THE HABIRO RING OF A NUMBER FIELD

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Abstract. We introduce the Habiro ring of a number field $\mathbb{K}$ and modules over it graded by $K_3(\mathbb{K})$. Elements of these modules are collections of power series at each complex root of unity that arithmetically glue with each other after applying a Frobenius endomorphism, and after dividing at each prime by a collection of series that depends solely on an element of the Bloch group. We prove that the perturbative Chern–Simons invariants of knots and 3-manifolds are elements of these modules and identify these elements with expansions of certain admissible series of Kontsevich–Soibelman at roots of unity, suggesting that some Donaldson-Thomas invariants have arithmetic meaning and that some elements of the Habiro ring of a number field have enumerative meaning.

Contents

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1. Introduction

1.1. **Integrality and congruence experiments.** Early experiments by two of the authors on perturbative invariants of Chern–Simons theory revealed unexpected integrality and gluing properties of functions near roots of unity. The explanation of this phenomenon was the original motivation for this work. One of the experiments in question involved the power series $\Phi(h)$ that appear in the asymptotic expansion of the Kashaev invariant of the 4_1 knot, whose first few terms are given (after multiplication by an overall eighth root of unity) by [21, Eqn.(3)]

$$\Phi(h) \;=\; \frac{1}{\sqrt[4]{-3}}\Big(1+\frac{11}{24\cdot 3\sqrt{-3}}\,h+\frac{697}{1152\cdot(3\sqrt{-3})^2}\,h^2+\frac{724351}{414720\cdot(3\sqrt{-3})^3}\,h^3+\cdots\Big)\,. \quad (1)$$

At the time, 100 terms of the series $\Phi(h)$ were known, and the denominator of the coefficient of h^{100} was given by

$$2^{397}\cdot 3^{298}\cdot 5^{40}\cdot 7^{22}\cdot 11^{12}\cdot 13^{9}\cdot 17^{6}\cdot 19^{5}\cdot 23^{4}\cdot 29^{3}\cdot 31^{3}\cdot 37^{2}\cdot 41^{2}\cdot 43^{2}\cdot 47^{2}\cdot 53\cdot 59\cdot 61\cdot 67\cdot 71\cdot 73\cdot 79\cdot 83\cdot 89\cdot 97\cdot 101$$

in agreement with Theorem 9.1 of [21]. However, the symmetrised series $\Phi(h)\Phi(-h)$, after a change of variable $q=e^h$ starts with

$$\Phi(h)\Phi(-h) \;=\; \frac{1}{\sqrt{-3}}\big(1-\frac{1}{3^3}(q-1)^2+\frac{1}{3^3}(q-1)^3-\frac{4}{3^5}(q-1)^4-\frac{1}{3^5}(q-1)^5+O((q-1)^6)\big)\,, \quad (2)$$

with all coefficients being integral away from 3, e.g. the denominator of the coefficient of $(q-1)^{100}$ being 3^{146}.

But that's not all. The asymptotics of the Kashaev invariant at other complex roots of unity ζ (other than 1 discussed above) involve a power series $\Phi_\zeta(h)$ discussed in [21], whose constant term is in the ring $\mathbb{Z}[\zeta_6,\zeta,1/3]$. (These series were denoted by $\Phi_\alpha(h)$ in [21], where $\zeta=e^{2\pi i\alpha}$, with $\alpha\in\mathbb{Q}$ and ζ_6 a primitive 6th root of unity). On the other hand the series (2), which lies in $\mathbb{Z}[1/\sqrt{-3}][\![q-1]\!]$, can be evaluated when $q=\zeta_p$ is a primitive p-th root of unity (with p a prime $p\neq 3$), the result being a well-defined element of $\mathbb{Z}_p[1/\sqrt{-3},\zeta_p]$, because ζ_p-1 is p-adically small. These two numbers matched, up to sign, with the constant term of $\Phi_{\zeta_p}(h)\Phi_{\zeta_p}(-h)$. The sign depended on the prime p and was equal to 1 when $p=1 \bmod 3$ and -1 when $p=2 \bmod 3$. In more invariant terms, the sign was given by the Legendre symbol $(\frac{p}{3})$.

Both parts of the experiment were reminiscent of properties that elements of the Habiro ring have, and this was a motivation for the current paper. It turned out that the symmetrised series considered above are concrete elements of an abstractly defined Habiro ring of a number field (in this case $\mathbb{Q}(\sqrt{-3})$) whose definition was originally motivated by the question whether q-de Rham cohomology admits a definition over the Habiro ring. In this general definition, the unexpected Legendre symbol originates from the p-Frobenius endomorphism of a p-completed ring.

This answer pushed the question further to find a home for the unsymmetrised collections of power series at roots of unity, which after all is the output of perturbative Chern–Simons theory. It turned out that such collections are elements of rank one modules over the Habiro ring of the number field indexed by the third algebraic K-group of the number field.

Quite unexpectedly, another source of elements of these modules comes from the admissible series introduced and studied by Kontsevich–Soibelman [30]. These are generating series of cohomological Hall algebras that appear in enumerative algebraic geometry such as Donaldson–Thomas theory.

These two sources of elements gave two complementary clues about the definition of the rank one modules. Combining both views with a correction coming from p-adic analysis gave the definition of the promised modules.

A corollary of our work is that the Donaldson-Thomas invariants have arithmetic meaning and that the elements of the Habiro ring of a number field have enumerative meaning, both being related to the comparison of the K-theory of local and global fields and to the ideas of q-de Rham cohomology.

1.2. **The original Habiro ring.** The Habiro ring, given by the remarkably short definition [22]

$$\mathcal{H} \;:=\; \varprojlim_n \; \mathbb{Z}[q]/(q;q)_n\mathbb{Z}[q] \tag{3}$$

(where $(q;q)_n = (1-q)\dots(1-q^n)$ is the q-Pochhammer symbol), originated in Quantum Topology as a natural home for quantum invariants of knots and 3-manifolds such as the Kashaev invariant of a knot [26] or the Witten-Reshetikhin-Turaev invariant of an integer homology 3-sphere [23]. A detailed discussion of this ring is given in the recent work [17].

Let us recall some basic properties of $\mathcal{H}$ whose proofs can be found in Habiro [22]. From its very definition, it follows that it consists of elements of the form

$$f(q) \;=\; \sum_{n=1}^{\infty} P_n(q)(q;q)_{n-1}, \qquad P_n(q) \;\in\; \mathbb{Z}[q]\,. \tag{4}$$

(The shift of index by 1 will be very convenient in Section 5 and reflect multiplicative properties of n.) This makes it easy to construct elements of $\mathcal{H}$. In fact, those quantum invariants of knots and three-manifolds often arise as expressions of this form. An example coming from the Kashaev invariant of the trefoil (3_1) knot is given by [27, Eqn.(4.7)]

$$f_{3_1}(q) \;=\; \sum_{n=0}^{\infty} (q;q)_n \;\in\; \mathcal{H}\,. \tag{5}$$

This particular element of $\mathcal{H}$ was studied years later by Kontsevich and Zagier (cf. [39]), without knowing its topological provenance, from the point of view of its modularity properties (with respect to τ, where $q = e^{2\pi i\tau}$).

A different property of the Habiro ring will play the central role in this paper. Observe that if $q = \zeta_m$ is a primitive m-th root of unity, then $(q;q)_n = 0$ for all $m > n$; thus an element $f(q) \in \mathcal{H}$ can be evaluated at $q = \zeta_m$ and the result is an element of $\mathbb{Z}[\zeta_m]$. More generally, when $q = \zeta_m + x$, we have $(q;q)_n \in x^{\lfloor n/m \rfloor}\mathbb{Z}[\zeta_m][x]$. This implies that if $f(q)$ is an element of the Habiro ring, then $f(\zeta_m + x) \in \mathbb{Z}[\zeta_m][\![x]\!]$, giving rise to a ring homomorphism $\iota_m : \mathcal{H} \to \mathbb{Z}[\zeta_m][\![x]\!]$. Putting these homomorphisms together, we obtain a map

$$\iota : \mathcal{H} \to \prod_{m\geq 1} \mathbb{Z}[\zeta_m][\![x]\!]\,, \quad \text{such that} \quad \iota(f) \;=\; (f_m(x))_{m\geq 1}\,, \qquad f_m(x) \;:=\; f(\zeta_m + x)\,, \tag{6}$$

where ζ_m for each m is a fixed primitive m-th root of unity. We will always suppose that the collection (ζ_m) is a compatible collection of roots of unity of order m, that is a collection satisfying

$$\zeta_{mm'} \;=\; \zeta_m\zeta_{m'}, \qquad (m, m') \;=\; 1, \qquad (\zeta_{p^r})^p \;=\; \zeta_{p^{r-1}} \tag{7}$$

for all positive integers m and m', primes p and positive integers r. For example, one can think of $\zeta_m = \exp(2\pi i \sum_{p \text{ prime}} p^{-v_p(m)})$, with $v_p(m)$ being the p-adic valuation of m. We note that our choice of compatible collection differs from the traditional choice $\omega_m = e^{2\pi i/m}$, which does not satisfy (7), but instead satisfies $\omega_{mm'}^m = \omega_{m'}$. However, our choice satisfies the important property that $v_p(\zeta_{pm} - \zeta_m) > 0$ for all positive integers m and all primes p, and this will be more convenient for us.

For a ring R, the map ι motivates the notion of Galois invariant R-valued "functions near roots of unity". By this, we mean elements of the ring

$$\mathcal{P}_R \;=\; \Big(\prod_{\zeta\in\mu_\infty} R[\zeta][\![x]\!]\Big)^{\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})} \;\cong\; \prod_{m\geq 1} R[\zeta_m][\![x]\!]\,. \tag{8}$$

(Here the f_m suffice to expand $F(q)$ around any root of unity, and not just around our distinguished collection of roots of unity, using Galois invariance.) We will denote a typical element of $\mathcal{P}_R$ by $f(q) = (f_m(x))$, where $q = \zeta_m + x$, rather than the more cumbersome $(f_m(x))_{m\geq 1}$. We see that equation (6) now gives a map $\iota : \mathcal{H} \to \mathcal{P}_{\mathbb{Z}}$. Note that $f_m(x)$ are formal series with no convergence properties assumed. In fact, in examples where $R \subseteq \mathbb{C}$ coming from perturbative knot invariants, the associated series will be factorially divergent series over the complex numbers, whereas at the same time they are always p-adically convergent on a disc of radius $p^{-p/((p-1)(p-2))}$, as follows from [21, Thm.9.1, Thm.9.2]. The series $f_m(x)$ are related to "functions near $\mathbb{Q}$" defined in the previous work [21] and more specifically the series $\Phi_\alpha(h)$, where $\zeta_m + x = e^{2\pi i\alpha - h/m}$ and $\alpha \in \mathbb{Q}$.

In [22] Habiro proved that the map (6) (and in fact, each of its factors $\iota_m : \mathcal{H}_{\mathbb{Z}} \to \mathbb{Z}[\zeta_m][\![x]\!]$) is injective and that its image consists of the collection of power series $f_m(x) = (\iota_m f)(x)$ that arithmetically "glue", generalising earlier work of Ohtsuki [33]. Let us explain how gluing works. The map (6) satisfies the formal substitution property

$$f_m(x + \zeta_{pm} - \zeta_m) \;=\; f_{pm}(x) \tag{9}$$

for all positive integers m and all prime numbers p. The only issue is that this equation requires one to re-expand the left hand side, which is a power series in x, after shifting x to $x + \zeta_{pm} - \zeta_m$. This is possible using the binomial theorem on x and $\zeta_{pm} - \zeta_m$ if the shift $\zeta_{pm} - \zeta_m$ is *small*, and indeed it is in the completion $\mathbb{Z}[\zeta_{pm}]^\wedge_p$ of $\mathbb{Z}[\zeta_{pm}]$, since $\zeta_{pm} - \zeta_m$ has positive p-valuation. Here $R^\wedge_p = \varprojlim_n R/(p^n)$, which is isomorphic to $R \otimes \mathbb{Z}_p$ and will often be denoted simply by R_p, denotes the p-completion of a ring R. Then, equation (9) holds as an identity in the ring $\mathbb{Z}[\zeta_{pm}]^\wedge_p[\![x]\!]$, and this compatibility property is what we meant by "gluing".

Summarising, an element $f(q)$ of the Habiro ring can be identified with a collection of power series $(f_m(x))_{m\geq 1} \in \prod_{m\geq 1} \mathbb{Z}[\zeta_m][\![x]\!]$ that satisfy the gluing property (9) for all primes p.

The gluing property implies that the entire collection $(f_m(x))_{m\geq 1}$ is uniquely determined by any one of its coordinate functions $f_m(x)$, indeed in many different ways, which of course are consistent with each other. For instance $f_6(x)$ can be determined from $f_1(x)$ via the chain $f_1(x) \to f_3(x) = f_1(x + \zeta_3 - 1) \to f_6(x) = f_3(x + \zeta_6 - \zeta_3)$ or via the chain $f_1(x) \to f_2(x) = f_1(x + \zeta_2 - 1) \to f_6(x) = f_2(x + \zeta_6 - \zeta_2)$:

$$\begin{array}{ccc} & f_1(x) & \\ \swarrow & & \searrow \\ f_3(x) = f_1(x+\zeta_3-1) & & f_1(x+\zeta_2-1) = f_2(x) \\ \searrow & & \swarrow \\ & f_3(x+\zeta_6-\zeta_3) = f_6(x) = f_2(x+\zeta_6-\zeta_2) & \end{array} \quad . \tag{10}$$

Notice, however, that the equalities here hold in different completions. We can phrase this gluing using the language of p-adic power series and their corresponding functions. Namely, an element of the Habiro ring consists of a collection of series $f_m(x) \in \mathbb{Z}[\zeta_m][\![x]\!]$ that satisfy the following property. For every prime p, $f_m(x)$ is convergent on the open disc $|x|_p < 1$ that contains $\zeta_{pm} - \zeta_m$, and its re-expansion $f_m(x + \zeta_{pm} - \zeta_m)$, which is therefore defined but a priori has potentially transcendental coefficients in $\mathbb{Z}_p[\zeta_{pm}]$, has algebraic coefficients (in fact, in $\mathbb{Z}[\zeta_{pm}]$) and agrees with $f_{pm}(x)$. In other words, for each fixed prime p, the collection of functions $(f_m(x))$ can be used to define a single p-adic analytic function on the *disconnected* domain $\{q \in \mathbb{C}_p \mid |q|_p = 1\}$, which is the union of the (not all distinct) open unit discs centred at roots of unity. (Of course this set is totally disconnected anyway in the p-adic topology, but here we mean that no two of the discs in question are p-adically close, so that the analytic functions on them cannot be continued from one to the other.) The remarkable fact is that, given these series come from something global, this "disconnectedness" disappears once we start varying the prime p, in the sense that the small neighbourhoods of any two roots of unity can be connected by neighbourhoods overlapping with respect to varying p-adic metrics. It is this property that makes the Habiro ring work.

