ARITHMETIC PROPERTIES OF THE HERGLOTZ FUNCTION

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ABSTRACT. In this paper we study two functions $F(x)$ and $J(x)$, originally found by Herglotz in 1923 and later rediscovered and used by one of the authors in connection with the Kronecker limit formula for real quadratic fields. We discuss many interesting properties of these functions, including special values at rational or quadratic irrational arguments as rational linear combinations of dilogarithms and products of logarithms, functional equations coming from Hecke operators, and connections with Stark's conjecture. We also discuss connections with 1-cocycles for the modular group $\mathrm{PSL}(2, \mathbb{Z})$.

CONTENTS

1. INTRODUCTION

Consider the function

$$J(x) \;=\; \int_0^1 \frac{\log(1+t^x)}{1+t}\,dt\,, \tag{1}$$

defined for $x > 0$. Some years ago, Henri Cohen [1] showed one of the authors the identity

$$J(1+\sqrt{2}) \;=\; -\frac{\pi^2}{24} + \frac{\log^2(2)}{2} + \frac{\log(2)\log(1+\sqrt{2})}{2}\,.$$

In this note we will give many more similar identities, like

$$J(4+\sqrt{17}) \;=\; -\frac{\pi^2}{6} + \frac{\log^2(2)}{2} + \log(2)\log(4+\sqrt{17})$$

and

$$J\left(\frac{2}{5}\right) \;=\; \frac{11\pi^2}{240} + \frac{3\log^2(2)}{4} - 2\log^2\left(\frac{\sqrt{5}+1}{2}\right).$$

We will also investigate the connection to several other topics, such as the Kronecker limit formula for real quadratic fields, Hecke operators, Stark's conjecture, and cohomology of the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

The function $J(x)$ can be expressed via the formula

$$J(x) \;=\; F(2x) - 2F(x) + F(x/2) + \frac{\pi^2}{12x} \tag{2}$$

[1] who learned it from H. Muzzafar (Montreal), see [2, Ex.60, p. 902–903]

in terms of the function

$$F(x) \;=\; \sum_{n=1}^{\infty} \frac{\psi(nx) - \log(nx)}{n} \qquad \bigl(x \in \mathbb{C}' := \mathbb{C} \smallsetminus (-\infty, 0]\,\bigr)\,, \tag{3}$$

which is therefore in some sense more fundamental. Here $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. The function F was introduced in [18], where it was shown, in particular, that it satisfies the two functional equations

$$\begin{aligned} F(x) \;-\; F(x+1) \;-\; F\Bigl(\frac{x}{x+1}\Bigr) \;&=\; -F(1) \;+\; \mathrm{Li}_2\Bigl(\frac{1}{1+x}\Bigr)\,, \\ F(x) \;+\; F\Bigl(\frac{1}{x}\Bigr) \;&=\; 2F(1) \;+\; \frac{\log^2 x}{2} \;-\; \frac{\pi^2(x-1)^2}{6x}\,, \end{aligned} \tag{4}$$

for all $x \in \mathbb{C}' = \mathbb{C} \smallsetminus (-\infty, 0]$, where $\mathrm{Li}_2(x)$ is Euler's dilogarithm function. A function essentially equivalent to $F(x)$ was also studied by Herglotz in [5] and for this reason we will call F the Herglotz function. In fact, Herglotz also introduced the function $J(x)$ and found explicit evaluations [5, Eq.(70a-c)] of $J(x)$ for $x = 4 + \sqrt{15}$, $6 + \sqrt{35}$, and $12 + \sqrt{143}$. Several other identities of this kind were found by Muzzafar and Williams [13], together with some sufficient conditions on n under which one can evaluate $J(n + \sqrt{n^2 - 1})$. We give a more systematic treatment of identities of this type in Section 6.

2. Elementary properties

Integral representations. Using the well-known integral formula for $\psi(x)$ (see [17, p. 247])

$$\psi(x) \;=\; \int_0^\infty \Bigl(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\Bigr)\, dt\,,$$

we see that the function F also has the integral representations

$$F(x) \;=\; \int_0^\infty \Bigl(\frac{1}{1-e^{-t}} - \frac{1}{t}\Bigr) \log(1-e^{-tx})\, dt \;=\; \int_0^1 \Bigl(\frac{1}{1-y} + \frac{1}{\log y}\Bigr) \log(1-y^x)\, \frac{dy}{y}\,. \tag{5}$$

By differentiating the last integral and using the obvious identity $\frac{d}{dx}\mathrm{Li}_m(y^x) = \mathrm{Li}_{m-1}(y^x)\log y$ we get also the following expression for the n-th derivative of F

$$F^{(n)}(x) \;=\; -\int_0^1 \Bigl(\frac{1}{1-y} + \frac{1}{\log y}\Bigr) \log^n(y)\, \mathrm{Li}_{1-n}(y^x)\, \frac{dy}{y}\,. \tag{6}$$

We will give another useful integral representation of F in Section 7.3.

Asymptotic expansions. The Herglotz function has the asymptotic expansion

$$F(x) \;=\; F(1) \;-\; \mathrm{Li}_2(1-x) \;+\; \sum_{n=1}^{\infty} \frac{B_n\,\zeta(n+1)}{n} (x-1)^n (1-x^{-n})$$

as $x \to 1$, the asymptotic expansion

$$F(x) \;\sim\; -\frac{\pi^2}{12x} \;-\; \sum_{n=2}^{\infty} \frac{B_n\,\zeta(n+1)}{n}\, x^{-n}\,,$$

as $x \to \infty$, and the asymptotic expansion

$$F(x) \;\sim\; -\frac{\zeta(2)}{x} \;+\; \frac{\log^2 x}{2} \;+\; 2\zeta(2) + 2F(1) \;+\; \sum_{n=1}^{\infty} \frac{B_n\,\zeta(n+1)}{n}\, x^n\,,$$

as $x \to 0$. In terms of the formal power series

$$\mathfrak{Z}(x) \;=\; \sum_{n=1}^{\infty} \zeta(1-n)\,\zeta(1+n)\, x^n \;\in \mathbb{R}[[x]]\,,$$

these can be rewritten as

$$F(x) \sim \begin{cases} -\mathfrak{Z}(x) + \dfrac{\log^2 x}{2} + 2F(1) - \zeta(2)(x-2+x^{-1}), & x \to 0, \\ \mathfrak{Z}\Big(\dfrac{1-x}{x}\Big) - \mathfrak{Z}(1-x) - \mathrm{Li}_2(1-x) + F(1), & x \to 1, \\ \mathfrak{Z}\Big(\dfrac{1}{x}\Big), & x \to \infty. \end{cases} \tag{7}$$

The expansion at $x \to \infty$ follows from the asymptotic expansion for the digamma function $\psi(z) \sim \log z + \sum_{n=1}^{\infty} \zeta(1-n) z^{-n}$, while the two other expansions can be derived easily from the expansion at ∞ using the functional equations (4).

Analytic continuation. The function F is analytic in the cut complex plane $\mathbb{C}'$, because the series (3) converges locally uniformly there. From the formula $\psi'(z) = \sum_{m=0}^{\infty} (z+m)^{-2}$ we get a convergent series representation for F' in $\mathbb{C}'$

$$F'(x) = \sum_{n=1}^{\infty} \Big(\sum_{m=0}^{\infty} \frac{1}{(nx+m)^2} - \frac{1}{nx} \Big). \tag{8}$$

Let us look at this from a different point of view. The change of variables $y \mapsto x = (\frac{1-y}{1+y})^2$ maps the unit disk $\{y\colon |y| < 1\}$ bijectively and conformally onto $\mathbb{C}'$. The analyticity of F in $\mathbb{C}'$ together with the second formula in (7) then implies that the power series

$$G(y) := F\Big(\Big(\frac{1-y}{1+y}\Big)^2\Big) - F(1) + \mathrm{Li}_2\Big(\frac{4y}{(1+y)^2}\Big) = \mathfrak{Z}\Big(\frac{4y}{(1-y)^2}\Big) - \mathfrak{Z}\Big(\frac{4y}{(1+y)^2}\Big) \tag{9}$$

has radius of convergence ≥ 1. The power series $\mathfrak{Z}(y)$ is factorially divergent, but we can show directly that the difference of two $\mathfrak{Z}$'s appearing on the right-hand side of (9) is convergent in the unit disk by the following argument. First, note that $\mathfrak{Z}(x) + \zeta(2)x$ is equal to $-\sum_{n\geq 1} \frac{\gamma_0(x/n)}{n}$. Here $\gamma_0(x) = \sum_{n\geq 1} \frac{B_m}{m} x^m$ is the formal power series discussed in detail in [22, (A.13)], where it was shown that it satisfies

$$\gamma_0\Big(\frac{x}{1-x}\Big) - \gamma_0(x) = \log(1-x) + x.$$

From this we obtain

$$G(y) + \zeta(2)\frac{16y^2}{(1-y^2)^2} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\Big(\frac{\frac{4}{n} y}{1 - 2(\frac{2j}{n} - 1)y + y^2} \Big), \qquad \varphi(x) := -\log(1-x) - x. \tag{10}$$

Using $\varphi(x) = \int_0^1 (\frac{x}{1-tx} - x) dt$ and the generating series $\sum_{m\geq 0} U_m(t)\, x^m = \frac{1}{1-2tx+x^2}$ for Chebyshev polynomials of the second kind, we obtain

$$[y^{m+1}]\Big(G(y) + \zeta(2)\frac{16y^2}{(1-y^2)^2} \Big) = \sum_{n=1}^{\infty} \frac{2}{n} \Big[\int_{-1}^{1} U_m(t)\, dt - \frac{2}{n} \sum_{j=0}^{n-1} U_m\Big(\frac{2j}{n} - 1\Big) \Big].$$

Now breaking up the integral into n integrals over intervals of length $2/n$ and using the estimate $|U_m'(t)| \leq 4m(m+1)$, $t \in [-1,1]$, we find that the right hand side is $O(m^2)$, and hence that the series $G(y)$ indeed has radius of convergence at least 1.

Relation to the Dedekind eta function. The analytic extension of F to $\mathbb{C}'$ satisfies another important functional equation. To state it, and for later purposes, it is convenient to introduce the slightly modified function

$$\mathcal{F}(x) = F(x) - F(1) + \frac{\pi^2}{12}(x - 2 + x^{-1}) - \frac{\log^2 x}{4}. \tag{11}$$

Proposition 1. *For any $z \in \mathfrak{H} = \{z\colon \mathrm{Im}(z) > 0\}$ we have*

$$\mathcal{F}(-z) - \mathcal{F}(z) = 2\pi i \log((z/i)^{1/4}\eta(z)),$$

where $\eta(z) = e^{\pi i z/12}\prod_{n\geq 1}(1 - e^{2\pi i n z})$ is the Dedekind eta function, and the branch of $\log((z/i)^{1/4}\eta(z))$ is chosen to be real-valued on the imaginary axis.

