{dw}{du} = - \frac{r+1}{2} u^2\, w^j \frac{dw}{du} + u\, w^{j+1} ,$$

which implies (76). The lemma is proved. □

We note that the above relations (76) and (77) determine $f_{r,j}(T) \in 1 + T\,\mathbb{C}[r,j][[T]]$ uniquely for all $r, j \in \mathbb{C}$. To prove this, we may assume that $j \geq 0$ and $r \geq 2$ are integers (since these countably many values define the polynomial coefficients uniquely). Then (76) implies by induction that $f_1, f_2, f_3, \ldots, f_r$ are uniquely determined by f_0 and then (77) with $j = 0$ gives a differential equation for f_0, which has a unique solution in $1 + T\,\mathbb{C}[r,j][[T]]$. (In fact, this proof shows that the $f_{j,r}(T)$ are uniquely determined by just (76) and the equation $f_0 = f_r$.)

We now define an odd Laurent series (again algebraic if r is rational)

$$X = \frac{1}{u} - \frac{r-1}{6} u + \frac{(r-1)(2r+1)(r-3)}{360} u^3 + \cdots \in \frac{1}{u}\,\mathbb{Q}[r][[u^2]]$$

by the equation

$$\frac{(X+1)^{r+1} - (X-1)^{r+1}}{2\,(r+1)} = \frac{1}{u^r} , \tag{79}$$

and define coefficients $\widetilde{C}_n(r,i,j) \in \mathbb{Q}[r,i,j]$ by

$$- (X+1)^i\, (X-1)^j \frac{dX}{du} = \sum_{n\geq 0} \widetilde{C}_n(r,i,j)\, u^{n-i-j-2} . \tag{80}$$

The first two $\widetilde{C}_n(r,i,j)$ are given by

$$\widetilde{C}_0(r,i,j) = 1 , \quad \widetilde{C}_1(r,i,j) = i - j . \tag{81}$$

Taking $i = j = 0$ in (80), we have

$$- \frac{dX}{du} = \sum_{n\geq 0} \widetilde{C}_n(r,0,0)\, u^{n-2} . \tag{82}$$

Integrating this equality with respect to u we find that

$$-X \;=\; -\frac{1}{u} + \sum_{n\ge 2} \frac{1}{n-1}\, \widetilde{C}_n(r,0,0)\, u^{n-1}\,. \tag{83}$$

Lemma 2. *The numbers $\widetilde{C}_n(r,i,j)$ satisfy the following relations:*

$$\widetilde{C}_n(r,i+1,j) \,-\, \widetilde{C}_n(r,i,j+1) \;=\; 2\,\widetilde{C}_{n-1}(r,i,j)\,, \tag{84}$$

$$\widetilde{C}_n(r,i+r+1,j) \,-\, \widetilde{C}_n(r,i,j+r+1) \;=\; 2\,(r+1)\,\widetilde{C}_{n-1}(r,i,j)\,. \tag{85}$$

Proof. We have from the defining equation (80) that

$$-\,(X+1)^{i+1}\,(X-1)^{j}\,\frac{dX}{du} \;=\; (X+1)\sum_{n\ge 0} \widetilde{C}_n(r,i,j)\, u^{n-i-j-2}\,,$$

$$-\,(X+1)^{i}\,(X-1)^{j+1}\,\frac{dX}{du} \;=\; (X-1)\sum_{n\ge 0} \widetilde{C}_n(r,i,j)\, u^{n-i-j-2}\,.$$

Subtracting these two identities, using again (80), and comparing the coefficients of powers of u gives (84). Similarly, multiplying $(X+1)^i(X-1)^j$ to both sides of the defining equation (79), using (80), and comparing the coefficients of powers of u, we obtain (85). □

Proposition 3. *We have the following identities: $\forall\, i,j\in\mathbb{C}$,*

$$f_i(T)\,f_j(-T) \;=\; \sum_{n\ge 0} \left(1+\tfrac{n-i-j-1}{r}\right)_n \widetilde{C}_n(r,i,j)\,\left(\frac{r\,T}{2}\right)^n, \tag{86}$$

where $(s)_n = s(s+1)\cdots(s+n-1)$ denotes the ascending Pochhammer symbol.

Proof. Write

$$f_i(T)\,f_j(-T) \;=\; \sum_{n\ge 0} \left(1+\tfrac{n-i-j-1}{r}\right)_n P_{i,j,n}(r)\,\left(\frac{r\,T}{2}\right)^n. \tag{87}$$

It then suffices to show that the $P_{i,j,n}(r)$ satisfy the same recursion relation as $\widetilde{C}_n(r,i,j)$. This can be verified straightforwardly using (75)–(77). The proposition is proved. □

Let us now assume that r is a positive integer and use the wave-function-pair approach (see Appendix A) to compute the residue of $L^{k/r}$, where we remind the reader that

$$L \;=\; \partial^r \,+\, Cx\,.$$

First we construct a particular pair of wave functions (ψ,ψ^*) of L. Start with solving the equation $L\psi = z^r\psi$, that is,

$$\big(\partial^r \,-\, C\,(z^r/C - x)\big)\,\psi \;=\; 0\,. \tag{88}$$

Denote $X = z^r/C - x$. Then we have

$$\big((-\partial_X)^r \,-\, CX\big)\,\psi \;=\; 0\,. \tag{89}$$

Using the formal saddle point method, we know that this linear ODE has a unique formal Puiseux series solution of the form:

$$P_1(X) \;=\; e^{-\frac{r}{r+1}C^{\frac{1}{r}}\,X^{\frac{r+1}{r}}}\,X^{-\frac{r-1}{2r}} \sum_{m\ge 0} \frac{c_m}{X^{\frac{(r+1)m}{r}}}\,, \quad c_0 \;:=\; 1\,. \tag{90}$$

Therefore, any wave function $\psi(x,z)$ for L can be expressed as $\alpha_1(z)\,P_1(X)$ for some $\alpha_1(z)$. Although as we know that the choice of $\alpha_1(z)$ is not unique, let us show that the following function gives a particular choice:

$$\alpha_1^{\rm bisp}(z) \;:=\; \frac{1}{e^{-\frac{r}{r+1}C^{\frac{1}{r}}X^{\frac{r+1}{r}}}X^{-\frac{r-1}{2r}}}\bigg|_{x=0} \;=\; C^{-\frac{r-1}{2r}}\,e^{\frac{r}{r+1}C^{-1}z^{r+1}}\,z^{\frac{r-1}{2}}\,. \tag{91}$$

Indeed, define

$$\psi \;=\; \psi(x,z) \;:=\; \alpha_1^{\rm bisp}(z)\,P_1(X)\,. \tag{92}$$

Then it is easy to see that ψ has the form $\psi = \Phi_1(e^{xz})$, where $\Phi_1 = \sum_{k\geq 0}\phi_{1,k}(x)\,\partial^{-k}$ with $\phi_{1,0}\equiv 1$. Hence the function ψ constructed by (92) is indeed a wave function of L. We call this choice of $\alpha_1(z)$ the *bispectral* one. Similarly, denote by

$$P_2(X) \;:=\; e^{\frac{r}{r+1}C^{\frac{1}{r}}X^{\frac{r+1}{r}}}X^{-\frac{r-1}{2r}}\sum_{m\geq 0}\frac{c_m^*}{X^{\frac{(r+1)m}{r}}}\,, \quad c_0^* \;:=\; 1 \tag{93}$$

the unique formal solution to the linear ODE

$$\big(\partial_X^r \,-\, CX\big)\,\psi^* \;=\; 0\,. \tag{94}$$

Define

$$\alpha_2^{\rm bisp}(z) \;:=\; C^{-\frac{r-1}{2r}}\,e^{-\frac{r}{r+1}C^{-1}z^{r+1}}\,z^{\frac{r-1}{2}}\,, \tag{95}$$

and construct

$$\psi^* \;=\; \psi^*(x,z) \;:=\; \alpha_2^{\rm bisp}(z)\,P_2(X)\,. \tag{96}$$

Then it is easy to see that ψ^* can be written as $\psi^* = \Phi_2(e^{-xz})$, where $\Phi_2 = \sum_{k\geq 0}\phi_{2,k}(x)\,\partial^{-k}$ with $\phi_{2,0}\equiv 1$.

Proposition 4. *The above ψ,ψ^* form a particular pair of wave and dual wave functions of L.*

Proof. Since we already know that ψ is a wave function and ψ^* is a dual wave function, we are left to show that ψ, ψ^* form a pair. From the definition it then suffices to show that $\Phi_1\circ\Phi_2^*\equiv 1$. Note that for any two pseudo-differential operators P,Q, $\mathrm{res}_z\,P(e^{xz})\,Q(e^{-xz})\,dz = \mathrm{res}\;P\circ Q^*$. Taking $P=\partial^i\circ\Phi_1$ and $Q=\Phi_2$ in this identity we find $\mathrm{res}\;\partial^i\circ\Phi_1\circ\Phi_2^* \;=\; \mathrm{res}_z\;\partial^i\circ\Phi_1(e^{xz})\,\Phi_2(e^{-xz})\,dz$. Therefore, showing $\Phi_1\circ\Phi_2^*\equiv 1$ is further equivalent to showing that for all $i\geq 0$,

$$\mathrm{res}_z\;\partial^i\big(\psi(z,x)\big)\,\psi^*(z,x)\,dz \;=\; 0\,. \tag{97}$$

Before continuing the proof let us do preparations. Following [14, 20] (see also [78]), introduce the following linear operators S_z and S_z^*:

$$S_z \;:=\; \frac{C}{rz^{r-1}}\partial_z \,-\, \frac{r-1}{2r}Cz^{-r} \,-\, z \;=\; \frac{C}{r}z^{-\frac{r-1}{2}}\circ\partial_z\circ z^{-\frac{r-1}{2}} \,-\, z\,, \tag{98}$$

$$S_z^* \;:=\; -\frac{C}{rz^{r-1}}\partial_z \,+\, \frac{r-1}{2r}Cz^{-r} \,-\, z \;=\; \frac{C}{r}z^{-\frac{r-1}{2}}\circ(-\partial_z)\circ z^{-\frac{r-1}{2}} \,-\, z\,. \tag{99}$$

Then we have the following lemma.

Lemma 3. *For any $i\geq 0$, we have*

$$\partial^i\big(\psi(x,z)\big) \;=\; \big(-S_z\big)^i\,\big(\psi(x,z)\big)\,, \qquad \partial^i\big(\psi^*(x,z)\big) \;=\; \big(S_z^*\big)^i\,\big(\psi^*(x,z)\big)\,. \tag{100}$$

Proof. By straightforward calculations using the definitions (92), (96). □

As in Appendix A, define

$$c(z) \;:=\; \psi(0,z)\,, \qquad c^*(z) \;:=\; \psi^*(0,z)\,.$$

We then further have the following lemma.

Lemma 4. *The ψ and ψ^* defined by (92) and (96) have the expressions:*

$$\psi(x,z) \;=\; \sum_{i\geq 0} \frac{(-1)^i}{i!} S_z^{\,i}\big(c(z)\big)\, x^i \;=\; \sum_{i\geq 0} \frac{(xz)^i}{i!} f_i\big(\tfrac{C/r}{z^{r+1}}\big)\,, \tag{101}$$

$$\psi^*(x,z) \;=\; \sum_{i\geq 0} \frac{1}{i!}\, (S_z^*)^i\big(c^*(z)\big)\, x^i \;=\; \sum_{i\geq 0} (-1)^i \frac{(xz)^i}{i!} f_i\big(\tfrac{-C/r}{z^{r+1}}\big)\,, \tag{102}$$

where we recall that f_i are given by (75).

Proof. Performing the Taylor expansion of ψ with respect to x at $x=0$ and using Lemma 3 we immediately get the first equality of (101). By using (90) we find that $c(z)$ has the form

$$c(z) \;=\; \sum_{m\geq 0} C^{\frac{(r+1)m}{r}} \frac{c_m}{z^{(r+1)m}}\,, \quad c_0 \;=\; 1\,. \tag{103}$$

Define

$$\tilde f_i \;:=\; z^{-i}\,(-S_z)^i(c(z))\,.$$

Then by using the definition of S_z we observe that $\tilde f_i \in \mathbb{C}[[1/z^{r+1}]]$. Write

$$\tilde f_i \;=\; \tilde f_i(T)\,, \quad T \;=\; \frac{C}{r\, z^{r+1}}\,.$$

Using again the definition of S_z as well as (88), we find that

$$\tilde f_{j+1}(T) \;=\; \Big(1 \,+\, \Big(\frac{r-1}{2} - j\Big)\, T \,+\, (r+1)\, T^2 \frac{d}{dT}\Big)\, \tilde f_j(T)\,,$$
$$\tilde f_{j+r}(T) \;=\; \tilde f_j(T) \,-\, r\, j\, T\, \tilde f_{j-1}(T)\,.$$

Comparing these with (76)–(77) and using the uniqueness of the recursion we conclude that $f_i = \tilde f_i$. This proves (101). The proof of (102) is similar. The lemma is proved. □

End of the proof of Proposition 4. Using the above Lemma 4 and (86), we have

$$\begin{aligned}
&\operatorname{res}_z \partial_x^i\big(\psi(z,x)\big)\, \psi^*(z,x)\, dz \\
&= \operatorname{res}_z \sum_{m\geq 0} \frac{z^{m+i} x^m}{m!} f_{m+i}\big(\tfrac{C/r}{z^{r+1}}\big) \sum_{\ell\geq 0} (-1)^\ell \frac{(xz)^\ell}{\ell!} f_\ell\big(\tfrac{-C/r}{z^{r+1}}\big)\, dz \\
&= \operatorname{res}_z \sum_{m,\ell,q\geq 0} (-1)^\ell\, \frac{z^{m+\ell+i-q(r+1)}\, x^{m+\ell}}{m!\,\ell!} \Big(1 + \frac{q-i-m-\ell-1}{r}\Big)_q \widetilde C_q(r,m+i,\ell) \Big(\frac{r}{2}\Big)^q dz \;=\; 0\,.
\end{aligned}$$

The proposition is proved. □

Corollary 2. *Let $L = \partial^r + Cx$ and $z_k(x)$ be defined by* (67)*. We have*

$$z_k(0) \;=\; \begin{cases} (-1)^{(r+1)n}\big(1 + \frac{n-1}{r}\big)_n\, \widetilde C_n(r,0,0)\, \frac{C^n}{2^n}\,, & \text{if } \ k = -1 + (r+1)n \ \text{ with } \ n \geq 0\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{104}$$

Proof. According to (277) and (276), and using Lemma 4, we have

$$\sum_{k\geq -1} (-1)^{k+1}\, z_k(0)\, z^{-(k+1)} \;=\; f_0\big(\tfrac{C/r}{z^{r+1}}\big)\, f_0\big(-\tfrac{C/r}{z^{r+1}}\big)\,. \tag{105}$$

Here $z_{-1}(0) := 1$. Substituting (86) in this equality we find

$$\sum_{n\geq 0}\bigl(1+\tfrac{n-1}{r}\bigr)_n \frac{\widetilde{C}_n(r,0,0)\,C^n}{2^n z^{n(r+1)}} \;=\; 1+\sum_{k\geq 0}(-1)^{k-1}\,z_k(0)\,z^{-(k+1)}\,. \tag{106}$$

The corollary is proved by comparing the coefficients of powers of z^{-1}. □

5. The A series

In this section, we will prove Theorem 5, which generalizes all parts of Theorem 2 (except for (i) and (ii), which were generalized to all $\mathfrak{g}$, not just A_l, in Section 3) from A_4 to A_{r-1} for all r. We will also prove the integrality result for b_g and c_g in Theorem 1, which we have obtained for $r=5$ only. The odd-looking numbering is meant to correspond to the various parts of Theorem 2.

Theorem 5. *Let $m := \bigl[(2g-1)/r\bigr]$. The following statements are true:*

(iii) [algebraicity] *Define $\tilde{a}_g(r)$ from the generating function*

$$y(x) \;=\; \sum_{g\geq 0}\tilde{a}_g(r)\,(2x)^{2g} \;=\; 1-\frac{r-1}{6}\,x^2+\frac{(r-1)(r-3)(2r+1)}{360}\,x^4+\cdots\,, \tag{107}$$

where $y(x)$ is the unique solution in $1+x^2\mathbb{Q}[r][[x^2]]$ of the polynomial equation

$$\frac{(y+x)^{r+1}\,-\,(y-x)^{r+1}}{2\,x}\,-\,r\,-\,1\;=\;0\,. \tag{108}$$

Then for all $g\geq 1$ with $2g-1\not\equiv 0\,(\mathrm{mod}\,r)$,

$$\tau_{A_{r-1}}(g) \;=\; \frac{(-1)^{m+g}\;r^{1-g}}{\bigl(\bigl\{\frac{2g-1}{r}\bigr\}\bigr)_m}\,\tilde{a}_g(r)\,. \tag{109}$$

(iv) [closed formula] *Define $c_{p,j}(r)$ as in (68). Then for all $g\geq 0$, we have*

$$\tau_{A_{r-1}}(g) \;=\; \frac{(-1)^{g+m-1}\,r^{-g}}{\bigl(\bigl\{\frac{2g-1}{r}\bigr\}\bigr)_{2g+m+1}}\sum_{p=0}^{2g}\left(\frac{2g(r+1)-1}{r}\right)^{-}_{p+2g} c_{p,2g}(r)\,. \tag{110}$$

(v) [product formula] *Define $C_n(r,j)\in\mathbb{Q}[r,j]$ by (71)–(73) and $f_j(T)$ by (75), i.e., $f_j(T) = \sum_{k\geq 0}(2k+1)!!\,C_{2k}(r,j)\,(-T)^k$. Then the following identity holds true:*

$$f_0(T)\,f_0(-T) \;=\; \sum_{g\geq 0}\Bigl(1-\frac{1-2g}{r}\Bigr)_{2g}(1-2g)\,r^{2g}\,\tilde{a}_g(r)\,T^{2g}\,. \tag{111}$$

(vi) [finite hypergeometric sum] *Set $R=[r/2]$. For all $g\geq 1$ with $2g-1\not\equiv 0\,(\mathrm{mod}\,r)$, we have*

$$\tau_{A_{r-1}}(g) \;=\; \frac{(-1)^{g+m}}{\bigl(\bigl\{\frac{2g-1}{r}\bigr\}\bigr)_m}\,\frac{r^{1-g}}{1-2g}\sum_{d\geq 0}\binom{\frac{2g-1}{r}}{d}K_{g,d}\,, \tag{112}$$

where

$$K_{g,d} \;=\; \frac{1}{4^g\,(r+1)^d}\sum_{\substack{m_1+m_2+\cdots+m_R=d\\ m_1+2m_2+\cdots+R\,m_R=g}}\binom{d}{m_1,\ldots,m_R}\prod_{i=1}^{R}\binom{r+1}{2i+1}^{m_i}. \tag{113}$$

(vii) [asymptotics][2] *For $r \geq 3$, as $g \to \infty$ with $2g-1 \not\equiv 0 \,(\mathrm{mod}\, r)$, we have*

$$\tau_{A_{r-1}}(g) \;\sim\; \frac{r\sqrt{\pi}}{\sqrt{r+1}^{\frac{r-2}{r}}} \, \frac{1}{\Gamma\bigl(1-\bigl\{\frac{2g-1}{r}\bigr\}\bigr)} \, \frac{1}{\Gamma(\frac{2g-1}{r})} \, g^{-\frac{3}{2}} \Bigl(4\, r\, (r+1)^{\frac{2}{r}} \, \sin^2\bigl(\tfrac{\pi}{r}\bigr)\Bigr)^{-g}. \tag{114}$$

For $r=2$ (the A_1 case) the asymptotic formula is the same with an extra factor of $1/2$.