1.3. **The Habiro ring of a number field.** An abstract definition of the Habiro ring of a number field was motivated by the ideas of q-de Rham cohomology in [36]. Our first task is to give a power series realisation of this ring. As in the case of the Habiro ring, the new definition involves $\mathcal{O}_{\mathbb{K}}$-valued functions near the roots of unity, where $\mathcal{O}_{\mathbb{K}}$ is the ring of integers of a number field $\mathbb{K}$. However, there are two new features that emerge, both of which already appeared in the numerical experiments described in Section 1.1, namely:

- one has to invert the primes dividing some integer Δ,
- the gluing now involves a Frobenius twist.

Both features cause subtleties, as we will see shortly. If $\Delta \neq 1$, the Habiro ring of $\mathcal{O}_{\mathbb{K}}[1/\Delta]$ is no longer an integral domain, while the twist makes it unclear how to construct elements that glue in the required way.[1]

We now give the promised definition. Let $\mathbb{K}$ be a number field with ring of integer $\mathcal{O}_{\mathbb{K}}$ and Δ a positive integer divisible by the discriminant of $\mathbb{K}$.[2] Below, we usually take Δ to also be divisible by 6. The Habiro ring[3] $\mathcal{H}_R$ associated to the finite étale map

$$\mathbb{Z}[1/\Delta] \to R, \qquad R \;=\; \mathcal{O}_{\mathbb{K}}[1/\Delta] \tag{11}$$

is defined as follows:

Definition 1.1. The Habiro ring $\mathcal{H}_R$ is the subset of $\mathcal{P}_R$ consisting of elements

$$f(q) \;=\; (f_m(x))_{m\in\mathbb{Z}_{>0}} \;\in\; \mathcal{P}_R \tag{12}$$

that satisfy the gluing property

$$f_m(x+\zeta_{pm}-\zeta_m) \;=\; (\varphi_p f_{pm})(x) \;\in\; R_p^\wedge[\zeta_{pm}][\![x]\!] \tag{13}$$

for primes p and all positive integers m, where φ_p is the Frobenius endomorphism of $R_p^\wedge$, lifted to an endomorphism of $R_p^\wedge[\zeta_{pm}][\![x]\!]$ fixing both ζ_{pm} and x.

Note that if p divides Δ, the above equation is trivial since the completed ring is trivial. We also define $\mathcal{H}_R|_\Delta$ to be the same ring where we restrict to m prime to Δ, and more generally $\mathcal{H}_R|_\gamma$ to be the ring where we restrict to m prime to any positive integer γ.

Note that the two equations (9) and (13) are nearly identical, the crucial difference being the presence of φ_p in (13). The reason this was not seen in the previous case was that $\varphi_p = \mathrm{id}$ for $R = \mathbb{Z}[1/\Delta]$. The Frobenius twist makes it unclear if there is any natural R-module structure on $\mathcal{H}_R$, and moreover how to construct nontrivial elements in $\mathcal{H}_R$. However, one can equivalently define $\mathcal{H}_R$ as a finite étale $\mathcal{H}_{\mathbb{Z}[1/\Delta]}$-algebra by gluing the naive base changes $R[\zeta_m][\![q-\zeta_m]\!]$ over $\mathbb{Z}[\zeta_m][\![q-\zeta_m]\!]$ along Frobenius twists after p-adic completion. This abstract definition of the Habiro ring implies in particular that $\mathcal{H}_R$ is a finitely generated projective $\mathcal{H}_{\mathbb{Z}[1/\Delta]}$-module of rank $r = [\mathbb{K} : \mathbb{Q}]$. (This follows from the corresponding fact after p-completion, which itself follows from the fact that $R_p[\zeta_m][\![x]\!]$ has rank r over $\mathbb{Z}_p[\zeta_m][\![x]\!]$.)

There is another way to view these conditions involving p-completions. Specifically, we have natural ring isomorphisms

$$(\mathcal{H}_R)_p \cong \left\{\begin{matrix}\text{collections of } R_p^\wedge[\zeta_m][\![x]\!]\text{-series for } m\geq 1 \text{ that}\\ \text{glue w.r.t. re-expansion in } x\mapsto x+\zeta_{mp}-\zeta_m\end{matrix}\right\} \cong \prod_{\substack{m\geq 1\\(m,p)=1}} R_p^\wedge[\zeta_m][\![x]\!] \cong \mathcal{H}_{R_p}\,, \tag{14}$$

[1] We remark that both problems disappear when $\mathbb{K}$ is abelian over $\mathbb{Q}$. Indeed, in this case we can define an R-module structure on $\mathcal{H}_R$ from the embedding $R \to \mathcal{H}_R$ given by $a \mapsto (\varphi_m a)_m$, where $\varphi_m = \prod_p \varphi_p^{v_p(m)}$ is the product of lifts from $\mathbb{K}_p$ to $\mathbb{K}$ of the Frobenius automorphisms φ_p, which exists for fields $\mathbb{K}$ abelian over $\mathbb{Q}$. This embedding also gives a canonical isomorphism $\mathcal{H}_R \cong \mathcal{H}_{\mathbb{Z}[1/\Delta]} \otimes_{\mathbb{Z}} R$ for such $\mathbb{K}$.

[2] One could also let Δ be a non-zero element of $\mathcal{O}_{\mathbb{K}}$ divisible by all prime ideals whose square divides the discriminant, but for simplicity we will always choose Δ in $\mathbb{Z}$.

[3] Here, we use $\mathcal{H}_R$ rather than the more accurate $\mathcal{H}_{\mathbb{Z}[1/\Delta]\to\mathcal{O}_{\mathbb{K}}[1/\Delta]}$. We will also refer to this ring as the Habiro ring of the field $\mathbb{K}$, rather than of the ring R.

because there are no gluing constraints between the expansions around different roots of unity of order prime to p. Moreover, for every prime p, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_R & \longrightarrow & \mathcal{H}_{R_p^\wedge} \\ \cap & & \cap \\ \mathcal{P}_R & \stackrel{\varphi}{\longrightarrow} & \mathcal{P}_{R_p^\wedge} \end{array}, \tag{15}$$

where the bottom map is the composite of the obvious inclusion with the Frobenius endomorphism $\varphi_p^{v_p(m)}$ acting on $R_p^\wedge[\zeta_m][\![x]\!]$, and the Habiro ring is the universal ring associated to the combinations over p of these diagrams, i.e., if we combine all of the p-completions $R_p^\wedge$ into $\widehat{R} = \varprojlim_N R/MR \cong \prod_{p \text{ prime}} R_p^\wedge$, then $\mathcal{H}_{\widehat{R}} \cong \prod_p \mathcal{H}_{R_p^\wedge}$ and $\mathcal{H}_R$ is the intersection in $\mathcal{P}_{\widehat{R}}$ of $\mathcal{P}_R$ and $\mathcal{H}_{\widehat{R}} \cong \mathcal{H}_{\mathbb{Z}[1/\Delta]} \otimes_{\mathbb{Z}} \widehat{R}$.

Remark 1.2. Note that the ring $\mathcal{H}_R$ is not an integral domain, since the gluing property (13) is nontrivial only for primes p not dividing Δ, otherwise $R_p^\wedge = 0$. Instead, $\mathcal{H}_R$ is a product of (in general infinitely many) integral domains indexed by the equivalence classes of $\mathbb{N}$ under the equivalence relation generated by $m \sim_\Delta pm$ for primes p not dividing Δ. Among those integral domains is $\mathcal{H}_R|_\Delta$ (corresponding to the equivalence class of all positive integers prime to Δ), whose elements are collections of power series at roots of unity of order prime to Δ that satisfy the gluing condition (13) for all primes p not dividing Δ. It is still possible that there is a definition of $\mathcal{H}_R$ when R is equal to $\mathcal{O}_\mathbb{K}$ rather than to $\mathcal{O}_\mathbb{K}[1/\Delta]$ that is an integral domain, although the definition could not be exactly along the lines of the definition for $\mathcal{O}_\mathbb{K}[1/\Delta]$.

1.4. **Modules over the Habiro ring of $\mathbb{K}$ indexed by $K_3(\mathbb{K})$.** In this section we define a collection of modules $\mathcal{H}_{R,\xi}$ over the Habiro ring $\mathcal{H}_R$ (with R given in (11)) labelled by elements of $\xi \in K_3(\mathbb{K})$. (Philosophically, the modules ought to be labelled by $K_3(R)$. However, given the isomorphism $K_{2r-1}(R) \simeq K_{2r-1}(\mathbb{K})$ for $r > 1$ (see e.g., Sec.5.2 of Weibel [37] as well as [9, Eqn.(33)]), we choose to label our modules by $K_3(\mathbb{K})$.) These modules turn out to be the home of perturbative quantum knot invariants, as we will see in Section 1.6.

Before explicitly defining these modules, we recall the original motivation for their existence. In Section 1.1, we recalled from [21] a series $\Phi(h)$ coming from the figure eight knot 4_1. Its properties become much better when one passes to a "completion" given by $\widehat{\Phi}(h) = \exp(i\mathrm{Vol}(4_1)/h)\,\Phi(h)$, where $\mathrm{Vol}(4_1)$ is given by $2\,\mathrm{Im}(\mathrm{Li}_2(e^{\pi i/3})) = 2.0299\cdots$. This dilogarithm is actually equal to a particular value of a regulator, which is a function from $K_3(\mathbb{Q}(\sqrt{-3}))$ to $\mathbb{C}/(2\pi i)^2\mathbb{Z}$. This already indicates a link between this series and algebraic K-theory of the field $\mathbb{Q}(\sqrt{-3})$, but work of Calegari and two of the authors [9] showed that this link is even stronger. The asymptotics of the same Kashaev invariant at different roots of unity gives similar series, but now the constant term at $q = \zeta_m$ (at least if $3|m$) contains the m-th root of a unit in $\mathbb{Q}(\sqrt{-3}, \zeta_m)$ which up to m-th powers depends only on the class $\xi \in K_3(\mathbb{Q}(\sqrt{-3})$ giving the volume. We want our module $\mathcal{H}_{R,\xi}$ to contain an element whose expansion at $q = e^{-h}$ is $\Phi(h)$. In this section, we will give such a definition. Its elements have power series expansions at roots of unity with subtle integrality properties that the series $\Phi(h)$ (and all the other series $\Phi_\zeta(h)$) indeed satisfy. To achieve this, since our point of view is p-adic for varying p, we will have to replace the completion $\widehat{\Phi}(h)$ by a p-adic version in which the usual dilogarithm is replaced with a p-adic dilogarithm. This motivates the Definition 1.3 given below.

The elements of $\mathcal{H}_{R,\xi}$ will give collections of power series at roots of unity whose constant terms contain the m-th root of the above mentioned unit $\varepsilon_m(\xi)$ defined in [9, Thm.1.5]. This unit comes from the Chern class map c_{ζ_m} of algebraic K-theory

$$\varepsilon_m \;=\; c^2_{\zeta_m} : K_3(\mathbb{K}) \to \mathbb{K}(\zeta_m)^\times/(\mathbb{K}(\zeta_m)^\times)^m \tag{16}$$

and satisfies the property that $(\sigma_\gamma\varepsilon_m(\xi))^\gamma/\varepsilon_m(\gamma)$ is *canonically* an m-th power for every integer γ prime to m, where σ_γ is the automorphism sending ζ_m to ζ_m^γ. In [9] this property is called χ^{-1}-equivariance. In more invariant terms, this is a map from the set $K_3(\mathbb{K})$ towards the groupoid of μ_m-torsors over $\mathbb{K}(\zeta_m)$ (with this χ^{-1}-equivariance datum); in fact, these μ_m-torsors extend (necessarily uniquely) to $\mathcal{O}_\mathbb{K}[\zeta_m]$. Indeed, the mod m étale Chern character gives a canonical map

$$c_{\zeta_m} : K_3(\mathbb{K}) \cong K_3(\mathcal{O}_\mathbb{K}) \to H^1(\mathcal{O}_\mathbb{K},\mathbb{Z}/m(2)) \to H^1(\mathcal{O}_\mathbb{K}[\zeta_m],\mathbb{Z}/m(2))^{(\mathbb{Z}/m\mathbb{Z})^\times} \cong H^1(\mathcal{O}_\mathbb{K}[\zeta_m],\mu_m)^{\chi^{-1}}, \tag{17}$$

where the last isomorphism is induced by cup product with $\zeta_m \in H^0(\mathcal{O}_\mathbb{K}[\zeta_m],\mu_m)^\chi$. In particular, the units $\varepsilon_m(\xi)$ can be chosen to be integral at any given finite set of places, and we will always implicitly fix such a choice at the relevant places. We remind the reader that $K_3(\mathbb{K})$, at least after tensoring with $\mathbb{Q}$, is isomorphic to the Bloch group $B(\mathbb{K})$ and that in this interpretation ξ is represented as a formal linear combination $\sum_i [x_i]$, with $x_i \in \mathbb{K}^\times \smallsetminus \{1\}$, satisfying $\sum_i x_i \wedge (1-x_i) = 0 \in \wedge^2 \mathbb{K}^\times$. We will use both points of views in this paper.

To give a first impression of the modules $\mathcal{H}_{R,\xi}$ over $\mathcal{H}_R$, we note that their base change to $\mathbb{K}[\zeta_m][\![x]\!]$ is induced by the μ_m-torsor described above (along the map from μ_m-torsors to $\mathbb{G}_m$-torsors, i.e. line bundles), noting that the μ_m-torsor deforms uniquely to the power series ring. In concrete terms, this is the collection of power series

$$f_m(x) \;\in\; \varepsilon_m(\xi)^{1/m}\, \mathbb{K}[\zeta_m][\![x]\!] \tag{18}$$

whose leading term contains the m-th root $\varepsilon_m(\xi)^{1/m}$ as a factor.

This determines the line bundle (described as usual by its collection of power series) over $R \otimes_\mathbb{Z} \mathbb{Q} = \mathbb{K}$. In order to descend to R, we need to have a p-adic description. For this we use the p-adic dilogarithm map

$$D_p : K_3(\mathbb{K}_p) \to \mathbb{K}_p\,, \qquad \mathbb{K}_p \;=\; R_p^\wedge[\tfrac{1}{p}] \;=\; \mathbb{K} \otimes_\mathbb{Q} \mathbb{Q}_p \tag{19}$$

of Coleman [10], which coincides with the p-adic regulator map as shown in Besser–de Jeu [6, Thm.1.6(2)] and is discussed in detail in Section 3.1. With these maps we can define $\mathcal{H}_{R_p^\wedge,\xi}$ via globally invertible group-like sections, which can be thought of as an upgrading to power series of the collection of units defined in [9].