Proof. This follows from $\psi(-x)-\psi(x)=x^{-1}+\pi\cot(\pi x)$. Compare [16, p. 27]. $\square$

As a corollary, $\mathcal{F}(x)$ cannot be extended analytically across the negative real axis either from above or from below.

Relation to the Rogers dilogarithm. The functional equation (4) already shows that F is intimately related to the dilogarithm function Li_2. If we use the modified function $\mathcal{F}$ from (11) instead, we get nicer identities involving the Rogers dilogarithm. Recall that the Rogers dilogarithm is defined by

$$L(x) \;=\; \mathrm{Li}_2(x)+\frac{1}{2}\log(x)\log(1-x)\,, \quad x\in[0,1]\,, \tag{12}$$

and extended to $\mathbb{R}$ by

$$L(x) \;=\; \begin{cases} \frac{\pi^2}{3}-L(\frac{1}{x})\,, & x>1\,,\\ -L(\frac{x}{x-1})\,, & x<0\,. \end{cases} \tag{13}$$

The key property of L is that it defines a continuous map from $\mathbb{P}^1(\mathbb{R})$ to $\mathbb{R}/\frac{\pi^2}{2}\mathbb{Z}$ that satisfies

$$L(x)+L(y)-L(xy)-L\Big(\frac{x(1-y)}{1-xy}\Big)-L\Big(\frac{y(1-x)}{1-xy}\Big) \;=\; 0 \pmod{\pi^2/2}\,. \tag{14}$$

Now, if we use $\mathcal{F}$ instead of F, then we get "clean" versions of the functional equation (4) with the Rogers dilogarithm in place of Li_2: for $x>0$ we have

$$\begin{aligned} \mathcal{F}(x)\;-\;\mathcal{F}(x+1)\;-\;\mathcal{F}\Big(\frac{x}{x+1}\Big) \;&=\; L(1/2)-L\Big(\frac{x}{x+1}\Big)\,,\\ \mathcal{F}(x)\;+\;\mathcal{F}\Big(\frac{1}{x}\Big)\;&=\;0\,. \end{aligned} \tag{15}$$

3. Functional equations related to Hecke operators

As it turns out, as well as the two functional equations (4) the Herglotz function satisfies infinitely many further 1-variable functional equations, one for each Hecke operator T_n. Before we make this statement precise, we look at a related elementary problem about relations satisfied by cotangent products.

3.1. Identities between cotangent products.

Let

$$C(x) \;=\; \begin{cases} \cot\pi x\,, & x\in\mathbb{C}\smallsetminus\mathbb{Z}\,,\\ 0\,, & x\in\mathbb{Z}\,, \end{cases}$$

so that C is periodic, vanishes on $\mathbb{Z}$, and is holomorphic away from $\mathbb{Z}$. Define

$$\mathscr{C}(x,y) \;=\; C(x)C(y)+1\,,$$

which will be our main interest in this subsection. Note that it is even, symmetric, and satisfies

$$\mathscr{C}(x,y)\;-\;\mathscr{C}(x,x+y)\;-\;\mathscr{C}(x+y,y)\;=\;0 \tag{16}$$

whenever $xy(x+y)\neq 0$. (See [20] for an interpretation of this type of functional equation in terms of the cohomology of $\mathrm{SL}(2,\mathbb{Z})$.) We are interested in more general relations of the form

$$\sum_i \lambda_i\mathscr{C}(a_ix+b_iy,c_ix+d_iy) \;=\; \text{const}\,,$$

where a_i,b_i,c_i,d_i are integers with $a_id_i-b_ic_i\neq 0$.

Let Γ be the group $\mathrm{PSL}_2(\mathbb{Z})$, which is generated by the matrices $S=\big(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\big)$ and $U=\big(\begin{smallmatrix}1&-1\\1&0\end{smallmatrix}\big)$, with $S^2=U^3=1$, and let $T=US=\big(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\big)$. For $n\in\mathbb{N}$, let $\mathcal{M}_n$ be the set of 2×2 integer matrices of determinant n, modulo $\{\pm1\}$, and let $\mathcal{R}_n=\mathbb{Q}[\mathcal{M}_n]$. We denote both the element of $\mathcal{M}_n$ corresponding to a matrix $\big(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\big)$ of determinant n and the corresponding basis element of $\mathcal{R}_n$ by $\big[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\big]$. Define $\mathcal{M}=\bigcup_{n\in\mathbb{N}}\mathcal{M}_n$ and set $\mathcal{R}=\mathbb{Q}[\mathcal{M}]=\bigoplus_{n\in\mathbb{N}}\mathcal{R}_n$, which is a graded ring.

This ring acts on the right on the vector space of even meromorphic functions on $\mathbb{C}^2$ by the formula

$$\Big(f \circ \sum_i \lambda_i [\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}]\Big)(x,y) \;=\; \sum_i \lambda_i\, f(a_i x + b_i y, c_i x + d_i y)\,.$$

The same formula defines the action of $\mathcal{R}$ on the space $\mathcal{V}$ of even complex-valued functions on $\mathbb{C}^2/\mathbb{Z}^2$. We also define $\delta_1, \delta_2 \in \mathcal{V}$ by

$$\delta_1(x,y) \;=\; C(x)\delta(y)\,, \qquad \delta_2(x,y) \;=\; \delta(x)\delta(y)\,,$$

where $\delta = \delta_{\mathbb{Z}}\colon \mathbb{C} \to \{0,1\}$ is the characteristic function of $\mathbb{Z}$.

The functional equations that we are interested in are elements of the set

$$\mathcal{I} \;=\; \{\xi \in \mathcal{R} \mid \mathscr{C} \circ \xi = \text{const}\}\,,$$

which is a right ideal in $\mathcal{R}$. First, we give an algebraic description of $\mathcal{I}$.

Proposition 2. *An element ξ of $\mathcal{R}$ lies in the ideal $\mathcal{I}$ if and only if*

$$(1-S)\xi \;\in\; \ker(\Phi_1) \cap \ker(\Phi_2)\,,$$

where $\Phi_i\colon \mathcal{R} \to \mathbb{Q}(u,v) \otimes \mathcal{V}$, $i = 1,2$, are homomorphisms defined for $\gamma = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \mathcal{M}$ by

$$\Phi_1(\gamma) \;=\; \frac{1}{cu+dv} \otimes \delta_1 \circ \gamma\,. \qquad \Phi_2(\gamma) \;=\; \frac{c\det(\gamma)^{-1}}{cu+dv} \otimes \delta_2 \circ \gamma\,,$$

Proof. From the definitions of C and δ it follows that

$$C(x+\varepsilon) \;=\; \frac{\delta(x)}{\pi\varepsilon} + C(x) + O(\varepsilon)\,, \qquad x \in \mathbb{C}/\mathbb{Z},\;\; \varepsilon \to 0\,,$$

where we recall that $C|_{\mathbb{Z}} = 0$. If we fix $\xi_0 = (x_0, y_0)$ and set $(x,y) = (x_0 + \varepsilon u, y_0 + \varepsilon v)$, then

$$\mathscr{C}(ax+by, cx+dy) \;=\; \frac{1}{\pi^2 v \varepsilon^2}\, \Phi_2\big((1-S)\gamma\big)(x_0,y_0) + \frac{1}{\pi\varepsilon}\, \Phi_1\big((1-S)\gamma\big)(x_0,y_0) + O(1)\,.$$

Thus the function $f \coloneqq \mathscr{C} \circ \xi$ is continuous if and only if $(1-S)\xi \in \ker(\Phi_1) \cap \ker(\Phi_2)$. When this happens, the function f for fixed y is holomorphic, 1-periodic, and bounded when $|\operatorname{Im}(x)| \to \infty$, and hence constant by Liouville's theorem. Applying the same argument to f as a function of y, we get that $f(x,y)$ is constant. □

We now use the criterion in Propostion 2 to write down nontrivial elements in the ideal of relations $\mathcal{I}$ using Hecke operators. Following [1], we say that an element $\widetilde{T}_n \in \mathcal{R}_n$ ($n \in \mathbb{N}$) "acts like the n-th Hecke operator on periods" if it satisfies

$$(1-S)\widetilde{T}_n \;=\; T_n^{\infty}(1-S) \;+\; (1-T)Y\,, \tag{17}$$

for some $Y \in \mathcal{R}_n$, where T_n^{∞} is the usual representative of the n-th Hecke operator

$$T_n^{\infty} \;=\; \sum_{ad=n} \sum_{0 \le b < d} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \;\in\; \mathbb{Q}[\mathcal{M}_n]\,.$$

It was shown in [1] that such elements exist for every $n \in \mathbb{N}$ and satisfy $r_f|_{2-k}\widetilde{T}_n = r_{f|_k T_n}$ for the period polynomial r_f of any holomorphic modular form of weight k, whence the name.

Theorem 1. *Suppose that $\widetilde{T}_n \in \mathcal{R}_n$ acts like the n-th Hecke operator on periods. Then*

$$(\mathscr{C} \circ \widetilde{T}_n)(x,y) \;-\; \sum_{l|n} l\,\mathscr{C}(lx, ly) \;=\; c(\widetilde{T}_n)\,, \tag{18}$$

where $c\colon \mathcal{R} \to \mathbb{Z}$ is the group homomorphism defined on generators by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\mapsto\; \begin{cases} 1 - \operatorname{sgn}(a+b)\,\operatorname{sgn}(c+d)\,, & a+b \neq 0\,,\;\; c+d \neq 0\,, \\ 1 - \operatorname{sgn}(a+b)\,\operatorname{sgn}(d)\,, & a+b \neq 0\,,\;\; c+d = 0\,, \\ 1 - \operatorname{sgn}(b)\,\operatorname{sgn}(c+d)\,, & a+b = 0\,,\;\; c+d \neq 0\,. \end{cases}$$