Proof. Let us start from recalling the r-spin Witten conjecture [114] (the Faber–Shadrin–Zvonkine theorem [63]). Introduce the Lax operator

$$L \;=\; \partial^r \,+\, \sum_{\alpha=1}^{r-1} u_\alpha \, \partial^{r-1-\alpha}. \tag{115}$$

The Gelfand–Dickey hierarchy [34] is the following infinite family of PDEs for $(r-1)$ unknown functions $u_1, \dots, u_{r-1}$:

$$\frac{\partial L}{\partial s^{\alpha,q}} \;=\; \Bigl[\bigl(L^{k/r}\bigr)_+ \,,\, L\Bigr], \quad q \geq 0, \tag{116}$$

where $k = \alpha + rq$. Similarly as before, denote $\mathbf{s} := (s^{\alpha,q})_{\alpha=1,\dots,r-1,\, q\geq 0}$. Since

$$\frac{\partial u_\alpha}{\partial s^{1,0}} \;=\; \frac{\partial u_\alpha}{\partial x},$$

we identify $s^{1,0}$ with x. Denote by $Z(\mathbf{t})$ the r-spin partition function defined by (34) with Ω being the Witten class. Then the r-spin Witten conjecture [114] is stated as follows: the partition function $Z = Z(\mathbf{t}(\mathbf{s}))$ is a particular tau-function for the Gelfand–Dickey hierarchy (116), where

$$t^{\alpha,q} \;=\; (-1)^{q+1} \, (\sqrt{-r})^{\frac{3k}{r+1}+1} \left(\frac{\alpha}{r}\right)_{q+1} s^{\alpha,q}, \quad q \geq 0; \tag{117}$$

moreover, the partition function Z satisfies the string equation (40) with $\eta_{\alpha\beta} = \delta_{\alpha+\beta,r}$. Note that the string equation (40) written in terms of the time variables $s^{\alpha,q}$ reads

$$\sum_{\alpha=1}^{l} \sum_{q\geq 0} (\alpha + rq + r) \, s^{\alpha,q+1} \frac{\partial Z}{\partial s^{\alpha,q}} \,+\, \frac{1}{2} \sum_{\alpha=1}^{l} \alpha\,(r-\alpha)\, s^{\alpha,0} s^{l+1-\alpha,0} \, Z \;=\; \frac{\partial Z}{\partial s^{1,0}}. \tag{118}$$

It is known that the differential polynomials h_k defined by

$$h_k \;:=\; \frac{(-1)^{q+1}}{(\sqrt{-r})^{3\frac{1+k}{r+1}} \, (\frac{\alpha}{r})_{q+1}} \, \mathrm{res}\, L^{\frac{k}{r}} \qquad \bigl(k \in \mathbb{Z}_{>0} \setminus r\mathbb{Z}_{>0}\bigr) \tag{119}$$

are the tau-symmetric Hamiltonian densities for the Gelfand–Dickey hierarchy (see [34, 114]; also cf. [14, 23, 24, 56]). Here $k = \alpha + rq$. Denote by $u_\alpha(\mathbf{s})$ the solution corresponding to the particular tau-function $Z(\mathbf{t}(\mathbf{s}))$. The Lax operator L given by (115) is now also subjected to this solution. We have

$$h_k(\mathbf{s}(\mathbf{t})) \;=\; \frac{\partial^2 \log Z(\mathbf{t})}{\partial t^{\alpha,q} \partial t^{1,0}}. \tag{120}$$

Then using the string equation together with a Miura-type transformation we find that the initial Lax operator $L|_{s^{\alpha,q}=x\,\delta^{\alpha,1}\delta^{q,0}}$, still denoted by L, has the following explicit expression [14]:

$$L \;=\; \partial^r \,+\, r\,x. \tag{121}$$

Now by using the above (36), (119), (120) we have

$$\tau_{A_{r-1}}(g) \;=\; (-1)^{q+g} \, \frac{z_{2g(r+1)-1}(0)}{r^{3g}\,(\frac{\alpha}{r})_{q+2}}, \quad \forall\, g \geq 0, \tag{122}$$

[2]The precise form of part (vii) was written by D.Y. and D.Z. after the first author B.D. passed away.

where $\alpha \in \{1,\dots,l\}$, $q \geq 0$ are the unique integers satisfying (20), and we recall that $z_k(x) := \operatorname{res} L^{k/r}$. Therefore, by using (104), Proposition 3, and the corollary to Proposition 2 from the previous section, we arrive at parts (iii)–(v). We remark in passing that equation (108) multiplied out, takes the form

$$\sum_{i=0}^{r/2} \binom{r+1}{2i+1} x^{2i}\, y^{r-2i} \;-\; r-1 \;=\; 0\,, \tag{123}$$

which for $r=5$ agrees with the equation in (5) (part (iii) of Theorem 2) if we set $z = -\,20\,x^2$.

To show (112), we will prove a more general statement (suitable for the later as well). Denote

$$\lambda(p;s) \;=\; p^r \;+\; \sum_{i=1}^{r-1} s_i\, p^{r-1-i}\,,$$

where $s=(s_1,\dots,s_l)$. Consider the following algebraic equation for p:

$$\lambda(p;s) \;=\; \xi^r\,. \tag{124}$$

This equation has a unique formal solution $p(\xi;s)$ in $\xi+\mathbb{Q}[s][[\xi^{-1}]]$. Write

$$p(\xi;s) \;=\; \xi \;+\; \sum_{k\geq 1} u_k(s)\,\xi^{-k} \tag{125}$$

with $u_k(s)\in\mathbb{Q}[s]$, $k\geq 1$. Differentiating (125) with respect to ξ, we find

$$\frac{dp(\xi;s)}{d\xi} \;=\; 1 \;-\; \sum_{k\geq 1} k\, u_k(s)\,\xi^{-k-1}\,. \tag{126}$$

Therefore,

$$k\,u_k(s) \;=\; \operatorname{res}_{\xi=\infty} \frac{dp(\xi;s)}{d\xi}\xi^k d\xi \;=\; \operatorname{res}_{\xi=\infty} \rho^k\, dp(\xi;s) \;=\; \operatorname{res}_{p=\infty} f(p;s)^{\frac{k}{r}}\, dp\,. \tag{127}$$

Noting that

$$\Bigl(p^r \;+\; \sum_{i=1}^{r-1} s_i\, p^{r-1-i}\Bigr)^{\frac{k}{r}} \;=\; p^k \sum_{j\geq 0} \binom{\frac{k}{r}}{j} \sum_{i_1,\dots,i_j=1}^{r-1} s_{i_1}\dots s_{i_j}\, p^{-j-i_1-\dots-i_j}\,, \tag{128}$$

we obtain the following expressions for $u_k(s)$:

$$k\,u_k(s) \;=\; -\sum_{j\geq 0}\binom{\frac{k}{r}}{j} \sum_{\substack{1\leq i_1,\dots,i_j\leq r-1\\ j+i_1+\dots+i_j=k+1}} s_{i_1}\dots s_{i_j} \tag{129}$$

$$\;=\; -\sum_{\substack{n_1,\dots,n_{r-1}\geq 0\\ 2n_1+3n_2+\dots+rn_{r-1}=k+1}} \binom{\frac{k}{r}}{\sum_i n_i}\binom{\sum_i n_i}{n_1,\dots,n_{r-1}} s_1^{n_1}\dots s_{r-1}^{n_{r-1}}\,. \tag{130}$$

Now specializing the above to $s=s^*$ given by

$$s^*_{2i}=0\,,\quad s^*_{2i-1} = \frac{1}{4^i(r+1)}\binom{r+1}{2i+1}\,,$$

we obtain

$$(1-2g)\,u_{2g-1}(s^*) \;=\; \sum_{d\geq 0} \frac{\binom{\frac{2g-1}{r}}{d}}{4^g\,(r+1)^d} \sum_{\substack{m_1+\dots+m_R=d\\ m_1+2m_2+\dots+R\,m_R=g}} \binom{d}{m_1,\dots,m_R} \prod_{i=1}^{R} \binom{r+1}{2i+1}^{m_i}\,. \tag{131}$$

Here $m_i = n_{2i-1}$. Combining with (108) we obtain (112).

It remains to prove part (vii). In view of (109), (114) is equivalent to the asymptotic formula

$$\tilde{a}_g(r) \sim \frac{(r+1)^{\frac{1}{r}-\frac{1}{2}}}{\sqrt{\pi g^3}} \, \frac{\sin\bigl(\frac{2g-1}{r}\pi\bigr)}{\bigl(-4\,(r+1)^{\frac{2}{r}}\sin^2\bigl(\frac{\pi}{r}\bigr)\bigr)^g}\,, \qquad r>2 \tag{132}$$

for the coefficients $\tilde{a}_g(r)$ defined in (107). By a standard principle, this is in turn equivalent to studying the asymptotic properties (to lowest order) of the generating function $y(x)$ near its singularities of smallest absolute value. Let $P = P(x,y)$ be the polynomial on the left-hand side of (108) and denote by P_x and P_y its partial derivatives with respect to x and y, respectively. The singularities of $y(x)$ are located at the points where its graph becomes vertical, i.e., where both P and P_y vanish (and where P_x doesn't vanish, but one sees easily that these three polynomials have no common zeros). Since $P_y/(r+1) = ((y+x)^r - (y-x)^r)/(2x)$, we see that the curve $\{P_y = 0\}$ has $(r-1)$ components, parametrized by the non-trivial rth roots of unity ζ and given by $\frac{y+x}{y-x} = \zeta$. Setting $(x,y) = c\bigl(\frac{\zeta-1}{2}, \frac{\zeta+1}{2}\bigr)$ with $c\in\mathbb{C}$, and substituting this into the equation $P(x,y)=0$, we find that $c^r = r+1$. Thus, setting $c_0 = (r+1)^{1/r}$, we have that the $r(r-1)$ common points of $P=0$ and $P_y=0$ are given parametrically by $(x,y) = c_0\bigl(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}\bigr)$ with (α,β) ranging over all pairs of distinct rth roots of unity. Of these, the ones with x nearest to the origin are those with $\alpha/\beta = \zeta_r^{\pm1}$, where $\zeta_r = e^{2\pi i/r}$, i.e., they are the $2r$ points $(\pm x_j, y_j)$, where $j\in\mathbb{Z}/r\mathbb{Z}$ and $(x_j,y_j) := \frac{c_0}{2}\bigl(\zeta_r^j - \zeta_r^{j-1}, \zeta_r^j + \zeta_r^{j-1}\bigr)$, all of them with $|x_j| = c_0\sin(\pi/r)$.

To complete the proof, we must (a) compute the potential contribution from each nearest singularity and (b) see that only the four singularities $(\pm x_0, y_0)$ and $(\pm x_1, y_1)$ belong to the subset of $\{P=0\}$ that is parametrized by $x\mapsto (x, y(x))$ in the closed disk $|x|\le c_0\sin(\pi/r)$. For (a), we note that from the Taylor expansion of P near the singularity (x_j,y_j) we find that for a point $(x,y) = (x_j-\varepsilon, y_j+\delta)$ near (x_j,y_j) lying on the curve $P=0$ we must have $P_x(x_j,y_j)\,\varepsilon \approx P_{yy}(x_j,y_j)\,\delta^2/2$ or $y \approx y_j + C_j\sqrt{1-x/x_j}$ with $C_j = \pm\sqrt{(2\,x\,P_x/P_{yy})(x_j,y_j)}$. A short computation gives that $x_jP_x(x_j,y_j) = r\,(r+1)$ and $P_{yy}(x_j,y_j) = -\,\zeta_r^{1-2j}\,r\,(r+1)^{2-2/r}$, so $C_j = \pm\, i\,\sqrt{2}\,\zeta_r^{j-\frac{1}{2}}\,(r+1)^{\frac{1}{r}-\frac{1}{2}}$. The contribution from this singularity to the coefficient $2^{2g}\tilde{a}_g(r)$ of x^{2g} in $y(x)$ is then asymptotically equal to $C_j\binom{1/2}{2g}x_j^{-2g}$, and of course the contribution from $(-x_j,y_j)$ is the same since P is even in X. Since $\binom{1/2}{2g}$ is asymptotically equal to $-1/\sqrt{32\pi g^3}$, we find that the sum of the four contributions coming from $(\pm x_0,y_0)$ and $(\pm x_1,y_1)$ indeed gives the asymptotic formula stated in (132). For (b), we first note that since P is a polynomial in x^2 and y having degree r in y, the map $\pi : (x,y)\mapsto X = x^2$ represents the Riemann surface $\{P=0\}$ as an r-sheeted branched cover of $\mathbb{P}^1(\mathbb{C})$. By the above calculations, this map is unramified over the open disc $D = \{|X| < |x_j^2| = c_0^2\sin^2(\pi/r)\}$, but has r ramification points $(X_j = x_j^2, y_j)$, at each of which exactly two of the r sheets come together, over the closure $\overline{D}$. The inverse image of the disk D consists of r disjoint disks D_ζ, indexed by the rth roots of unity ζ, where D_ζ is the component of $\pi^{-1}(D)$ containing $(0,\zeta)$ and is parametrized by $y = \zeta y(x/\zeta)$ with $x\in D$. If we write D_j $(j\in\mathbb{Z}/r\mathbb{Z})$ for $D_{\zeta_r^j}$, then it is not hard to see that $\overline{D}_j$ meets $\overline{D}_{j-1}$ at the point (X_j,y_j) and, of course (replacing j by $j+1$) also meets $\overline{D}_{j+1}$ at (X_{j+1},y_{j+1}), and that the closed disks $\overline{D}_j$ have no other points in common. In particular, the component D_0 parametrized by $y=y(x)$ contains the two ramification points (X_0,y_0) and (X_1,y_1) in its closure, and no others, and this means that the two points nearest to the origin where the series $y(\sqrt{X})$ is not analytic are X_0 and X_1, as claimed. For $r=2$, this proof of the asymptotics also applies, but the situation on how the various branches over the closed disk $\overline{D}$ meet at their boundaries is slightly different, and we get the extra factor $1/2$. This can also be seen directly, since $y = \sqrt{1-x^2/3}$ in this case, but in any case there is no need to do any of this since the well-known formula $\tau_{A_1}(g)=\frac{1}{24^g\,g!}$ immediately gives the asymptotics.

This completes the proof of all parts of Theorem 5. □

Remark 7. A different approach for computing the r-spin intersection numbers, using the theory of matrix models, was obtained by Brézin–Hikami [16, 17]. For the one-point numbers $\tau_{A_{r-1}}(g)$, Brézin–Hikami discovered the following integral formula:

$$\tau_{A_{r-1}}(g) \;=\; \frac{(-1)^g\, r^{1-g}}{1-\frac{\alpha}{r}}\,\bigl[t^{2g}\bigr]\int_0^\infty \exp\biggl(-\frac{(s+t/2)^{r+1}-(s-t/2)^{r+1}}{(r+1)\,t}\biggr)\,ds\,, \qquad (133)$$

where $g\geq 0$ with $2g-1\equiv\alpha\,(\mathrm{mod}\,r)$, $\alpha=1,\dots,r-1$. We note that formula (112) of part (vi) of Theorem 5 can also be proved by using (133). Indeed, expanding the integrand suitably and integrating term-by-term, one can obtain that

$$\tau_{A_{r-1}}(g) \;=\; \frac{(-1)^g\, r^{-g}}{4^g\,\Gamma\bigl(1-\bigl\{\frac{2g-1}{r}\bigr\}\bigr)}\sum_{d\geq 0}(-1)^d\,\frac{\Gamma(d-\frac{2g-1}{r})}{(r+1)^d}\sum_{\substack{m_1+m_2+\cdots+m_R=d\\ m_1+2m_2+\cdots+R\,m_R=g}}\prod_{i=1}^{R}\frac{\binom{r+1}{2i+1}^{m_i}}{m_i!}\,, \qquad (134)$$

which by Euler's formula is equivalent to (112). Liu–Vakil–Xu [87] obtained formula (134) by using the integral formula (133); this was also our original derivation of (112). Therefore, part (vi) of Theorem 5 is not new; however, our current proof for (112) using the wave-function-pair approach from the theory of integrable systems is a self-contained one. We also note that the integral formula (133) in principle could also be used together with the method of steepest descent to give a different proof for part (vii) of Theorem 5. (The recursion given by the dual topological ODE, cf. Theorem 4, could give a third proof of part (vii) of Theorem 5 with the constant term in the right-hand side of (132) undetermined.)

By using (107)–(109) one immediately gets the first few values of $\tau_{A_{r-1}}(g)$:

$$\tau_{A_{r-1}}(1) \;=\; \frac{r-1}{24}\,,\quad r\geq 2\,, \qquad (135)$$

$$\tau_{A_{r-1}}(2) \;=\; \begin{cases} 1/1152\,, & r=2\,,\\ \frac{(r-3)(r-1)(2r+1)}{5760\,r}\,, & r\geq 3\,,\end{cases} \qquad (136)$$

$$\tau_{A_{r-1}}(3) \;=\; \begin{cases} 1/82944\,, & r=2\,,\\ 1/31104\,, & r=3\,,\\ 3/20480\,, & r=4\,,\\ \frac{(r-5)(r-1)(2r+1)(8r^2-13r-13)}{2903040\,r^2}\,, & r\geq 5\,.\end{cases} \qquad (137)$$

Corollary 3. *For $r\geq 2g$, the value of $\tau_{A_{r-1}}(g)$ is a Laurent polynomial in r. Moreover, the value of this Laurent polynomial at $r=-1$ is equal to $\frac{B_{2g}}{2g}$, where B_n is the nth Bernoulli number.*

Proof. If $r\geq 2g$, then $m=0$, so the right-hand side of (109) reduces to $-\tilde a_g(r)/(-r)^{g-1}$. The first statement then follows from the fact that $\tilde a_g(r)\in\mathbb{Q}[r]$. Now by taking the $r\to -1$ limit in (108) we find the unique solution $y=\frac{x}{\tanh x}$. The second statement then follows. □

According to Witten [113, 114] the second statement in the corollary should give a new proof of the Harer–Zagier formula [74, 101] on the orbifold Euler characteristic of $\mathcal{M}_{g,1}$ (see also [17, 87]).

We now give the proof of the three integrality statements for $\tau_{A_4}(g)$ stated in the introduction.

Proof of Theorem 1. The algebraicity of the generating function of the numbers a_g as given in part (iii) of Theorem 2 implies their integrality away from the primes 2, 3, and 5. To prove the integrality of b_g and c_g, we use (11). It follows from (11) that

$$c_g \;=\; 6^{-g} \sum_{0\le s\le g/2} 2^{-2s}\,(-9)^s\,c_g^{[s]}\,, \tag{138}$$

where the numbers $c_g^{[s]}$ are given according to the value of $g\,(\mathrm{mod}\,5)$ by

$$c_{5n}^{[s]} := \frac{5^{-2s}(\frac35)_n(\frac45)_n(\frac15)_{3n-s}}{s!\,(5n-2s)!}\,, \qquad c_{5n-1}^{[s]} := \frac{5^{-2s}(\frac25)_n(\frac45)_n(\frac35)_{3n-1-s}}{s!\,(5n-1-2s)!}\,,$$
$$c_{5n-3}^{[s]} := \frac{5^{-2s}(\frac35)_{n-1}(\frac15)_n(\frac25)_{3n-2-s}}{s!\,(5n-3-2s)!}\,, \qquad c_{5n-4}^{[s]} := \frac{5^{-2s}(\frac15)_n(\frac25)_{n-1}(\frac45)_{3n-3-s}}{s!\,(5n-4-2s)!}\,.$$

To show that c_g belongs to $\mathbb{Z}\big[1/30\big]$ we will actually show the stronger statement that each $c_g^{[s]}$ belongs to $\mathbb{Z}[1/5]$.