Definition 1.3. Fix $\xi \in K_3(\mathbb{K})$ and a prime p. An invertible $L_p(\xi)$-section is a collection

$$f(q) \;=\; (f_m(x))_{m\geq 1,(m,p)=1}, \qquad f_m(x) \;\in\; \varepsilon_m(\xi)^{1/m}\,(R_p^\wedge[\zeta_m]^\times + x\mathbb{K}_p[\zeta_m][\![x]\!]) \tag{20}$$

of power series (here $x = q - \zeta_m$ as usual) that satisfies

$$\log\left(\frac{\varphi_p \widehat{f}(q^p)}{\widehat{f}(q)^p}\right) \;\in \prod_{m\geq 1\,,\,(m,p)=1} \frac{p}{x} R_p^\wedge[\zeta_m][\![x]\!]\,. \tag{21}$$

Here $\widehat{f}(q) = (\widehat{f}_m(x))_m$ is the formal completion f defined by

$$\log(\widehat{f}_m(x)) \;=\; \frac{D_p(\xi)}{m^2 \log(q)} + \log(f_m(x)) \quad\in\quad x^{-1}\,\mathbb{K}_p[\zeta_m][\![x]\!]\,. \tag{22}$$

We denote the $\mathcal{H}_{R_p^\wedge}$-span of invertible $L_p(\xi)$-sections by $\mathcal{H}_{R_p^\wedge,\xi}$.

The construction of such invertible sections can be done via telescoping sums. Note also that the condition (21) is a logarithmic version of that used in Dwork's lemma (see Lemma 3.4 of Section 3.2). The existence of invertible sections and Dwork's lemma makes it easy to see that $\mathcal{H}_{R_p^\wedge,\xi}$ is a rank one module over $\mathcal{H}_{R_p^\wedge}$. This is summarised in the following theorem, whose proof will be given in Section 3.2.

Theorem 1. *Let $\mathbb{K}$ be a number field, and Δ a number divisible by its discriminant and by 6. For $\xi \in K_3(\mathbb{K})$ and p a prime with $(p,\Delta)=1$, there exists invertible $L_p(\xi)$-sections. Moreover, $\mathcal{H}_{R_p^\wedge,\xi}$ is a free rank 1 module over $\mathcal{H}_{R_p^\wedge}$.*

In Section 3.2 we will construct some natural and explicit invertible sections. These sections will depend on an expression of ξ as a $\mathbb{Z}_p$-linear combination of symbols associated to roots of unity ζ in $\mathbb{K}_p$, and on the expansion of the infinite Pochhammer symbol $(q^{1/2}\zeta;q)_\infty$. While these collections are only defined at ζ_m with m prime to p, they have a unique extension to all roots of unity satisfying integrality and gluing conditions, as shown in Corollary 3.7 of Section 3.2.

We can patch together these modules over the p-completed ring to define modules over the global one. This is completely analogous to the description of the global Habiro ring from the p-completed Habiro ring via the diagram (15). These collections are motivated and modelled on series coming from perturbative Chern–Simons theory at roots of unity.

Definition 1.4. Fix $\xi \in K_3(\mathbb{K})$. The $\mathcal{H}_R$-module $\mathcal{H}_{R,\xi}$ consists of collections

$$f(q) \;=\; (f_m(x))_{m\geq 1}, \qquad f_m(x) \;\in\; \varepsilon_m(\xi)^{1/m}\,\mathbb{K}[\zeta_m][\![x]\!] \tag{23}$$

such that under the canonical map $\mathbb{K}\to\mathbb{K}_p$ we have $(f_m(x))_{m\geq 1,(m,p)=1} \in \mathcal{H}_{R_p^\wedge,\xi}$, and the following gluing condition for $\gamma\in\mathbb{Z}_{>0}$ is satisfied:

$$f(q^\gamma)^\gamma f(q^{-1}) \;\in\; \mathcal{H}_{R[1/\gamma]}|_\gamma\,, \tag{24}$$

where $\mathcal{H}_{R[1/\gamma]}|_\gamma$ denotes the restriction of $\mathcal{H}_{R[1/\gamma]}$ to collections of power series at roots of unity ζ_m with m prime to γ, as introduced after Definition 1.1.

Note that the function $(\gamma^* f)(q) := f(q^\gamma)$ behaves like an element of the module $\mathcal{H}_{R[1/\gamma],\xi/\gamma}|_\gamma$ (where again $|_\gamma$ denotes the restriction to roots of unity of order prime to γ). The issue is that one must make sense of ξ/γ as an element of K_3. We also note that the χ^{-1}-equivariance of ε_m makes the a priori ambiguity of their m-th roots disappear in equation (24). For the moment we assume that p is prime to 6 (cf. the remark in Section 1.5), because only then does the above condition yield the correct gluing condition: one needs that there is some γ prime to p such that γ^2-1 is also prime to p (see the end of the proof of Lemma 3.6 of Section 3.2). Moreover, notice that $f(q^\gamma)^\gamma f(q^{-1})$ determines $f(q)$ for q near all roots of unity of order prime to 6 as we vary $\gamma\in\mathbb{Z}_{>0}$. More generally, one can consider the combinations $f(q^{\gamma_1})\cdots f(q^{\gamma_n}) f(q^{-\gamma'_1})\cdots f(q^{-\gamma'_{n'}})$, where $\sum_k \gamma_k^{-1} - \sum_k \gamma_k'^{-1} = 0$, the previous case being

$n = \gamma$, $\gamma_i = \gamma$, $n' = 1$ and $\gamma'_1 = 1$. The important property of these combinations is that including the polar part of equation (22) has no effect on the expansions at roots of unity with order prime to $\prod_k \gamma_k \prod_k \gamma'_k$. For example, taking $q = \zeta_m + x$ with $(m, \gamma) = 1$ we find that $f(q^\gamma)^\gamma f(q^{-1}) = \widehat{f}(q^\gamma)^\gamma \widehat{f}(q^{-1})$.

The module $\mathcal{H}_{R,\xi}$ associated to the global ring is not necessarily free, but it is of rank one and locally free, i.e., invertible. This was observed numerically in our early experiments. In these experiments, we compared power series coming from two knots giving the same class in $K_3(\mathbb{K})$ (in particular, the 5_2 and $(-2,3,7)$-pretzel knots with the field $\mathbb{K}$ being the cubic field of discriminant -23 with R being its ring of integers with 23 inverted). We observed that these series seemed to satisfy a linear dependence over $\mathcal{H}_R$. This numerical observation is indeed true and is implied by Theorem 5 below and the following:

Theorem 2. *We have $\mathcal{H}_{R,0} = \mathcal{H}_R$. In general, multiplication gives a canonical isomorphism*

$$\mathcal{H}_{R,\xi} \otimes_{\mathcal{H}_R} \mathcal{H}_{R,\xi'} \;\cong\; \mathcal{H}_{R,\xi+\xi'}\,. \tag{25}$$

In particular $\mathcal{H}_{R,\xi}$ is an invertible $\mathcal{H}_R$-module, and $\xi \mapsto \mathcal{H}_{R,\xi}$ defines a homomorphism of abelian groups

$$K_3(\mathbb{K}) \;\to\; \mathrm{Pic}(\mathcal{H}_R)\,. \tag{26}$$

Here, $\mathrm{Pic}(\mathcal{H}_R)$ denotes as usual the abelian group of line bundles on $\mathrm{Spec}(\mathcal{H}_R)$, i.e., the group under tensor product of invertible $\mathcal{H}_R$-modules. This map is actually a homomorphism from the abelian group $K_3(\mathbb{K})$ (in the category of sets) to the abelian group of line bundles on $\mathcal{H}_R$ (in the category of groupoids), i.e., to every element in $K_3(\mathbb{K})$ we obtain an invertible $\mathcal{H}_R$-module as opposed to just an isomorphism class.

We observe that, just as for $\mathcal{H}_R$, the definition of the module $\mathcal{H}_{R,\xi}$ becomes simpler when $\mathbb{K}$ is abelian over $\mathbb{Q}$. Indeed, in that case the field $\mathbb{K}$ can be embedded in $\mathbb{Q}(\zeta)$ for some root of unity ζ. Assume that $\xi \in \mathcal{B}(\mathbb{K})$ can be represented as a $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{K})$-invariant combination $\xi = \sum_j n_j[\zeta^j]$. (We do not know whether this is always the case, at least after tensoring with $\mathbb{Q}$, but it certainly holds in many examples, e.g., the example coming from the figure eight knot, where $\xi = 2[\zeta_6]$.) Then $\mathcal{H}_{R,\xi}$ contains the element $\prod_j (q^{1/2}\zeta^j;q)_\infty^{n_j}$ and is freely generated by this element as an $\mathcal{H}_R$-module. Compare Theorem 10 of Section 3.2.

The next proposition summarises some of the properties of these $\mathcal{H}_R$-modules and the operation, defined after Definition 1.4, $f \mapsto \gamma^* f$. Its proof, along with the proof of Theorem 2, are given in Section 3.3.

Proposition 1.5. *We have:*
(*a*) *If $f \in \mathcal{H}_{R,\xi}$, then $\tau f \in \mathcal{H}_{R,-\xi}$, where $\tau f := (-1)^* f$ is the involution $f(q) \mapsto f(q^{-1})$.*
(*b*) *If $f, g \in \mathcal{H}_{R,\xi}$, then $f \cdot \tau g \in \mathcal{H}_R$. Moreover, there exist $a, b \in \mathcal{H}_R$ such that $af = bg$.*
(*c*) *If $f \in \mathcal{P}_R$ and $\gamma, \gamma' \in \mathbb{Z}$ are non-zero then $(\gamma\gamma')^* f \;=\; \gamma^*(\gamma'^* f)$.*
(*d*) *If $f \in \mathcal{H}_{R,\xi}$ and γ a positive integer, then $(\gamma^* f)^\gamma \in \mathcal{H}_{R[\gamma^{-1}],\xi}|_\gamma$ (see Definitions 1.1 and 1.4).*
(*e*) *If $f \in \mathcal{H}_{R,\xi}$ and $m \in \mathbb{Z}_{>0}$ and $f_m(x) = 0$ then $f_{pm}(x) = 0$ for all primes p not diving Δ.*
(*f*) *If $f(q) \in \mathcal{H}_{R,\xi}$, then $f_m(0) \in R[\zeta_m, \varepsilon_m^{1/m}]$ for $(m, \Delta) = 1$.*

Remark 1.6. The operation τ corresponds geometrically to the orientation reversal of an oriented manifold, i.e., the quantum invariants of an oriented three-manifold with the opposite orientation are equal to the operation τ applied to the original invariants.

1.5. **Admissible series and formal Gaussian integration.** It was conjectured by Nahm that the modularity of certain q-hypergeometric functions is intimately related to the vanishing of certain associated classes in algebraic K-theory, as was indeed later seen in a variety of works [31, 40, 19, 9]. The asymptotics of these q-hypergeometric functions as q appraoches roots of unity can be defined, using formal Gaussian integration [40, 11, 12]. Modularity would then imply some triviality of these asymptotic series. We will see that this link goes even deeper, and that the integrality of the formal series (as series in $q-\zeta$) is also linked to the vanishing of the same classes in algebraic K-theory.

As we will see, there are two sources of these collections of power series at roots of unity, with complementary properties. One of these is the "admissible series" of Kontsevich–Soibelman [30, Sec.6], which arise as generating series of cohomological Hall algebras that appear in enumerative algebraic geometry such as Donaldson–Thomas theory.

Definition 1.7. For $N \in \mathbb{Z}_{>0}$ and $t = (t_1, \dots, t_N)$, an admissible series is a power series $F(t,q) \in \mathbb{Q}(q)[\![t]\!]$ satisfying $F(t,q) = 1 + \mathrm{O}(t)$ and

$$\log F(t,q) \;=\; -\sum_{0\neq n\in(\mathbb{Z}_{\geq 0})^N} \sum_{\ell\geq 1} \frac{L_n(q^\ell)}{\ell\,(1-q^\ell)} t_1^{\ell n_1}\cdots t_N^{\ell n_N}\,, \qquad \text{where} \qquad L_n(q) \;\in\; \mathbb{Z}[q^{\pm 1}]. \tag{27}$$

An equivalent way to describe admissible series is via their factorisation into Pochhammer symbols. In particular, a series $F(t,q)$ is admissible if and only if the unique integers $c_{n,i}$ (generalised Donaldson–Thomas invariants), defined by the equality

$$F(t,q) \;=\; \prod_{0\neq n\in(\mathbb{Z}_{\geq 0})^N} \prod_{i\in\mathbb{Z}} (q^i t_1^{n_1}\cdots t_N^{n_N};q)_\infty^{c_{n,i}}, \tag{28}$$

vanish for all but finitely many i for each fixed n. Here, as usual $(x;q)_n = \prod_{j=0}^{n-1}(1-q^j x)$ and $(x;q)_\infty^{-1} = \sum_{k=0}^\infty x^k/(q;q)_k$. The equivalence follows from the elementary but key identity (45) for the infinite Pochhammer symbol, which implies the relation

$$L_n(q) \;=\; \sum_{i\in\mathbb{Z}} c_{n,i}\, q^i \;\in\; \mathbb{Z}[q^{\pm 1}] \tag{29}$$

between the exponents $c_{n,i}$ and the Laurent polynomials $L_n(q)$.

An admissible series $F(t,q)$ can be expanded at each root of unity and defines a collection of series in $t^{1/m} = (t_1^{1/m}, \dots, t_N^{1/m})$ given by

$$f_m(t,x) \;=\; F(t^{1/m}, \zeta_m + x)\,. \tag{30}$$

The properties of these collections of series at roots of unity are explained in detail in Section 2.2 below.

Certain admissible series are easy to construct, compute and analyse. A main theorem of [30, Sec. 6.1, Thm. 9] (see also Efimov [13]) is that multidimensional q-hypergeometric

Nahm sums[4]

$$F_A(t,q) \;=\; \sum_{n\in\mathbb{Z}^N_{\geq 0}} \frac{(-1)^{\mathrm{diag}(A)\cdot n}\, q^{\frac{1}{2}(n^tAn+\mathrm{diag}(A)\cdot n)}}{(q;q)_{n_1}\dots(q;q)_{n_N}} t_1^{n_1}\dots t_N^{n_N} \tag{31}$$

defined from a symmetric, integral $N\times N$ matrix A, are always admissible series. These series encode the Poincaré polynomials of representations of a quiver with symmetric potential and arbitrary dimension vector.