Proof. We apply the criterion from Proposition 2. Since $\delta_1 \circ (1-T) = \delta_2 \circ (1-T) = 0$, the subspace $(1-T)\mathcal{R}_n$ lies in both $\ker \Phi_1$ and $\ker \Phi_2$, and therefore $\Phi_i((1-S)\widetilde{T}_n) = \Phi_i(T_n^\infty(1-S))$. Next, we calculate

$$\begin{aligned} n^{-1} \sum_{ad=n} \sum_{0\le b<d} \delta(ax+by)\delta(dy) \ &= \sum_{ad'd''=n} (ad')^{-1}\mathrm{den}(y,d')\delta(ad'x) \\ &= \sum_{d''|n} d'' n^{-1}\delta(xn/d'')\delta(yn/d'') \ = \ \sum_{l|n} l^{-1}\delta(lx)\delta(ly) \end{aligned}$$

and

$$\begin{aligned} \sum_{ad=n} \sum_{0\le b<d} d^{-1}C(ax+by)\delta(dy) \ &= \sum_{ad'd''=n} \mathrm{den}(y,d')C(ad'x) \\ &= \sum_{d''|n} C(xn/d'')\delta(yn/d'') \ = \ \sum_{l|n} C(lx)\delta(ly)\,, \end{aligned}$$

where $\mathrm{den}(x,d)$ is the function that equals 1 if x is a rational number with denominator d, and 0 otherwise. (We have used the identity $\sum_{j=0}^{m-1} C(x+j/m) = mC(mx)$.) Using these we compute

$$\begin{aligned} \Phi_1\big(T_n^\infty(1-S)\big) \ &= \sum_{ad=n} \sum_{0\le b<d} \Big(\frac{d^{-1}C(ax+by)\delta(dy)}{v} - \frac{d^{-1}C(bx-ay)\delta(dx)}{u}\Big) \\ &= \sum_{l|n} l \Big(\frac{C(lx)\delta(ly)}{lv} - \frac{C(-ly)\delta(lx)}{lu}\Big) \ = \ \Phi_1\Big(\sum_{l|n} l\left(\begin{smallmatrix} l & 0 \\ 0 & l\end{smallmatrix}\right)(1-S)\Big)\,. \end{aligned}$$

and

$$\begin{aligned} \Phi_2\big(T_n^\infty(1-S)\big) \ &= \ n^{-1}\sum_{ad=n} \sum_{0\le b<d} \Big(-\frac{\delta(bx-ay)\delta(dx)}{u}\Big) \ = \ \sum_{l|n} l^{-1}\Big(-\frac{\delta(-ly)\delta(lx)}{u}\Big) \\ &= \ \Phi_2\Big(\sum_{l|n} l\left(\begin{smallmatrix} l & 0 \\ 0 & l\end{smallmatrix}\right)(1-S)\Big) \end{aligned}$$

This implies that the left-hand side of (18) is constant. To get the stated value of this constant we set $x = it$, $y = i(1+\varepsilon)t$, and take the limit of the LHS in (18) as $t \to +\infty$, $\varepsilon \to 0+$. □

3.2. **Functional equations for $\mathcal{F}$.** We will now use the cotangent functional equations of Theorem 1 to obtain functional equations for the Herglotz function. Recall the standard slash action of $\mathcal{R}$ in even weight k, defined on the generators $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathcal{M}$ by

$$(f|_k\gamma)(x) \ = \ \frac{(ad-bc)^{k/2}}{(cx+d)^k}\, f\Big(\frac{ax+b}{cx+d}\Big)\,.$$

Theorem 2. *Suppose that $\widetilde{T}_n = \sum_\gamma \nu_\gamma\gamma$ acts like the n-th Hecke operator on periods and that all matrices in $\widetilde{T}_n$ have nonnegative entries. Then the modified Herglotz function $\mathcal{F}$ satisfies*

$$(19)\qquad \big(\mathcal{F}\big|_0\widetilde{T}_n\big)(x) \,-\, \sigma(n)\mathcal{F}(x) \,-\, \sum_{\gamma(\infty)\ne\infty} \nu_\gamma L\big(\gamma(x)/\gamma(\infty)\big) \ = \ A_n\log x + \mathrm{const} \qquad (x>0)\,,$$

where $L(x)$ is the Rogers dilogarithm defined in equation (12) *and* $A_n = \frac{1}{2}\sum_{l|n} l\log(l^2/n)$.

Proof. We use the same method as in [18], namely, to characterize $F'(x)$ by the equation

$$A(x,s) \ = \ \frac{x^{-1}}{s-1} \,+\, \Big(F'(x) - \frac{\log x}{x} - \frac{\zeta(2)}{x^2}\Big) \,+\, O(s-1)\,,$$

where $A(x,s)$ is the function defined for $\mathrm{Re}\,x>0$, $\mathrm{Re}\,s>1$ by

$$A(x,s) \ = \ \int_0^\infty \frac{t^s\,dt}{(e^{xt}-1)(e^t-1)}\,.$$

Now, for $x, y > 0$ we calculate

$$\begin{aligned} -\frac{(2\pi)^{s+1}}{4}\int_0^\infty \mathscr{C}(ixt,iyt)\,t^s\,dt \;&=\; y^{-s-1}A(x/y,s)+\tfrac{1}{2}\Gamma(s+1)\,\zeta(s+1)\,(x^{-s-1}+y^{-s-1}) \\ &=\; \frac{(xy)^{-1}}{s-1}+y^{-2}F'\Big(\frac{x}{y}\Big)+\frac{\pi^2}{12}\Big(\frac{1}{y^2}-\frac{1}{x^2}\Big)-\frac{\log x}{xy}+O(s-1) \\ &=\; \frac{(xy)^{-1}}{s-1}+y^{-2}\mathcal{F}'\Big(\frac{x}{y}\Big)-\frac{\log(xy)}{2xy}+O(s-1)\,. \end{aligned}$$

(Note that $\mathscr{C}(ixt, iyt)$ is exponentially small for t large.) Since all entries of matrices in $\widetilde{T}_n$ are nonnegative, we have $c(\widetilde{T}_n) = 0$ (where c is the homomorphism defined in Theorem 1), and equation (18) then implies that for $x > 0$

$$\sum_{\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathcal{M}_n} \nu_\gamma\left((\mathcal{F}'|_2\gamma)(x)-n\frac{\log((ax+b)(cx+d))}{2(ax+b)(cx+d)}\right) \;=\; \sum_{l|n}\frac{n}{l}\Big(\mathcal{F}'(x)-\frac{\log(l^2x)}{2x}\Big)\,.$$

Next, we integrate the above identity. The primitive of $(\mathcal{F}'|_2\gamma)(x)$ is $\mathcal{F}(\gamma x)$, while for the logarithmic term we use (assuming that $ad - bc = n$)

$$n\int\frac{\log((ax+b)(cx+d))}{2(ax+b)(cx+d)}\,dx \;=\; L\Big(\frac{x+b/a}{x+d/c}\Big)+\frac{1}{4}\log^2(n(x+b/a))-\frac{1}{4}\log^2(n(x+d/c))$$

for $c > 0$ and its variant

$$n\int\frac{\log((ax+b)d)}{2(ax+b)d}\,dx \;=\; \frac{1}{4}\log^2(n(x+b/a))$$

for $c = 0$ (both statements are easily checked by differentiation). Thus, modulo constants, the left-hand side of (19) is equal to

$$\frac{1}{4}\bigg(\sum_\gamma \nu_\gamma\log^2\big(n(x-\gamma^{-1}(0))\big)-\sum_{\gamma^{-1}(\infty)\neq\infty}\nu_\gamma\log^2\big(n(x-\gamma^{-1}(\infty))\big)-\sum_{l|n}\frac{n}{l}\log^2(l^2x)\bigg)\,.$$

A simple calculation using (17) shows that the homomorphism $\varphi\colon \mathcal{R} \to \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]$ defined on the generators by $\gamma \mapsto [\gamma^{-1}(0)] - [\gamma^{-1}(\infty)]$ satisfies $\varphi(\widetilde{T}_n) = \sigma(n)([0] - [\infty])$. Thus, the above expression simplifies to

$$\frac{1}{4}\Big(\sigma(n)\log^2(nx)-\sum_{l|n}\frac{n}{l}\log^2(l^2x)\Big) \;=\; A_n\log x+\text{const}\,,$$

as claimed. □

The following proposition shows that a $\widetilde{T}_n$ consisting of matrices with nonnegative entries exist for all $n \in \mathbb{N}$. Similar elements were constructed by Manin [10] and others.

Proposition 3. *The element $\widehat{T}_n \in \mathcal{R}_n$ defined by*

$$\widehat{T}_n \;=\; \sum_{\substack{0\le c<a,\ 0\le b<d\\ ad-bc=n}}\begin{bmatrix} a & b\\ c & d\end{bmatrix} \tag{20}$$

acts like the n-th Hecke operator on periods, i.e., there exists an element $Y_n \in \mathcal{R}_n$ such that

$$(1-S)\widehat{T}_n \;=\; T_n^\infty(1-S)+(1-T)Y_n\,.$$

Proof. Let us denote $\mathcal{P}_n = \{\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right] \in \mathcal{M}_n \mid 0 < c < a,\, 0 \le b < d\}$. It is easy to check that the map $\iota\colon \mathcal{P}_n \to \mathcal{P}_n$ given by

$$\begin{bmatrix} a & b\\ c & d\end{bmatrix} \;\mapsto\; \begin{bmatrix} \lceil a/c\rceil d-b & \lceil a/c\rceil c-a\\ d & c\end{bmatrix}$$

is a well-defined involution. Working modulo $(1-T)\mathcal{R}_n$, which amounts to introducing the equivalence relation $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} a+kc & b+kd \\ c & d \end{smallmatrix}\right]$, $k \in \mathbb{Z}$, we calculate

$$(1-S)\widehat{T}_n - T_n^\infty(1-S) \;\equiv\; \sum_{\substack{0<c<a\\0\le b<d}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \sum_{\substack{0\le c<a\\0<b<d}} \begin{bmatrix} -c & -d \\ a & b \end{bmatrix} = \sum_{\substack{0<c<a\\0\le b<d}} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} -b & -a \\ d & c \end{bmatrix} \right),$$

where all the sums are over matrices of determinant n, and to get the last equality we made the substitution $a \leftrightarrow d, b \leftrightarrow c$ in the second sum. Since $\iota(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) \equiv \left[\begin{smallmatrix} -b & -a \\ d & c \end{smallmatrix}\right]$, the terms in the last sum cancel in pairs (modulo $(1-T)\mathcal{R}_n$), and we conclude that $(1-S)\widehat{T}_n - T_n^\infty(1-S) \in (1-T)\mathcal{R}_n$. ☐

An essentially equivalent version of $\widehat{T}_n$ (with inequalities for rows instead of columns) was given by Merel in [11]. Here are the first three nontrivial examples

$$\widehat{T}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \widehat{T}_3 = \sum_{j=0}^{2} \begin{bmatrix} 1 & j \\ 0 & 3 \end{bmatrix} + \sum_{j=0}^{2} \begin{bmatrix} 3 & 0 \\ j & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$$\widehat{T}_4 = \sum_{j=0}^{3} \begin{bmatrix} 1 & j \\ 0 & 4 \end{bmatrix} + \sum_{j=0}^{3} \begin{bmatrix} 4 & 0 \\ j & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}.$$

For the special case $\widetilde{T}_n = \widehat{T}_n$, Theorem 2 gives the following explicit functional equation.