Consider first the case $g=5n$. For each prime $p\neq 5$, the p-adic valuation of $c_{5n}^{[s]}$ is given by

$$\nu_p\big(c_{5n}^{[s]}\big) \;=\; \nu_p\big((\tfrac35)_n\big) \,+\, \nu_p\big((\tfrac45)_n\big) \,+\, \nu_p\big((\tfrac15)_{3n-s}\big) \,-\, \nu_p(s!) \,-\, \nu_p((5n-2s)!)\,.$$

Namely,

$$\nu_p\big(c_{5n}^{[s]}\big) \;=\; \sum_{k\ge1}\Big[u\big((\tfrac35)_n,p^k\big) \,+\, u\big((\tfrac45)_n,p^k\big) \,+\, u\big((\tfrac15)_{3n-s},p^k\big) \,-\, u\big(s!,p^k\big) \,-\, u\big((5n-2s)!,p^k\big)\Big]\,,$$

where $u\big((a/5)_n,p^k\big)$ denotes the number of elements in $\{a,a+5,\dots,a+5n-5\}$ that are divisible by p^k ($a=1,\dots,5$, $k\ge1$). Counting these numbers we find

$$\begin{aligned} &u\big((\tfrac35)_n,p^k\big) \,+\, u\big((\tfrac45)_n,p^k\big) \,+\, u\big((\tfrac15)_{3n-s},p^k\big) \,-\, u\big(s!,p^k\big) \,-\, u\big((5n-2s)!,p^k\big)\\ &\qquad =\; f_j\big(\tfrac{n}{p^k},\tfrac{s}{p^k}\big) \quad \text{if } p^k\equiv j\,(\mathrm{mod}\,5),\ 1\le j\le 4\,, \end{aligned} \tag{139}$$

where the functions $f_j:\mathbb{R}^2\to\mathbb{Z}$ are defined by

$$f_1(x,y) \;:=\; [x+\tfrac25] \,+\, [x+\tfrac15] \,+\, [3x-y+\tfrac45] \,-\, [y] \,-\, [5x-2y]\,, \tag{140}$$
$$f_2(x,y) \;:=\; [x+\tfrac15] \,+\, [x+\tfrac35] \,+\, [3x-y+\tfrac25] \,-\, [y] \,-\, [5x-2y]\,, \tag{141}$$
$$f_3(x,y) \;:=\; [x+\tfrac45] \,+\, [x+\tfrac25] \,+\, [3x-y+\tfrac35] \,-\, [y] \,-\, [5x-2y]\,, \tag{142}$$
$$f_4(x,y) \;:=\; [x+\tfrac35] \,+\, [x+\tfrac45] \,+\, [3x-y+\tfrac15] \,-\, [y] \,-\, [5x-2y]\,. \tag{143}$$

Therefore it suffices to show that each of f_1,f_2,f_3,f_4 is nowhere negative on $\mathbb{R}^2$. To show this, we observe that each $f_\alpha(x,y)$ is periodic in both variables and is also piecewise constant, with jumps only along finitely many lines in the unit square $[0,1]^2$, so one only has to compute the values of $f_\alpha(x,y)$ in each component of the complement of the union of these lines. (The value of f_i at a point lying on one of these lines is always equal to the value of f_i in one of the adjacent open regions.) This is most easily seen graphically, as illustrated in Figure 1, which shows that each of f_1 and f_2 assumes only the values 0, 1, and 2, and this also gives the non-negativity of f_3 and f_4 as well since $f_3(x,y)=2-f_2(-x,-y)$ and $f_4(x,y)=2-f_1(-x,-y)$.

The proof for each other residue class of $g\,(\mathrm{mod}\,5)$ is exactly similar and again reduces to the non-negativity of four piecewise continuous functions on $\mathbb{R}^2/\mathbb{Z}^2$, but when one goes through the details it turns out that these four are a permutation of the same four functions f_j as above in each case, so that we do not need to make new graphs. The proofs for b_g are also similar, and

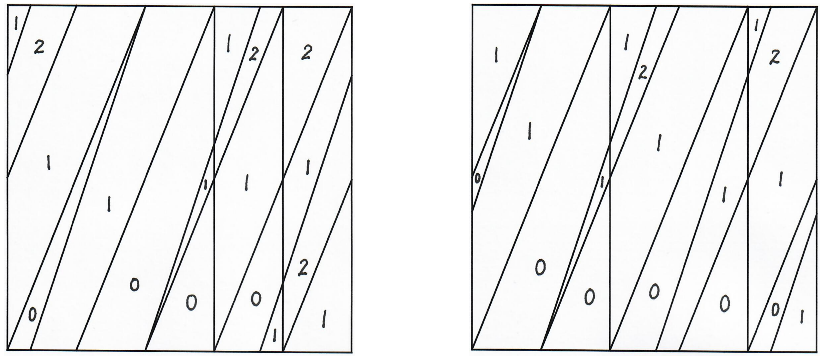

FIGURE 1. The values of f_1 and f_2 in $[0,1]^2$

again a priori involve the non-negativity of 16 periodic functions on $\mathbb{R}^2$, which again turn out to be repetitions of only four functions:

$$g_1(x,y) \;=\; [x+\tfrac{1}{5}] \,+\, [x+\tfrac{4}{5}] \,+\, [3x-y+\tfrac{4}{5}] \,-\, [y] \,-\, [5x-2y]\,, \tag{144}$$

$$g_2(x,y) \;=\; [x+\tfrac{3}{5}] \,+\, [x+\tfrac{2}{5}] \,+\, [3x-y+\tfrac{2}{5}] \,-\, [y] \,-\, [5x-2y]\,, \tag{145}$$

$$g_3(x,y) \;=\; [x+\tfrac{2}{5}] \,+\, [x+\tfrac{3}{5}] \,+\, [3x-y+\tfrac{3}{5}] \,-\, [y] \,-\, [5x-2y]\,, \tag{146}$$

$$g_4(x,y) \;=\; [x+\tfrac{4}{5}] \,+\, [x+\tfrac{1}{5}] \,+\, [3x-y+\tfrac{1}{5}] \,-\, [y] \,-\, [5x-2y]\,. \tag{147}$$

Their non-negativity can be proved as for the f_j. The theorem is proved. □

Note that the the first part of Theorem 1, concerning the integrality of the numbers a_g away from the primes 2, 3 and 5, is generalized by Theorem 5 to the integrality of the numbers $\tilde{a}_g(r)$ away from a finite number of primes depending only on r (in fact, away from $r(r+1)$) for any r, as a direct consequence of the algebraicity of the power series $y(x)$ in (107). In fact, as was mentioned briefly in Remark 3 for the $r=5$ case, for any positive value of r we have the stronger statment that the numbers $\frac{\tilde{a}_g(r)}{2g-r-1}$ are integral away from a finite set of primes (in fact, $\frac{(r+2)\tilde{a}_g(r)}{2g-r-1}$ is integral away from $r(r+1)$). To see this, define a second algebraic power series $y_1 = y_1(x)$ by

$$y_1 \;=\; \frac{(y+x)^{r+1} \,+\, (y-x)^{r+1}}{2\,(r+1)}\,, \tag{148}$$

with $y=y(x)$ as in (107). Then it is straightforward to verify that

$$x\,y_1' \,-\, (r+1)\,y_1 \;=\; (r+1)\,x\,y' \,-\, y \tag{149}$$

or equivalently that the coefficient of $(2x)^{2g}$ in y_1 is equal to $\frac{2(r+1)g-1}{2g-r-1}\,\tilde{a}_g(r)$ for $2g \neq r+1$, and the divisibility statement follows. An even stronger divisibility statement, saying that the numbers $\tilde{a}_g(r)/(2g-kr-1)$ are integral away from a finite number of primes for any r and any fixed $k \geq 0$, will be discussed in the subsequent publication [116]. In the case $r=5$ this follows easily from Theorem 1.

6. THE D SERIES

In this section, we derive explicit formulas for the numbers $\tau_{\mathfrak{g}}(g)$ with $\mathfrak{g}$ being D_l $(l \geq 4)$. Denote by $r = 2l-2$ the Coxeter number of $\mathfrak{g}$, and recall that $\tau_{D_l}(g) = \langle \tau_{\alpha,q} \rangle$ are the one-point FJRW invariants of D_l-type, where α, q are determined by (42). Denote by $Z = Z(\mathbf{t})$ the partition function for the FJRW invariants of D_l-type as in (34).

Theorem 6. *Let $l\ge 4$ be an integer and $r=2l-2$. Set $m:=[(2g-1)/r]$. Then the following statements are true:*

(iii) [algebraicity] *Define $n_g(r)$ from the generating function*

$$y(t) \;=\; \sum_{g\ge 0} n_g(r)\,t^{2g} \;=\; 1-\frac{r+2}{24}\,t^2+\frac{(r+2)(r-6)(2r+1)}{5760}\,t^4+\cdots\,, \tag{150}$$

where $y(t)$ is the unique solution in $1+t^2\,\mathbb{Q}[r][[t^2]]$ to the algebraic equation

$$\sum_{j=0}^{r/2}\frac{1}{2j+1}\binom{j+r/2}{2j}\,y^{r-2j}\,t^{2j}\;-\;1\;=\;0 \tag{151}$$

or alternatively t and $y=y(t)$ are related algebraically by

$$t/y=Z-Z^{-1}\,,\quad t^r\,\frac{Z^{r+1}-Z^{-r-1}}{r+1}\;=\;\bigl(Z-Z^{-1}\bigr)^{r+1}\,. \tag{152}$$

Then $n_g(r)$, $g\ge 0$, are integral away from a finite number of primes, and for all $g\ge 1$ we have

$$\tau_{D_l}(g)\;=\;\frac{(-1)^{m+g}\,r^{1-g}}{\bigl(\{\frac{2g-1}{r}\}\bigr)_m}\,n_g(r)\,. \tag{153}$$

(iv) [closed formula] *For all $g\ge 1$, we have*

$$\tau_{D_l}(g)\;=\;\frac{(-1)^{g+m-1}\,r^{-g}}{\bigl(\{\frac{2g-1}{r}\}\bigr)_{2g+m+1}}\sum_{j=0}^{2g}\sum_{p=0}^{j}c_{p,j}(r)\left(\frac{2g(r+1)-1}{r}\right)^{-}_{2g+p}\binom{-\frac12}{2g-j}. \tag{154}$$

(v) [product formula] *Define $C_n(r,j)\in\mathbb{Q}[r,j]$ by (71)–(73) and $f_j(T)$ by (75), i.e., $f_j(T)=\sum_{k\ge 0}(2k+1)!!\,C_{2k}(r,j)\,(-T)^k$. Then the following identity holds true:*

$$\Bigl[f_{\frac12}(T)\,f_{-\frac12}(-T)\Bigr]_{\rm even}\;=\;\sum_{g\ge 0}\Bigl(1-\frac{1-2g}{r}\Bigr)_{2g}\,(1-2g)\,r^{2g}\,n_g(r)\,T^{2g}\,. \tag{155}$$

Here for a power series $v(T)$, $[\,v(T)\,]_{\rm even}$ means taking the even degree part of $v(T)$.

(vi) [terminating hypergeometric sum] *For all $g\ge 1$,*

$$\tau_{D_l}(g)\;=\;\frac{(-1)^{g+m}}{\bigl(\{\frac{2g-1}{r}\}\bigr)_m}\,\frac{r^{1-g}}{1-2g}\sum_{d\ge 0}\binom{\frac{2g-1}{r}}{d}K_{g,d}\,, \tag{156}$$

where

$$K_{g,d}\;=\;\sum_{\substack{m_1+m_2+\cdots+m_{l-1}=d\\ m_1+2m_2+\cdots+(l-1)\,m_{l-1}=g}}\binom{d}{m_1,\dots,m_{l-1}}\prod_{i=1}^{R}\left[\frac{\binom{\frac{r+i+1}{2}}{i+1}}{i+2}\right]^{m_i}. \tag{157}$$

(vii) [asymptotics][3] *For $l>4$, as $g\to\infty$, $\tau_{D_l}(g)$ is given asymptotically by*

$$\tau_{D_l}(g)\;\sim\;\frac{r\,\sqrt{\pi}\,\cos(\frac{\pi}{r})}{\sqrt{r+1}^{\frac{r-2}{r}}}\,\frac{1}{\Gamma\bigl(1-\{\frac{2g-1}{r}\}\bigr)}\,\frac{1}{\Gamma(\frac{2g-1}{r})}\,g^{-\frac32}\Bigl(4\,r\,(r+1)^{\frac{2}{r}}\,\sin^2\bigl(\tfrac{\pi}{r}\bigr)\Bigr)^{-g}. \tag{158}$$

For $l=4$ the formula is the same with an extra factor of 3.

[3] The precise form of part (vii) was written by D.Y. and D.Z. after the first author B.D. passed away.

Proof. Recall that the DS hierarchy of D_l-type admits the following scalar Lax operator [38]:

$$L \;=\; \partial^r \,+\, \partial^{-1}\circ \sum_{\alpha=1}^{l-1}\bigl(u^\alpha\circ\partial^{2\alpha-1}+\partial^{2\alpha-1}\circ u^\alpha\bigr) \,+\, \partial^{-1}\circ u^l\circ\partial^{-1}\circ u^l\,. \tag{159}$$

For $\alpha=1,\dots,l-1$, the corresponding DS flows can be written in terms of L by

$$\frac{\partial L}{\partial s^{\alpha,q}} \;=\; \Bigl[\bigl(L^{k/r}\bigr)_+\,,\,L\Bigr]\,,\quad q\geq 0\,, \tag{160}$$

where $k=m_\alpha+rq$. Denote $\mathbf{s}:=(s^{\alpha,q})_{\alpha=1,\dots,l,\,q\geq 0}$. For $\alpha=l$, the corresponding flows could also be obtained in terms of L via the so-called negative flows [91, 110] or by definition from the original Drinfeld–Sokolov's matrix Lax system [38]. Since $\partial u^\alpha/\partial s^{1,0}=\partial u^\alpha/\partial x$, we identify $s^{1,0}$ with x. Witten's ADE conjecture [64, 66, 114] for the D_l case (the Fan-Jarvis-Ruan theorem [64, 66]; see also [73, 79, 90]) can then be stated as follows: the partition function $Z:=Z(\mathbf{t}(\mathbf{s}))$ is a particular tau-function for the DS hierarchy of D_l-type, where

$$t^{\alpha,q} \;=\; (-1)^{q+1}\,(\sqrt{-r})^{\frac{3k}{r+1}+1}\left(\frac{m_\alpha}{r}\right)_{q+1} s^{\alpha,q}\,,\quad q\geq 0\,; \tag{161}$$

moreover, $Z(\mathbf{t})$ satisfies the string equation (40) with $\eta_{\alpha\beta}$ in (40) given by

$$(\eta_{\alpha\beta}) \;=\; \begin{pmatrix} 0 & \dots & 0 & 1 & 0\\ \vdots & \iddots & \iddots & 0 & 0\\ 0 & 1 & \iddots & \iddots & \vdots\\ 1 & 0 & \iddots & \iddots & 0\\ 0 & 0 & \dots & 0 & 1\end{pmatrix}. \tag{162}$$

One can alternatively write the string equation (40) using the variables $s^{\alpha,q}$ from (161) as

$$\sum_{\alpha=1}^{l}\sum_{q\geq0}(m_\alpha+rq+r)\,s^{\alpha,q+1}\frac{\partial Z}{\partial s^{\alpha,q}}+\sum_{\alpha=1}^{l-1}\frac{m_\alpha(r-m_\alpha)}{2}\,s^{\alpha,0}s^{l-\alpha,0}Z+\frac{m_l^2}{2}(s^{l,0})^2Z \;=\; \frac{\partial Z}{\partial s^{1,0}}\,. \tag{163}$$

Denote by $u^\alpha=u^\alpha(\mathbf{s})$ the solution corresponding to the particular tau-function Z, the so-called topological solution. The Lax operator L given by (159) is now also subjected to this solution. Similarly as in the proof for the A case, we note that

$$h_k \;:=\; \frac{(-1)^{q+1}}{(\sqrt{-r})^{3\frac{1+k}{r+1}}(\frac{m_\alpha}{r})_{q+1}}\,\mathrm{res}\,L^{\frac{k}{r}} \qquad (k>0 \text{ odd}) \tag{164}$$

are a part of the tau-symmetric Hamiltonian densities for the DS hierarchy of D_l-type. Here $k=m_\alpha+rq$. Therefore,

$$h_k(\mathbf{s}(\mathbf{t})) \;=\; \frac{\partial^2\log Z(\mathbf{t})}{\partial t^{\alpha,q}\partial t^{1,0}}\,. \tag{165}$$

The initial Lax operator $L|_{s^{\alpha,q}=x\,\delta^{\alpha,1}\delta^{q,0}}$ will again be denoted by L. We have the following lemma.

Lemma 5. *We have*

$$L \;=\; \partial^r \,+\, r\,x-\frac{1}{2}\,r\,\partial^{-1}\,. \tag{166}$$

Proof. For the topological solution, the corresponding normal coordinates r_α satisfy

$$r_\alpha(\mathbf{t}) \;=\; \frac{\partial^2\log Z(\mathbf{t})}{\partial t^{\alpha,0}\partial t^{1,0}}\,. \tag{167}$$

Dividing the string equation (40) by $Z(\mathbf{t})$, then differentiating it with respect to $t^{\alpha,0}$, and finally taking $t^{\alpha,q} = t^{1,0}\,\delta^{\alpha,1}\delta^{q,0}$ we obtain that

$$r_\alpha(\mathbf{t})|_{t^{\alpha,q}=t^{1,0}\,\delta^{\alpha,1}\delta^{q,0}} \;=\; \eta_{\alpha 1}\,t^{1,0} \;=\; \delta_{\alpha,l-1}\,t^{1,0}\,. \tag{168}$$

Then, as in [13, 14], by using the Miura-type transformation between the normal coordinates and the u-coordinates together with a degree argument we find that

$$u^\alpha|_{s^{\beta,q}=x\,\delta^{\beta,1}\delta^{q,0}} \;=\; \frac{r}{2}\,x\,\delta^{\alpha,1}\,.$$

The lemma is proved. □

As before, denote $z_k(x) := \operatorname{res} L^{k/r}$. Using (165), (36) and (42) we obtain that

$$\tau_{D_l}(g) \;=\; (-1)^{q+g}\,\frac{z_{2g(r+1)-1}(0)}{r^{3g}\,(\frac{m_\alpha}{r})_{q+2}}\,. \tag{169}$$

Lemma 6. *The following formula holds true:*

$$L \;=\; \partial^{-\frac12}\circ\left(\partial^r \,+\, r\,x\right)\circ\partial^{\frac12}\,. \tag{170}$$

Proof. RHS $=\; \partial^r \,+\, r\,\partial^{-\frac12}\circ x\circ\partial^{\frac12} \;=\;$ LHS. □

Using (69) and (170), we have

$$\begin{aligned}
&\Big(\partial^r \,+\, r\,x \,-\, \frac12\,r\,\partial^{-1}\Big)^\lambda \\
&\quad=\; \partial^{-\frac12}\circ\left(\partial^r \,+\, r\,x\right)^\lambda\circ\partial^{\frac12} \\
&\quad=\; \sum_{\substack{0\le p\le j\\ s\ge 0}} r^{j+s}\,c_{p,j}(r)\,(\lambda)^-_{s+j+p}\,\sum_{v=0}^{s}\binom{-\frac12}{v}\,\frac{x^{s-v}}{(s-v)!}\,\partial^{r(\lambda-s)-(r+1)j-v}\,.
\end{aligned}$$

Combining with (169) we obtain (154).