An elementary proof of the admissibility of $F_A(t,q)$ that uses the system of linear q-difference equations (33) is given in Section 2.3. The proof also leads to a notion of a level m admissible series defined in Section 2.4 (abbreviated by m-admissible series throughout this paper), the prototypical example being given by exactly the same sums as in equation (31) with n restricted to an m-congruence class $k \in \{0,\dots,m-1\}^N$

$$F_{A,m,k}(t,q) \;=\; \sum_{n\,\in\, k+m\mathbb{Z}^N_{\geq 0}} \frac{(-1)^{\mathrm{diag}(A)\cdot(n-k)} q^{\frac{1}{2}(n^tAn-k^tAk)+\frac{1}{2}\mathrm{diag}(A)\cdot(n-k)}}{(q^{k_1+1};q)_{n_1-k_1}\dots(q^{k_N+1};q)_{n_N-k_N}}\, t_1^{n_1-k_1}\cdots t_N^{n_N-k_N}. \tag{32}$$

On the other hand, we also have a collection of power series at roots of unity $f^{\mathrm{FGI}}_{A,m}(t,x)$ associated, by formal Gaussian integration, to an integer symmetric matrix A. These series were previously defined in [40, 11, 12] for the special case $t=(1,\dots,1)$ giving series $\Phi_{A,m}(h) = f^{\mathrm{FGI}}_{A,m}(1,x)$, expressed there with respect to the variable $h=\log(1+\zeta_m^{-1}x)$. This definition mimics the perturbation expansions of complex Chern–Simons theory given in terms of state-integrals, and is discussed in detail in Section 2.5. Both collections $f_{A,m}(t,x)$ and $f^{\mathrm{FGI}}_{A,m}(t,x)$ are in $\mathbb{Q}[\zeta_m](\!(x)\!)[\![t^{1/m}]\!]$, for every positive integer m.

Our two main results are that these two seemingly independent collections of power series agree and that their values at $t=(1,\dots,1)$ belong to explicit Habiro modules. In fact, these series can be defined by the admissible series of Kontsevich–Soibelman and this gives a new perspective to the arithmetic properties of DT-invariants and the enumerative properties of the elements of the Habiro ring.

Remark 1.8. The primes $p=2$ and 3 require special consideration. This is true for the Bloch group itself, which has a uniform definition up to 2-torsion and agrees with the algebraic third K group of a field away from some primes that always include 2 and 3. There are further difficulties with the primes 2 and 3 due to the presence of a $\frac{1}{24}$ whose origin can be seen to come from the Dedekind η-function and the famous equation of Euler $\mathrm{Li}_2(1) = -\frac{1}{24}(2\pi i)^2$. Therefore, throughout the paper we will only consider primes $p>3$ and always assume that $2,3|\Delta$ unless otherwise stated. However, a more refined approach should be able to cover the remaining cases as well.

1.6. **The main theorems.** We now state our main results. The first result identifies the two collections $f_{A,m}(t,x)$ and $f^{\mathrm{FGI}}_{A,m}(t,x)$ of power series at roots of unity and draws complementary conclusions about them. A key ingredient is the fact that both collections are solutions to

[4]The original Nahm sums were given for positive definite A by the special case when $t=(-1)^{\mathrm{diag}(A)}q^b$ for some $b\in\mathbb{Z}^N$.

the q-holonomic system of equations

$$F_A(t,q) - F_A(\sigma_j t, q) \;=\; (-1)^{A_{j,j}} t_j q^{A_{j,j}} F_A\Big(\prod_{i=1}^N \sigma_i^{A_{i,j}} t, q\Big), \qquad j = 1,\dots,N\,, \tag{33}$$

where $\sigma_j t$ shifts t_j to $q t_j$ and keeps $t_{j'}$ for $j' \neq j$ fixed.

After setting $q = 1$ and replacing the operators σ_j by $z_j(t)$, the equations (33) become the t-deformed Nahm equations (34)

$$1 - z_j(t) \;=\; (-1)^{A_{j,j}} t_j \prod_{i=1}^N z_j(t)^{A_{i,j}}, \qquad j \;=\; 1,\dots,N\,, \qquad z_j(0) \;=\; 1\,. \tag{34}$$

These equations in turn define the ring

$$S \;=\; \mathbb{Z}[t^{\pm 1}, z^{\pm 1}(t), \delta(t)^{-1/2}]/(1 - z(t) - (-1)^A t\, z(t)^A)\,, \tag{35}$$

whose relations are a shorthand of the Equations (34), where $z(t) = (z_1(t),\dots,z_N(t))$ and

$$\delta(t) \;:=\; \prod_{j=1}^N z_j(t)^{-A_{jj}} \det(\mathrm{diag}(1 - z(t))A + \mathrm{diag}(z(t))) \tag{36}$$

is the discriminant of the t-deformed Nahm equations, so that (after inverting 2) S is an étale $\mathbb{Z}[t]$-algebra. The Equations (34) are a t-deformation of Nahm's equations and appear both in the admissible series (see Section 2.2) as well as in the formal Gaussian integration series (see Section 2.5), and play a key role in identifying the two collections of series.

More generally, for a positive integer m, we define

$$S^{(m)} \;=\; \mathbb{Z}[\zeta_m, t^{\pm 1/m}, z^{\pm 1}(t), \delta(t)^{-1/2}]/(1 - z(t) - (-1)^A t\, z(t)^A) \tag{37}$$

and note that $S^{(1)} = S$.

Theorem 3. *For every symmetric matrix A with integer entries we have:*

$$f_A(t,q) \;=\; f_A^{\mathrm{FGI}}(t,q)\,. \tag{38}$$

Theorem 4. *Fix a symmetric matrix A with integer entries, a positive integer m and an m-congruence class $k \in \{0,\dots,m-1\}^N$. Then, for every prime p with $(m,p) = 1$, we have:*

$$\log(F_{A,m,k}(t^{p/m}, q^p)) - p\log(F_{A,m,k}(t^{1/m}, q)) \;\in\; \frac{p}{x} S^{(m)}[z(t)^{1/m}]_p^{\wedge}[\![x]\!]\,, \qquad q \;=\; \zeta_m + x\,. \tag{39}$$

These theorems are proved in Section 2.7.

Our main theorem below, which follows from the above theorems by specialisation to $t = 1$, states that the perturbative complex Chern–Simons invariants of knots and 3-manifolds are elements of the Habiro modules. The $t = 1$ specialisation refers to the ring homomorphism

$$\begin{array}{c} S \;=\; \mathbb{Z}[t^{\pm}, z^{\pm 1}(t), \delta(t)^{-1/2}]/(1 - z(t) - (-1)^A t\, z(t)^A) \\ \downarrow \\ R[\delta^{-1/2}] \;=\; \mathbb{Z}[z^{\pm 1}, \delta^{-1/2}]/(1 - z - (-1)^A z^A)\,, \end{array} \tag{40}$$

where we again use shorthand in the definition of $R[\delta^{-1/2}]$ with $z = (z_1, \dots, z_N)$ and the relations given by the Nahm equations

$$1 - z_j \;=\; (-1)^{A_{j,j}} \prod_{i=1}^{N} z_j^{A_{i,j}}, \qquad j \;=\; 1, \dots, N\,. \tag{41}$$

The specialisation $t = 1$ is compatible with the Frobenius endomorphism $\varphi_p(t) = t^p$ of the p-completions of the above rings. Below, we will fix an irreducible component of the Equations (41) that is non-degenerate, i.e., an isolated solution z with $\delta \neq 0$.

Such data gives rise to an element $\xi = \sum_{j=1}^{N}[z_j] \in B(\mathbb{K})$ in the Bloch group of the number field $\mathbb{K}$ generated by z. In geometry, the data (A, z) comes from an ideal triangulation $\mathcal{T}$ of a 3-manifold with torus boundary components, together with a solution z of the gluing (i.e., the Neumann–Zagier [32]) equations, as was explained in [11]. Thus, we denote the specialisation of $f_A(t, q)$ to $t = 1$, after removing the principle part of the logarithm, by $f_{A,z}(q)$.

Fixing a prime p prime to Δ, $V(t)$ specialises under $t = 1$ to $D_p(\xi)$. When $t = 1$, $V(t^p) = \varphi_p V(t)$ specialises to $D_p(\varphi_p \xi) = \varphi_p D_p(\xi)$, where φ_p now reduces to the Frobenius endomorphism of $\mathbb{K}_p$.

To state the next theorem we need to introduce some technical assumptions regarding primes of bad reduction. Recall that in Definition 1.4 of Section 1.4 we used collections of series at all roots of unity. However, below we will restrict to the subset of roots of unity with order prime to Δ. To this end, extending the notation of Definition 1.1 of Section 1.3 and the remark of Section 1.3 we define $\mathcal{H}_{R,\xi}|_\Delta$ to denote the set of collections of series indexed by roots of unity of order prime to Δ that satisfy (13) for all positive integers m and primes p such that $(pm, \Delta) = 1$. When $\xi = 0$, we have $\mathcal{H}_{R,0}|_\Delta = \mathcal{H}_R|_\Delta$.

Theorem 5. *Fix a symmetric matrix A with integer entries and a non-degenerate solution z of the Nahm equations with associated ξ. Then we have:*

$$f_{A,z}(q) \;\in\; \mathcal{H}_{R[\delta^{-1/2}],\xi}|_\Delta\,. \tag{42}$$

Remark 1.9. A stronger statement of the above theorem is probably true in $\mathcal{H}_{R[\delta^{-1/2}],\xi}$ without including Δ, and in fact follows if Theorem [9, Thm. 1.6] holds for any $m \in \mathbb{Z}$. Moreover, under the action of the Galois automorphism of $R[\delta^{-1/2}]$ sending $\sqrt{\delta}$ to $-\sqrt{\delta}$, the element $f_{A,z}$ lives in the -1-eigenspace.

All the results of this section can be strengthen to rational A by extending the proofs using m-admissible series, where m will include factors from the denominator of A. This will consequentially introduce more factors in Δ.

The above theorem combined with part (f) of Proposition 1.5 of Section 1.4 implies:

Corollary 1.10. *The constant term of the expansion of $f_{A,z}(q)$ at m-th roots of unity ζ_m of order prime to Δ is in $R[\zeta_m]$.*

This is by no means an obvious fact. In the case of the series associated to the 4_1 knot, it asserts the integrality of the following sum (see, e.g., [21, Eqn.(95)])

$$\frac{1}{\sqrt{m}} \sum_{k \in \mathbb{Z}/m\mathbb{Z}} (\zeta_m \theta; \zeta_m)_k (\zeta_m^{-1} \theta^{-1}; \zeta_m^{-1})_k \;\in\; \mathbb{Z}[\zeta_{6m}], \qquad \theta^m \;=\; \zeta_6\,, \qquad \text{for } (m, 6) \;=\; 1\,. \tag{43}$$

The above theorems allow us to construct elements of the ring $\mathcal{H}_R$ by considering torsion elements of the Bloch group, or by symmetrising elements of $\mathcal{H}_{R,\xi}$ under $q \mapsto q^{-1}$.

Corollary 1.11. *Fix A and z as in Theorem 5.*
(*a*) *We have*

$$f_{A,z}(q)\, f_{A,z}(q^{-1}) \;\in\; \mathcal{H}_R\,. \tag{44}$$

(*b*) *If $r\,\xi = 0 \in K_3(\mathbb{K})$ for some positive integer r, then $f_{A,z}(q)^r \in \mathcal{H}_{R[\delta^{-1/2}]}$.*

In fact, one can obtain even more elements of $\mathcal{H}_R$ by applying “descendants” to the above theorem, see Theorem 12 of Section 4.1, as well as elements of the modules $\mathcal{H}_{R,\xi}$ by specialising $t = q^\nu$; see the Remark in Section 3.3. Keeping in mind that the Habiro ring $\mathcal{H}_R$ of a number field $\mathbb{K}$ is a rank r module over $\mathcal{H}_{\mathbb{Z}[1/\Delta]}$, where $r = [\mathbb{K} : \mathbb{Q}]$, the above constructions presumably give a spanning set for $\mathcal{H}_R$ at the generic point of $\mathcal{H}$. We illustrate this with examples in Section 4.

The above corollary gives an if and only if statement: Let us say that an orbit of the Nahm equation is Bloch-torsion if the associated element of the Bloch group is torsion. Then an orbit is Bloch-torsion if and only if the series $f_{A,z}(q)^{2r} \in \mathcal{H}_R$, where r is the order of the torsion element of the Bloch group. One direction is given in the above corollary. The converse direction follows from the fact that the vanishing of the Bloch-Wigner dilogarithms of all complex embeddings of an element of the Bloch group implies torsion, or from the fact that the triviality of the unit $\varepsilon_m(\xi)$ for all but finitely m implies torsion.

1.7. **Future extensions.** We conclude this introduction by discussing several natural extensions of our paper that we hope to come back to.

• **Bad primes.** The current definition of $\mathcal{H}_R$ is a collection of power series associated to all roots of unity that satisfies the gluing condition (13), which allows to re-expand the series at ζ_{pm} from the series at ζ_m as long as the prime p is prime to Δ. In other words, the series at ζ_1 does not determine the series at any ζ_m when m is not prime to Δ. This implies that $\mathcal{H}_R$ is not an integral domain, but instead a product of integral domains, see the remark of Section 1.3. On the other hand, the collections $f_{A,z}(q)$ of power series associated to a symmetric matrix A together with a non-degenerate solution z to the Nahm equations assigns a collection of power series $f_{A,z,m}(x)$ at all roots of unity, not just those that are prime to an integer Δ; see Theorem 5 of Section 1.6. Understanding how these series at the bad primes determine the series at the remaining primes is an interesting question whose answer could potentially improve the current definition of $\mathcal{H}_R$.

• **Relative Bloch group.** The Habiro ring of the étale map $\mathbb{Z}[t] \to S$ (with S given in (35)) can be defined and is being studied in the upcoming thesis of Ferdinand Wagner. With additional work, one can define modules over this Habiro ring that are graded by a relative Bloch group (defined like the usual Bloch group with $t \wedge S = 0$) that ought to be identified with some relative K-theory group, and that the collection of series $f_A(t,q)$ is an element of such a module.

• **Higher weight modules.** One can define higher weight modules over the Habiro ring of a number field, indexed by the odd K groups of a number field, which we hope to discuss in a subsequent publication.

• **Line bundles on $A_{\rm inf}$.** The line bundles defined in the present paper can also be defined over the period ring $A_{\rm inf}$ of p-adic Hodge theory, for all elements of $K_3^{\rm cont}(\mathbb{C}_p)$. In this picture, the functions f_A define sections of this line bundle on $\mathrm{Spa}(A_{\rm inf})$ away from a disc around $\{q^{1/p}=1\}$.

• **Relation with holomorphic quantum modular forms.** A fifth extension concerns the quantum modularity properties of the collection of the series associated to knots over $\mathbb{C}$, and especially the extension of matrices of holomorphic functions on $\mathbb{C}\smallsetminus\mathbb{R}$ to the cut plane $\mathbb{C}'=\mathbb{C}\smallsetminus(-\infty,0]$, or more generally $\mathbb{C}_\gamma=\mathbb{C}\smallsetminus\{\tau\in\mathbb{R}\,|\,c\,\tau+d\le 0\}$ for an element $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$; see [21, Sec.5] and [20, Sec.5]. There are some tantalising parallels between the p-adic picture of the preceding bullet and this complex picture, similar to the parallels between the Fargues-Fontaine curve and the so called twistor-$\mathbb{P}^1$.