Corollary. *For all $n \in \mathbb{N}$ and $x > 0$ the modified Herglotz function $\mathcal{F}$ satisfies*

$$(21) \qquad \big(\mathcal{F}\big|_0\widehat{T}_n\big)(x) \;-\; \sigma(n)\mathcal{F}(x) \;=\; \sum_{\substack{0<c<a\\0\le b<d\\ad-bc=n}} \left[L\!\left(\frac{x+b/a}{x+d/c}\right) - L\!\left(\frac{1+b/a}{1+d/c}\right) \right] \;+\; A_n \log x\,,$$

where $L(x)$ is the Rogers dilogarithm, and $A_n = \frac{1}{2}\sum_{l|n} l \log(l^2/n)$.

Proof. By Theorem 2 equation (21) holds up to an additive constant. Setting $x = 1$ and using the functional equation $\mathcal{F}(x)+\mathcal{F}(1/x) = 0$ together with the fact that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\widehat{T}_n\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \widehat{T}_n$ shows that (21) holds for $x = 1$, and hence the constant is 0. ☐

The functional equations corresponding to the three $\widehat{T}_n$'s given above are

$$(22a) \qquad \mathcal{F}(2x) + \mathcal{F}(\tfrac{x}{2}) + \mathcal{F}(\tfrac{x+1}{2}) + \mathcal{F}(\tfrac{2x}{x+1}) - 3\mathcal{F}(x) \;=\; L(\tfrac{x}{x+1}) - L(\tfrac{1}{2}) + \tfrac{1}{2}\log(2)\log(x)\,,$$

$$(22b) \qquad \begin{aligned} &\mathcal{F}(3x) + \mathcal{F}(\tfrac{3x}{x+1}) + \mathcal{F}(\tfrac{3x}{2x+1}) + \mathcal{F}(\tfrac{x}{3}) + \mathcal{F}(\tfrac{x+1}{3}) + \mathcal{F}(\tfrac{x+2}{3}) + \mathcal{F}(\tfrac{2x+1}{x+2}) - 4\mathcal{F}(x) \\ &\qquad = L(\tfrac{2x+1}{2x+4}) + L(\tfrac{x}{x+1}) + L(\tfrac{2x}{2x+1}) - L(1) - L(\tfrac{2}{3}) + \log(3)\log(x)\,, \end{aligned}$$

$$(22c) \qquad \begin{aligned} &\textstyle\sum_{j=0}^{3} \big(\mathcal{F}(\tfrac{x+j}{4}) + \mathcal{F}(\tfrac{4x}{jx+1})\big) + \mathcal{F}(\tfrac{2x}{x+2}) + \mathcal{F}(\tfrac{2x+1}{2}) + \mathcal{F}(\tfrac{2x+2}{x+3}) + \mathcal{F}(\tfrac{3x+1}{2x+2}) - 6\mathcal{F}(x) \\ &= \textstyle\sum_{j=1}^{3} L(\tfrac{jx}{jx+1}) + L(\tfrac{x}{x+2}) + L(\tfrac{x+1}{x+3}) + L(\tfrac{3x+1}{3x+3}) - 4L(1) + L(\tfrac{2}{3}) + 3\log(2)\log(x)\,. \end{aligned}$$

We will make use of the first and the third of these in Section 6 to obtain explicit evaluations of the function $J(x)$ for certain quadratic units x.

We remark that there is no loss of generality in passing from the apparently more general functional equation (19) to the special case (21), because one can show that the functional equation coming from any other choice of $\widetilde{T}_n$ would be the same up to a linear combination of specializations of the functional equations (15).

4. Special values at positive rationals

The special values of the Herglotz function at positive rationals turn out to always be expressible in terms of dilogarithms. Here it is more convenient to use the original function $F(x)$ instead of $\mathcal{F}(x)$.

Theorem 3. *For any positive rational x the value $F(x)-F(1)$ is a rational linear combination of Li_2 with arguments belonging to the cyclotomic field $\mathbb{Q}(e^{2\pi ix}, e^{2\pi i/x})$.*

Proof. From the second integral representation in (5) we get

$$F\Big(\frac{P}{Q}\Big)-F(1) \;=\; F\Big(\frac{P}{Q}\Big)-F\Big(\frac{P}{P}\Big) \;=\; \int_0^1\Big(\frac{Q}{1-t^Q}-\frac{P}{1-t^P}\Big)\log(1-t^P)\,\frac{dt}{t}\,.$$

Therefore, using the distribution relations

$$\log(1-t^P) \;=\; \sum_{\beta^P=1}\log(1-\alpha t) \qquad \text{and} \qquad \frac{Q}{1-t^Q} \;=\; \sum_{\alpha^Q=1}\frac{1}{1-\beta t}\,,$$

we see that for all positive integers P, Q one has

$$F\Big(\frac{P}{Q}\Big)-F(1) \;=\; \sum_{\alpha^P=1}\Bigg(\sum_{\beta^Q=1,\,\beta\neq 1} f(\alpha,\beta) \;-\; \sum_{\beta^P=1,\,\beta\neq 1} f(\alpha,\beta)\Bigg)\,,$$

where

$$f(\alpha,\beta) \;=\; \int_0^1\frac{\log(1-\alpha t)}{t(1-\beta t)}\,dt\,.$$

The statement of the theorem follows using the identity

$$f(\alpha,\beta) \;=\; \mathrm{Li}_2\Big(\frac{\beta}{\beta-1}\Big) \;-\; \mathrm{Li}_2\Big(\frac{\alpha-\beta}{1-\beta}\Big)\,,$$

which can be proved easily by differentiating both sides in α. □

We give three concrete examples of the theorem.

1. Setting $(P,Q)=(1,n)$ in the above proof and using $\mathrm{Li}_2(\frac{x}{x-1})+\mathrm{Li}_2(x)=-\frac{1}{2}\log^2(1-x)$ gives

$$F(n)-F(1) \;=\; -\frac{(n-1)(n-2)\pi^2}{24n} \;+\; \frac{\log^2 n}{2} \;+\; \frac{1}{2}\sum_{j=1}^{n-1}\log^2\Big(2\sin\frac{\pi j}{n}\Big)\,. \tag{23}$$

Notice that this statement is stronger than Theorem 3, since we only get products of logarithms rather than dilogarithms. Conjecturally, the only positive rationals x for which $F(x)-F(1)$ can reduce to products of logarithms are $x=n$ or $x=1/n$, since for other values of x the corresponding formal combination of arguments of Li_2 does not lie in the Bloch group of $\overline{\mathbb{Q}}$.

2. Combining (23) with the 3-term relation (4) gives $F(\frac{n}{n+1})-\mathrm{Li}_2(\frac{n}{n+1})$ as a bilinear combination of logarithms of elements of $\mathbb{Q}(\zeta_n,\zeta_{n+1})$, where $\zeta_n=e^{2\pi i/n}$.

3. As a further example, which we will generalize in Section 7.2, we have

$$F\Big(\frac{2}{5}\Big)-F(1) \;=\; \frac{1}{2}\mathrm{Li}_2\Big(\frac{4}{5}\Big)-\frac{\pi^2}{5} \;+\; \log^2\Big(2\sin\frac{\pi}{5}\Big) \;+\; \log^2\Big(2\sin\frac{2\pi}{5}\Big) \;+\; \log(2)\log\Big(\frac{2}{5}\Big)\,. \tag{24}$$

An argument similar to the one used in the proof of Theorem 3 allows one to compute also the derivative of F at rational points.

Proposition 4. *For any coprime $p,q>0$ the difference $\frac{p}{q}F'(\frac{p}{q})-(1+\log(p))$ is a linear combination of Li_2 at p-th or q-th roots of unity with coefficients in $\mathbb{Q}(\zeta_p,\zeta_q)$.*

Proof. Replacing the integral reprsentation (5) by (6) in the previous proof we obtain

$$\frac{P}{Q}F'\Big(\frac{P}{Q}\Big)-F'(1) \;=\; \sum_{\alpha^P=1}\Bigg(\sum_{\beta^Q=1,\,\beta\neq 1} g(\alpha,\beta) \;-\; \sum_{\beta^P=1,\,\beta\neq 1} g(\alpha,\beta)\Bigg)\,,$$

where

$$g(\alpha,\beta) \;=\; -\int_0^1\frac{\alpha\log(y)}{(1-\alpha y)(1-\beta y)}\,dy \;=\; \begin{cases}\alpha\dfrac{\mathrm{Li}_2(\alpha)-\mathrm{Li}_2(\beta)}{\alpha-\beta}\,, & \alpha\neq\beta\,,\\ \mathrm{Li}_1(\alpha)\,, & \alpha=\beta\,.\end{cases}$$

The result then follows by noting that $F'(1)=1$, and that $\sum_{\alpha^P=1,\alpha\neq 1}\mathrm{Li}_1(\alpha)=-\log(P)$. □

Similarly, one can show that the value $(p/q)^kF^{(k)}(p/q)-F^{(k)}(1)+(-1)^k(k-1)!\log(p)$ is a combination of Li_m, $m=2,\ldots,k+1$ at p-th or q-th roots of unity with coefficients in $\mathbb{Q}(\zeta_p,\zeta_q)$.