To prove (151), let us use the wave-function-pair approach (see Appendix A and Section 4). We first construct a particular pair of wave and dual wave functions for L. Start with solving $L\psi = z^r\psi$, i.e.,

$$\Big(\partial^r \,+\, (r\,x - z^r) - \frac{r}{2}\,\partial^{-1}\Big)\,\psi \;=\; 0\,. \tag{171}$$

Denote $X = z^r/r - x$. We have

$$\Big((-\partial_X)^r \,-\, r\,X \,+\, \frac{r}{2}\,\partial_X^{-1}\Big)\,\psi \;=\; 0\,. \tag{172}$$

Similarly as in Section 4 we find that this equation has a unique formal solution of the form

$$P_1(X) \;=\; e^{-\frac{1}{r+1}r^{\frac{r+1}{r}}X^{\frac{r+1}{r}}}\,X^{-\frac12}\sum_{m\ge0}\frac{c_m}{X^{\frac{(r+1)m}{r}}}\,, \qquad c_0 \;:=\; 1\,. \tag{173}$$

Therefore, $\psi(x,z) = \alpha_1(z)\,P_1(X)$ for some $\alpha_1(z)$ to be determined. The choice of $\alpha_1(z)$ is not unique. We will use the particular choice of $\alpha_1(z)$ given by

$$\alpha_1^{\rm bisp}(z) \;:=\; \frac{1}{e^{-\frac{r}{r+1}r^{\frac1r}X^{\frac{r+1}{r}}}\,X^{-\frac12}}\Bigg|_{x=0} \;=\; r^{-\frac12}\,e^{\frac{r}{r+1}C^{-1}z^{r+1}}\,z^{\frac{r}{2}}\,. \tag{174}$$

We call this particular choice the *bispectral* one. Namely, we construct

$$\psi \;=\; \psi(x,z) \;:=\; \alpha_1^{\rm bisp}(z)\,P_1(X)\,. \tag{175}$$

Similarly, denote by

$$P_2(X) \;:=\; e^{\frac{r}{r+1} r^{\frac{1}{r}} X^{\frac{r+1}{r}}} X^{-\frac{1}{2}+\frac{1}{r}} \sum_{m\geq 0} \frac{c^*_m}{X^{\frac{(r+1)m}{r}}}\,, \qquad c^*_0 \;:=\; 1 \tag{176}$$

the unique formal solution to the following linear equation:

$$\Big(\partial_X^r \,-\, r\,X - \frac{r}{2}\,\partial_X^{-1}\Big)\,\psi^* \;=\; 0\,. \tag{177}$$

Define

$$\alpha_2^{\rm bisp}(z) \;:=\; r^{-\frac{1}{2}+\frac{1}{r}}\, e^{-\frac{1}{r+1} z^{r+1}}\, z^{\frac{r}{2}-1}\,, \tag{178}$$

and construct

$$\psi^* \;=\; \psi^*(x,z) := \alpha_2^{\rm bisp}(z)\, P_2(X)\,. \tag{179}$$

Proposition 5. *The ψ, ψ^* form a particular pair of wave and dual wave functions of L.*

Proof. It is easy to verify that ψ is a wave function and ψ^* is a dual wave function for L. Write $\psi = \Phi_1(e^{xz})$ and $\psi^* = \Phi_2(e^{-xz})$, where

$$\Phi_1 \;=\; \sum_{k\geq 0} \phi_{1,k}(x)\,\partial^{-k}\,, \quad \Phi_2 \;=\; \sum_{k\geq 0} \phi_{2,k}(x)\,\partial^{-k}$$

with $\phi_{1,0} = \phi_{2,0} \equiv 1$. To show ψ and ψ^* form a pair, it suffices to show $\Phi_1 \circ \Phi_2^* = 1$. Similarly as before, we know that this is equivalent to show that for all $i \geq 0$,

$$\mathrm{res}_z\; \partial^i\big(\psi(z,x)\big)\, \psi^*(z,x)\, dz \;=\; 0\,. \tag{180}$$

Before continuing the proof of the proposition let us prove two lemmas.

Lemma 7. *Introduce two linear operators R_z and R^*_z:*

$$R_z \;=\; \frac{1}{z^{r-1}}\,\partial_z \,-\, \frac{r}{2}\, z^{-r} \,-\, z\,, \qquad R^*_z \;=\; -\frac{1}{z^{r-1}}\,\partial_z \,+\, \frac{r-2}{2}\, z^{-r} \,-\, z\,. \tag{181}$$

Then we have for any $i \geq 0$,

$$\partial_x^i\big(\psi(x,z)\big) \;=\; \big(-R_z\big)^i\,\big(\psi(x,z)\big)\,, \qquad \partial_x^i\big(\psi^*(x,z)\big) \;=\; \big(R^*_z\big)^i\,\big(\psi^*(x,z)\big)\,. \tag{182}$$

Proof. By direct calculations. □

Lemma 8. *The ψ and ψ^* have the following expressions:*

$$\psi(x,z) \;=\; \sum_{i\geq 0} \frac{(-1)^i}{i!}\, R_z^i\big(c(z)\big)\, x^i \;=\; \sum_{i\geq 0} \frac{(xz)^i}{i!}\, f_{i-1/2}\big(\tfrac{1}{z^{r+1}}\big)\,, \tag{183}$$

$$\psi^*(x,z) \;=\; \sum_{i\geq 0} \frac{1}{i!}\, (R^*_z)^i\big(c^*(z)\big)\, x^i \;=\; \sum_{i\geq 0} (-1)^i\, \frac{(xz)^i}{i!}\, f_{i+1/2}\big(\tfrac{-1}{z^{r+1}}\big)\,. \tag{184}$$

Here, $c(z) := \psi(0,z)$, $c^*(z) := \psi^*(0,z)$, and $f_{i\pm 1/2}$ are given by (75).

Proof. Doing the Taylor expansion of ψ with respect to x at $x = 0$ and using Lemma 7 we immediately get the first equality in (183). From (173) we find that $c(z)$ has the form

$$c(z) \;=\; \sum_{m\geq 0} r^{\frac{(r+1)m}{r}}\, \frac{c_m}{z^{(r+1)m}}\,, \qquad c_0 = 1\,. \tag{185}$$

Define

$$\tilde{f}_i \;:=\; z^{-i}\,(-R_z)^i\big(c(z)\big)\,.$$

From the definition of S_z one easily finds that $\tilde f_i \in \mathbb{C}[[z^{-(r+1)}]]$. Write

$$\tilde f_i \;=\; \tilde f_i(T)\,, \quad T \;=\; \frac{1}{z^{r+1}}\,,$$

and we find from the definition of R_z and (171) that

$$\tilde f_{j+1}(T) \;=\; \Bigl(1 \,+\, \Bigl(\frac{r}{2}-j\Bigr)\,T \,+\, (r+1)\,T^2\frac{d}{dT}\Bigr)\,\tilde f_j(T)\,,$$

$$\tilde f_{j+r}(T) \;=\; \tilde f_j(T) \,-\, r\,\Bigl(j-\frac{1}{2}\Bigr)\,T\,\tilde f_{j-1}(T)\,.$$

Comparing with (76)–(77) and using the uniqueness argument that is similar to the one given after the proof of Lemma 1, we conclude that $\tilde f_i = f_{i-1/2}$. This proves (183). The proof of (184) is similar. The lemma is proved. □

End of the proof of Proposition 5. Using Lemma 8 and identity (86) we have

$$\begin{aligned}
&\mathrm{res}_z\, \partial^i\bigl(\psi(z,x)\bigr)\,\psi^*(z,x)\,dz\\
&=\; \mathrm{res}_z \sum_{m\ge 0} \frac{z^{m+i}\,x^m}{m!}\, f_{m+i-1/2}\bigl(\tfrac{1}{z^{r+1}}\bigr) \sum_{\ell\ge0}(-1)^\ell\frac{(xz)^\ell}{\ell!}\, f_{\ell+1/2}\bigl(\tfrac{-1}{z^{r+1}}\bigr)\,dz\\
&=\; \mathrm{res}_z \sum_{m,\ell,q\ge 0} (-1)^\ell\, \frac{z^{m+\ell+i-q(r+1)}x^{m+\ell}}{m!\,\ell!}\Bigl(1+\frac{q-i-m-\ell-1}{r}\Bigr)_q \widetilde C_q\Bigl(r, m+i-\frac12, \ell+\frac12\Bigr)\Bigl(\frac{r}{2}\Bigr)^q dz\\
&=\; 0\,.
\end{aligned}$$

Proposition 5 is proved. □

Using formulas (169), (276), (277), Proposition 5, Lemma 8 and Proposition 3 we obtain that

$$\tau_{D_l}(g) \;=\; \frac{(-1)^{g+m-1}}{\bigl(\{\frac{2g-1}{r}\}\bigr)_{m+1}}\,\frac{\widetilde C_{\frac12,-\frac12,2g}(r)}{2^{2g}\,r^{3g}}\,. \tag{186}$$

Lemma 9. *For any $g\ge 0$, we have*

$$n_g \;=\; -\frac{\widetilde C_{2g}(r,\frac12,-\frac12)}{2^{2g}\,(2g-1)}\,. \tag{187}$$

Proof. From the definition of $\widetilde C_n(r,i,j)$ and the fact that X is an odd Laurent series in u^{-1} one easily obtains the equality

$$\sum_{g\ge0} \widetilde C_{2g}\Bigl(r,\frac12,-\frac12\Bigr)\,u^{2g} \;=\; -\frac{u^2}{2}\Biggl(\sqrt{\frac{X+1}{X-1}}+\sqrt{\frac{X-1}{X+1}}\Biggr)\frac{dX}{du} \;=\; \frac{-u^2X}{\sqrt{X^2-1}}\,\frac{dX}{du}\,. \tag{188}$$

Then it suffices to show that the series P defined by

$$P \;:=\; \sum_{g\ge0}\widetilde C_{2g}\Bigl(r,\frac12,-\frac12\Bigr)\,\frac{u^{2g}}{1-2g} \;=\; 1\,-\,\frac{r+2}{6}\,u^2\,+\,\cdots \tag{189}$$

satisfies

$$\sum_{j\ge0}\binom{j+r/2}{2j}\,\frac{(2u)^{2j}\,P^{r-2j}}{2j+1} \;=\; 1\,. \tag{190}$$

(Note that since in our case $r=2l-2$ is a non-negative even integer, the above sum terminates at $j=l-1$ and expresses P as an algebraic function of u, but even if r is a complex or formal variable the identity makes sense and will be shown to be true.) Comparing (188) and (189) we find

$$\frac{d}{dX}\Big(\frac{P}{u}\Big) \;=\; \frac{d(P/u)/du}{dX/du} \;=\; \frac{X}{\sqrt{X^2-1}}\,.$$

Therefore,

$$P \;=\; u\sqrt{X^2-1}\,.$$

Combining with (79) it suffices to show

$$\sum_{j\geq 0} \frac{2^{2j}}{2j+1}\binom{j+r/2}{2j}\,(X^2-1)^{r/2-j} \;=\; \frac{(X+1)^{r+1}-(X-1)^{r+1}}{2\,(r+1)}\,,$$

which is an elementary exercise (also equivalent to (152)). □

Identity (155) follows directly from (186). Formulas (186) and (187) yield (153). Formulas (153) and (130) yield (156).

Finally, the proof of the asymptotic formula (158) is very similar to that of the corresponding statement in Theorem 5, and will only be sketched. In view of the relation (153) we see that it is equivalent to the asymptotic formula

$$n_g(r) \;\sim\; \cos\big(\frac{\pi}{r}\big)\,\frac{(r+1)^{\frac{1}{r}-\frac{1}{2}}}{\sqrt{\pi g^3}}\,\frac{\sin((2g-1)\frac{\pi}{r})}{\big(-4\,(r+1)^{\frac{2}{r}}\sin^2(\frac{\pi}{r})\big)^g} \qquad (l>4) \tag{191}$$

for the coefficients $n_g(r)$ defined in (150). Writing (151) in the form $\big(\frac{Z^2}{Z^2-1}\big)^{r+1}-\big(\frac{1}{Z^2-1}\big)^{r+1}=\frac{r+1}{t^r}$ and setting the derivative of this expression with respect to Z equal to 0, we find that Z^2 is a non-trivial rth root of unity at all singularities of $y=y(t)$, and substituting this back into (151) we find that the singularities are given by $(t,Z^2)=((r+1)^{1/r}(\alpha-\beta),\alpha/\beta))$ where α and β are distinct rth roots of unity. In particular, the singular points of $y(t)$ of smallest absolute value are equal to $2\,(r+1)^{1/r}\sin(\pi/r)$ times $(2r)$th roots of unity. The rest of the proof is exactly along the lines of the proof of part (vii) of Theorem 5 and will be omitted. Just as in the A case for $l=1$, the proof has to be modified slightly if $l=4$ because the way that the sheets above the closed disk $\{|t|\leq 2\,(r+1)^{1/r}\sin(\pi/r)\}$ meet at their boundary is slightly different from what happens for $l>4$; we leave the details as an exercise.

This completes the proof of all parts of Theorem 6. □

Remark 8. It seems worth observing that the factors 1/2 and 3 appearing in part (vii) of Theorems 5 and 6 respectively are related to the symmetries of the corresponding Dynkin diagrams. Indeed, $|\mathrm{Sym}(A_l)|=2$ for $l>1$, 1 for $l=1$; $|\mathrm{Sym}(D_l)|=2$ for $l>4$, 6 for $l=4$.

We note that the polynomial P on the left-hand side of (151) also has the following expression:

$$P \;=\; \frac{2\,y^{r+1}\sinh\big[(r+1)\,\mathrm{arcsinh}\,\big(\frac{t}{2y}\big)\big]}{(r+1)\,t} \;-\; 1\,. \tag{192}$$

Using the formulas (150)–(153) one can compute the first few values of $\tau_{D_l}(g)$:

$$\tau_{D_l}(1) \;=\; \frac{r+2}{24}\,, \tag{193}$$

$$\tau_{D_l}(2) \;=\; \frac{(r+2)(r-6)(2r+1)}{5760\,r}\,, \tag{194}$$

$$\tau_{D_l}(3) \;=\; \frac{(r+2)(2r+1)(8r^3-77r^2+196r+188)}{2903040\,r^2}\,. \tag{195}$$

Here we note that $r \geq 6$ (as $l \geq 4$). For $r = 6$, our formulas agree with the explicit computations in [12]. More precisely, the explicit expression of the dual topological ODE of D_4-type was computed in [12], which reduces to a second order ODE for ϕ_3 (in our current notation):

$$108\,x^2\,\phi_3'' \;-\; \bigl(104\,x^8 + 108\,x\bigr)\,\phi_3' \;-\; \bigl(4\,x^{14} + 260\,x^7 + 39\bigr)\,\phi_3 \;=\; 0\,. \tag{196}$$

One could then give an alternative proof of Theorem 6 for D_4 by using (196). We leave this as an exercise because in the next section we will prove the algebracity for E_6 in this way.

Similarly as in the A case, we have the following corollary.

Corollary 4. *For all $r \geq 2g$, the value of $\tau_{D_l}(g)$ is a Laurent polynomial in r. Moreover, the value of this Laurent polynomial at $r = -1$ is equal to $\frac{(1-2^{1-2g})\,B_{2g}}{2g}$.*

Proof. If $r \geq 2g$, then $m = 0$, so the right-hand side of (153) reduces to $-n_g(r)/(-r)^{g-1}$. The first statement then follows from the fact that $n_g(r) \in \mathbb{Q}[r]$. By taking the $r \to -1$ limit in (152) we find the unique solution $y = \frac{t/2}{\sinh(t/2)}$, which yields the second statement of the corollary. □

It might be of interest to see whether the coefficients or the values of the Laurent polynomials in r occurring in the above corollary and the corollary to Theorem 5 have any topological meaning.

7. The E_6 case[4]

In the proof of Theorems 5 and 6 for the A_l, D_l cases in the previous two sections, we used the equivalent scalar Lax representation of the corresponding DS hierarchy [38]. For the E_6 case, as far as we know, the existence of such a representation is an open question. However, following Proposition 1, we can get the higher-genera one-point invariants by computing the dual topological ODE of E_6-type. (Of course, for any fixed l, as it was mentioned above (see (196)), we could alternatively have used the corresponding dual topological ODE to give a different proof of Theorems 5–6; this deserves a further study.) In this section we will use this to prove the algebraicity of a generating function of the τ-numbers for E_6.

We first compute the dual topological ODE of E_6-type. We use the 27-dimensional representation [49] of $\mathfrak{g} = E_6$. Recall that the Coxeter number and the exponents for this case read as follows:

$$r = 12\,, \quad m_1 = 1\,, \quad m_2 = 4\,, \quad m_3 = 5\,, \quad m_4 = 7\,, \quad m_5 = 8\,, \quad m_6 = 11\,,$$

and that the dimension of this simple Lie algebra is 78. Denote

$$\begin{aligned}
X_1 &= E_{6,7} + E_{8,9} + E_{10,11} + E_{12,14} + E_{15,17} + E_{26,27}\,,\\
X_2 &= E_{4,5} + E_{6,8} + E_{7,9} - E_{18,20} - E_{21,22} - E_{23,24}\,,\\
X_3 &= E_{4,6} + E_{5,8} + E_{11,13} + E_{14,16} + E_{17,19} + E_{25,26}\,,\\
X_4 &= E_{3,4} - E_{8,10} - E_{9,11} - E_{16,18} - E_{19,21} + E_{24,25}\,,\\
X_5 &= E_{2,3} - E_{10,12} - E_{11,14} - E_{13,16} + E_{21,23} + E_{22,24}\,,\\
X_6 &= E_{1,2} + E_{12,15} + E_{14,17} + E_{16,19} + E_{18,21} + E_{20,22}\,.
\end{aligned}$$

[4] This section is by D.Y. and D.Z. only, who found the proof in the E_6 case after B.D. passed away.