2. Two complementary collections of power series at roots of unity

In this section we will describe two beautiful consequences of a formal power series satisfying a simple system of linear q-difference equations. On one hand these series will have a subtle integrality property called *admissibility* — introduced by Kontsevich-Soibelman [30] — and on the other hand they will satisfy *algebraicity*. The combination of these two properties will provide the proofs of our main theorems. We will begin by explaining these results for the case of the Pochhammer symbol. Then we will describe some generalities on admissible series and an elementary proof that our series are admissible. This will lead us to discuss the generalisation to level m admissible series, which come from the restriction to congruence sums of the original admissible series. After that we will give a different formula for the same objects using formal Gaussian integration and an example of how the WKB (Wentzel–Kramers–Brillouin) method can be used to prove a similar statement. We will end this section with proofs of Theorem 3 and Theorem 4 of Section 1.6, which require combining all of the previous results.

2.1. The infinite Pochhammer symbol.

A key role in our paper is played by the infinite Pochhammer symbol $(t;q)_\infty = \prod_{n\ge0}(1-q^nt)$. In this section we collect some elementary, although remarkable, properties of it. There are two well-known complementary expansions of the logarithm of $(t;q)_\infty$

$$\log(t;q)_\infty \;=\; -\sum_{\ell\ge1}\frac{t^\ell}{\ell(1-q^\ell)} \tag{45}$$

$$\;=\; \sum_{k\ge0}\frac{B_k}{k!}\mathrm{Li}_{2-k}(t)\log(1+x)^{k-1}\,, \tag{46}$$

where $\mathrm{Li}_n(t)=\sum_{k\ge1}\frac{t^k}{k^n}$ is the nth polylogarithm and $B_k=B_k(0)$ are the Bernoulli numbers given by the generating series (59) with $B_0=1$, $B_1=-1/2$, etc.

These two expansions in turn imply two complementary views of the Dwork-type difference, namely for every prime p, and for $q=1+x$, we have

$$\log(t^p;q^p)_\infty - p\log(t;q)_\infty \;=\; p\sum_{\ell\ge 1,\, p\nmid \ell}\frac{t^\ell}{\ell(1-q^\ell)} \tag{47}$$

$$\;=\; -p\sum_{k\ge 0}\frac{B_k}{k!}\mathrm{Li}^{(p)}_{2-k}(t)\log(1+x)^{k-1}\,, \tag{48}$$

where

$$\mathrm{Li}^{(p)}_n(t) \;=\; \mathrm{Li}_n(t) - \frac{1}{p^n}\mathrm{Li}_n(t^p) \;=\; \sum_{k\ge 1,\, p\nmid k}\frac{t^k}{k^n} \tag{49}$$

is the p-version of the polylogarithm [10, Prop.6.2].

On the one hand, (47) implies that the coefficient of x^{k-1} in the Dwork-type difference is in $p\mathbb{Z}_{(p)}[\![t]\!]$, and on the other hand, (48) implies that it is a $\mathbb{Q}$-linear combination of the series $\mathrm{Li}^{(p)}_{2-s}(t)$ for $s=0,\dots,k$. But in fact, more is true. In [10, Prop.6.2] Coleman proves (working with the variable $1/t$) that $\mathrm{Li}^{(p)}_n(t)\in\mathbb{Z}_{(p)}[\![t]\!]$ is convergent for $|t/(1-t)|<p^{(p-1)^{-1}}$ and consequently lies in the ring $\mathbb{Z}[\frac{t}{1-t}]^\wedge_p$ of all series in $\mathbb{Z}_p[\![t]\!]$ that converge for $|t/(1-t)|\le 1$. In the present paper, we will only use a weaker statement with an elementary proof.

Lemma 2.1. For every integer n and prime number p we have

$$\mathrm{Li}^{(p)}_n(t) \;\in\; \mathbb{Z}[t,\tfrac{1}{1-t}]^\wedge_p\,. \tag{50}$$

Proof. For $r=1,\dots,p-1$ we have

$$\frac{1}{r+pk} \;=\; r^{-1}\sum_{\ell=0}^{\infty}(-1)^\ell(r^{-1}pk)^\ell \;\in\; \mathbb{Z}_p\,. \tag{51}$$

This implies that for all positive integers n and N we have

$$\frac{1}{(r+pk)^n} \;\equiv\; \frac{1}{(r+p(k+p^N))^n} \pmod{p^N}\,. \tag{52}$$

Using equation (49) and separating the summation over the positive integers prime to p into congruence classes modulo p^N, we obtain that

$$\mathrm{Li}^{(p)}_n(t) \;\equiv\; \sum_{r=1}^{p-1}\sum_{k=0}^{p^N-1}\sum_{\ell=0}^{\infty}\frac{t^{r+pk+p^N\ell}}{(r+pk)^n} \;=\; \frac{1}{1-t^{p^N}}\sum_{r=1}^{p-1}\sum_{k=0}^{p^N-1}\frac{t^{r+pk}}{(r+pk)^n} \pmod{p^N}\,. \tag{53}$$

Moreover, $\frac{1}{1-t^{p^N}}\in\mathbb{Z}[t,\frac{1}{1-t}]^\wedge_p$, which completes the proof. □

Proposition 2.2. (*a*) *For every prime p we have*

$$\log(t^p;q^p)_\infty - p\log(t;q)_\infty \;\in\; \frac{p}{x}\mathbb{Z}[t,\tfrac{1}{1-t}]^\wedge_p[\![x]\!]\,, \quad q \;=\; 1+x\,. \tag{54}$$

(*b*) *Fix a root of unity $\zeta\in\mathbb{C}_p\smallsetminus\{1\}$ of order not a power of p. The function*

$$\log(\zeta^p;q^p)_\infty - p\log(\zeta;q)_\infty \;\in\; p\frac{\mathrm{Li}^{(p)}_2(\zeta)}{x} + p\mathbb{Z}_{(p)}[\zeta][\![x]\!]\,, \quad q \;=\; 1+x \tag{55}$$

is meromorphic on the disc $|x|<1$ with a simple pole at $x=0$ and the residue as given.

Proof. Equation (54) follows from (47), Lemma 2.1, and the elementary identity

$$\mathbb{Z}[t, \tfrac{1}{1-t}]^\wedge_p[\tfrac{1}{p}] \cap \mathbb{Z}_{(p)}[\![t]\!] \;\subseteq\; \mathbb{Z}[t, \tfrac{1}{1-t}]^\wedge_p\,. \tag{56}$$

Part (b) follows trivially from part (a) by applying the natural map

$$\mathbb{Z}[t, \tfrac{1}{1-t}]^\wedge_p \to \mathbb{Z}_p[\zeta], \qquad t \mapsto \zeta\,, \tag{57}$$

which is compatible with the Frobenius endomorphism $t \mapsto t^p$. □

The power series expansion in equation (46) was centred about $q = 1$, but it has an extension at all roots of unity given by

$$\log(t;q)_\infty \;=\; \sum_{j=0}^{m-1} \log(q^j t; q^m)_\infty \;=\; \sum_{j=0}^{m-1}\sum_{k=0}^{\infty} \frac{B_k(j/m)m^{k-1}}{k!}\mathrm{Li}_{2-k}(\zeta_m^j t)\log(1+x/\zeta_m)^{k-1}\,, \tag{58}$$

where $q = \zeta_m + x$ and the Bernoulli polynomials $B_k(y) \in \mathbb{Q}[y]$ are defined by the generating series

$$\frac{xe^{xy}}{e^x - 1} \;=\; \sum_{k=0}^{\infty} \frac{B_k(y)}{k!}x^k \tag{59}$$

with $B_0(y) = 1$ and $B_1(y) = y - 1/2$.

2.2. **Admissible series.** The admissible series introduced by Kontsevich–Soibelman [30] arise as generating series of Poincaré polynomials of Cohomological Hall Algebras. The latter are algebra structures that one can define on the homology of a moduli stack parametrising objects in abelian categories satisfying appropriate conditions.

By their very definition, the product of two admissible series is admissible, and since the choice of the integers $c_{n,i}$ in (28) is arbitrary (as long as they have finite support for each n), it follows that admissible series belong to an uncountable group of formal power series. In contrast with q-series with integer coefficients, which are typically defined only for $|q| < 1$,

$$F(t,q) \quad \text{is admissible if and only if} \quad F(t,q^{-1}) \quad \text{is admissible}\,. \tag{60}$$

Another property of admissible series $F(t,q)$ is that the ratio (which is often considered in the literature) and the symmetrisation (which is less common, but useful for us)

$$G(t,q) \;=\; F(qt,q)/F(t,q), \qquad F^{\mathrm{sym}}(t,q) \;=\; F(t,q)F(t,q^{-1}) \tag{61}$$

are integral i.e., both lie in $\mathbb{Z}[q^{\pm 1}][\![t]\!]$. This follows easily from equation (28) combined with the two identities

$$(1-t)(qt;q)_\infty \;=\; (t;q)_\infty, \qquad (t;q^{-1})_\infty \;=\; \frac{1}{(qt;q)_\infty} \tag{62}$$

in $\mathbb{Q}(q)[\![t]\!]$, which follow from the known t-series expansions of both sides or from the fact that they satisfy the same first order linear q-difference equation. More precisely, equation (28) implies that

$$G(t,q) \;=\; \prod_{0\neq n\in(\mathbb{Z}_{\geq 0})^N}\prod_{i\in\mathbb{Z}}(q^i t^n;q)_{n_1+\cdots+n_N}^{-c_{n,i}}, \qquad F^{\mathrm{sym}}(t,q) \;=\; \prod_{0\neq n\in(\mathbb{Z}_{\geq 0})^N}\prod_{i\in\mathbb{Z}}(q^{1-i}t^n;q)_{2i-1}^{-c_{n,i}}\,. \tag{63}$$

Admissible series initially appear to treat q near 0 or infinity. As already mentioned, we can also consider q near 1, or better yet, near a primitive m-th root of unity ζ_m. Doing so, we are led to associate to an admissible series $F(t,q)$ a collection of completed series $f(t,q) = (f_m(t,x))_{m\geq 1}$ defined for $q = \zeta_m + x$ via the very short definition (30). The next lemma unravels this definition. Its proof follows directly from the definition of the admissibility (27), and among other things explains the necessity for the rescaling $t^{1/m}$ of the t (multi-)variable. To state it, recall $L_n(q)$ from equation (27) and set

$$V(t) \;=\; \sum_{0\neq n\in(\mathbb{Z}_{\geq 0})^N} L_n(1)\mathrm{Li}_2(t^n) \;\in\; \mathbb{Q}[\![t]\!]\,, \tag{64}$$

$$\delta(t) \;=\; \exp\Big(\sum_{0\neq n\in(\mathbb{Z}_{\geq 0})^N} L_n(1)\mathrm{Li}_1(t^n) - 2\sum_{0\neq n\in(\mathbb{Z}_{\geq 0})^N} L_n'(1)\mathrm{Li}_1(t^n)\Big) \;\in\; \mathbb{Q}[\![t]\!]\,, \tag{65}$$

with constant terms $V(0) = 0$ and $\delta(0) = 1$. Furthermore, for every integer $m \geq 1$, we let $U_m(t) = e^{u_m(t)}$, where

$$\begin{aligned} u_m(t) \;=\; &-\frac{m-1}{2m}\sum_{0\neq n\in(\mathbb{Z}_{\geq 0})^N} L_n(1)\mathrm{Li}_1(t^n) + \frac{1-m}{m}\sum_{0\neq n\in(\mathbb{Z}_{\geq 0})^N} L_n'(1)\mathrm{Li}_1(t^n) \\ &-\sum_{j=1}^{m-1}\sum_{0\neq n\in(\mathbb{Z}_{\geq 0})^N}\frac{L_n(\zeta_m^j)}{1-\zeta_m^j}\sum_{k\in\mathbb{Z}/m\mathbb{Z}}\frac{\zeta_m^{-kj}}{m}\mathrm{Li}_1(\zeta_m^k t^{n/m}) \quad\in\; t\,\mathbb{Q}[\![t]\!]\,. \end{aligned} \tag{66}$$

Lemma 2.3. For every m, we have

$$f_m(t,x) \in e^{\frac{V(t)}{m^2\log(1+x/\zeta_m)}}\frac{1}{\sqrt{\delta(t)}}U_m(t)(1 + x\,t^{1/m}\,\mathbb{Q}[\zeta_m][\![t^{1/m}]\!][\![x]\!]), \qquad f_m(0,x) \;=\; 1\,. \tag{67}$$

When $F(t,q) = F_A(t,q)$ is given by (31), we will denote the corresponding series $f_A(t,q)$, $V_A(t)$, $\delta_A(t)$ and $U_{A,m}(t)$, respectively.

Proof. To simplify the notation, we assume $N = 1$. Fix m, and observe that the right hand side of (27) has a simple pole at $q = \zeta_m$ coming from the terms with ℓ divisible by m, for which

$$\operatorname*{Res}_{q=\zeta_m}\frac{L_n(q^\ell)}{\ell\,(1-q^\ell)} \;=\; -\frac{L_n(1)\,\zeta_m}{\ell^2}\,. \tag{68}$$

Setting $\ell = mr$ with $r \geq 1$, we see that the residue of the right hand side of (27) is given by

$$\sum_{n,r\geq 1}\frac{L_n(1)\,\zeta_m}{m^2r^2}\,t^{nmr} \;=\; \frac{1}{m^2}\sum_{n\geq 1}L_n(1)\,\mathrm{Li}_2(t^{nm}) \;=\; \frac{1}{m^2}V(t^m)\,. \tag{69}$$

This explains the need for the rescaling $t \mapsto t^{1/m}$ in order to match with equation (67). The remaining terms of (27) can be expanded into power series in $t^{1/m}$ with coefficients in $\mathbb{Q}[\zeta_m]$.

When we compute the constant term of the x-series expansion of $f_m(t,x)$ there are two contributions to $u_m(t)$ from $\log F(t,q)$ in (27), depending whether or not m divides ℓ. When m divides ℓ, with $q = \zeta_m + x$, we have

$$-\frac{L_n(q^\ell)}{\ell(1-q^\ell)} \;=\; \zeta_m\frac{L_n(1)}{\ell^2}\frac{1}{x} + \Big(\frac{L_n(1)}{2\ell^2} - \frac{L_n(1)}{2\ell} + \frac{L_n'(1)}{\ell}\Big) + \mathrm{O}(x) \tag{70}$$

Setting $m = r\ell$ and summing over $n \geq 1, r \geq 1$ and rescaling t to $t^{1/m}$ gives the volume term $V(t)/(2m^2)$ and the first two terms in equation (66).