5. Kronecker limit formula for real quadratic fields

The function F appeared in [18] in a formulation of a so-called Kronecker limit formula for real quadratic fields. (The original Kronecker limit formula was the corresponding statement for imaginary quadratic fields.) This is a formula expressing the value

$$\varrho(\mathcal{B}) \;=\; \lim_{s\to1}\left(D^{s/2}\zeta(\mathcal{B},s)\;-\;\frac{\log\varepsilon}{s-1}\right),$$

where $\mathcal{B}$ is an element of the narrow class group [2] of the quadratic order $\mathcal{O}_D=\mathbb{Z}+\mathbb{Z}\frac{D+\sqrt{D}}{2}$ of discriminant $D>0$, $\zeta(\mathcal{B},s)$ is the corresponding partial zeta function (the sum of $N(\mathfrak{a})^{-s}$ over all invertible $\mathcal{O}_D$-ideals $\mathfrak{a}$ in the class $\mathcal{B}$), and $\varepsilon=\varepsilon_D$ is the smallest unit >1 in $\mathcal{O}_D$ of norm 1. The numbers $\varrho(\mathcal{B})$ are of interest because the value at $s=1$ of the Dirichlet series $L_K(s,\chi)=\sum_{\mathfrak{a}}\chi(\mathfrak{a})/N(\mathfrak{a})^s$ equals $D^{-1/2}\sum_{\mathcal{B}}\chi(\mathcal{B})\varrho(\mathcal{B})$ for any character χ on the narrow ideal class group. The formula proved in [18] is that

$$\varrho(\mathcal{B}) \;=\; \sum_{w\in\mathrm{Red}(\mathcal{B})}P(w,w'),\tag{25}$$

where the function $P(x,y)$ for $x>y>0$ is defined as

$$P(x,y) \;=\; F(x)-F(y)+\mathrm{Li}_2\Big(\frac{y}{x}\Big)-\frac{\pi^2}{6}+\log\frac{x}{y}\,\Big(\gamma-\frac12\log(x-y)+\frac14\log\frac{x}{y}\Big),$$

and $\mathrm{Red}(\mathcal{B})$ is the set of larger roots $w=\frac{-b+\sqrt{D}}{2a}$ of all reduced primitive quadratic forms $Q(X,Y)=aX^2+bXY+cY^2$ $(a,c>0,\;a+b+c<0)$ of discriminant D which belong to the class $\mathcal{B}$. Recall that narrow ideal classes correspond to $\mathrm{PSL}_2(\mathbb{Z})$-orbits on the set of integral quadratic forms: if $\mathfrak{b}=\mathbb{Z}w_1+\mathbb{Z}w_2\in\mathcal{B}$ with $\frac{w_1w_2'-w_2w_1'}{\sqrt{D}}>0$, then the quadratic form $Q(X,Y)=\frac{N(Xw_1+Yw_2)}{N(\mathfrak{b})}$ is in the corresponding orbit. The set $\mathrm{Red}(\mathcal{B})$ is finite and every element $w\in\mathrm{Red}(\mathcal{B})$ satisfies $w>1>w'>0$. Let us also denote $l(\mathcal{B})=\#\mathrm{Red}(\mathcal{B})$.

The set $\mathrm{Red}(\mathcal{B})$ can also be understood in terms of continued fractions. To any element $w\in K$ with $w>w'$ (here we assume that K is embedded into $\mathbb{R}$) one can associate a continued fraction

$$w \;=\; b_1-\cfrac{1}{b_2-\cfrac{1}{\ddots}},$$

which by the standard theory of continued fractions is eventually periodic. The sequence $\{b_j\}_{j\ge1}$ is periodic (without a pre-period) if and only if the number w is *reduced* (i.e., $w>1>w'>0$), and then has period $l=l(\mathcal{B})$. We call the l-tuple $(\!(b_1,\dots,b_l)\!)$ the cycle associated to w. If we fix a narrow ideal class $\mathcal{B}$ and take any $\mathfrak{b}\in\mathcal{B}^{-1}$ such that $\mathfrak{b}=\mathbb{Z}+\mathbb{Z}w$ with w reduced, then the equivalence class of $(\!(b_1,\dots,b_l)\!)$ modulo cyclic permutations depends only on $\mathcal{B}$. The set $\mathrm{Red}(\mathcal{B})$ is then simply $\{w_1,\dots,w_l\}$, where

$$w_j \;=\; b_j-\cfrac{1}{b_{j+1}-\cfrac{1}{\ddots}},$$

and both b_j and w_j depend only on $j \pmod l$. Thus we can restate (25) as

$$\varrho(\mathcal{B}) \;=\; \sum_{j \pmod l}P(w_j,w_j').\tag{26}$$

[2] more precisely, the group of narrow ideal classes of invertible fractional $\mathcal{O}_D$-ideals, where two ideals belong to the same narrow class if their quotient is a principal ideal λO_D with $N(\lambda)>0$. For more details see [19] or [4]

Using the fact that $\sum_{j \pmod l}\bigl(w_j + 1/w_j\bigr) = \sum_{j \pmod l}\bigl(w'_j + 1/w'_j\bigr)$ and $\prod_{j \pmod l} w_j = \varepsilon$, we can rewrite (26) as

$$\varrho(\mathcal{B}) \;=\; 2\gamma \log \varepsilon \;-\; l(\mathcal{B})\frac{\pi^2}{6} \;+\; \sum_{j \pmod l} \mathcal{P}(w_j, w'_j)\,, \tag{27}$$

where the function $\mathcal{P}(x,y)$, $x > y > 0$ is now defined by a simpler formula

$$\mathcal{P}(x,y) \;=\; \mathcal{F}(x) \;-\; \mathcal{F}(y) \;+\; L\Bigl(\frac{y}{x}\Bigr)\,. \tag{28}$$

One can use the Kronecker limit formula (25) to prove various properties of $\varrho(\mathcal{B})$. For example, the functional equation (4) was used in [18] to prove Meyer's formula

$$\varrho(\mathcal{B}) - \varrho(\mathcal{B}^*) \;=\; -\frac{\pi^2}{6}\bigl(l(\mathcal{B}) - l(\mathcal{B}^*)\bigr)\,,$$

where $\mathcal{B}^* = \Theta\mathcal{B}$, and Θ is the narrow ideal class of principal ideals of negative norm. The key observation used in that proof is equivalent to the identity

$$\mathcal{P}(x,y) \;=\; \widetilde{\mathcal{P}}(x-1, 1-y) \;-\; \widetilde{\mathcal{P}}\Bigl(1 - \frac{1}{x}, \frac{1}{y} - 1\Bigr)\,, \qquad x > 1 > y > 0\,, \tag{29}$$

where $\widetilde{\mathcal{P}}(x,y) = \mathcal{F}(x) - \mathcal{F}(y) + L(-y/x)$, that allows one to rewrite $\varrho(\mathcal{B}) - 2\gamma\log\varepsilon + \frac{\pi^2}{6} l(\mathcal{B})$ in terms of the continued fraction associated to the wide ideal class containing $\mathcal{B}$. For details we refer to [18], but we do state here a version of the Kronecker limit formula for wide ideal classes, since we will use it in Section 6 below. A number $x \in K$ is called *reduced in the wide sense* if $x > 1$, $0 > x' > -1$, and to any such number one associates a cycle $[\![a_1, \ldots, a_m]\!]$, which is simply the periodic part of the regular continued fraction of x, i.e., $x = a_1 + 1/(a_2 + 1/(a_3 + \ldots))$ with $a_i = a_{i \,(\mathrm{mod}\ m)}$ for $i \in \mathbb{Z}$. We define $\mathrm{Red}^W(\mathcal{A})$ to be the set of reduced numbers x (in the wide sense) such that $\mathbb{Z}x + \mathbb{Z}$ is a fractional ideal in $\mathcal{A}$. Then $\mathrm{Red}^W(\mathcal{A}) = \{x_1, \ldots, x_m\}$, where

$$x_i \;=\; a_i + \cfrac{1}{a_{i+1} + \cfrac{1}{\ddots}}\,.$$

Proposition 5 ([18, Corollary 2, p. 179]). *Let $\mathcal{A}$ be a wide ideal class in a real quadratic field K, $\varepsilon_0 > 1$ the fundamental unit of K, $x_1, \ldots, x_m$ the elements of K satisfying $x_i > 1$, $0 > x'_i > -1$ and such that $\{1, x_i\}$ is a basis of some ideal in $\mathcal{A}$, and $[\![a_1, \ldots, a_m]\!]$ the corresponding cycle of integers. Then*

$$\varrho(\mathcal{A}) \;=\; 2\sum_{i=1}^{m} \widetilde{\mathcal{P}}(x_i, -x'_i) \;+\; 4\gamma\log\varepsilon_0 \;-\; \frac{\pi^2}{6}\sum_{i=1}^{m}(a_i - 1) \tag{30}$$

with $\widetilde{P}$ as above.

6. Special values at quadratic units

While there does not seem to be a general formula, akin to that of Theorem 3, that would express in closed form the individual values $\mathcal{F}(x)$ for x in real quadratic fields, there are many rational linear combinations of these values that can be evaluated. One way to obtain such identities is to specialize functional equations satisfied by $\mathcal{F}(x)$, for example the 3-term relation (15), or the more complicated functional equations from Section 3. A completely different source of identities, as was explained in [18, §9], stems from the fact that if χ is a genus character on the narrow class group of a real quadratic field of discriminant D, then the special value $L_K(1,\chi) = D^{-1/2}\sum_{\mathcal{B}} \chi(\mathcal{B})\varrho(\mathcal{B})$ is equal to a product $L_{D_1}(1)L_{D_2}(1)$, where the splitting $D = D_1 D_2$ corresponds to the genus character χ, and $L_D(s)$ is the Dirichlet series

$$L_D(s) = \sum_{n \geq 1} \Bigl(\frac{D}{n}\Bigr) n^{-s}\,.$$

n	n^2-1	$4\left(\mathcal{J}(n+\sqrt{n^2-1})-n\frac{\pi^2}{24}\right)$
2	3	$2S_{1,12}$
3	2^3	$6S_{1,8}$
4	$3\cdot 5$	$2S_{1,15}+4S_{5,12}$
5	$2^3\cdot 3$	$3S_{1,24}+2S_{8,12}$
6	$5\cdot 7$	$2S_{1,140}+4S_{5,28}$
7	$2^4\cdot 3$	$7S_{1,12}+2S_{8,24}$
8	$3^2\cdot 7$	$2S_{1,28}+4S_{12,21}$
9	$2^4\cdot 5$	$22S_{1,5}+4S_{8,40}$
11	$2^3\cdot 3\cdot 5$	$3S_{1,120}+2S_{8,60}+2S_{12,40}+2S_{5,24}$
12	$11\cdot 13$	$2S_{1,572}+4S_{13,44}$
13	$2^3\cdot 3\cdot 7$	$3S_{1,168}+2S_{8,21}+S_{12,56}+S_{24,28}$
14	$3\cdot 5\cdot 13$	$2S_{1,780}+4S_{5,156}+4S_{13,60}$
16	$3\cdot 5\cdot 17$	$2S_{1,1020}+4S_{12,85}+4S_{5,204}$
19	$2^3\cdot 3^2\cdot 5$	$6S_{1,40}+12S_{5,8}+2S_{12,120}+2S_{24,60}$
21	$2^3\cdot 5\cdot 11$	$3S_{1,440}+2S_{8,220}+2S_{5,88}+2S_{40,44}$
22	$3\cdot 7\cdot 23$	$2S_{1,1932}+2S_{28,69}+2S_{21,92}$
23	$2^4\cdot 3\cdot 11$	$4S_{1,33}+2S_{8,264}+S_{12,44}+S_{24,88}$
27	$2^3\cdot 7\cdot 13$	$3S_{1,728}+2S_{8,364}+2S_{28,104}+2S_{13,56}$
29	$2^3\cdot 3\cdot 5\cdot 7$	$3S_{1,840}+S_{12,280}+2S_{5,168}+S_{24,140}+S_{28,120}+2S_{21,40}+S_{56,60}$
34	$3\cdot 5\cdot 7\cdot 11$	$2S_{1,4620}+4S_{5,924}+2S_{28,165}+2S_{60,77}+2S_{21,220}$
36	$5\cdot 7\cdot 37$	$2S_{1,5180}+4S_{5,1036}+4S_{37,140}$
41	$2^4\cdot 3\cdot 5\cdot 7$	$4S_{1,105}+2S_{8,840}+S_{12,140}+2S_{5,21}+S_{24,280}+S_{28,60}+2S_{40,168}+S_{56,120}$
43	$2^3\cdot 3\cdot 7\cdot 11$	$3S_{1,1848}+2S_{8,924}+S_{12,616}+S_{24,77}+S_{28,264}+S_{44,168}+S_{33,56}+S_{21,88}$
56	$3\cdot 5\cdot 11\cdot 19$	$2S_{1,12540}+4S_{12,1045}+4S_{5,2508}+2S_{44,285}+2S_{76,165}$
61	$2^3\cdot 3\cdot 5\cdot 31$	$3S_{1,3720}+S_{12,1240}+2S_{5,744}+S_{24,620}+2S_{40,93}+S_{60,1032}+S_{120,124}$
69	$2^3\cdot 5\cdot 7\cdot 17$	$3S_{1,4760}+2S_{8,2380}+2S_{5,952}+2S_{28,680}+2S_{40,476}+2S_{56,85}+2S_{136,140}$
77	$2^3\cdot 3\cdot 13\cdot 19$	$3S_{1,5928}+2S_{8,741}+S_{12,1976}+S_{24,988}+2S_{13,456}+S_{76,312}+S_{152,156}$
83	$2^3\cdot 3\cdot 7\cdot 41$	$3S_{1,6888}+2S_{8,861}+3S_{12,2296}+S_{24,1148}+S_{28,984}+S_{56,492}+4S_{21,328}$
131	$2^3\cdot 3\cdot 5\cdot 11\cdot 13$	$3S_{1,17160}+S_{12,5720}+2S_{5,3432}+S_{24,2860}+2S_{40,429}+S_{44,1560}+2S_{13,1320}$ $+S_{60,1144}+S_{88,780}+2S_{104,165}+S_{120,572}+S_{156,440}+S_{220,312}$