Kostant's sl_2-subalgebra of $\mathfrak{g}$ can be given by

$$I^- = 16X_1^T + 22X_2^T + 30X_3^T + 42X_4^T + 30X_5^T + 16X_6^T \,, \tag{197}$$
$$I^+ = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \,, \tag{198}$$
$$\rho^\vee = \frac{1}{2}\big(I^+I^- - I^-I^+\big)\,. \tag{199}$$

Using the above data one can compute explicitly the dual topological ODE of E_6-type, i.e., equation (48) for $G \in \mathfrak{g}$, or equivalently (50) for the vector $\phi = (\phi_1, \ldots, \phi_6) \in \mathbb{C}^6$. We denote by $(\phi_{\alpha;\beta})_{\beta=1,\ldots,6}$ $(\alpha = 1, \ldots, 6)$ the six linearly independent vector solutions that were introduced in Section 3. (Here β labels the components of the vector.) Recall that each $\phi_{\alpha;6}$ has the form (52). Following the principle of Theorem 4, namely, by reducing the ODE (50) for the vector-valued function ϕ to a scalar ODE for the top component ϕ_6 (the highest weight vector of $\mathfrak{g}$), we find the following fourth-order one:

$$\begin{aligned} 0 \;=\;& 2985984\, x^4\big(37\, x^{39} - 2775\, x^{26} - 36960\, x^{13} + 11520\big)\, \phi_6'''' \\ &- 466560\, x^3\big(2331\, x^{52} - 162985\, x^{39} - 2985600\, x^{26} - 4951296\, x^{13} + 811008\big)\, \phi_6''' \\ &- 27\, x^2\big(6545189\, x^{65} - 1276342935\, x^{52} + 10115971680\, x^{39} - 127523831040\, x^{26} \\ &\qquad + 860446310400\, x^{13} - 52110950400\big)\, \phi_6'' \\ &- 27\, x\,\big(23310\, x^{78} - 8293439\, x^{65} - 3559160940\, x^{52} - 153887586840\, x^{39} \\ &\qquad + 1228034776320 x^{26} + 236111616000\, x^{13} + 49235558400\big)\, \phi_6' \\ &+ \big(37x^{91} - 3464310\, x^{78} + 2278737540\, x^{65} + 114309996390\, x^{52} \\ &\qquad + 10889113435200\, x^{39} - 60840963615600\, x^{26} \\ &\qquad - 15770999462400\, x^{13} - 328914432000\big)\, \phi_6\,. \end{aligned} \tag{200}$$

This means that every $\phi_{\alpha;6}$ must satisfy (200). From the discussion in Section 3, we know that each $\phi_{\alpha;6}$ has the form

$$\phi_{\alpha;6} \;=\; x^{1+\frac{13}{12}m_\alpha}\, f_\alpha\big(x^{13}\big) \tag{201}$$

for some power series $f_\alpha(u) \in \mathbb{Q}[[u]]$ (resp. $u^{-1}\mathbb{Q}[[u]]$ for $\alpha = 6$). But if we use the Frobenius method, then we find that the indicial equation for (200) at $x = 0$ has only four roots 25/12, 77/12, 103/12, $-1/12$. This implies that $\phi_{2;6}$ and $\phi_{5;6}$ must both vanish (which was not obvious from their original definition), while the expansions of the other $\phi_{\alpha;6}$, if we normalize to make the power series monic, are given by (201) with $1 + \frac{13}{12}m_\alpha = \frac{26\alpha-1}{12}$ and

$$\begin{aligned} f_1(u) &= 1 + \frac{4235}{2^9\, 3^1\, 13}\, u + \frac{23102233}{2^{18}\, 3^2\, 13^2}\, u^2 + \frac{381109489145}{2^{29}\, 3^3\, 13^3}\, u^3 + \ldots\,, \\ f_3(u) &= 1 + \frac{4613}{2^{10}\, 13}\, u + \frac{340813583}{2^{19}\, 3^2\, 5^1\, 13^2}\, u^2 + \frac{1468738987769}{2^{28}\, 3^2\, 13^3\, 17}\, u^3 + \ldots\,, \\ f_4(u) &= 1 + \frac{34829}{2^8\, 5^1\, 7^1\, 13}\, u + \frac{112497481}{2^{20}\, 3^2\, 13^2}\, u^2 + \frac{45611422760339}{2^{28}\, 3^2\, 5^1\, 7^1\, 13^3\, 19}\, u^3 + \ldots\,, \\ u\, f_6(u) &= 1 + \frac{435}{2^8\, 13}\, u + \frac{330276383}{2^{19}\, 3^2\, 11^1\, 13^2}\, u^2 + \frac{7178883185}{2^{27}\, 3^1\, 13^3}\, u^3 + \ldots\,. \end{aligned}$$

According to Proposition 1, the numbers $\tau_{E_6}(g)$ with $g \geq 1$ in the particular normalizations that we are using, can be expressed in terms of $\phi_{\alpha;6}$ as follows: for $g \not\equiv 2 \,(\mathrm{mod}\, 3)$,

$$\tau_{E_6}(g) \;=\; \frac{c_\alpha}{2^{6m}\, 3^{4m}}\, \big[u^m\big](f_\alpha)\,, \tag{202}$$

where $\alpha \in \{1,3,4,6\}$ and m are determined by $2g-1 = m_\alpha + 12m$ and

$$c_1 = \frac{1}{4}\,, \quad c_3 = \frac{5}{1152}\,, \quad c_4 = \frac{25}{27648}\,, \quad c_6 = \frac{1}{2^6 3^4}\,; \tag{203}$$

otherwise $\tau_{E_6}(g) = 0$. For simplicity we set $\tau_{E_6}(0) = 1$ which also agrees with (202). It should be noted that here the freedom of normalizations is fixed in such a way that it agrees with the explicit Frobenius manifold potential of E_6-type given by Klemm–Theisen–Schmidt [81] (see (241) below) as well as with the genus 1 formula of Dubrovin–Zhang [54] (cf. [49, 66]). For the reader's convenience, let us list the first few values of $\tau_{E_6}(g)$ in the following table:

g	0	1	2	3	4	5	6	7	8	9	10
$\tau_{E_6}(g)$	1	$\frac{1}{4}$	0	$\frac{5}{1152}$	$\frac{25}{27648}$	0	$\frac{145}{5750784}$	$\frac{4235}{414056448}$	0	$\frac{23065}{79498838016}$	$\frac{174145}{3338951196672}$

One-point FJRW invariants of E_6-type

Theorem 7. *Set $B = 2^{1/12}\big(3+2\sqrt{3}\big)^{1/4}$. Define A_k, $k \ge -1$ from the generating series*

$$U(V) \;=\; \frac{1}{B} \sum_{k\ge -1} A_k\, V^{-k/12}\,,$$

where $U(V)$ is the unique solution in

$$\frac{1}{B}\, V^{1/12} \,+\, \mathbb{C}\big[\big[V^{-1/12}\big]\big]$$

to the polynomial equation

$$\begin{aligned} &U^{24} \,-\, 36\, U^{22} \,+\, 540\, U^{20} \,-\, 4488\, U^{18} \,+\, 22992\, U^{16} \,-\, 76032\, U^{14} \\ &+\, \Big(5\,V + \frac{2140032}{13}\Big)\, U^{12} \,-\, 2\,\Big(19\,V + \frac{1501248}{13}\Big)\, U^{10} \,+\, 108\,\Big(V + \frac{24320}{13}\Big)\, U^{8} \\ &-\, 32\,\Big(5\,V + \frac{41104}{13}\Big)\, U^{6} \,+\, 32\,\Big(5\,V + \frac{9696}{13}\Big)\, U^{4} \,-\, 96\,\Big(V + \frac{192}{13}\Big)\, U^{2} \\ &-\, \frac{1}{108}\,\Big(V^2 - \frac{34560}{13}\, V - \frac{442368}{169}\Big) \;=\; 0\,, \end{aligned} \tag{204}$$

of degree 12 in U^2 and 2 in V. Then for all $g \ge 0$ with $g \not\equiv 2 \,(\mathrm{mod}\, 3)$ we have

$$\tau_{E_6}(g) \;=\; \psi_\rho\, 2^{-\frac{13}{6}g}\, 3^{-\frac{7}{6}g}\, \frac{A_{2g-1}}{\big(\big\{\frac{2g-1}{12}\big\}\big)_m}\,, \tag{205}$$

where $\rho \in \{1,5,7,11\}$ and m are such that $2g-1 = \rho + 12\,m$, and ψ_ρ are given by

$$\psi_1 = 2^{\frac{1}{2}}\, 3^{\frac{5}{12}}\,, \quad \psi_5 = 2^{1}\, 3^{\frac{3}{4}}\, \varepsilon_3^{-\frac{1}{2}}\,, \quad \psi_7 = -2^{\frac{3}{2}}\, 3^{\frac{2}{3}}\, \varepsilon_3^{-\frac{1}{2}}\,, \quad \psi_{11} = -12\,, \tag{206}$$

where $\varepsilon_3 = 2 + \sqrt{3}$.

Proof. The differential equation of ϕ_6 translates into recursions for the coefficients of f_α with $\alpha = 1,3,4,6$. On the other hand, any algebraic function satisfies a linear differential equation and hence its coefficients satisfy a recursion. One verifies by computer that the recursions and the initial conditions agree. □

The first few values of A_k are given by

$$A_{-1} = 1\,, \quad A_1 = 2^{-\frac{1}{3}}\, 3^{\frac{3}{4}}\,, \quad A_5 = 2^{-\frac{3}{2}}\, 3^{\frac{3}{4}}\, 5^1\, \varepsilon_3^{\frac{1}{2}}\,, \quad A_7 = -2^{-\frac{17}{6}}\, 3^1\, 5^2\, \varepsilon_3^{\frac{1}{2}}\,. \tag{207}$$

In general, the numbers A_k are (up to fractional powers of 2, 3 and ε_3) algebraic integers belonging to $\mathbb{Q}(\sqrt{3})$, so that Theorem 7 implies an integrality statement for the $\tau_{E_6}(g)$ similar to the first one in Theorem 1. Another consequence of the theorem is an asymptotic formula for the numbers $\tau_{E_6}(g)$ similar to the ones we found for the A and D cases:

Corollary 5. *As $g\to\infty$ with $g\not\equiv 2\,(\mathrm{mod}\,3)$, we have*

$$\tau_{E_6}(g) \;\sim\; 13^{-5/12}\,\sqrt{\pi}\,\frac{\theta_\rho}{\Gamma\big(1-\{\frac{2g-1}{12}\}\big)}\,\frac{1}{\Gamma\big(\frac{2g-1}{12}\big)}\,g^{-\frac{3}{2}}\left(\frac{\sqrt{3+2\sqrt{3}}}{2^1\,3^{\frac{7}{6}}\,13^{\frac{1}{6}}}\right)^{g}, \tag{208}$$

where $\rho\in\{1,5,7,11\}$ and m are such that $2g-1=\rho+12\,m$, and θ_ρ are constants defined by

$$\theta_1 \;=\; 2^1\,3^{\frac{5}{12}}\,\big(1+\sqrt{3}\big)\,,\quad \theta_5 \;=\; 2^2\,3^{\frac{3}{4}}\,,\quad \theta_7 \;=\; 2^{\frac{5}{2}}\,3^{\frac{2}{3}}\,,\quad \theta_{11} \;=\; 2^{\frac{5}{2}}\,3^1\,\big(1+\sqrt{3}\big)\,. \tag{209}$$

Proof. Using (205) we find that it suffices to show that as $g\to\infty$ with $g\not\equiv 2\,(\mathrm{mod}\,3)$,

$$A_{2g-1} \;\sim\; \frac{13^{-5/12}}{\sqrt{\pi}}\,\chi_\rho\,Q^{g/6}\,g^{-3/2} \tag{210}$$

with $Q=2^7\,\big(3+2\sqrt{3}\big)^3/13$ and with

$$\chi_1 \;=\; -\chi_{11} \;=\; 1\,,\quad \chi_5 \;=\; -\chi_7 \;=\; \varepsilon_3\,. \tag{211}$$

The corollary can then be proved again by using the similar argument as for the A case. □

Remark 9. We expect that Theorem 7 (and its corollary) will have an analogue for E_7 and E_8, and actually also for all non-simply-laced simple Lie algebras.

8. Explicit relation for FJRW invariants of $\mathfrak{g}$-type

In this section, based on the results in the previous sections and on the theory of Frobenius manifolds (see [43, 46, 47, 75, 94, 111]), we generalize Theorem 3 from A_4 to A_l, D_l and E_6.

According to Kontsevich–Manin [84, 94] (cf. Appendix B), a homogeneous CohFT of charge d gives a formal Frobenius manifold of charge d with the formal Frobenius potential $F=F(v)$ given by (27). Denote by B the domain of convergence of F. Let $\mathfrak{g}$ be a simply-laced simple Lie algebra of rank l. As before, denote by $\Omega_{g,n}$ the FJRW CohFT of $\mathfrak{g}$-type and by r the Coxeter number. It follows from the quasi-homogeneity (38) that the formal Frobenius potential F is a polynomial. Therefore, the convergence domain B is the whole of $\mathbb{C}^l$. We call the Frobenius manifold B constructed from $\Omega_{g,n}$ *the Frobenius manifold of $\mathfrak{g}$-type*. Note that the Frobenius structure on B given by (305)–(306) can alternatively be constructed from the orbit space of the Coxeter group of $\mathfrak{g}$-type ([43, 45, 106, 125]), or else from the miniversal deformation of a simple singularity of $\mathfrak{g}$-type ([22, 36, 37, 46, 75, 104]). The charge and the spectrum of B are

$$d \;=\; \frac{r-2}{r}\,,\quad \mu_\alpha=\frac{m_\alpha}{r}-\frac{1}{2}\,,\quad R \;=\; 0\,, \tag{212}$$

and the Euler vector field of B is

$$E \;=\; \sum_{\alpha=1}^{l}\frac{r+1-m_\alpha}{r}\,v^\alpha\partial_\alpha\,. \tag{213}$$

Before stating and proving the generalized theorem, we first prove a useful lemma.

Lemma 10. *If $\mathfrak{g}$ is a simply-laced simple Lie algebra, there exists a unique choice of calibration $\{\theta_{\alpha,m}\}_{m\geq 0}$ for the Frobenius manifold of $\mathfrak{g}$-type.*

Proof. The existence of $\theta_{\alpha,m}$ that satisfy (299)–(304) is known. (For the meaning of calibration and the proof of the existence see Appendix B.) We only need to prove the uniqueness. First of all, observing that set of the differences $\{\mu_\alpha - \mu_\beta\}$ does not contain positive integers, we know from [43, 46] that equations (288)–(289) and (293)–(295) determine $\Theta(v;z)$ uniquely. Therefore, $\theta_{\alpha,m}$ are uniquely determined by (296) and (297) up to constants only, i.e., if $\{\tilde\theta_{\alpha,m}\}_{m\geq 0}$ also satisfies (299)–(302), then $\tilde\theta_{\alpha,m}(v) - \theta_{\alpha,m}(v) = b_{\alpha,m}$ with $b_{\alpha,m}$ being constants. The uniqueness then follows immediately from (304). The lemma is proved. □

Lemma 10 is true for the Frobenius manifold associated to any finite Coxeter group with the same proof as above.

The following theorem gives the main result of this section.

Theorem 8. *Let $\mathfrak{g}$ be a simple Lie algebra of type A_l, D_l, or E_6, and let $(B, \langle\,,\,\rangle, \cdot)$ denote the Frobenius manifold of $\mathfrak{g}$-type. Take $v = (v_1, \dots, v_l)$ the flat coordinates on B and $(\theta_{\alpha,m})_{\alpha=1,\dots,l,m\geq 0}$ the associated unique calibration. Denote $v^*_\alpha = \langle \tau_{\alpha,m_\alpha}\rangle$. Then the following identity holds:*

$$\langle \tau_{\alpha,m_\alpha+(r+1)m}\rangle \;=\; \theta_{\alpha,m}(v^*)\,, \quad \forall\, m \geq 0\,. \tag{214}$$

Here, we recall that $\langle\tau_{\alpha,q}\rangle$ denote the one-point FJRW invariants of $\mathfrak{g}$-type.

Proof. Let us prove this theorem case by case from A to E.

Start with the A_l case. Consider the Frobenius manifold of A_l-type. In this case, $r = l+1$ is the Coxeter number. Define

$$\lambda \;=\; \lambda(p;s) \;=\; p^{l+1} + \sum_{\beta=1}^{l} s_\beta\, p^{l-\beta}\,. \tag{215}$$

Here $s = (s_1, \dots, s_l) \in \mathbb{C}^l$. The reader who is familiar with the theory of Frobenius manifolds recognizes λ as the superpotential, and the Frobenius structure on B could be defined by the standard formulas [43, 46, 47] (see also [36, 59, 60, 104, 105]) using the superpotential under some carefully chosen normalization factors, and the flat coordinates v are related to s by

$$v^\alpha \;=\; -\frac{(\sqrt{-r})^{\frac{3\alpha}{r+1}-1}}{r-\alpha}\,\mathrm{res}_{p=\infty}\,\lambda^{\frac{r-\alpha}{r}}\,dp\,. \tag{216}$$

It is not difficult to verify that $\det\bigl(\frac{\partial v^\alpha}{\partial s_\beta}\bigr) \neq 0$, so the map $s \mapsto v$ given by (216) is indeed an invertible coordinate transformation. By a degree argument, this coordinate change (216) actually has the triangular nature.

Lemma 11. *For the A_l case, the unique functions $\theta_{\alpha,m}(v)$, $m \geq 0$ have the following expressions:*

$$\theta_{\alpha,m}(v) \;=\; -\frac{(-1)^{m+1}}{(\sqrt{-r})^{3\frac{1+\alpha+rm}{r+1}}\,(\frac{\alpha}{r})_{m+1}}\,\mathrm{res}_{p=\infty}\,\lambda^{\frac{\alpha+rm}{r}}\,dp\,. \tag{217}$$

Proof. We prove this lemma by showing that the right-hand side of (217), denoted by $\tilde\theta_{\alpha,m}$, satisfies the defining conditions (299)–(304). The conditions (302) with $z = 0$ and (299) are known to hold true. Indeed, they follow from the Laplace-type transform between the deformed flat coordinates and λ-periods of the Frobenius manifold, with λ being the corresponding superpotential [4, 36, 37, 43, 46, 60, 61, 104, 105] and with the careful choosing of the rescaling factor to match with Witten's normalization of times [114]. Here, the definitions of the product and the invariant flat metric can be found *ibid.*, and we also normalize the metric such that $\eta_{\alpha\beta} = \delta_{\alpha+\beta,l+1}$. The equality (303) can be easily proved by computing the residues, which also implies (301). Since equality (302) is a rewritting of (295), we know that (302) is actually a consequence of (299)–(301). To show (300), introduce the following extended Euler operator:

$$\widetilde{E} \;:=\; E + \frac{1}{r}\, p\, \frac{\partial}{\partial p}\,, \tag{218}$$

where E in this case is given by (213) with $m_\alpha = \alpha$. From (215) and (216) we obtain

$$\widetilde{E}\,\lambda \;=\; \lambda\,. \tag{219}$$

The required quasi-homogeneity (300) for the gradients of $\theta_{\alpha,m}$ then holds true for the gradients of $\tilde{\theta}_{\alpha,m}$. It remains to show (304). Observe from the definition (217)–(219) that the $\tilde{\theta}_{\alpha,m}$ themselves satisfy the following quasi-homogeneity condition:

$$E\,\theta_{\alpha,m} \;=\; \Big(m + \mu_\alpha + \frac{1}{r} + \frac{1}{2}\Big)\,\theta_{\alpha,m}\,, \quad \forall\, m \geq 0\,. \tag{220}$$

Hence the $\tilde{\theta}_{\alpha,m}$, $m \geq 0$, which are polynomials of v, do not contain constant terms. This gives (304). The lemma is proved. □

We are ready to show (214). For $m = 0$, the validity of (214) is obvious, because from the definition we know that $\theta_{\alpha,0} = v_\alpha$. Let us now prove the validity of (214) for an arbitrary $m \geq 0$. Following [43, 46, 47], consider the polynomial equation

$$\lambda(p; s) \;=\; \xi^r\,. \tag{221}$$

We know that equation (221) has a unique solution $p = p(\xi; s)$ in $\xi + \mathbb{C}[s][[\xi^{-1}]]$. Write

$$p(\xi; s) \;=\; \xi \,+\, \sum_{k\geq 1} u_k(s)\, \xi^{-k}\,, \tag{222}$$

where $u_k(s) \in \mathbb{Q}[s]$, $k \geq 1$. Using similar arguments to those in the previous sections we have

$$k\, u_k(s) \;=\; \mathrm{res}_{p=\infty}\; \lambda(p; s)^{\frac{k}{r}}\, dp\,. \tag{223}$$

Comparing this equality with (217) we obtain that

$$\theta_{\alpha,m}(v) \;=\; -\,\frac{(-1)^{m+1}(\alpha + mr)}{(\sqrt{-r})^{3\frac{1+\alpha+rm}{r+1}}\,(\frac{\alpha}{r})_{m+1}}\, u_{\alpha+mr}(s)\,, \quad \forall\, m \geq 0\,. \tag{224}$$

Now define

$$s^*_\beta \;=\; \begin{cases} \binom{r}{2i}\frac{a^{2i}}{1+2i}\,, & \beta = 2i-1 \ (i = 1, \dots, [r/2])\,; \\ 0\,, & \text{otherwise.} \end{cases} \tag{225}$$

And we restrict our discussion to the particular point $s = s^*$ on B. Here

$$a \;:=\; \sqrt{-r}^{\frac{2-r}{r+1}}/2\,. \tag{226}$$

From (215), (221) and (225), we know that the series $p(\xi; s^*)$ satisfies

$$\sum_{i=0}^{[r/2]} \binom{r}{2i} a^{2i}\, \frac{p(\xi; s^*)^{r-2i}}{1+2i} \;=\; \xi^r\,.$$

Comparing this equation with formula (107)–(109) (cf. (123)) we get

$$\tau_{A_{r-1}}(g) \;=\; \frac{(-1)^{m+g}\, r^{1-g}}{2^{2g}\, a^{2g}\, (\frac{\alpha}{r})_m}\, u_{2g-1}\,,$$

where g, α, m are related by $2g - 1 = \alpha + rm$. Substituting (224) in this equality and using (20) and (226), we obtain (214). This completes the proof for the A case.