When m does not divide ℓ, then $\ell = rm + j$ for some $r \geq 0$ and $1 \leq j \leq m-1$. Then, again with $q = \zeta_m + x$, we have

$$-\frac{L_n(q^\ell)}{\ell(1-q^\ell)} \;=\; -\frac{L_n(\zeta_m^j)}{(mr+j)(1-\zeta_m^j)} + \mathrm{O}(x) \tag{71}$$

and after rescaling t, we find that this contribution is

$$-\sum_{j=1}^{m-1} \sum_{n\geq 1,\, r\geq 0} \frac{L_n(\zeta_m^j)}{(mr+j)(1-\zeta_m^j)} t^{n(mr+j)/m} \;=\; -\sum_{j=1}^{m-1}\sum_{n\geq 1} \frac{L_n(\zeta_m^j)}{1-\zeta_m^j} \sum_{k\in\mathbb{Z}/m\mathbb{Z}} \frac{\zeta_m^{-kj}}{m} \mathrm{Li}_1(\zeta_m^k t^{n/m})\,, \tag{72}$$

which is the last term of equation (66). The lemma follows. □

Equations (64) and (66) imply the following corollary, which in a sense is a gluing property of the collection of power series $f(t,q)$.

Corollary 2.4. *Let $F^{(1)}$ and $F^{(2)}$ be admissible series with the same number of variables t, and $L_n^{(i)}$, $V^{(i)}(t)$, $\delta^{(i)}(t)$, and $U_m^{(i)}(t)$ $(i = 1,2)$ be as defined before Lemma 2.3. Then*

- (a) *$V^{(1)} = V^{(2)}$ if and only if $L_n^{(1)}(1) = L_n^{(2)}(1)$ for all $n \geq 1$.*
- (b) *$F^{(1)} = F^{(2)}$ if and only if $V^{(1)} = V^{(2)}$, $\delta^{(1)} = \delta^{(2)}$ and $U_m^{(1)} = U_m^{(2)}$ for all $m \geq 1$.*
- (c) *$F^{(1)} = F^{(2)}$ if and only $f_m^{(1)} = f_m^{(2)}$ for some (and hence for every) $m \geq 1$.*

Proof. Only the if direction of (b) is not obvious. To show it, we note that equation (66) and the fact that $\mathrm{Li}_s(t) = t + O(t^2)$ imply that $V(t)$ and $\delta(t)$ determine $L_n(1)$ and $L_n'(1)$ for all n, and hence with $U_m(t)$ determine $s_m(t) := \sum_{j=1}^{m-1}\sum_{n\geq 1} \frac{L_n(\zeta_m^j)}{1-\zeta_m^j} \sum_{k\in\mathbb{Z}/m\mathbb{Z}} \frac{\zeta_m^{-kj}}{m}\mathrm{Li}_1(\zeta_m^k t^{n/m})$. Now use $\sum_{k\in\mathbb{Z}/m\mathbb{Z}} \frac{\zeta_m^{-kj}}{m}\mathrm{Li}_1(\zeta_m^k t^{n/m}) = t^{jn/m}/j + \cdots$ to deduce that the coefficient of $t^{1/m}$ in $s_m(t)$ is $L_1(\zeta_m)/(1-\zeta_m)$ for all m. Thus $s_m(t)$ determines $L_1(q) \in \mathbb{Z}[q^{\pm 1}]$. Subtracting this first contribution from $s_m(t)$, we find that the coefficient of $t^{2/m}$ is now $L_2(\zeta_m)/(2(1-\zeta_m))$ and hence is again determined by $s_m(t)$. Continuing by induction on n, we deduce that $s_m(t)$ determines $L_n(q)$ for all $n \geq 1$. □

We next discuss a Dwork-type quotient of admissible series for a prime p. The product formula (28) and equation (27) for admissible series lead immediately to the equality

$$\log\Big(\frac{F(t^p,q^p)}{F(t,q)^p}\Big) \;=\; p \sum_{0\neq n\in(\mathbb{Z}_{\geq 0})^N} \sum_{\substack{\ell>0\\(p,\ell)=1}} \frac{L_n(q^\ell)}{\ell(1-q^\ell)} t^{\ell n}\,. \tag{73}$$

Expanding near $q = \zeta_m + x$ as in (30) with $(m,p) = 1$, we see that the left-hand side, even after division by p, is p-integral. (This strong integrality property will be important.) This proves:

Lemma 2.5. If $F(t,q)$ is admissible, then for all primes p and positive integers m not divisible by p we have

$$\log(F(t^{p/m},q^p)) - p\log(F(t^{1/m},q)) \;\in\; \frac{p}{x}\mathbb{Z}_{(p)}[t^{1/m},\zeta_m][\![t,x]\!]\,, \qquad q \;=\; \zeta_m + x\,. \tag{74}$$

2.3. **Admissibility of q-hypergeometric series.** Our main interest is in the admissible series $F_A(t,q)$ defined by a q-hypergeometric sum that is determined by an integral symmetric matrix A as in (31). These admissible series are in several respects special. Most importantly for us, they are q-holonomic, i.e., satisfy the system (33) of linear q-difference equations with respect to t that, together with the initial condition $F_A(0,q)=1$, uniquely determine $F_A(t,q)$. The proof that $F_A(t,q)$ satisfies the system of equations (33) follows easily from the fact that $F_A(t,q)$ is a sum of proper q-hypergeometric series, and in more elementary terms, it is an easy consequence of the fact that the q-Pochhammer symbol $(q;q)_n$ satisfies the relation $(q;q)_{n+1}=(1-q^{n+1})(q;q)_n$ for all non-negative integers n.

The condition for an admissible series to be q-holonomic is a delicate intersection between additive and multiplicative properties, and for the case of the series $F_A(t,q)$, the shape of the q-difference equations (33) depends on the matrix A and in fact determines it.

We next discuss the algebraic system of t-deformed Nahm equations (34) that we encountered in the introduction. It is clear that the latter has a unique formal power series solution $z(t)=(z_1(t),\dots,z_N(t))$ with $z(0)=1\in\mathbb{Z}^N$, which in fact satisfies $z(t)\in\mathbb{Z}[\![t]\!]$. But more is true. Namely, $z(t)$ is given by a hypergeometric series, as was discovered by Rodriguez Villegas [35, Sec. 4.1] (see also [41, Sec. 7]). For example, in the rank one case when $A\in\mathbb{Z}$, the unique solution of the equation $1-z=t\,(-z)^A$ in $\mathbb{Z}[\![t]\!]$ with $z(0)=1$ is given by the hypergeometric series [35, Eqn.(3.0.8)]

$$z(t) \;=\; \sum_{k=0}^{\infty}\frac{(-1)^{(A+1)k}\binom{Ak}{k}}{(A-1)k+1}t^k \tag{75}$$

(as follows from Lagrange inversion) and more generally, $z(t)^s$ as well as $\log(z(t))$ have hypergeometric series expansions [35, p.5]

$$z(t)^s \;=\; s\sum_{k=0}^{\infty}\frac{(-1)^{(A+1)k}\binom{Ak+s-1}{k}}{(A-1)k+s}t^k, \qquad \log(z(t)) \;=\; \sum_{k=1}^{\infty}\frac{(-1)^{(A+1)k}\binom{Ak}{k}}{Ak}t^k\,. \tag{76}$$

Note that $z(t)$ given in (75) is an algebraic function of t in $\mathbb{CP}^1$ with three singularities at $t=0,\infty,(-1)^A(A-1)^{A-1}/A^A$.

The involution (60) for the admissible series $F_A(t,q)$ becomes the identity

$$F_A(t,q) \;=\; F_{I-A}(t,q^{-1})\,, \tag{77}$$

which follows easily from the sum definition of $F_A(t,q)$, equation (31) and the elementary fact that $(q^{-1};q^{-1})_n=(-1)^nq^{-n(n+1)/2}(q;q)_n$.

Next we give a simple proof that the q-hypergeometric series $F_A(t,q)$ of equation (31) are admissible. For this we will need the following lemma whose proof we defer to Section 2.7, after Theorem 8 of that section.

Lemma 2.6. For every integral symmetric $N\times N$ matrix A, there exists a series $V(t)=V_A(t)\in\mathbb{Q}[\![t]\!]$ such that for all $m\in\mathbb{Z}_{>0}$ we have

$$\log(F_A(t,\zeta_m+x)) \;=\; \frac{\zeta_m V(t^m)}{m^2x}+O(x^0)\,. \tag{78}$$

This lemma essentially follows from the q-difference equations and the WKB algorithm, or alternatively from the identification of the series $F_A(t,q)$ with the one $F_A^{\mathrm{FGI}}(t,q)$ that comes from formal Gaussian integration.

The power series $V_A(t)$ is effectively computable (and is given explicitly in Equation (115)), the example of $A \in \mathbb{Z}_{>0}$ being given by the following:

$$V_A(t) \;=\; -\mathrm{Li}_2(1-z(t)) - \frac{A}{2}\log(z(t))^2 \;=\; \sum_{k=1}^{\infty} \frac{(-1)^{(A+1)k}\binom{Ak}{k}}{Ak^2} t^k \,. \tag{79}$$

Assuming Lemma 2.6, and using the q-holonomic system of equations that F_A satisfies, we can give an alternative proof of the admissibility of $F_A(t,q)$.

Theorem 6. [13, 30] *Suppose that A is an integral symmetric $N \times N$ matrix. Then the unique $c_{n,i} \in \mathbb{Z}$ such that*

$$F_A(t,q) \;=\; \sum_{n \in \mathbb{Z}_{\geq 0}^N} \frac{(-1)^{\mathrm{diag}(A)\cdot n} q^{\frac{1}{2}(n^t A n + \mathrm{diag}(A)\cdot n)}}{(q;q)_{n_1} \dots (q;q)_{n_N}} t_1^{n_1} \dots t_N^{n_N} \;=\; \prod_{0 \neq n \in (\mathbb{Z}_{\geq 0})^N} \prod_{i \in \mathbb{Z}} (q^i t^n; q)_\infty^{c_{n,i}} \tag{80}$$

have finite support, i.e. for fixed n all but finitely many i satisfy $c_{n,i} = 0$.

Proof. To begin with, every $F(t,q) \in \mathbb{Z}((q))[\![t]\!]$ satisfying $F(0,q) = 1$ has a unique expansion of the form (28) for integers $c_{n,i}$, where for every $n \in \mathbb{Z}_{\geq 0}^N - \{0\}$, $c_{n,i} = 0$ is zero for sufficiently negative integers i. This follows easily by induction on the total degree of $q^i t^n$.

So, it suffices to show that when $F = F_A$, for each fixed n, there are only finitely many non-zero $c_{n,i}$. To make the idea clear, assume $N = 1$. Then $F = F_A$ satisfies the linear q-difference equation

$$F(t,q) - F(qt,q) \;=\; (-q)^A t F(q^A t, q) \tag{81}$$

Consider the ratio

$$G(t,q) \;=\; \frac{F(qt,q)}{F(t,q)} \;=\; \prod_{n>0} \prod_{i \in \mathbb{Z}} (q^i t^n; q)_n^{-c_{n,i}} \,. \tag{82}$$

It recovers F by telescoping product $F(t,q) = G(t,q)^{-1} G(qt,q)^{-1} \dots$. Moreover, G satisfies the Ricatti equation

$$1 - G(t,q) \;=\; (-q)^A t \prod_{j=0}^{A-1} G(q^j t, q) \tag{83}$$

and it follows by induction on the powers of t that $G(t,q) \in \mathbb{Z}[q^{\pm 1}][\![t]\!]$. Therefore, we find as in equation (27) that

$$-\log(F(t,q)) \;=\; \sum_{\ell,n=1}^{\infty} \frac{L_n(q^\ell)}{\ell(1-q^\ell)} t^{n\ell} \,, \tag{84}$$

with $L_n(q) \in \frac{1-q}{1-q^n}\mathbb{Z}[q^{\pm 1}]$. Suppose for induction that $L_n(q) \in \mathbb{Z}[q^{\pm 1}]$ for $n < N$. This implies that the $V(t)$ from Lemma 2.6 is given by

$$V(t) \;=\; \sum_{n=1}^{N-1} L_n(1) \mathrm{Li}_2(t^n) + O(t^N) \,. \tag{85}$$

If $L_N(q)$ does not belong to $\mathbb{Z}[q^{\pm 1}]$, it must have a pole at ζ_a for some $a|N$ and $a > 1$ with residue $\alpha \neq 0$. Notice that for $ab = N$ this would imply that the coefficient of x^{-1} in $\log(F(t, \zeta_a + x))$ would be equal to

$$\frac{\zeta_a}{a^2} \sum_{n=1}^{N-1} L_n(1) \mathrm{Li}_2(t^{an}) + \frac{\alpha}{1-\zeta_a} t^{ab} + O(t^{N+1}) \,. \tag{86}$$

This contradicts Lemma 2.6 and completes the proof for $N = 1$. For the case of general $N \geq 1$, $F(t,q)$ has an expansion of the form (28), where for each fixed $n \in \mathbb{Z}_{\geq 0}^N - \{0\}$, $c_{n,i} = 0$ is zero for sufficiently negative integers i. Define

$$G_j(t,q) \;=\; \frac{F(t_1, \cdots, qt_j, \cdots, t_N, q)}{F(t,q)} \;=\; \prod_{0 \neq n \in (\mathbb{Z}_{\geq 0})^N} \prod_{i \in \mathbb{Z}} (q^i t^n; q)_{n_j}^{-c_{n,i}}, \qquad j \;=\; 1, \dots, N \,. \tag{87}$$

Since F satisfies (33), it follows that the power series $G_j(t,q)$ for $j = 1, \dots, N$ satisfy the system of equations

$$1 - G_j(t,q) \;=\; (-q)^{A_{j,j}} t_j \prod_{k=0}^{A_{j,j}-1} G_j\Big(\sigma_j^k \prod_{\substack{i=1 \\ i \neq j}}^{N} \sigma_i^{A_{i,j}} t, q\Big) \,, \tag{88}$$

where $\prod_{k=0}^{-m} a_k = a_{-1}^{-1} \cdots a_{-m}^{-1}$ for $m > 0$. This system of equations has a unique solution expanded in power series in t and we claim that $G_j(t,q) \in \mathbb{Z}[q^{\pm 1}][\![t]\!]$. This follows easily by induction. Then we find that $\log(F(t,q))$ has the expression in equation (27) with $L_n(q) \in \bigcap_{i=1}^N \frac{1-q}{1-q^{n_i}} \mathbb{Z}[q^{\pm 1}]$. Then applying a similar contradiction argument as for $N = 1$, and again using Lemma 2.6, we can conclude the proof. □

2.4. **Level m admissible series.** The proof of Theorem 5 of Section 1.6 requires an extension of admissibility, which is modelled on the congruence sums of equation (32) and equation (125). We introduce this notion here and then prove that the above-mentioned sums give examples of level m admissible series. This will be crucial in proving the integrality of Theorem 4 of Section 1.6.