TABLE 1. Values of $\mathcal{J}(n+\sqrt{n^2-1})$

Since $L_D(1)$ can be evaluated explicitly as

$$L_D(1) \;=\; \begin{cases} \frac{2h\log\varepsilon}{\sqrt{|D|}}\,, & D>0\,,\\ \frac{2\pi h}{w\sqrt{|D|}}\,, & D<0\,, \end{cases}$$

and $\varrho(\mathcal{B})$ can be evaluated in terms of $\mathcal{F}$ via the Kronecker limit formula (27), this leads to nontrivial identities for $\mathcal{F}(x)$. We will discuss different realizations of this idea in this section, looking first at the case where each genus consists of only one narrow ideal class.

6.1. **Explicit formulas when there is one class per genus.** In Table 1 (resp. Table 2) we collect all numbers n less than 100 000 (and conjecturally all n) such that each genus of binary quadratic forms of discriminant $4(n^2-1)$ (resp. $4(n^2+1)$) contains exactly one narrow

n	n^2+1	$2\left(\mathcal{J}(n+\sqrt{n^2+1})-n\frac{\pi^2}{24}\right)$
1	2	$S_{1,8}$
2	5	$4S_{1,5}$
3	$2\cdot 5$	$S_{1,40}+2S_{5,8}$
4	17	$2S_{1,17}$
5	$2\cdot 13$	$S_{1,104}+2S_{8,13}$
7	$2\cdot 5^2$	$3S_{1,8}+4S_{5,40}$
8	$5\cdot 13$	$2S_{1,65}+4S_{5,13}$
11	$2\cdot 61$	$S_{1,488}+2S_{8,61}$
13	$2\cdot 5\cdot 17$	$S_{1,680}+2S_{8,85}+2S_{5,136}$
17	$2\cdot 5\cdot 29$	$S_{1,1160}+2S_{5,232}+2S_{29,40}$
19	$2\cdot 181$	$S_{1,1448}+2S_{8,181}$
23	$2\cdot 5\cdot 53$	$S_{1,2120}+2S_{5,424}+2S_{40,53}$
31	$2\cdot 13\cdot 37$	$S_{1,3848}+2S_{13,296}+2S_{37,104}$
37	$2\cdot 5\cdot 137$	$S_{1,5480}+2S_{8,685}+4S_{5,1096}$
47	$2\cdot 5\cdot 13\cdot 17$	$S_{1,8840}+4S_{5,1768}+2S_{40,221}+2S_{13,680}+2S_{85,104}$
73	$2\cdot 5\cdot 13\cdot 41$	$S_{1,21320}+2S_{5,4264}+2S_{40,533}+2S_{13,1640}+2S_{104,205}$

TABLE 2. Values of $\mathcal{J}(n+\sqrt{n^2+1})$

equivalence class. In each case we give an identity for $\mathcal{J}(n+\sqrt{n^2\pm 1})$, where

$$\mathcal{J}(x) \;=\; J(x)-\frac{\log^2(2)}{2}+\frac{\pi^2}{24}\Big(x-\frac{1}{x}\Big)$$

is related to $\mathcal{F}(x)$ by

$$\mathcal{J}(x)=\mathcal{F}(2x)-2\mathcal{F}(x)+\mathcal{F}(x/2)\,,$$

parallel to the relation (2) between $J(x)$ and $F(x)$. The numbers $S_{p,q}$ are defined as

$$S_{p,q} \;=\; \log(\varepsilon_p)\log(\varepsilon_q)\,,$$

where ε_d for d a fundamental discriminant is the fundamental unit of $\mathbb{Q}(\sqrt{d})$ if $d>1$, and ε_d is defined to be 2 if $d=1$.

Theorem 4. *The values of* $\mathcal{J}(n+\sqrt{n^2\pm 1})$ *for the* 45 *known one-class-per-genus cases are as given in Table 1 and Table 2.*

We observe that, with the exception of $n=7$, 9, 23, and 41, the cases listed in Table 1 also follow from Theorem 6 (for n even) or from Theorems 8 and 9 (for n odd) of [13]. The identities listed in Table 2 appear to be new.

As already indicated, we will deduce Theorem 4 from the following general expression for $\mathcal{J}(n+\sqrt{n^2\pm 1})$ as a combination of Kronecker limits $\varrho(\mathcal{B})$.

Theorem 5. *For all* $n\geq 1$ *the value* $\mathcal{J}(n+\sqrt{n^2\pm 1})$ *is a rational linear combination of* $\zeta(2)$, $\log(2)\log(n+\sqrt{n^2\pm 1})$, *and the values* $\varrho(\mathcal{B})$ *for at most four narrow classes* $\mathcal{B}$ *of quadratic forms of discriminant* $4^a(n^2\pm 1)$, *with* $a=0,\pm 1$.

Proof. We first consider the case of $\mathcal{J}(u)$ for $u=n+\sqrt{n^2-1}$. Define $\mathcal{B}_1$ and $\mathcal{B}_2$ to be the narrow classes with cycles $(\!(2n)\!)$ and $(\!(n+1,2)\!)$, respectively, and if n is odd also define $\mathcal{B}_3$ and $\mathcal{B}_4$ to be the narrow classes corresponding to the cycles $\big(\!\big(\frac{n+3}{2},2,2,2\big)\!\big)$ and $\big(\!\big(\frac{n+1}{2},4\big)\!\big)$ respectively. Then we claim that

$$2\mathcal{J}(u) \;=\; \varrho(\mathcal{B}_1)-\varrho(\mathcal{B}_2)+\log(2)\log(u)\,,\qquad n \text{ even}\,, \tag{31}$$

$$4\mathcal{J}(u) \;=\; \varrho(\mathcal{B}_1)+\varrho(\mathcal{B}_2)-\varrho(\mathcal{B}_3)-\varrho(\mathcal{B}_4)+3\log(2)\log(u)\,,\qquad n \text{ odd},\ n\neq 7\,. \tag{32}$$

We have

$$\mathrm{Red}(\mathcal{B}_1) \;=\; \{u\}\,, \quad \mathrm{Red}(\mathcal{B}_2) \;=\; \Big\{\frac{u+1}{2},\,\frac{2u}{u+1}\Big\}\,,$$

and for n odd also

$$\mathrm{Red}(\mathcal{B}_3) \;=\; \Big\{\frac{4u}{3u+1},\frac{3u+1}{2u+2},\frac{2u+2}{u+3},\frac{u+3}{4}\Big\}\,, \qquad \mathrm{Red}(\mathcal{B}_4) \;=\; \Big\{\frac{4u}{u+1},\frac{u+1}{4}\Big\}\,.$$

Since $u'=1/u$, we can use the Kronecker limit formula (27) to rewrite equation (31) as

$$2\mathcal{J}(u) \;=\; \mathcal{P}\big(u,\tfrac{1}{u}\big)-\mathcal{P}\big(\tfrac{u+1}{2},\tfrac{u+1}{2u}\big)-\mathcal{P}\big(\tfrac{2u}{u+1},\tfrac{2}{u+1}\big)+\zeta(2)+\log(2)\log(u)\,,$$

where we recall that $\mathcal{P}(x,y)=\mathcal{F}(x)-\mathcal{F}(y)+L(y/x)$. This identity, in fact, holds for all real $u>1$, as can be derived easily using the functional equation (22a) and the following simple relation for the Rogers dilogarithm

$$2L\big(\tfrac{u}{u+1}\big)\;+\;2L\big(\tfrac{1}{u}\big)\;-\;L\big(\tfrac{1}{u^2}\big)\;=\;2L(1)\,.$$

Similarly, (32) is equivalent to

$$\begin{aligned}4\mathcal{J}(u) \;&=\mathcal{P}\big(u,\tfrac{1}{u}\big)\;+\;2\mathcal{P}\big(\tfrac{u+1}{2},\tfrac{u+1}{2u}\big)\;-\;2\mathcal{P}\big(\tfrac{u+1}{4},\tfrac{u+1}{4u}\big)\\ &\quad-\;2\mathcal{P}\big(\tfrac{4u}{3u+1},\tfrac{4}{u+3}\big)\;-\;2\mathcal{P}\big(\tfrac{3u+1}{2u+2},\tfrac{u+3}{2u+2}\big)\;+\;3\zeta(2)\;+\;3\log(2)\log(u)\,.\end{aligned}$$

Again, this identity holds for all $u>1$, and can be derived using the functional equations (22a) and (22c) for $\mathcal{F}(x)$. Finally, for $n=7$ we have

$$4\mathcal{J}(7+\sqrt{48}) \;=\; \varrho(\mathcal{B}_1)-\varrho(\mathcal{B}_2')+2\zeta(2)+\tfrac{7}{2}\log(2)\log(7+\sqrt{48})\,,$$

where $\mathcal{B}_2'$ corresponds to the cycle $(\!(6,2,2)\!)$. This can again be derived using (22a) and (22c).