We continue to prove the statement for the D case. In this case, the Coxeter number $r = 2l-2$. Similarly as in the A case, introduce the superpotential [36]:

$$\lambda \;=\; p^r \,+\, \sum_{\beta=1}^{l-1} s_\beta\, p^{r-2\beta} \,+\, s_l^2\, p^{-2}\,. \tag{227}$$

The flat coordinates in our normalization are given by

$$v^\alpha \;:=\; -\,\frac{\left(\sqrt{-r}\right)^{\frac{3m_\alpha}{r+1}-1}}{r-m_\alpha}\,\mathrm{res}_{p=\infty}\lambda^{\frac{r-m_\alpha}{r}}\,dp\,,\quad \alpha=1,\dots,l-1 \tag{228}$$

and

$$v^l \;:=\; -\frac{2\sqrt{l-1}}{\sqrt{-r}^{\frac{3}{2}\frac{r+2}{r+1}}}\,s_l\,. \tag{229}$$

It is not difficult to verify again that $\det\bigl(\partial v^\alpha/\partial s_\beta\bigr)\neq 0$, and the map $s\mapsto v$ given by (228)–(229) indeed gives an invertible coordinate transformation.

Lemma 12. *The unique functions $\theta_{\alpha,m}(v)$, $m\geq 0$ have the following expressions:*

$$\theta_{\alpha,m}(v) \;=\; -\frac{(-1)^{m+1}}{\left(\sqrt{-r}\right)^{3\frac{1+m_\alpha+rm}{r+1}}\left(\frac{m_\alpha}{r}\right)_{m+1}}\,\mathrm{res}_{p=\infty}\,\lambda^{\frac{m_\alpha}{r}+m}dp\,,\quad \alpha=1,\dots,l-1\,, \tag{230}$$

$$\theta_{l,m}(v) \;=\; \frac{(-1)^{m+1}\sqrt{l-1}}{\left(\sqrt{-r}\right)^{3\frac{l+rm}{r+1}}\left(\frac{1}{2}\right)_{m+1}}\,\mathrm{res}_{p=0}\,\lambda^{\frac{1}{2}+m}dp\,. \tag{231}$$

Proof. The proof is almost identical with that of Lemma 11, so we only give some brief indications here. Note that the invariant metric η in our normalization is given by

$$\eta \;=\; \begin{pmatrix} 0 & \dots & 0 & 1 & 0\\ \vdots & \iddots & \iddots & 0 & 0\\ 0 & \iddots & \iddots & \iddots & \vdots\\ 1 & 0 & \iddots & \iddots & 0\\ 0 & 0 & \dots & 0 & 1\end{pmatrix}.$$

The extension of the Euler vector field is again given by (218). The lemma is proved. □

Following [43, 46, 47], we consider the following polynomial equation:

$$\lambda(p;s) \;=\; \xi^r\,. \tag{232}$$

It is easy to see that (232) has a unique solution $p=p^+(\xi;s)$ in $\xi+\mathbb{C}[s][[\xi^{-1}]]$ as well as a unique solution $p=p^-(\xi;s)$ in $s_l\,\xi^{-r/2}+\xi^{-(r+2)/2}\mathbb{C}[s][[\xi^{-1}]]$. Write

$$p^+(\xi;s) \;=\; \xi \,+\, \sum_{k\geq1} u_k(s)\,\xi^{-k}\,,\quad p^-(\xi;s) \;=\; \sum_{m\geq 0} v_m(s)\,\xi^{-m-\frac{r}{2}}\,. \tag{233}$$

Here $v_0(s)=s_l$. Differentiating both equations in (233) with respect to ξ, we find

$$\frac{dp^+}{d\xi} \;=\; 1 \,-\, \sum_{k\geq1} k\,u_k(s)\,\xi^{-k-1}\,,\quad \frac{dp^-}{d\xi} \;=\; -\sum_{m\geq0}\Bigl(m+\frac{r}{2}\Bigr)\,v_m(s)\,\xi^{-m-\frac{r}{2}-1}\,. \tag{234}$$

Therefore,

$$k u_k(s) \;=\; \mathrm{res}_{\xi=\infty} \frac{dp^+(\xi;s)}{d\xi} \xi^k d\xi \;=\; \mathrm{res}_{p=\infty} \lambda(p;s)^{\frac{k}{r}} dp\,, \tag{235}$$

$$\Big(m+\frac{r}{2}\Big) v_m(s) \;=\; \mathrm{res}_{\xi=\infty} \frac{dp^-(\xi;s)}{d\xi} \xi^{rm+\frac{r}{2}} d\xi \;=\; -\,\mathrm{res}_{p=0} \lambda(p;s)^{m+\frac{1}{2}} dp\,. \tag{236}$$

Comparing (235) and (236) with (230) and (231), respectively, we have for $\alpha = 1,\dots,l-1$,

$$\theta_{\alpha,m}(v) \;=\; -\frac{(-1)^{m+1}(m_\alpha+mr)}{(\sqrt{-r})^{3\frac{1+m_\alpha+rm}{r+1}} \, (\frac{m_\alpha}{r})_{m+1}} u_{m_\alpha+mr}(s)\,, \quad \forall\, m\ge 0\,, \tag{237}$$

and for $\alpha = l$,

$$\theta_{l,m}(v) \;=\; -\sqrt{l-1}\frac{(-1)^{m+1}(l-1+mr)}{(\sqrt{-r})^{3\frac{l+rm}{r+1}} \, (\frac{1}{2})_{m+1}} v_m(s)\,, \quad \forall\, m\ge 0\,. \tag{238}$$

Define

$$s^*_\beta \;=\; \begin{cases} \binom{l-1+\beta}{2\beta}\frac{a^{2\beta}}{1+2\beta}\,, & \beta=1,\dots,l-1\,, \\ 0\,, & \text{otherwise,} \end{cases} \tag{239}$$

and we now restrict the discussion to the particular point $s = s^*$ on B. Here

$$a \;:=\; \sqrt{-r}^{\frac{2-r}{r+1}}\,. \tag{240}$$

From (227), (232) and (239), we know that the series $p^+(\xi;s^*)$ and $p^-(\xi;s^*)$ satisfy

$$\sum_{i=0}^{l-1} \binom{i+l-1}{2i} \frac{a^{2i}}{1+2i} \, p^+(\xi;s^*)^{r-2i} \;=\; \xi^r\,, \qquad p^-(\xi;s^*) \;=\; 0\,.$$

Comparing with (150)–(153) we get

$$\tau_{D_l}(g) \;=\; \frac{(-1)^{m+g}\, r^{1-g}}{a^{2g}\,(\frac{m_\alpha}{r})_m}\, u_{2g-1}\,,$$

where g is related with α, m by $2g-1 = m_\alpha + rm$. Substituting (237) in this identity and using (42), and (226), we find (214). This completes the proof for the D case.

Finally, let us prove the statement for the E_6 case. Let $B = \mathbb{C}^6$ be the Frobenius manifold of E_6-type. The spectrum data (μ, R) are now given by

$$\mu_1 = -\frac{5}{12}\,, \quad \mu_2 = -\frac{1}{6}\,, \quad \mu_3 = -\frac{1}{12}\,, \quad \mu_4 = \frac{1}{12}\,, \quad \mu_5 = \frac{1}{6}\,, \quad \mu_6 = \frac{5}{12}\,, \quad R = 0\,.$$

According to Klemm–Theisen–Schmidt [81], the Frobenius potential has the explicit expression:

$$\begin{aligned} F \;=\; & \frac{1}{2}\,(v^1)^2\,v^6 \,+\, v^1\,v^2\,v^5 \,+\, v^1\,v^3\,v^4 \,+\, \frac{1}{2}\,(v^2)^2\,v^3 \\ & +\, \frac{1}{2}\,(v^2)^2\,v^4\,v^6 \,+\, \frac{1}{2}\,v^2\,(v^4)^2\,v^5 \,+\, \frac{1}{6}\,(v^3)^3\,v^6 \,+\, \frac{1}{4}\,(v^3)^2\,(v^5)^2 \\ & +\, \frac{1}{2}\,v^2\,v^3\,v^5\,(v^6)^2 \,+\, \frac{1}{6}\,v^2\,(v^5)^3\,v^6 \,+\, \frac{1}{2}\,v^3\,v^4\,(v^5)^2\,v^6 \,+\, \frac{1}{12}\,v^4\,(v^5)^4 \,+\, \frac{1}{12}\,(v^4)^4\,v^6 \\ & +\, \frac{1}{6}\,v^2\,v^4\,v^5\,(v^6)^3 \,+\, \frac{1}{24}\,(v^2)^2\,(v^6)^4 \,+\, \frac{1}{4}\,(v^4)^2\,(v^5)^2\,(v^6)^2 \,+\, \frac{1}{6}\,v^3\,(v^4)^2\,(v^6)^3 \\ & +\, \frac{1}{60}\,(v^3)^2\,(v^6)^5 \,+\, \frac{1}{24}\,v^4\,(v^5)^2\,(v^6)^5 \,+\, \frac{1}{24}\,v^3\,(v^5)^2\,(v^6)^4 \,+\, \frac{1}{24}\,(v^5)^4\,(v^6)^3 \\ & +\, \frac{1}{252}\,(v^4)^2\,(v^6)^7 \,+\, \frac{1}{576}\,(v^5)^2\,(v^6)^8 \,+\, \frac{(v^6)^{13}}{185328}\,. \end{aligned} \tag{241}$$

Here, $(v^1,\dots,v^6)\in B$ is a system of flat coordinates satisfying that ∂_1 is the identity vector field. The invariant flat metric and the Euler vector filed in this coordinate system are given by $\eta_{\alpha\beta}=\delta_{\alpha+\beta,7}$, and

$$E \;=\; v^1\,\partial_1 \,+\, \frac{3}{4}\,v^2\,\partial_2 \,+\, \frac{2}{3}\,v^3\,\partial_3 \,+\, \frac{1}{2}\,v^4\,\partial_4 \,+\, \frac{5}{12}\,v^5\,\partial_5 \,+\, \frac{1}{6}\,v^6\,\partial_6\,. \tag{242}$$

The first few values of the unique calibration $\theta_{\alpha,p}$ for B are

$$\begin{aligned} \theta_{\alpha,0} \;&=\; v_\alpha\,, \\ \theta_{1,1} \;&=\; v^1\,v^6 \,+\, v^3\,v^4 \,+\, v^2\,v^5\,, \\ \theta_{2,1} \;&=\; v^1\,v^5 \,+\, v^2\,v^3 \,+\, v^2\,v^4\,v^6 \,+\, \frac{1}{2}\,(v^4)^2\,v^5 \\ &\quad +\, \frac{1}{2}\,v^3\,v^5\,(v^6)^2 \,+\, \frac{1}{6}\,v^4\,v^5\,(v^6)^3 \,+\, \frac{1}{6}\,(v^5)^3\,v^6 \,+\, \frac{1}{12}\,v^2\,(v^6)^4\,. \end{aligned}$$

To prove (214), similarly as in the A and D cases, we will use the corresponding superpotential. Following Eguchi–Yang [62] (cf. [19, 82, 86]), introduce three polynomials Q_1, P_1, P_2:

$$\begin{aligned} Q_1 \;=\; & 270\,p^{15} \,+\, \big(171+57\sqrt{3}\big)\,t_{10}\,p^{13} \,+\, \big(54+27\sqrt{3}\big)\,t_{10}^2\,p^{11} \\ & +\, \big(126+84\sqrt{3}\big)\,t_7\,p^{10} \,+\, \Big(\Big(\frac{35}{4}+\frac{175}{36}\sqrt{3}\Big)\,t_{10}^3 + \big(144+72\sqrt{3}\big)\,t_6\Big)\,p^9 \\ & +\, \Big(\frac{135}{2}+\frac{81}{2}\sqrt{3}\Big)\,t_7\,t_{10}\,p^8 \\ & +\, \Big(\Big(\frac{225}{4}+\frac{125}{4}\sqrt{3}\Big)\,t_6\,t_{10} + \Big(\frac{345}{384}+\frac{35}{96}\sqrt{3}\Big)\,t_{10}^4 + \big(135+81\sqrt{3}\big)\,t_4\Big)\,p^7 \\ & +\, \Big(\big(126+72\sqrt{3}\big)\,t_3 + \Big(10+\frac{35}{6}\sqrt{3}\Big)\,t_7\,t_{10}^2\Big)\,p^6 \\ & +\, \Big(\Big(\frac{63}{4}+9\sqrt{3}\Big)\,t_7^2 + \big(36+21\sqrt{3}\big)\,t_4\,t_{10} + \Big(\frac{11}{768}+\frac{19\sqrt{3}}{2304}\Big)\,t_{10}^5 + \Big(\frac{21}{4}+3\sqrt{3}\Big)\,t_6\,t_{10}^2\Big)\,p^5 \\ & +\, \Big(\Big(\frac{33}{2}+\frac{19}{2}\sqrt{3}\Big)\,t_3\,t_{10} + \Big(\frac{19}{48}+\frac{11}{48}\sqrt{3}\Big)\,t_7\,t_{10}^3 + \big(24+14\sqrt{3}\big)\,t_6\,t_7\Big)\,p^4 \\ & -\, \Big(\frac{11}{8}+\frac{19}{24}\sqrt{3}\Big)\,t_7^3\,t_{10}\,p^3 \,+\, \Big(\frac{45}{4}+\frac{13}{2}\sqrt{3}\Big)\,t_3\,t_7\,p \,+\, \Big(\frac{5}{8}+\frac{13}{36}\sqrt{3}\Big)\,t_7^3\,, \end{aligned} \tag{243}$$

$$P_1 \;=\; 78\,p^{10} \,+\, \big(30+10\sqrt{3}\big)\,t_{10}\,p^8 \,+\, \Big(\frac{14}{3}+\frac{7}{3}\sqrt{3}\Big)\,t_{10}^2\,p^6 \,+\, \Big(\frac{33}{2}+11\sqrt{3}\Big)\,t_7\,p^5$$

$$
\begin{aligned}
&+\ \Big(\Big(\frac{1}{4}+\frac{5}{36}\sqrt{3}\Big)\,t_{10}^3+\big(16+8\sqrt{3}\big)\,t_6\Big)\,p^4\ +\ \Big(\frac{25}{12}+\frac{5}{4}\sqrt{3}\Big)\,t_7\,t_{10}\,p^3\\
&+\ \Big(\big(5+3\sqrt{3}\big)\,t_4+\Big(\frac{7}{3456}+\frac{1}{864}\sqrt{3}\Big)\,t_{10}^4+\Big(\frac{3}{4}+\frac{5}{12}\sqrt{3}\Big)\,t_6\,t_{10}\Big)\,p^2\\
&-\ \Big(\frac{7}{2}+2\sqrt{3}\Big)\,t_3\,p\ -\ \Big(\frac{7}{12}+\frac{1}{3}\sqrt{3}\Big)\,t_7^2\,, \qquad (244)
\end{aligned}
$$

$$
\begin{aligned}
P_2\ =\ &12\,p^{10}\ +\ \big(6+2\sqrt{3}\big)\,t_{10}\,p^8\ +\ \Big(\frac{4}{3}+\frac{2}{3}\sqrt{3}\Big)\,t_{10}^2\,p^6\ +\ \big(6+4\sqrt{3}\big)\,t_7\,p^5\\
&+\ \Big(\big(8+4\sqrt{3}\big)\,t_6+\Big(\frac{1}{8}+\frac{5}{72}\sqrt{3}\Big)\,t_{10}^3\Big)\,p^4\ +\ \Big(\frac{5}{3}+\sqrt{3}\Big)\,t_7\,t_{10}\,p^3\\
&+\ \Big(\big(10+6\sqrt{3}\big)\,t_4+\Big(\frac{7}{1728}+\frac{1}{432}\sqrt{3}\Big)\,t_{10}^4+\Big(\frac{3}{2}+\frac{5}{6}\sqrt{3}\Big)\,t_6\,t_{10}\Big)\,p^2\\
&+\ \big(14+8\sqrt{3}\big)\,t_3\,p\ +\ \Big(\frac{7}{12}+\frac{1}{3}\sqrt{3}\Big)\,t_7^2\,, \qquad (245)
\end{aligned}
$$

and define

$$
\lambda(p;t)\ =\ \frac{1}{270+156\sqrt{3}}\left(-u_0\ +\ \frac{Q_1+P_1\sqrt{P_2}}{p^3}\right), \qquad (246)
$$

where

$$
\begin{aligned}
u_0\ =\ &-\big(270+156\sqrt{3}\big)\,t_0-\frac{1}{16}\big(19+11\sqrt{3}\big)\,t_4\,t_{10}^2-\frac{1}{576}\big(33+19\sqrt{3}\big)\,t_6\,t_{10}^3\\
&-\frac{1}{4}\big(21+12\sqrt{3}\big)\,t_6^2-\frac{1}{16}\big(33+19\sqrt{3}\big)\,t_{10}\,t_7^2\,. \qquad (247)
\end{aligned}
$$

Here, $(t_0,t_3,t_4,t_6,t_7,t_{10})$ gives the flat coordinates of Eguchi–Yang, which relate to the flat coordinates $(v^1,\dots,v^6)$ of Klemm–Theisen–Schmidt via a rescaling:

$$
v^1\ =\ \kappa_6\,t_0\,,\quad v^2\ =\ \kappa_5\,t_3\,,\quad v^3\ =\ \kappa_4\,t_4\,,\quad v^4\ =\ \kappa_3\,t_6\,,\quad v^5\ =\ \kappa_2\,t_7\,,\quad v^6\ =\ \kappa_1\,t_{10}\,, \qquad (248)
$$

where

$$
\begin{aligned}
&\kappa_1\ =\ -\frac{1}{2\sqrt{3}}\,,\quad \kappa_2\ =\ (\sqrt{3}-1)^{\frac{1}{2}}\,,\quad \kappa_3\ =\ -(\sqrt{3}-1)\,,\\
&\kappa_4\ =\ 2(\sqrt{3}-1)\,,\quad \kappa_5\ =\ -2\sqrt{3}\,(\sqrt{3}-1)^{\frac{1}{2}}\,,\quad \kappa_6\ =\ 8\sqrt{3}\,.
\end{aligned}
$$

Lemma 13. *We have*

$$
\theta_{\alpha,m}(v)\ =\ -\kappa_\alpha\,\frac{(8\sqrt{3})^m}{(\frac{m_\alpha}{12})_{m+1}}\,\mathrm{res}_{p=\infty}\,\lambda^{\frac{m_\alpha+12m}{12}}\,dp\,,\quad \alpha=1,\dots,6,\ m\ge 0\,. \qquad (249)
$$