To state the definition we need the following:

Lemma 2.7. Every $F(t,q) \in 1 + t\,\mathbb{Z}[\frac{1}{m}](\!(q)\!)[\![t]\!]$ can be written uniquely in the form

$$F(t,q) \;=\; \exp\Big(- \sum_{n=1}^{\infty} \sum_{\substack{\ell=1 \\ (\ell,m)=1}}^{\infty} \frac{L_n(q^\ell)}{\ell(1-q^{m\ell})} t^{n\ell} \Big) \tag{89}$$

with $L_n(q) \in \mathbb{Z}[\frac{1}{m}](\!(q)\!)$, and conversely.

Proof. The existence and uniqueness of $L_n(q) \in \mathbb{Q}(\!(q)\!)$ that satisfy (89) is clear by removing one $L_n(q)$ at a time and induction on powers of t. To show that $L_n(q) \in \mathbb{Z}[\frac{1}{m}](\!(q)\!)$, we use the fact that

$$\exp\Big(- \sum_{\substack{\ell=1 \\ (\ell,m)=1}}^{\infty} \frac{q^{k\ell}}{\ell(1-q^{m\ell})} t^{n\ell} \Big) \;\in\; 1 + t\,\mathbb{Z}[\tfrac{1}{m}](\!(q)\!)[\![t]\!] \,, \tag{90}$$

which itself follows from the fact that by inclusion/exclusion on the divisors of m, equation (90) can be written as a product of Pochhammer symbols raised to the power of $\frac{1}{m}$. Indeed, we find that

$$\exp\Bigg(-\sum_{\substack{\ell=1\\(\ell,m)=1}}^{\infty}\frac{q^{k\ell}}{\ell(1-q^{m\ell})}t^{n\ell}\Bigg) \;=\; \prod_{d|m}(q^{dk}t^{dn};q^{dm})_{\infty}^{\mu(d)/d} \;\in\; 1+t\,\mathbb{Z}[\tfrac{1}{m}](\!(q)\!)[\![t]\!]\,, \tag{91}$$

where μ is the Möbius function. □

Definition 2.8. A series $F(t,q)\in 1+t\,\mathbb{Z}[\frac{1}{m}](\!(q)\!)[\![t]\!]$ is called level m admissible (and abbreviated by m-admissible) if, for the unique $L_n(q)\in\mathbb{Z}[\frac{1}{m}](\!(q)\!)$ from equation (89), we have

$$L_n(q) \;\in\; \mathbb{Z}[\tfrac{1}{m},q^{\pm 1},\Phi_d(q)^{-1}\,|\,d\not\equiv 0 \pmod{m}] \qquad\text{and}\qquad L_n(\zeta_m)\;\in\;\mathbb{Z}[\tfrac{1}{m}]\,, \tag{92}$$

where Φ_d denotes the d-th cyclotomic polynomial.

Of course level 1 admissible series are simply the admissible series of Kontsevich–Soibelman. As in Section 2.2, we can expand the logarithm of a m-admissible series for q near a ζ_{mn} root of unity for any $n\in\mathbb{Z}_{>0}$. This will be a Laurent series with simple polar part given by a series

$$\frac{V(t)}{m^2} \;=\; \sum_{n=1}^{\infty}\sum_{\substack{\ell=1\\(\ell,m)=1}}^{\infty}\frac{L_n(\zeta_m^{\ell})}{m\ell^2}t^{n\ell} \;\in\; \mathbb{Q}[\![t]\!]\,. \tag{93}$$

With this definition we can state the generalisation to $m>1$ of Lemma 2.6 of Section 2.3 and Theorem 6 of Section 2.3 for the congruence sums $F_{A,m,k}$ in equation (32).

Lemma 2.9. Let A be a symmetric integral $N\times N$ matrix and $V(t)=V_A(t)\in\mathbb{Q}[\![t]\!]$ as in Lemma 2.6 of Section 2.3. Then for all $m>0$ and all $k\in\{0,\dots,m-1\}^N$, we have that

$$\log(F_{A,m,k}(t,\zeta_m+x)) \;=\; \frac{\zeta_m V(t^m)}{m^2 x}+O(x^0)\,. \tag{94}$$

This lemma will be proved after Theorem 8 of Section 2.7. Assuming this lemma, we have the following:

Theorem 7. *Fix an integral symmetric $N\times N$ matrix A, a positive integer m and a residue class $k\in\{0,\dots,m-1\}^N$. Then the series*

$$F_{A,m,k}(t^{1/m},q) \;\in\; 1+t\,\mathbb{Z}[\tfrac{1}{m}](\!(q)\!)[\![t]\!] \tag{95}$$

is m-admissible.

Proof. We will give the details of the proof when $N=1$, while for $N>1$ we use similar methods to the proof of Theorem 6 of Section 2.3. The series

$$H_{A,m,k}(t,q) \;=\; \frac{(-1)^{Ak}q^{\frac{1}{2}Ak(k+1)}}{(q;q)_k}t^k F_{A,m,k}(t,q) \tag{96}$$

satisfies the linear q-difference equation

$$\sum_{\ell=0}^{m}(-1)^{\ell}q^{-\ell(\ell-1)/2}\binom{m}{\ell}_{q^{-1}}H_{A,m,k}(q^{\ell}t,q) \;=\; q^{Am(m+1)/2}t^m H_{A,m,k}(q^{Am}t,q)\,, \tag{97}$$

where $\binom{m}{\ell}_{q^{-1}} = \frac{(q^{-1};q^{-1})_m}{(q^{-1};q^{-1})_\ell (q^{-1};q^{-1})_{m-\ell}}$ is the q^{-1}-binomial coefficient. Consider the ratio

$$G_{A,m,k}(t,q) \;=\; \frac{H_{A,m,k}(qt,q)}{H_{A,m,k}(t,q)} \;=\; q^k + \sum_{n=1}^{\infty} a_n(q) t^{mn}\,. \tag{98}$$

This satisfies the non-linear equation q-difference equation

$$\sum_{\ell=0}^{m} (-1)^\ell q^{-\ell(\ell-1)/2} \binom{m}{\ell}_{q^{-1}} \prod_{j=0}^{\ell-1} G_{A,m,k}(q^j t, q) \;=\; q^{Am(m+1)/2} t^m \prod_{j=0}^{Am-1} G_{A,m,k}(q^j t, q)\,. \tag{99}$$

This gives an integral recursion for $a_n(q)$ multiplied by

$$\begin{aligned} \sum_{\ell=0}^{m} (-1)^\ell q^{-\ell(\ell-1)/2} \binom{m}{\ell}_{q^{-1}} \sum_{j=0}^{\ell-1} q^{nmj} \;&=\; \sum_{\ell=0}^{m} (-1)^\ell q^{-\ell(\ell-1)/2} \binom{m}{\ell}_{q^{-1}} \frac{1-q^{nm\ell}}{1-q^{nm}} \\ &=\; \frac{(q^{1-m};q)_m - (q^{nm+1-m};q)_m}{1-q^{nm}} \;=\; -(q^{nm+1-m};q)_{m-1}\,. \end{aligned} \tag{100}$$

It follows that

$$-(q^{nm+1-m};q)_{m-1} a_n(q) \;\in\; \mathbb{Z}[q^{\pm 1}, a_1(q), \dots, a_{n-1}(q)]\,. \tag{101}$$

Notice that $-(q^{mn+1-m};q)_{m-1}$, never contains $\Phi_d(q)$ with $d \equiv 0 \pmod{m}$ as a factor and therefore, by induction, we see that

$$G_{A,m,k}(t,q) \;\in\; q^k + t^m \mathbb{Z}[q^{\pm 1}, \Phi_d(q) : d \not\equiv 0 \pmod{m}][\![t^m]\!]\,. \tag{102}$$

Therefore, solving for the unique Laurent series $L_n(q)$ such that

$$\log(q^{-k} G_{A,m,k}(t,q)) \;=\; \sum_{n=1}^{\infty} \sum_{\substack{\ell=1 \\ (\ell,m)=1}}^{\infty} \frac{L_n(q^\ell)(1-q^{nm\ell})}{\ell(1-q^{m\ell})} t^{nm\ell}\,, \tag{103}$$

we find that $L_n(q) \in \frac{1-q^m}{1-q^{nm}} \mathbb{Z}[\frac{1}{m}, q^{\pm 1}, \Phi_d(q) \,|\, d \not\equiv 0 \pmod{m}]$. It follows that

$$\log\big(F_{A,m,k}(t,q)\big) \;=\; -\sum_{n=1}^{\infty} \sum_{\substack{\ell=1 \\ (\ell,m)=1}}^{\infty} \frac{L_n(q^\ell)}{\ell(1-q^{m\ell})} t^{nm\ell}\,. \tag{104}$$

Suppose by induction that for $n < n_0$, for some n_0, we have $L_n(q) \in \mathbb{Z}[\frac{1}{m}, q^{\pm 1}, \Phi_d(q) : d \not\equiv 0 \pmod{m}]$ (the case $n_0 = 1$ being obvious). If $L_{n_0}(q) \notin \mathbb{Z}[\frac{1}{m}, q^{\pm 1}, \Phi_d(q) : d \not\equiv 0 \pmod{m}]$ it must have a pole at $q = \zeta_{am}$ for some $1 < a$. From Lemma 2.9, the residues at ζ_c are given universally by $\zeta_c V(t^c)\, c^{-2}$ for all $c \in m\mathbb{Z}_{>0}$ and we therefore find that

$$V(t^m) \;=\; m \sum_{n=1}^{\infty} \sum_{\substack{\ell=1 \\ (\ell,m)=1}}^{\infty} \frac{L_n(\zeta_m^\ell)}{\ell^2} t^{nm\ell}\,, \tag{105}$$

and so

$$V(t^{am}) \;=\; -am \sum_{n=1}^{\infty} \sum_{\substack{\ell=1 \\ (\ell,m)=1}}^{\infty} \frac{L_n(\zeta_m^\ell)}{\ell^2} t^{amn\ell} + O(t^{amn_0})\,. \tag{106}$$

Assuming that $L_{n_0}(q)$ has a pole at ζ_{am} would imply that the logarithm expanded at $q = \zeta_{am} + x$ would have an additional residue containing a power t^{n_0}. Given that $V(t^{am})$ is the residue of $\log\big(F_{A,m,k}(t,q)\big)$ at $q = \zeta_{am}$ from Lemma 2.9, and $V(t^{am})$ is a power series in t^{am}, we see that this must imply that n_0 is a multiple of am. In particular, we find that $n_0 = abm$ for some integer $0 < b < n_0$. However, we see that this would also change the coefficient of t^{abm}, which would contradict the fact that the residue is $V(t^{am})$. This completes the proof. $\square$

We are especially interested in the integrality that is implied by admissibility. With this in mind, we state a generalisation of Lemma 2.5 of Section 2.2 to the case of m-admissible series.

Lemma 2.10. Fix an m-admissible series $F(t,q)$. For all primes p and positive integers m' with $(mm',p) = 1$, we have

$$\log(F(t^{p/m'},q^p)) - p\log(F(t^{1/m'},q)) \;\in\; \frac{p}{x}\mathbb{Z}_{(p)}[t^{1/mm'},\zeta_{mm'}][\![t,x]\!]\,, \qquad q \;=\; \zeta_{mm'} + x\,. \tag{107}$$

Proof. Notice that

$$\begin{aligned}\frac{F(t^p,q^p)}{F(t,q)^p} \;&=\; \exp\Big(p\sum_{n=1}^{\infty}\sum_{\substack{\ell=1\\(\ell,m)=1}}^{\infty}\frac{L_n(q^\ell)}{\ell(1-q^{m\ell})}t^{j\ell} - p\sum_{n=1}^{\infty}\sum_{\substack{\ell=1\\(\ell,m)=1}}^{\infty}\frac{L_n(q^{p\ell})}{p\ell(1-q^{mp\ell})}t^{jp\ell}\Big)\\ &=\; \exp\Big(p\sum_{n=1}^{\infty}\sum_{\substack{\ell=1\\(\ell,mp)=1}}^{\infty}\frac{L_n(q^\ell)}{\ell(1-q^{m\ell})}t^{j\ell}\Big)\,.\end{aligned} \tag{108}$$

Then expanding this sum at $q = \zeta_{mm'} + x$ concludes the proof. $\square$

2.5. **Formal Gaussian integration.** In this section we review a collection of formal power series at roots of unity that were defined by formal Gaussian integration in [11] for $\zeta = 1$ and in [12] for arbitrary complex roots of unity ζ. The input to define these power series was a Neumann–Zagier datum, which, in the case of perturbative complex Chern–Simons theory, comes from an ideal triangulation of a 3-manifold and a solution of its gluing equations. For the simpler case of a single matrix A and the Nahm equations, and for $\zeta = 1$, the same series were found in [40]. We will tailor the definition of these series for the purpose of our paper and match them with the associated series $f_A(t,q)$ of the admissible series $F_A(t,q)$ of (31). The most important point here is that formal Gaussian integration gives series with coefficients that are manifestly algebraic functions.