The case of $\mathcal{J}(v)$ for $v=n+\sqrt{n^2+1}$ is similar. Here we define $\mathcal{A}_1$ and $\mathcal{A}_2$ to be the wide ideal classes with cycles $[\![2n]\!]$ and $[\![n-1,1,1]\!]$ respectively. We claim that

$$4\mathcal{J}(v) \;=\; \varrho(\mathcal{A}_1)-\varrho(\mathcal{A}_2)+2\log(2)\log(v)+n\zeta(2)\,, \qquad\qquad n>2\,.$$

Once again, using

$$\mathrm{Red}_w(\mathcal{A}_1) \;=\; \{v\}\,, \qquad \mathrm{Red}_w(\mathcal{A}_2) \;=\; \big\{\tfrac{2v}{v+1},\,\tfrac{v+1}{v-1},\,\tfrac{v-1}{2}\big\}\,,$$

and the Kronecker limit formula (30) one can rewrite the identity as

$$2\mathcal{J}(v) \;=\; \widetilde{\mathcal{P}}\big(v,\tfrac{1}{v}\big)-\widetilde{\mathcal{P}}\big(\tfrac{2v}{v+1},\tfrac{2}{v-1}\big)-\widetilde{\mathcal{P}}\big(\tfrac{v+1}{v-1},\tfrac{v-1}{v+1}\big)-\widetilde{\mathcal{P}}\big(\tfrac{v-1}{2},\tfrac{v+1}{2v}\big)-\frac{1}{2}\zeta(2)+\log(2)\log(v)\,,$$

(with $\widetilde{\mathcal{P}}$ as in (29)) which follows easily from (22a) and the 3-term relation (15). Finally, the cases $n=1,2$ do not fit the above scheme, but can be derived directly using functional equations that $\mathcal{J}(n+\sqrt{n^2+1})$ is a linear combination of $\zeta(2)$ and $\log(2)\log(n+\sqrt{n^2+1})$. For instance, the case $n=1$ follows directly by substituting $x=1+\sqrt{2}$ into (22a) and applying the 3-term relation (15) with $x=\sqrt{2}$. The derivation for $n=2$ is slightly more complicated, but again only involves the functional equations (15) and (22a). □

Proof of Theorem 4. Since in all the cases listed in the tables there is one class per genus of quadratic forms, we can rewrite the corresponding linear combination of $\varrho(\mathcal{B}_i)$ as a rational linear combination of $D^{1/2}L_{\mathcal{O}_D}(1,\chi)$, where χ runs over genus characters of the corresponding class groups. In each case the identity then follows from the factorization of $L_{\mathcal{O}}(s,\chi)$ into a product of two Dirichlet L-functions (for general discriminants, see, for example, [7]).

There is a small subtlety in some cases when we need to consider a combination $\varrho(\mathcal{B})-\varrho(\mathcal{B}')$ with $\mathcal{B}$ of discriminant D and $\mathcal{B}'$ of discriminant $4D$ and a priori the resulting expression for $\mathcal{J}(n+\sqrt{n^2\pm1})$ may involve constant terms of $\zeta_{\mathcal{O}_D}(s)$ and $\zeta_{\mathcal{O}_{4D}}(s)$ at $s=1$. However, a simple but somewhat tedious calculation using the explicit expression for $\zeta_{\mathcal{O}}(s)$ (see, say [6]) and the relation between the class groups of discriminants D and $4D$ (see, e.g., [4, Cor. 5.9.9]) shows that in all these cases the nontrivial contributions coming from $\zeta_{\mathcal{O}}(s)$ cancel out. □

Note that to say that ε is a number of the form $n+\sqrt{n^2\pm1}$ is equivalent to saying that $\varepsilon>1$ is a quadratic unit with even trace (or a unit in an order of even discriminant). As far as we can ascertain, there are no similar formulas for $\mathcal{J}(\varepsilon)$ when ε is a unit of odd trace.

6.2. **General case.** The proof of Theorem 4 shows that $\mathcal{J}(n+\sqrt{n^2 \pm 1})$ can always be expressed as an algebraic linear combination of $\zeta(2)$, $\log(2)\log(n+\sqrt{n^2 \pm 1})$, and terms of the form $D^{1/2}L_{\mathcal{O}_D}(1,\chi)$, where $D = 4^a(n^2 \pm 1)$, $a \in \{0, \pm 1\}$, and χ is a narrow class group character.

Recall that the Stark conjecture predicts that the special value at $s = 1$ of the Artin L-function of a Galois representation ρ for a Galois extension E/F is a simple multiple (a power of π times an algebraic number) of the so-called Stark regulator (the determinant of a certain matrix of logarithms of units) in such a way that the factorization $\zeta_E(s) = \prod_\rho L_{E/F}(s,\rho)^{\dim(\rho)}$ of the Dedekind zeta function of E matches the factorization of the regulator of E obtained by decomposing the group of units of E (after extending scalars to $\overline{\mathbb{Q}}$) into irreducible $\mathrm{Gal}(E/F)$-representations. For the general formulation of the Stark conjecture, see [15, p. 25–28].

In the case of an abelian extension H/K (in our cases, a ring class field) of a real quadratic field K, all irreducible representations of $\mathrm{Gal}(H/K)$ are one-dimensional, all irreducible representations of $\mathrm{Gal}(H/\mathbb{Q})$ are one- or two-dimensional, and for any character χ of $\mathrm{Gal}(H/K)$ we have $L_{H/K}(s,\chi) = L_{H/\mathbb{Q}}(s,\rho)$, where ρ is the two-dimensional representation of $\mathrm{Gal}(H/\mathbb{Q})$ induced from χ. The Stark regulator for $L_{H/\mathbb{Q}}(1,\rho)$ is then a 2×2, 1×1, or 0×0 determinant of logarithms of units in H, depending on whether $\mathrm{tr}(\rho(\sigma))$ is 2, 0, or -2, where σ denotes the complex conjugation.

If H is the ring class field of a real quadratic field K corresponding to an order $\mathcal{O}_D \subset K$ and χ a character on the corresponding narrow ideal class group, then $L_K(s,\chi)$ is the L-function of the corresponding character on $\mathrm{Gal}(H/K)$, and since H is totally real, the Stark conjecture predicts that $L_K(1,\chi)$ should be an algebraic multiple of a 2×2 determinant of units in H (compare with [14, p. 61]). We therefore obtain the following result.

Proposition 6. *Let $\varepsilon > 0$ be a quadratic unit with even trace. Assume the abelian Stark conjecture for the real quadratic field $\mathbb{Q}(\varepsilon)$. Then $\mathcal{J}(\varepsilon)$ is a rational linear combination of $\zeta(2)$, $\log(2)\log(\varepsilon)$, and of 2×2 determinants of algebraic units in the narrow ring class field of the quadratic order $\mathbb{Z}[2\varepsilon]$.*

Here is an explicit numerical example. The field $K = \mathbb{Q}(\sqrt{257})$ has class number 3 and its Hilbert class field H is $K(\alpha)$, where α satisfies $\alpha^3 - 2\alpha^2 - 3\alpha + 1 = 0$. If we let $\alpha_1 < \alpha_2 < \alpha_3$ be the roots of $x^3 - 2x^2 - 3x + 1$, then to very high precision we find

$$\mathcal{J}(16+\sqrt{257}) \stackrel{?}{=} 4\zeta(2) + \log(2)\log(16+\sqrt{257}) + 2\begin{vmatrix} \log(-\alpha_1) & \log(3-\alpha_1) \\ \log(\alpha_3) & \log(3-\alpha_3) \end{vmatrix}.$$

As another example, this time with a non-fundamental discriminant, we find

$$2\,\mathcal{J}(10+3\sqrt{11}) \stackrel{?}{=} 5\zeta(2) + \log(2)\log(10+3\sqrt{11}) + \begin{vmatrix} \log(\beta_1) & \log(\gamma_1) \\ \log(\beta_2) & \log(\gamma_2) \end{vmatrix},$$

where $\beta_1 > \beta_2$ are the two largest real roots of $x^4 - 11x^3 + 24x^2 - 11x + 1$ and $\gamma_j = \frac{\beta_j^2 - 9\beta_j + 4}{1-\beta_j}$.

7. Cohomological aspects

Note that the 3-term relation

$$\mathcal{F}(x) - \mathcal{F}(x+1) - \mathcal{F}\Big(\frac{x}{x+1}\Big) = L(1/2) - L\Big(\frac{x}{x+1}\Big), \tag{33}$$

when viewed modulo the more elementary right-hand side is exactly the period relation satisfied by the component φ_S of a 1-cocycle $\{\varphi_\gamma\}_{\gamma\in\mathrm{PSL}_2(\mathbb{Z})}$ such that $\varphi_T = 0$. (Cf. [20] and also [9] for more discussion of this relationship.) In this section we will discuss various ways—all of them still somewhat provisional—in which one can relate $\mathcal{F}$ to 1-cocycles for $\mathrm{PSL}_2(\mathbb{Z})$.

7.1. **An interplay between the 5-term relation and the 3-term relation.** If we extend $\mathcal{F}$ to an even function on $\mathbb{R}\smallsetminus\{0\}$, then the weaker form of relation (33)

$$\mathcal{F}(x) - \mathcal{F}(x-1) + \mathcal{F}\Big(\frac{x-1}{x}\Big) = L(x) - L(1/2) \pmod{\zeta(2)} \tag{34}$$

now holds for all $x \in \mathbb{R} \smallsetminus \{0,1\}$. The function $\mathcal{P}(x,y)$ defined by (28) then becomes an even piecewise continuous function on all of $\mathbb{R}^2$ and satisfies the functional equation

$$(35)\qquad \mathcal{P}(x,y) - \mathcal{P}(x-1,y-1) + \mathcal{P}\Big(\frac{x-1}{x},\frac{y-1}{y}\Big) \;=\; 0 \pmod{\zeta(2)}$$

for all $x,y \in \mathbb{R} \smallsetminus \{0,1\}$. Conversely, if $\mathcal{F}$ is any function and we define L and $\mathcal{P}$ (modulo constants) by equations (34) and (28), respectively, then the left hand side of (35) becomes a sum of 15 $\mathcal{F}$'s that can be grouped into 5 L's and then becomes the famous 5-term relation (equivalent to (14)) for L:

$$(36)\qquad L\Big(\frac{y}{x}\Big) - L\Big(\frac{y-1}{x-1}\Big) + L\Big(\frac{1-1/y}{1-1/x}\Big) - L(y) + L(x) \;=\; 0 \pmod{\zeta(2)}\,.$$

The anti-invariance of $\mathcal{F}$ and L under inversion implies that $\mathcal{P}$ is also anti-invariant and this together with (35) implies that $\mathcal{P}$ defines a cocycle for the group $\mathrm{PSL}_2(\mathbb{Z})$ with values in the space of functions from $\mathbb{R}^2$ to $\mathbb{R}/\zeta(2)\mathbb{Z}$, by mapping T to 0 and S to $\mathcal{P}$.