Proof. The conditions (302) with $z=0$ and (299) have been verified by Eguchi–Yang [62], where we use different normalization constants κ_α from the ones in [62] to agree with Klemm–Theisen–Schmidt's normalization for the product and the metric. The proof is again similar to that of Lemma 11. For this case, the extended Euler operator is again defined by

$$
\widetilde{E}\ :=\ E\ +\ \frac{1}{r}\,p\,\frac{\partial}{\partial p}\,, \qquad (250)
$$

where E is given by (242). We have $\widetilde{E}\lambda=\lambda$ and the quasi-homogeneity (300) for the gradients of $\theta_{\alpha,m}$ follows. The condition (304) follows from the quasi-homogeneity (just as (220)) of $\theta_{\alpha,m}$. The lemma is proved. □

Following [43, 46, 47], we consider the polynomial equation

$$
\lambda(p;t)\ =\ \xi^{12}\,. \qquad (251)
$$

It is easy to see that (251) has a unique solution $p = p(\xi; t)$ in $\xi + \mathbb{C}[t][[\xi^{-1}]]$. Write

$$p(\xi;t) \;=\; \xi + \sum_{k\geq 1} u_k(t)\, \xi^{-k} \,. \tag{252}$$

By using the same arguments as before we have

$$k\, u_k(t) \;=\; \mathrm{res}_{p=\infty}\, \lambda(p;t)^{\frac{k}{12}} dp \,. \tag{253}$$

Comparing (253) with (249) we obtain

$$\theta_{\alpha,m}(v) \;=\; -\kappa_\alpha\, (8\sqrt{3})^m \, \frac{m_\alpha + 12m}{(\frac{m_\alpha}{12})_{m+1}} u_{m_\alpha+12m}(t)\,, \quad \forall\, m\geq 0\,. \tag{254}$$

Now define $(v^2)^* = (v^5)^* = 0$ and

$$(v^1)^* = \frac{145}{5750784}\,, \quad (v^3)^* = \frac{25}{27648}\,, \quad (v^4)^* = \frac{5}{1152}\,, \quad (v^6)^* = \frac{1}{4}\,, \tag{255}$$

and restrict the discussion to the particular point $v = v^*$ on B, which corresponds $t = t^*$. From (246), (251) we know that the series $p = p(\xi; t^*)$ satisfies

$$\frac{1}{270+156\sqrt{3}} \left(-u_0^* + \frac{Q_1^* + P_1^* \sqrt{P_2^*}}{p^3} \right) \;=\; \xi^{12}\,, \tag{256}$$

where Q_1^*, P_1^*, P_2^*, u_0^* are respectively Q_1, P_1, P_2, u_0 evaluated at $t = t^*$. Theorem 8 is proved by simplifying (256). □

Observe that on the left-hand side of (214), $g = \frac{m_\alpha + rm+1}{2}$. So if $m_\alpha + rm + 1$ is an odd number, then as a direct consequence of (214) the value $\theta_{\alpha,m}(v^*)$ must vanish.

We also have the following corollary.

Corollary 6. *For a FJRW CohFT of $\mathfrak{g}$-type with $\mathfrak{g}$ being A_l, D_l, or E_6, we have*

$$\langle \tau_{\alpha,\, m_\alpha+(r+1)m} \rangle \;=\; \sum_{n\geq 0} \sum_{\substack{1\leq \alpha_1,\dots,\alpha_n\leq l\\ 1\leq \beta_1,\dots,\beta_n\leq l}} \frac{\langle \tau_{\alpha,m-1}\tau_{\alpha_1,0}\cdots\tau_{\alpha_n,0}\rangle_0}{n!} \prod_{i=1}^n \eta^{\alpha_i\beta_i} \langle \tau_{\beta_i, m_{\beta_i}} \rangle\,, \quad \forall\, m\geq 1\,. \tag{257}$$

Proof. Following [43], consider the principal integrable hierarchy associated to B:

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} \;=\; \eta^{\alpha\gamma} \partial_{t^{1,0}} \left(\frac{\partial \theta_{\beta,q+1}}{\partial v^\gamma} \right), \quad q\geq 0 \tag{258}$$

and its initial value problem (viewing $t^{1,0}$ as the space variable) with the initial data:

$$v^\alpha|_{t^{\beta,q}=t^{1,0}\,\delta^{\beta,1}\,\delta^{q,0}} \;=\; \delta^{\alpha,1}\, t^{1,0}\,. \tag{259}$$

The defining equations (299)–(304) for $\theta_{\beta,q}$ and the axioms of Frobenius manifolds imply that the equations in (258) all commute, and thus the above initial value problem has a unique solution in $\mathbb{C}[[\mathbf{t}]]^{\otimes l}$, called the topological solution to the principal hierarchy, denoted by $v^{\mathrm{top}}(\mathbf{t}) = \big(v^{\mathrm{top},1}(\mathbf{t}), \dots, v^{\mathrm{top},l}(\mathbf{t})\big)$. It is shown in [43] that the following identity holds true:

$$\frac{\partial^2 \mathcal{F}_0(\mathbf{t})}{\partial t^{1,0} \partial t^{\beta,m}} \;=\; \theta_{\beta,m}\big(v^{\mathrm{top}}(\mathbf{t})\big)\,. \tag{260}$$

Taking $t^{\gamma,p} = t^{\gamma,0}\,\delta^{p,0}$ on both sides of this identity, and noticing that

$$v^{\mathrm{top},\alpha}(\mathbf{t})\big|_{t^{\gamma,p}=t^{\gamma,0}\,\delta^{p,0}} \;=\; t^{\alpha,0}\,, \tag{261}$$

we obtain

$$\left.\frac{\partial^2 \mathcal{F}_0(\mathbf{t})}{\partial t^{1,0} \partial t^{\beta,m}}\right|_{t^{\gamma,p}=t^{\gamma,0}\,\delta^{p,0}} = \theta_{\beta,m}\big(t^{1,0},\dots,t^{l,0}\big)\,. \tag{262}$$

On the other hand, the genus zero part of the string equation (40) reads as follows:

$$\sum_{p\geq 1} t^{\alpha,p} \frac{\partial \mathcal{F}_0(\mathbf{t})}{\partial t^{\alpha,p-1}} + \frac{1}{2}\,\eta_{\rho\sigma}\, t^{\rho,0} t^{\sigma,0} = \frac{\partial \mathcal{F}_0(\mathbf{t})}{\partial t^{1,0}}\,. \tag{263}$$

For $m\geq 1$, by differentiating (263) with respect to $t^{\beta,m}$ and taking $t^{\gamma,q}=t^{\gamma,0}\,\delta^{q,0}$ we get

$$\left.\frac{\partial \mathcal{F}_0(\mathbf{t})}{\partial t^{\beta,m-1}}\right|_{t^{\gamma,p}=t^{\gamma,0}\,\delta^{p,0}} = \left.\frac{\partial^2 \mathcal{F}_0(\mathbf{t})}{\partial t^{1,0} \partial t^{\beta,m}}\right|_{t^{\gamma,p}=t^{\gamma,0}\,\delta^{p,0}}. \tag{264}$$

Using the definition (33) for $\mathcal{F}_0(\mathbf{t})$, identities (262) and (264), as well as identity (214), we obtain (257). □

Note that due to (38) the sum "$\sum_{n\geq 0}$" in the right-hand side of (257) is actually a finite sum. We also note that identity (257) can be written alternatively as follows: for all $g\geq (r+2)/2$,

$$\tau_{\mathfrak{g}}(g) = \sum_{n\geq 0} \sum_{\substack{1\leq \alpha_1,\dots,\alpha_n\leq l\\ 1\leq \beta_1,\dots,\beta_n\leq l}} \frac{\langle \tau_{\alpha,m-1}\tau_{\alpha_1,0}\cdots\tau_{\alpha_n,0}\rangle_0}{n!} \prod_{i=1}^n \eta^{\alpha_i\beta_i}\,\langle \tau_{\beta_i,m_{\beta_i}}\rangle\,, \tag{265}$$

where α, m are such that $2g-1=m_\alpha+rm$.

Remark 10. For $\mathfrak{g}=E_7$ or E_8, we also expect the validity of identity (214) (or (257), (265)). It might be possible to get proofs in these cases by applying the constructions of the λ-periods for the orbit space of the Coxeter group of $\mathfrak{g}$-type ([43, 44, 45, 106]) (or for the simple singularity of $\mathfrak{g}$-type [4, 10, 46, 73, 75, 93, 104, 105, 107]) as well as their Laplace-type transforms [43, 46, 47] to compute the $\theta_{\alpha,m}$, and then matching these with the left-hand side $\tau_{\mathfrak{g}}(g)$ of (214), which can be read off from the coefficients of the top components $\phi_{\alpha;l}$ of the fundamental solutions to the dual topological ODE of $\mathfrak{g}$-type. Moreover, since we know that the λ-periods for the Frobenius manifold of $\mathfrak{g}$-type are algebraic, this method of proof, if it works, would also lead to algebraicity (as already mentioned in Remark 9) and therefore integrality of the renormalized numbers of $\tau_{\mathfrak{g}}(g)$. When $\mathfrak{g}$ is a non-simply-laced simple Lie algebra, in order for identity (214) (or (257), (265)) to remain valid, one may need to use the notion of the *partial CohFT* [90]. We hope to study these cases and other more general situations (semisimple or nonsemisimple CohFTs including the cases with the Novikov ring mentioned in Remark 5; cf. e.g. [19, 43, 51, 55, 124]) in later work.

Remark 11. We also observe that the particular point v^* on the Frobenius manifold of A-type that we use for $l\geq 2$ is different from the particular semisimple point used by Pandharipande, Pixton and Zvonkine [99, 100] for obtaining relations in the cohomology ring of $\overline{\mathcal{M}}_{g,n}$. It might be interesting to see whether their method can be applied also to our v^* to give further information connected with the topology of $\overline{\mathcal{M}}_{g,n}$.

We provide a few examples to illustrate the results of this section.

Example 1 (A_1)**.** The superpotential reads as follows:

$$\lambda = p^2 + s_1\,, \quad v^1 = \frac{s_1}{2}\,.$$

We have $\eta=1$ and $F=(v^1)^3/6$. For $m\geq 0$, we know that $\theta_{1,m}(v)=(v^1)^{1+m}/(1+m)!$. Recall again the well-known formula for the one-point invariants:

$$\langle \tau_{1,1+3m}\rangle_{g=1+m} = \frac{1}{24^{1+m}(1+m)!}\,. \tag{266}$$

From these explicit expressions we immediately see the validity of identity (214), where the particular point of the Frobenius manifold is given by $v_1^* = 1/24$.

Example 2 (A_2)**.** The superpotential reads

$$\lambda \;=\; p^3 \,+\, s_1\, p \,+\, s_2\,, \quad v^1 \;=\; \frac{s_2}{3\sqrt{-3}^{1/4}}\,, \quad v^2 \;=\; \frac{\sqrt{-3}^{1/2}}{3}\, s_1\,.$$

We have $F = (v^1)^2 v^2/2 + (v^2)^4/72$ and $\eta_{\alpha\beta} = \delta_{\alpha+\beta,3}$. The first few terms of the unique calibration can be read from

$$\theta_2(v;z) \;=\; v^1 \,+\, \left(\frac{(v^2)^3}{18} + \frac{(v^1)^2}{2}\right) z \,+\, \left(\frac{(v^1)^3}{6} + \frac{1}{18}\,(v^2)^3\, v^1\right) z^2 \,+\, \cdots\,,$$

$$\theta_1(v;z) \;=\; v^2 \,+\, v^1\, v^2\, z \,+\, \left(\frac{(v^2)^4}{36} + \frac{1}{2}\,(v^1)^2\, v^2\right) z^2 \,+\, \cdots\,.$$

The particular point v^* of the Frobenius manifold is given by $v_1^* = 1/12$ and $v_2^* = 0$.

Example 3 (A_4)**.** We have

$$\lambda \;=\; p^5 \,+\, s_1\, p^3 \,+\, s_2\, p^2 \,+\, s_3\, p \,+\, s_4\,,$$

$$v^1 \;=\; -\frac{s_1 s_2 - 5 s_4}{25\sqrt{-5}^{\frac{1}{2}}}\,, \quad v^2 \;=\; -\frac{s_1^2 - 5 s_3}{25}\,, \quad v^3 \;=\; \frac{1}{5}\sqrt{-5}^{\frac{1}{2}}\, s_2\,, \quad v^4 \;=\; \frac{\sqrt{-5}}{5} s_1\,,$$

$$F \;=\; \frac{1}{2}(v^1)^2(v^4) \,+\, v^1 v^2 v^3 \,+\, \frac{(v^2)^3}{6} \,+\, \frac{(v^4)^6}{15000} \,+\, \frac{1}{150}(v^3)^2(v^4)^3$$
$$+\, \frac{1}{20}(v^2)^2(v^4)^2 \,+\, \frac{1}{10} v^2 (v^3)^2 v^4 \,+\, \frac{(v^3)^4}{60}\,.$$

and $\eta_{\alpha\beta} = \delta_{\alpha+\beta,5}$. The first few terms of the unique calibration and the particular point v^* of the Frobenius manifold are already given in Theorem 3.

Example 4 (D_4)**.** The superpotential reads

$$\lambda \;=\; p^6 \,+\, s_1\, p^4 \,+\, s_2\, p^2 \,+\, s_3 \,+\, \frac{s_4^2}{p^2}\,,$$

and

$$p^+(\xi;s) \;=\; \xi \,-\, \frac{s_1}{6\xi} \,+\, \frac{s_1^2 - 4 s_2}{24\xi^3} \,+\, \frac{-7 s_1^3 + 36 s_2 s_1 - 216 s_3}{1296\xi^5}$$
$$+\, \frac{-55 s_1^4 + 360 s_2 s_1^2 - 864 s_3 s_1 - 432 s_2^2 - 5184 s_4^2}{31104\xi^7} \,+\, \cdots\,,$$

$$p^-(\xi;s) \;=\; \frac{s_4}{\xi^3} \,+\, \frac{s_3 s_4}{2\xi^9} \,+\, \frac{(3 s_3^2 + 4 s_2 s_4^2) s_4}{8\xi^{15}} \,+\, \frac{(8 s_1 s_4^4 + 20 s_2 s_3 s_4^2 + 5 s_3^3) s_4}{16\xi^{21}} \,+\, \cdots\,.$$

We have $\eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and

$$F \;=\; \frac{1}{2}\,(v^1)^2\, v^3 \,+\, \frac{1}{2}\, v^1\, (v^2)^2 \,+\, \frac{1}{36}\,(v^2)^3\, v^3 \,+\, \frac{1}{216}\,(v^2)^2\,(v^3)^3 \,+\, \frac{(v^3)^7}{272160}$$
$$+\, \left(\frac{v^1}{2} - \frac{1}{12} v^2 v^3 + \frac{(v^3)^3}{216}\right)(v^4)^2\,. \tag{267}$$

We have $\theta_{\alpha,0} = v_\alpha = \eta_{\alpha\beta} v^\beta$, and

$$\begin{aligned}
\theta_{1,1} &= \frac{(v^2)^2}{2} + \frac{(v^4)^2}{2} + v^1 v^3 , \\
\theta_{2,1} &= \frac{1}{108} v^2 (v^3)^3 + \frac{1}{12} (v^2)^2 v^3 - \frac{1}{12} v^3 (v^4)^2 + v^1 v^2 , \\
\theta_{3,1} &= \frac{(v^3)^6}{38880} + \frac{1}{72} (v^2)^2 (v^3)^2 + \frac{1}{72} (v^3)^2 (v^4)^2 + \frac{(v^2)^3}{36} - \frac{1}{12} v^2 (v^4)^2 + \frac{(v^1)^2}{2} , \\
\theta_{4,1} &= \frac{1}{108} v^4 (v^3)^3 - \frac{1}{6} v^2 v^3 v^4 + v^1 v^4 .
\end{aligned}$$

The particular point v^* of the Frobenius manifold is given by

$$v_1^* = \frac{1}{3} , \quad v_2^* = 0 , \quad v_3^* = \frac{13}{40824} , \quad v_4^* = 0 . \tag{268}$$

We list in the following table the first few of the numbers $\tau_{D_4}(g)$, again putting $\tau_{D_4}(0) = 1$.

g	0	1	2	3	4	5	6	7	8	9
$\tau_{D_4}(g)$	1	$\frac{1}{3}$	0	$\frac{13}{40824}$	$\frac{13}{122472}$	0	$\frac{1433}{16665989760}$	$\frac{253}{9999593856}$	0	$\frac{33917}{2041117097886720}$

One-point FJRW invariants of D_4-type

Appendix A. Wave functions

In this appendix we give a method of computing residues of pseudo-differential operators by means of wave functions. Denote by r a positive integer. Let $q_m(x)$ $(m \le r-1)$ be arbitrarily given power series of x, and L the pseudodifferential operator

$$L := \partial^r + \sum_{m=-\infty}^{r-1} q_m(x) \, \partial^m . \tag{269}$$

An element $\psi = \psi(x,z) \in \mathbb{C}[[x]][[z^{-1}]] \otimes e^{xz}$ of the form $\psi = \sum_{i=0}^\infty \phi_i(x) \, z^{-i} \, e^{xz}$ with $\phi_0(x) \equiv 1$ is called a *wave function* of L if

$$L \, \psi = z^r \, \psi . \tag{270}$$

Here we recall that $\partial^{-i}(e^{xz}) := e^{xz} z^{-i}$, $i \ge 0$, and that for all $f(x) \in \mathbb{C}[[x]]$,

$$\partial^m \circ f(x) := \sum_{\ell \ge 0} \binom{m}{\ell} f^{(\ell)}(x) \, \partial^{m-\ell} , \quad \forall \, m \in \mathbb{Z} .$$

Multiplying by e^{-xz} on both sides of (270) and comparing the coefficients of powers of z give

$$\sum_{\substack{j,\ell \ge 0 \\ j+\ell \le i}} q_{\ell+j+r-i}(x) \binom{\ell+j+r-i}{\ell} \phi_j^{(\ell)}(x) = \phi_i(x) , \quad i \ge 0 . \tag{271}$$

This leads to recursive relations for the power series $\phi_i(x)$, $i \ge 0$. The solution ψ depends on a sequence of arbitrary constants $C_1, C_2, \cdots$. Alternatively, we observe that if $\psi(x,z)$ is a wave function, then for an arbitrary power series $g(z)$ of z^{-1} with constant coefficients $g(z) = \sum_{i \ge 0} g_i \, z^{-i}$, $g_0 = 1$, the product $g(z) \, \psi(z,x)$ is again a wave function.

A pseudo-differential operator Φ of the form

$$\Phi = \sum_{i \ge 0} \widetilde{\phi}_i(x) \, \partial^{-i} , \quad \widetilde{\phi}_i(x) \in \mathbb{C}[[x]] , \ \widetilde{\phi}_0(x) \equiv 1 \tag{272}$$

is called a *dressing operator of* L if

$$\Phi \circ \partial^r \circ \Phi^{-1} \;=\; L\,. \tag{273}$$

For the given $q_m(x)$ $(m \le r-1)$, the dressing operator is not unique, its freedom being in one-to-one correspondence with the coefficients of $g(z)$ above. Indeed, there is a one-to-one correspondence between wave functions ψ and dressing operators by $\psi = \sum_i \phi_i(x)\, z^{-i}\, e^{xz} \leftrightarrow \Phi = \sum_i \phi_i(x)\, \partial^{-i}$. Define the formal adjoint operator Φ^* of Φ by

$$\Phi^* \;:=\; \sum_{i\ge 0} (-1)^i\, \partial^{-i} \circ \widetilde{\phi}_i(x)\,.$$

Fix ψ a wave function of L, and take Φ to be the corresponding dressing operator. Define ψ^* by

$$\psi^* \;=\; \psi^*(x,z) \;:=\; \bigl(\Phi^{-1}\bigr)^*\,\bigl(e^{-xz}\bigr)\,. \tag{274}$$

Clearly, $\psi^* \in \mathbb{C}[[x]][[z^{-1}]] \otimes e^{-xz}$, the product $\psi^* e^{xz}$ has leading term 1, and

$$L^*\,\psi^* \;=\; z^r\,\psi^*\,. \tag{275}$$

We call ψ^* *a dual wave function of* L associated to ψ, and we call (ψ,ψ^*) *a pair of wave and dual wave functions.*

Lemma 14. *Let* $\bigl(\psi,\psi^*\bigr)$ *and* $\bigl(\widetilde{\psi},\widetilde{\psi}^*\bigr)$ *be two pairs of wave and dual wave functions of* L. *Then*

$$\psi(x,z)\,\psi^*(x,z) \;=\; \widetilde{\psi}(x,z)\,\widetilde{\psi}^*(x,z)\,.$$

Proof. Let $g = g(x,z) := \widetilde{\psi}(x,z)/\psi(x,z)$, which must have the form

$$g \;=\; \sum_{i\ge 0} g_i(x)\, z^{-i} \in \mathbb{C}[[x]]\bigl[\bigl[z^{-1}\bigr]\bigr]\,, \quad g_0(x) \equiv 1\,.$$

It follows from (271) that $g_i'(x) = 0$, $i \ge 1$. Therefore, for $i \ge 1$, g_i are all constants. Let Φ, $\widetilde{\Phi}$ be the dressing operators corresponding to ψ, $\widetilde{\psi}$, respectively. We have $\widetilde{\psi} = g(z)\,\Phi\bigl(e^{xz}\bigr) = \Phi \circ G\bigl(e^{xz}\bigr)$, where $G := \sum_{i\ge 0} g_i\, \partial_x^{-i}$. Therefore, $\widetilde{\Phi} = \Phi \circ G$. It follows that $\bigl(\widetilde{\Phi}^{-1}\bigr)^* \circ G^* = \bigl(\Phi^{-1}\bigr)^*$. So $\bigl(\widetilde{\Phi}^{-1}\bigr)^*\bigl(g(z)\, e^{-xz}\bigr) = \bigl(\Phi^{-1}\bigr)^*\bigl(e^{-xz}\bigr)$. Namely, $g(z)\widetilde{\psi}^*(x,z) = \psi^*(x,z)$. The lemma is proved. □

Let ψ and ψ^* be a pair of wave and dual wave functions of L. Define

$$c(z) \;:=\; \bigl(e^{-xz}\,\psi(z,x)\bigr)\big|_{x=0}\,, \quad c^*(z) \;:=\; \bigl(e^{xz}\,\psi^*(z,x)\bigr)\big|_{x=0}\,, \quad H(z) \;:=\; c(z)\, c^*(z)\,. \tag{276}$$

It follow from Lemma 14 that the product $H(z) \in \mathbb{C}[[z^{-1}]]$ is uniquely determined by L, where we recall that L is defined by (269). We have the following lemma.

Lemma 15. *Define* $z_k(x) = \operatorname{res} L^{\frac{k}{r}}$ *for all* $k \ge 0$. *Then*

$$H(z) \;=\; 1 + \sum_{k\ge 1} (-1)^k\, z_{k-1}(0)\, z^{-k}\,. \tag{277}$$

Proof. Following Liu–Vakil–Xu [87], define $\Phi_- = \sum_{i\ge 0} \phi_{-,i}(x)\, \partial^{-i}$ with $\phi_{-,i}(x) \in \mathbb{C}[[x]]$ and $\widetilde{\phi}_{-,0}(x) \equiv 1$ as the particular dressing operator of L fixed by the additional conditions:

$$\phi_{-,i}(0) \;=\; 0\,, \quad i \ge 1\,.$$

Denote by (ψ_-, ψ_-^*) the corresponding pair of wave and dual wave functions, and define c_-, c_-^* as ψ_-, ψ_-^* evaluated at $x=0$, respectively. It is clear from the definition that $c_-(z)\equiv 1$. Therefore, $H(z)\equiv c_-^*(z)$. Now, on one hand, noticing that

$$\big(\Phi_-^{-1}\big)^* \;=\; \sum_{i\geq 0}(-\partial)^{-i}\circ\phi_{-,i}(x) \;=\; \sum_{k\geq 0}(-1)^k \sum_{\substack{i,\ell\geq 0\\ i+\ell=k}} \binom{k-1}{\ell}\,\phi_{-,i}^{(\ell)}(x)\,\partial^{-k}\,,$$

we find

$$\psi_-^*(x,z) \;=\; \big(\Phi_-^{-1}\big)^*\big(e^{-xz}\big) \;=\; \sum_{k\geq 0}(-1)^k \sum_{\substack{i,\ell\geq 0\\ i+\ell=k}} \binom{k-1}{\ell}\,\phi_{-,i}^{(\ell)}(x)\,z^{-k}\,e^{-xz}\,.$$

Hence

$$c_-^*(z) \;=\; \sum_{k\geq 0}(-1)^k \sum_{i=0}^{k} \binom{k-1}{i-1}\,\phi_{-,i}^{(k-i)}(0)\,z^{-k}\,. \tag{278}$$

On the other hand, from the definition of the dressing operator Φ_- we know that

$$z_k(x) \;=\; \operatorname{res}\big(\Phi_-\circ\partial^r\circ\Phi_-^{-1}\big)^{\frac{k}{r}} \;=\; \operatorname{res}\,\Phi_-\circ\partial^k\circ\Phi_-^{-1}\,.$$

Taking $x=0$ in this formula we find that

$$z_k(0) \;=\; \big(\operatorname{res}\,\partial^k\circ\Phi_-^{-1}\big)\big|_{x=0} \;=\; \left(\operatorname{res}\sum_{i\geq 0}\sum_{\ell=0}^{k}\binom{k}{\ell}\,\phi_{-,i}^{(k-\ell)}(x)\,\partial^{\ell-i}\right)\Bigg|_{x=0} \;=\; \sum_{\ell=0}^{k}\binom{k}{\ell}\phi_{-,\ell+1}^{k-\ell}(0)\,.$$

The lemma is proved by comparing this expression with (278) and using $c_-(z)\equiv 1$. □

We note that, in the above proof, the function $c_-(z)$ is very simple, being just the constant function 1, but the formula (278) for $c_-^*(z)$ might be very complicated. The main point of the wave-function-pair approach is the following: using Lemma 14 it is sometimes possible to find a particular choice of pair of wave functions such that $c(z)$ and $c^*(z)$ both have relatively simple expressions so that their product can be given in closed form, and this is the case in particular both for this paper and for [52, 53], where *bispectrality* [52, 53, 57] is used for fixing the particular choice. In other situations, the Sato tau-function, theta-functions, and etc. could also be used to construct a pair of wave functions; instead of giving the details we refer to [5, 11, 14, 20, 34, 40] for specific constructions.

Appendix B. Frobenius manifolds

In this appendix, we give a brief review of the theory of Frobenius manifolds [43, 46] (cf. also [47, 56, 71, 94, 111]). Recall that a *Frobenius algebra* is a triple $(V, \mathbb{1}, \langle\,,\,\rangle)$, where V is a commutative associative algebra over $\mathbb{C}$ with unity $\mathbb{1}$, and $\langle\,,\,\rangle : V\times V\to\mathbb{C}$ is a symmetric non-degenerate bilinear form satisfying $\langle x\cdot y, z\rangle \;=\; \langle x, y\cdot z\rangle$, $\forall\, x,y,z\in V$.

Definition 2 ([41, 43])**.** A *Frobenius structure* of charge d on a complex manifold M^l is a family of Frobenius algebras $\big(T_pM, \mathbf{1}_p, \langle\,,\,\rangle_p\big)$, $p\in M$ depending holomorphically on p and satisfying:

- **FM1** The metric $\langle\,,\,\rangle$ on M is flat, and $\nabla\mathbf{1}=0$, where ∇ is the Levi–Civita connection of $\langle\,,\,\rangle$ and $\mathbf{1}$ is the unity vector field.
- **FM2** Define a 3-tensor field c by $c(X,Y,Z) := \langle X\cdot Y, Z\rangle$, for any three holomorphic vector fields X, Y, Z on M. The 4-tensor field ∇c must be symmetric.

FM3 There exists a holomorphic vector field E on M, called the Euler vector field, satisfying

$$\nabla\nabla E \;=\; 0\,, \tag{279}$$

$$[E, X\cdot Y] - [E,X]\cdot Y \;-\; X\cdot[E,Y] \;=\; X\cdot Y\,, \tag{280}$$

$$E\,\langle X,Y\rangle \;-\; \langle [E,X],Y\rangle \;-\; \langle X,[E,Y]\rangle \;=\; (2-d)\,\langle X,Y\rangle\,. \tag{281}$$

A complex manifold endowed with a Frobenius structure is called a *Frobenius manifold*, with $\langle\,,\rangle$ being called the *invariant flat metric*.

Let M be a Frobenius manifold of complex dimension l. Following [43], define a one-parameter family of affine connections on M:

$$\widetilde{\nabla}_X Y \;:=\; \nabla_X Y \,+\, z\,X\cdot Y\,, \quad z\in\mathbb{C}\,. \tag{282}$$

This family of affine connections $\widetilde{\nabla}$ are all flat [43], and is called the *deformed flat connection*. Moreover, it can be extended to a flat affine connection [43] on $M\times\mathbb{C}^*$, still denoted by $\widetilde{\nabla}$, whose definition along the z-direction is given as follows:

$$\widetilde{\nabla}_{\partial_z} X \;:=\; \frac{\partial X}{\partial z} \,+\, E\cdot X \,-\, \frac{1}{z}\,\mu\,X\,, \quad \widetilde{\nabla}_{\partial_z}\partial_z \;:=\; 0\,, \quad \widetilde{\nabla}_X\partial_z \;:=\; 0\,, \tag{283}$$

for X being an arbitrary holomorphic vector field on $M\times\mathbb{C}^*$ with zero component along the z direction. Here, $\mu := \frac{2-d}{2} - \nabla E$. We call $\widetilde{\nabla}$ the *extended deformed flat connection*. A holomorphic function $f(v;z)$ on some open subset of $M\times\mathbb{C}^*$ is called $\widetilde{\nabla}$*-flat* if

$$\widetilde{\nabla} df \;=\; 0\,. \tag{284}$$

To understand the flat coordinates for the extended deformed flat connection $\widetilde{\nabla}$, let us take $v = \big(v^1,\dots,v^l\big)$ a system of flat coordinates with respect to $\langle\,,\rangle$. Denote $\partial_\alpha = \frac{\partial}{\partial v^\alpha}$, $\eta_{\alpha\beta} := \langle\partial_\alpha,\partial_\beta\rangle$, $\eta = (\eta_{\alpha\beta})$, and $\big(\eta^{\alpha\beta}\big) := \eta^{-1}$. By FM1 we choose v^1 satisfying $\partial_1 = \mathbf{1}$. For simplicity we will assume that ∇E is diagonalizable, so the flat coordinates are chosen such that $\mu = \mathrm{diag}(\mu_1,\dots,\mu_l)$, and we have

$$E \;=\; \sum_{1\le\alpha\le l} \Big(\big(1-\tfrac{d}{2}-\mu_\alpha\big)\,v^\alpha\,\partial_\alpha \,+\, r^\alpha\,\partial_\alpha\Big)\,. \tag{285}$$

The axioms FM1–FM3 imply the local existence of a holomorphic function F, called the *Frobenius potential* of M, satisfying

$$\frac{\partial^3 F}{\partial v^\alpha\partial v^\beta\partial v^\gamma} \;=\; c_{\alpha\beta\gamma} \;:=\; c\big(\partial_\alpha,\partial_\beta,\partial_\gamma\big)\,, \tag{286}$$

$$EF \;=\; (3-d)\,F \,+\, \text{a quadratic function of } v\,. \tag{287}$$

Clearly, F is uniquely determined by the Frobenius structure up to a quadratic function of v. The flatness of $\widetilde{\nabla}$ implies that there locally exist l independent $\widetilde{\nabla}$-flat holomorphic functions

$$\tilde v_1,\dots,\tilde v_l$$

on $M\times\mathbb{C}^*$, called the *deformed flat coordinates* for the Frobenius manifold. Let us give their construction. For a $\widetilde{\nabla}$-flat holomorphic function f on $M\times\mathbb{C}^*$, denote $y^\alpha = \eta^{\alpha\beta}\partial f/\partial v^\beta$ and $y = (y^1,\dots,y^l)^T$. By definition the column-vector-valued function y satisfies the following system of linear differential equations:

$$\frac{\partial y}{\partial v^\alpha} \;=\; z\,C_\alpha\,y\,, \tag{288}$$

$$\frac{dy}{dz} \;=\; \Big(\mathcal{U} + \frac{\mu}{z}\Big)y\,. \tag{289}$$

Here, $(C_\alpha)^\beta_\gamma := c^\beta_{\alpha\gamma}$ and $\mathcal{U}^\alpha_\beta := E^\rho c^\alpha_{\rho\beta}$. To fix a system of deformed flat coordinates we need to choose a basis of the solution space to (288)–(289). Observe that the ODE system (289) possesses a Fuchsian singular point at $z=0$ and an irregular singular point of Poincaré rank 1 at $z=\infty$. The axioms of Frobenius manifolds imply the isomonodromicity of (288)–(289). The monodromy data around $z=0$ is then given by two constant matrices μ, R, satisfying

$$R^* \;=\; -e^{\pi i \mu} R\, e^{-\pi i \mu} \tag{290}$$

$$z^\mu R z^{-\mu} \;=\; R_0 \,+\, R_1 z \,+\, R_2 z^2 \,+\, \dots \tag{291}$$

for some matrices R_k, $k \geq 0$ satisfying

$$[\mu, R_k] \;=\; k\, R_k\,, \quad k \geq 0\,. \tag{292}$$

Let us fix a choice of R. It is shown in [43, 46] (see also [71]) that locally there exists a fundamental solution matrix $\mathcal{Y} = \mathcal{Y}(v;z)$ around $z=0$ to (288)–(289) of the form

$$\mathcal{Y}(v;z) \;=\; \Theta(v;z)\, z^\mu z^R \;=\; \sum_{m\geq 0} \Theta_m(v)\, z^{m+\mu} z^R\,, \tag{293}$$

where $\Theta(v;z)$ is a matrix-valued analytic function on $M\times\mathbb{C}$ satisfying

$$\Theta(v;0) \;\equiv\; I\,, \tag{294}$$

$$\eta^{-1}\Theta(v;-z)^T \eta\, \Theta(v;z) \;\equiv\; I\,. \tag{295}$$

It can be easily checked that the matrix-valued function $\Theta(v;z)$ satisfies

$$\partial^\beta \Theta^\alpha_\gamma(v;z) \;=\; \partial^\alpha \Theta^\beta_\gamma(v;z)\,.$$

Then by the Poincaré lemma, there locally exist holomorphic functions $\theta_\gamma(v;z)$ of the form

$$\theta_\gamma(v;z) \;=\; \sum_{m\geq 0} \theta_{\gamma,m}(v)\, z^m\,, \tag{296}$$

such that

$$\eta^{\alpha\beta} \frac{\partial \theta_\gamma(v;z)}{\partial v^\beta} \;=\; \Theta^\alpha_\gamma(v;z)\,. \tag{297}$$

Hence the functions $\tilde v_1(v;z), \dots, \tilde v_n(v;z)$ defined by

$$\big(\tilde v_1(v;z), \dots, \tilde v_n(v;z)\big) \;=\; \big(\theta_1(v;z), \dots, \theta_l(v;z)\big)\, z^\mu z^R \tag{298}$$

give a system of deformed flat coordinates on $M\times\mathbb{C}^*$ (see [43, 46, 47, 56] for more details).

From the above construction we know that $\theta_{\alpha,m}$, $m\geq 0$, satisfy the following conditions:

$$\partial_\alpha \partial_\beta \big(\theta_{\gamma,m+1}\big) \;=\; c^\sigma_{\alpha\beta}\, \partial_\sigma\big(\theta_{\gamma,m}\big)\,, \quad m\geq 0\,, \tag{299}$$

$$E\left(\partial_\beta \theta_{\alpha,m}\right) \;=\; (p+\mu_\alpha+\mu_\beta)\, \partial_\beta\big(\theta_{\alpha,m}\big) \,+\, \sum_{1\leq k\leq m} (R_k)^\gamma_\alpha\, \partial_\beta\big(\theta_{\gamma,m-k}\big)\,, \quad m\geq 0\,, \tag{300}$$

$$\frac{\partial \theta_{\alpha,0}}{\partial v^\beta} \;=\; \eta_{\alpha\beta}\,, \tag{301}$$

$$\langle \nabla\theta_\alpha(v,z), \nabla\theta_\beta(v,-z)\rangle \;=\; \eta_{\alpha\beta}\,. \tag{302}$$

One can further normalize $\theta_{\alpha,m}(v)$ by requiring

$$\theta_{\alpha,0} \;=\; v_\alpha\,, \tag{303}$$

$$\partial_1\big(\theta_{\alpha,m+1}\big) \;=\; \theta_{\alpha,m}\,, \quad m\geq 0\,. \tag{304}$$

Indeed, (303) is obviously compatible with (299)–(302), and we leave the verification of the compatibility between (304) and (299)–(302) as an exercise to the reader. As in [50], we call a choice of $\{\theta_{\alpha,m}\}_{m\geq 0}$ satisfying (299)–(304) a *calibration* on M.

According to Kontsevich–Manin [84, 94], the genus zero part of a CohFT

$$\big(V^l, \langle,\rangle, \mathbb{1}, \{\Omega_{g,n}\}_{2g-2+n>0}\big)$$

gives a pre-Frobenius structure on the convergence domain B under the assumption (see Section 2). Here "pre" means the axiom FM3 is not required. (In some literature a pre-Frobenius structure is called a Frobenius structure.) Let us recall Kontsevich–Manin's construction of the Frobenius structure from the CohFT. As in Section 2, take $e_1 = \mathbb{1}, e_2, \dots, e_l$ a basis of V. Define a metric $\langle\,,\,\rangle$ on B and a multiplication "$\cdot$" on the tangent spaces of B by

$$\langle \partial_\alpha, \partial_\beta \rangle \;:=\; \eta_{\alpha\beta}\,, \tag{305}$$

$$\langle \partial_\alpha \cdot \partial_\beta, \partial_\gamma \rangle \;:=\; \frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma}\,, \tag{306}$$

where F is defined in (27). From (305), it is obvious that the metric $\langle\,,\,\rangle$ is flat and in fact $(v^1, \dots, v^n)$ is a system of flat coordinates for $\langle\,,\,\rangle$. The invariance of $\langle\,,\,\rangle$ with respect to the multiplication "$\cdot$" on each tangent space as well as the FM2 are also obviously true from the construction. The axioms of the CohFT imply [84] that the multiplication "$\cdot$" is associative and $\partial_1 = \mathbf{1}$ (unity vector field). So $(B, \langle\,,\,\rangle, \cdot, \partial_1)$ is a pre-Frobenius manifold. Using (31) we see further that a *homogeneous* CohFT $\big(V, \langle,\rangle, \mathbb{1}, \{\Omega_{g,n}\}_{2g-2+n>0}\big)$ of charge d endows B with a Frobenius structure of charge d with E given by (28) being the Euler vector field.

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