The above discussion thus gives an interesting connection, which we have not yet understood completely, between the 5-term relation and 1-cocycles for the group $\mathrm{PSL}_2(\mathbb{Z})$.

7.2. **A cocycle with values in $\Lambda^2(\mathbb{Q}^{\mathrm{ab}\,\times})$.** A different construction of a 1-cocycle can be obtained from the evaluation of F at positive rationals given by Theorem 3. The formula that we established in the proof has the form

$$F(\alpha) - F(1) \;=\; \mathrm{Li}_2(\xi_\alpha)\,,$$

for a certain $\xi_a \in \mathbb{Z}[\mathbb{Q}(e^{2\pi i\alpha}, e^{2\pi i/\alpha})]$. It is therefore interesting to know when a linear combination of ξ_α's lands in the Bloch group of $\overline{\mathbb{Q}}$ (see [21]). For this we have to calculate $\delta(\xi_\alpha)$, where $\delta\colon \mathbb{Z}[\overline{\mathbb{Q}}] \to \Lambda^2(\overline{\mathbb{Q}}^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the usual differential given by $\delta([x]) = x \wedge (1-x)$. We find that if $\alpha = p/q$, $(p,q)=1$, then

$$\delta(\xi_\alpha) \;=\; \beta(\alpha) - \beta(1/\alpha) - p \wedge q\,,$$

where $\beta\colon \mathbb{Q} \to \Lambda^2(\mathbb{Q}^{\mathrm{ab}\,\times}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is defined by

$$(37)\qquad \beta(p/q) \;=\; \sum_{\lambda^q=1} (1-\lambda^p) \wedge (1-\lambda) \qquad ((p,q)=1)\,.$$

We further note that the element ξ_α is Galois invariant, so whenever $\delta(\xi_\alpha)$ vanishes, by Galois descent we get an element in the Bloch group of $\mathbb{Q}$, which is a torsion group. Since $K_2(\mathbb{Q})$ is torsion (see Theorem 11.6 in [12]) the group $\Lambda^2(\mathbb{Q}^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$ is equal to $\delta(\mathbb{Q}[\mathbb{Q}])$. Therefore, whenever a linear combination $\sum_i c_i(\beta(\alpha_i) - \beta(1/\alpha_i))$ vanishes, there are rational numbers x_j and d_j such that $\sum_i c_i \xi_{\alpha_i} + \sum d_j[x_j]$ is in $\ker(\delta)$, and then the sum $\sum_i c_i F(\alpha_i) + \sum_j d_j \mathrm{Li}_2(x_j)$ vanishes modulo products of logarithms of algebraic numbers. A rather simple example of this was given in (23). As a slightly less trivial example, one can show that $\beta(\frac{n}{n^2+1}) = \beta(\frac{n^2+1}{n})$, which implies that $F(\frac{n}{n^2+1}) - F(1) - \frac{1}{2}\mathrm{Li}_2(\frac{n^2}{n^2+1})$ is a bilinear combination of logarithms of algebraic numbers, generalizing (24).

Two remarks about the function β appearing above are that it takes values in Λ^2 of the group of cyclotomic units and that it satisfies the two identities $\beta(-x) = \beta(x)$ and $\beta(x+1) = \beta(x)$. Because of this, sending $T \mapsto 0$, $S \mapsto \varphi$, where $\varphi(p/q) = \delta(\xi_{p/q}) + p \wedge q$ defines a $\mathrm{PSL}_2(\mathbb{Z})$-cocycle. This cocycle can also be lifted to a cocycle given by $T \mapsto 0$, $S \mapsto \xi$ with values in functions from $\mathbb{P}^1(\mathbb{Q})$ to $\mathbb{Q}[\mathbb{Q}^{\mathrm{ab}}]/(\mathbb{Q}[\mathbb{Q}] + \mathcal{C}(\mathbb{Q}^{\mathrm{ab}}))$, where $\mathcal{C}(F)$ denotes the subspace of $\mathbb{Q}[F]$ spanned by all specializations of the 5-term relation.

A further remark is that β itself can be used to construct infinitely many 1-cocycles, now with values in $\Lambda^2(\mathbb{Q}(\zeta_p)^\times)^{\mathbb{P}^1(\mathbb{F}_p)}$ for any prime p: the mapping $\beta_p\colon \mathbb{Z}/p\mathbb{Z} \to \Lambda^2(\mathbb{Q}(\zeta_p)^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$ given by $\beta_p(n) = \beta(n/p)$ satisfies the functional equations $\beta_p(n) = \beta_p(-n) = -\beta_p(1/n) = \beta_p(n+1) + \beta_p(n/(n+1))$, and thus sending $T \mapsto 0$, $S \mapsto \beta_p$ defines a $\mathrm{PSL}(2,\mathbb{F}_p)$-cocycle.

A final remark is that there is a formal similarity between the formula (37) and the classical Dedekind sums arising in the modular transformation behavior of the Dedekind eta function. This similarity can be made precise using the notion of generalized Dedekind symbols due to Fukuhara [3]: the mapping $(p,q) \mapsto \beta(p/q)$ is an even generalized Dedekind symbol with values in $\Lambda^2(\mathbb{Q}^{\mathrm{ab}\times})$, and the mapping $(p,q)\mapsto\delta(\xi_{p/q})+p\wedge q$ is its reciprocity function. Note that from this point of view the functional equations discussed in Section 3 (or, more precisely, the corresponding functional equations for β) are analogous to the functional equations discovered by Knopp [8] for the classical Dedekind sums.

7.3. **The Herglotz function and the weight 2 Eisenstein series.** In this final subsection, we will give an explanation of the cocycle nature of F by writing it as an Eichler-type integral. We slightly modify the definition of F to

$$F^\star(z) \;=\; \sum_{n\geq 1}\frac{\psi(nz)-\log(nz)+(2nz)^{-1}}{n} \;=\; F(z)\,+\,\frac{\pi^2}{12\,z}\,.$$

Using Binet's integral formula [17, p. 250] for the digamma function,

$$\psi(x) \;=\; \log x \,-\, \frac{1}{2x} \,-\, \int_0^\infty \frac{1}{e^{2\pi t}-1}\,\frac{2t\,dt}{t^2+x^2}\,,$$

we obtain the following integral representation, valid for $\mathrm{Re}(z)>0$:

$$F^\star(z) \;=\; -\int_0^\infty\biggl(\sum_{n=1}^\infty\frac{1}{n(e^{2\pi nt}-1)}\biggr)\frac{2t\,dt}{t^2+z^2} \;=\; -\int_0^{i\infty}H(\tau)\Bigl(\frac{1}{\tau+z}+\frac{1}{\tau-z}\Bigr)\,d\tau\,,$$

where $H(\tau)=\sum_{n=1}^\infty\sigma_{-1}(n)\,q^n$. (Here, as usual, we use q to denote $e^{2\pi i\tau}$ and $\sigma_\nu(n)$ for the sum of the ν-th powers of the positive divisors of n.) Note that $H(\tau)=\log(q^{1/24}/\eta(\tau))$, where η is the Dedekind eta function, and that its derivative is $2\pi i\,G_2^0(\tau)$, where $G_2^0(\tau)$ is the weight 2 Eisenstein series $G_2(\tau)=-\frac{1}{24}+\sum_{n=1}^\infty\sigma_1(n)\,q^n$ without its constant term. Integrating by parts, we therefore get

$$F^\star(z) \;=\; 2\pi i\int_0^{i\infty}G_2^0(\tau)\log\Bigl(1-\frac{\tau^2}{z^2}\Bigr)\,d\tau\,.$$

This formula is valid as it stands for $\mathrm{Re}(z)>0$ and gives yet another proof of the analytic continuation of F to $\mathbb{C}'$ by deforming the path of integration to remain to the left of z if z is in the second quadrant and to the right of $-z$ if z is in the third quadrant.

The above integral for $F^\star$ can be rewritten as $F^\star(z)=F^+(z)+F^-(z)$, where

$$F^\pm(z) \;=\; 2\pi i\int_0^{i\infty}G_2^0(\tau)\,\Bigl(\log\Bigl(1\mp\frac{\tau}{z}\Bigr)\pm\frac{\tau}{z}\Bigr)\,d\tau\,.$$

Note that both F^+ and F^- continue analytically to $\mathbb{C}'$. If we denote by $\mathfrak{H}^+$ and $\mathfrak{H}^-$ the upper and lower half-planes respectively, then by splitting the integral from 0 to $i\infty$ at z or $-z$ we get

$$F^\pm(z) \;\equiv\; H^\pm(z)\,-\,H^\pm(-1/z)\,+\,\log(z)H(\pm z)\,-\,U(z^{\pm1}) \qquad \bigl(z\in\mathfrak{H}^\pm\bigr)\,,$$

where "$\equiv$" means "modulo elementary functions", $U(z)=2\pi i\int_z^\infty G_2^0(\tau)\log(\tau)d\tau$, and

$$H^\pm(z) \;=\; \pm 2\pi i\int_z^{\pm i\infty}G_2^0(\pm\tau)\log(\tau-z)\,d\tau \qquad (z\in\mathfrak{H}^\pm)\,.$$

Since $H^\pm(z)$ is obviously a 1-periodic function in $\mathfrak{H}^\pm$ and $U(z)$ is essentially the primitive of $G_2(z)\log(z)$, this computation implies that $F^\pm(z)-F^\pm(z+1)-F^\pm(z/(z+1))=0$ modulo elementary functions, and that $T\mapsto 0$, $S\mapsto F^\pm$ defines a 1-cocycle with values in the space of suitably nice analytic functions modulo elementary functions.

The above calculations go through much the same way for the higher Herglotz functions

$$\mathscr{F}_k(z) \;=\; \sum_{n\geq 1}\frac{\psi(nz)}{n^{k-1}} \qquad (k>2)\,,$$

with H replaced by $\sum_{n\geq1}\sigma_{1-k}(n)q^n$, which for even k is essentially the Eichler integral of the Eisenstein series of weight k on the full modular group. The functions $\mathscr{F}_k$ were defined in [16] in connection with a higher Kronecker "limit" formula for $\zeta_K(\mathcal{B},s)$ at $s=k/2$ (for k even) rather than the limiting value at $s=1$. It is likely that most of the properties we have given have analogues for higher Herglotz functions. This could be a topic for future research.

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