We now recall the definition of $f_{A,m}$. Fix a symmetric $N\times N$ matrix Λ with integer entries. For a $\mathbb{Q}$-algebra R, formal Gaussian integration is a map $\langle\cdot\rangle : R[\![w, w^3h^{-1}, h]\!] \to R[\![h]\!]$, where $w = (w_1,\dots,w_N)$, defined by

$$\langle f(w,h)\rangle_\Lambda \;:=\; \exp\left(\frac{h}{2}\sum_{i,j=1}^{N}(\Lambda^{-1})_{i,j}\frac{\partial}{\partial w_i}\frac{\partial}{\partial w_j}\right) f(w,h)\Big|_{w=0} \;\in\; R[\![h]\!]\,. \tag{109}$$

This formally recovers the integral

$$\frac{\int e^{-\frac{1}{2h}w^t\Lambda\, w} f(w,h)\, dw}{\int e^{-\frac{1}{2h}w^t\Lambda\, w}\, dw} \;\in\; R[\![h]\!]\,. \tag{110}$$

Starting with the admissible series $F_A(t,q)$ of equation (31) and taking $q = \zeta_m e^h$, summing over congruences modulo m, and applying Poisson summation assuming that one term dominates, leads formally to the integral

$$\begin{aligned} &\frac{1}{(q;q)_\infty^N} \sum_{k\in(\mathbb{Z}/m\mathbb{Z})^N} (-1)^{\mathrm{diag}(A)\cdot k} q^{\frac{1}{2}(k^tAk+\mathrm{diag}(A)\cdot k)} t_1^{k_1}\cdots t_N^{k_N} \int \prod_{j=1}^N (q^{k_j+1}e^{w_j};q)_\infty \\ &\quad\times \exp\Big(\frac{1}{h}\Big(\frac{\mathrm{diag}(A)\cdot w\pi i}{m} + \frac{1}{2}w^tAw + w\cdot\log(t)\Big) + \big(\frac{\mathrm{diag}(A)}{2} + k^tA\big)\cdot w\Big)\frac{dw}{(mh)^N}\,. \end{aligned} \tag{111}$$

To define the integral in equation (111) precisely, we use the same coordinates as we did previously for the Habiro ring

$$q \;=\; \zeta_m e^h \;=\; \zeta_m + x\,, \qquad h \;=\; \log(1+x/\zeta_m)\,. \tag{112}$$

together with the expansion of the infinite Pochhammer symbol (see (58)) from which we remove four terms

$$\begin{aligned} \psi_{k,z,\zeta_m}(w,x) \;=\; &((\zeta_m+x)^{k+1}ze^w;\zeta_m+x)_\infty \cdot \exp\Big(-\frac{\mathrm{Li}_2(z^m)}{m^2\log(1+x/\zeta_m)} \\ &-\frac{\mathrm{Li}_1(z^m)}{m^2\log(1+x/\zeta_m)}w - \frac{\mathrm{Li}_0(z^m)}{m^2\log(1+x/\zeta_m)}w^2 + \sum_{\ell=0}^{m-1}\big(\frac{1}{2} - \frac{k+\ell+1}{m}\big)\log(1-\zeta_m^{k+\ell+1}z)\Big)\,, \end{aligned} \tag{113}$$

where $\psi_{k,z,\zeta_m}(w,x) \in 1 + x\mathbb{Q}(z,\zeta_m)[\![w, w^3x^{-1}, x]\!]$. The integral (111) is dominated by the leading term given by the exponential of h^{-1} times the function

$$\frac{1}{2}w^tAw + \sum_{j=1}^N -\frac{\mathrm{Li}_2(1-e^{mw_j})}{m^2} + \frac{w_j}{m}\log\Big(\frac{(-1)^A t_j^m}{1-e^{mw_j}}\Big)\,, \tag{114}$$

whose critical points are exactly $1/m$ times the logarithm of z, where $z = z(t) \in (\mathbb{Z}[\![t]\!])^N$ is the unique solution to the equations (34). The critical values of equation (114) are given by

$$V^{\mathrm{FGI}}(t) \;=\; -\sum_{j=1}^N \mathrm{Li}_2(1-z_j(t)) - \frac{1}{2}\sum_{i,j=1}^N A_{ij}\log(z_i(t))\log(z_j(t)) \;\in\; \mathbb{Q}[\![t]\!], \qquad V^{\mathrm{FGI}}(0) \;=\; 0\,. \tag{115}$$

The above discussion leads to the following definition.

Definition 2.11. We let:

$$f_{A,m}^{\mathrm{FGI}}(t,x) \;=\; \sum_{k\in(\mathbb{Z}/m\mathbb{Z})^N} I_{A,m,k}(t^{1/m},x)\,, \tag{116}$$

where $I_{A,m,k}(t,x)$ is given by a formal Gaussian integral

$$
\begin{aligned}
I_{A,m,k}(t,x) \;=\; & \frac{(-1)^{\mathrm{diag}(A)\cdot k} q^{\frac{1}{2}(k^tAk+\mathrm{diag}(A)\cdot k)} t_1^{k_1}\cdots t_N^{k_N} \exp\Big(\frac{V(t^m)}{m^2\log(1+x/\zeta_m)}\Big)}{\sqrt{m^N\det(-\Lambda(t^m))\prod_{j=1}^N(1-z_j(t^m)^{\frac{1}{m}})}} \\
& \times \prod_{j=1}^N \Big(\prod_{\ell=1}^{m-1-k_j} \Big(\frac{1-\zeta_m^{k_j+\ell} z_j(t^m)^{\frac{1}{m}}}{1-\zeta_m^{\ell+k_j}}\Big)^{\frac{1}{2}-\frac{k_j+\ell}{m}} \prod_{\ell=m+1-k_j}^{m} \Big(\frac{1-\zeta_m^{k_j+\ell} z_j(t^m)^{\frac{1}{m}}}{1-\zeta_m^{\ell+k_j}}\Big)^{\frac{1}{2}-\frac{k_j+\ell}{m}} (1-\zeta_m^{\ell+k_j}) \Big) \\
& \times \Big\langle \exp\Big(\Big(k^tA+\frac{1}{2}\mathrm{diag}(A)\Big)\Big(w+\frac{1}{m}\log z(t^m)\Big)\Big) \prod_{j=1}^N \psi_{k_j, z_j(t^m)^{\frac{1}{m}}, \zeta_m}(w_j,x) \Big\rangle_{\Lambda(t^m)}
\end{aligned}
\tag{117}
$$

with

$$
\Lambda(t) \;=\; -A - \mathrm{diag}\Big(\frac{z(t)}{1-z(t)}\Big)\,. \tag{118}
$$

An important and non-trivial property is that $I_{A,m,k}(t,x)$ is m-periodic in k, so that equation (116) makes sense. This follows from the definition of $I_{A,m,k}(t,x)$ and a simple change of coordinates in the Gaussian integration as done in [3] or [18].

Note that the exponential prefactor of all $I_{A,m,k}(t,x)$ is independent of k. Said differently, these formal Gaussian integrals are equi-peaked Gaussians. Consider

$$
\delta^{\mathrm{FGI}}(t) \;=\; \prod_{j=1}^N z_j(t)^{-A_{jj}}(1-z_j(t))\det(-\Lambda(t))\,, \qquad \delta^{\mathrm{FGI}}(0) \;=\; 1\,. \tag{119}
$$

and

$$
U_m^{\mathrm{FGI}}(t) \;=\; \frac{1}{m^N}\prod_{j=1}^N \frac{(1-z_j)D_{\zeta_m}(1)}{(1-z_j^{\frac{1}{m}})D_{\zeta_m}(z_j^{\frac{1}{m}})} \sum_{k\in(\mathbb{Z}/m\mathbb{Z})^N} \frac{\zeta_m^{\frac{1}{2}\mathrm{diag}(A)\cdot k} t^{\frac{k}{m}} z^{\frac{Ak}{m}+\mathrm{diag}(A)(\frac{1}{2m}-\frac{1}{2})}}{(-\zeta_m)^{-\frac{1}{2}k^tAk}\prod_{j=1}^N(\zeta_m z_j^{\frac{1}{m}};\zeta_m)_{k_j}}\,, \tag{120}
$$

where $z_j = z_j(t)$ and

$$
D_{\zeta_m}(z) \;=\; \prod_{\ell=1}^{m-1}(1-\zeta_m^\ell z)^{\frac{\ell}{m}}\,. \tag{121}
$$

Note that (120) is a well-defined power series in t and $U_m^{\mathrm{FGI}}(0) \;=\; 1$. Recall the ring $S^{(m)}$ defined in (37) and let $S_{\mathbb{Q}}^{(m)} = S^{(m)}\otimes\mathbb{Q}$.

Lemma 2.12. For every positive integer m we have:

$$
\log f_{A,m}^{\mathrm{FGI}}(t,x) \;\in\; \frac{V^{\mathrm{FGI}}(t)}{m^2\log(1+x/\zeta_m)} - \frac{1}{2}\log\delta^{\mathrm{FGI}}(t) + \log U_m^{\mathrm{FGI}}(t) + xS_{\mathbb{Q}}^{(m)}[\![x]\!] \tag{122}
$$

and $\delta^{\mathrm{FGI}}(t)\in S$ and $m^{Nm}U_m^{\mathrm{FGI}}(t)^{2m}\in S^{(m)}$.

Proof. A priori, the coefficients of x^k for $k>1$ in the LHS of equation (122) are in the bigger ring $\mathbb{Z}[\zeta_m, t^{\pm 1/m}, z^{\pm 1/m}, \frac{1}{\delta}]/(1-z-(-1)^Atz^A)$. The endomorphism γ_j that sends $z_j(t)^{1/m}$ to $\zeta_m z_j(t)^{1/m}$ and fixes $S^{(m)}$ satisfies

$$
\gamma_j I_{A,m,k}(t,x) \;=\; I_{A,m,k+\delta_j}(t,x)\,. \tag{123}
$$

This follows from applying γ_j and also a change of variables to sending $w \mapsto w+h$. Therefore, the coefficients in the LHS of Equation (122) is invariant under γ_j and hence an element of $S^{(m)}$. For $U_m^{\mathrm{FGI}}(t)^{2m}$, we see that it is in $m^{-2N}S^{(m)}$. To see that a factor of m cancels we use

$$D_{\zeta_m}(1)^{24m} \;=\; m^{12m}\,, \tag{124}$$

which follows from properties of the multiplier system of the Dedekind η-function [34]. □

We will now improve on the sets where the coefficients of the x-series expansions lie. For $m, m' \in \mathbb{Z}_{>0}$ with m' prime to m and $k \in (\mathbb{Z}/m\mathbb{Z})^N$ we define

$$\Omega^{\mathrm{FGI}}_{A,m,k}(t,q) \;=\; \sum_{\substack{\ell\in(\mathbb{Z}/mm'\mathbb{Z})^N\\ \ell\equiv_m k}} \frac{(q;q)_{k_1}\cdots(q;q)_{k_N} I_{A,mm',\ell}(t^{1/m},x)}{(-1)^{\mathrm{diag}(A)\cdot k} q^{\frac{1}{2}(k^tAk+\mathrm{diag}(A)\cdot k)} t_1^{k_1/m}\cdots t_N^{k_N/m}}\,, \qquad q \;=\; \zeta_{mm'}+x\,. \tag{125}$$

Notice that when $m=1$ we have

$$\Omega^{\mathrm{FGI}}_{A,1,0}(t,\zeta_{m'}+x) \;=\; f_A^{\mathrm{FGI}}(t^{m'},\zeta_{m'}+x)\,. \tag{126}$$

The sets that the coefficients of the expansions of $\log(\Omega^{\mathrm{FGI}}_{A,m,k})$ live in is improved by Dwork-like quotients. In particular the pole and the constant term in the expansion (117) becomes better.

Lemma 2.13. For all primes p and positive integers m with $(m,p)=1$, we have

$$\log(\Omega^{\mathrm{FGI}}_{A,m,k}(t^p,q^p)) - p\log(\Omega^{\mathrm{FGI}}_{A,m,k}(t,q)) \;\in\; x^{-1}S_p^{(m)}[\tfrac{1}{p}, z^{1/m}][\![x]\!]\,, \qquad q \;=\; \zeta_m + x\,. \tag{127}$$

Proof. To prove the result we need to check two conditions. For $V(t)$ of Equation (115) and a single term in the sum $U_m(t)$ of equation (120)

$$U_{m,k}(t) \;=\; z^{\frac{Ak}{m}+\mathrm{diag}(A)(\frac{1}{2m}-\frac{1}{2})} \prod_{j=1}^{N} \frac{(1-z_j)D_{\zeta_m}(1)}{m(1-z_j^{\frac{1}{m}})D_{\zeta_m}(z_j^{\frac{1}{m}})} \frac{(\zeta_m;\zeta_m)_{k_j}}{(\zeta_m z_j^{\frac{1}{m}};\zeta_m)_{k_j}}\,, \tag{128}$$

we need to show that

$$\frac{V(t^p)}{p} - pV(t) \;\in\; pS_p^{(1)} \tag{129}$$

and

$$\log(\delta(t^p)^{-1}U_{m,k}(t^p)^2) - p\log(\delta(t)^{-1}U_{m,k}(t)^2) \in pS_p^{(m)}[z^{1/m}]\,. \tag{130}$$

Equation (115) implies that

$$\frac{V(t^p)}{p} - pV(t) \;=\; \frac{1}{p}\varphi_p\mathrm{Li}_2(tz^A) - p\mathrm{Li}_2(tz^A) - \frac{1}{2p}\varphi_p\log(z)^tA\log(z) + \frac{p}{2}\log(z)^tA\log(z)\,. \tag{131}$$

Since z is a unit, there exists $\eta=\eta(z)\in S_p^{(1)}$ such that $\varphi_p(z) \;=\; z^pe^{p\eta}$. We have

$$\begin{aligned}\varphi_p\mathrm{Li}_2(tz^A) - p^2\mathrm{Li}_2(tz^A) \;&=\; \mathrm{Li}_2(t^pz^{pA}e^{p\eta}) - p^2\mathrm{Li}_2(tz^A)\\ &\in\; \mathrm{Li}_2^{(p)}(t^pz^{pA}) + \mathrm{Li}_1(t^pz^pz^{pA})^tpA\eta + S_p^{(1)}\,,\end{aligned} \tag{132}$$

which follows from $\mathrm{Li}_n(ze^x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\mathrm{Li}_{n-k}(z)$, and $\mathrm{Li}_n(z) \in \mathbb{Z}[z,(1-z)^{-1}]$ for $n \le 0$. Moreover, we have

$$\varphi_p \log(z)^t A \log(z) - p^2 \log(z)^t A \log(z) \;=\; 2p\log(z^p)^t A\eta + p^2 \eta^t A \eta \tag{133}$$

$$A\mathrm{Li}_1(t^p z^p z^{pA}) + A\log(z^p) \;=\; \mathrm{Li}_1((1-z)^p) - p\mathrm{Li}_1(1-z)A \;=\; pA\mathrm{Li}_1^{(p)}(1-z)\,. \tag{134}$$

Combining (132), (133) and (134) with $\mathrm{Li}_n^{(p)}(1-z) \in S_p^{(1)}$, we deduce that the right hand side of equation (131) is also an element of $S_p^{(1)}$. Finally, as $\delta(t)^{-m} U_{m,k}(t)^{2m}$ is a p-unit there is a $\beta \in S_p^{(1)}$ such that $\varphi_p(\delta(t)^{-m}U_{m,k}(t)^{2m}) = \delta(t)^{-mp}U_{m,k}(t)^{2mp}e^{p\beta}$, and a similar argument can be used to deduce the remaining statements. □

Remark 2.14. If $U_m^{\mathrm{FGI}}(t)$ is a p-unit then we can lift the statement of equation (127) to

$$\log(f_A^{\mathrm{FGI}}(t^p,q^p)) - p\log(f_A^{\mathrm{FGI}}(t,q)) \;\in\; x^{-1}S_p^{(m)}[\tfrac{1}{p}][\![x]\!]\,, \quad q \;=\; \zeta_m + x\,. \tag{135}$$

The final property we will need is that $\Omega_{A,m,k}^{\mathrm{FGI}}$ satisfies a system of q-difference equations.

Lemma 2.15. For a positive integer m and residue class $k \in \{0,\cdots,m-1\}^N$, the function $t^k\Omega_{A,m,k}^{\mathrm{FGI}}(t,q)$ defined in equation (125) satisfies the q-difference equations

$$\begin{aligned}&\sum_{\ell=0}^{m}(-1)^\ell q^{-\ell(\ell-1)/2}\binom{m}{\ell}_{q^{-1}}\sigma_j^\ell\Big(t^k\Omega_{A,m,k}^{\mathrm{FGI}}(t^m,q)\Big)\\ &\;=\;(-1)^{A_{j,j}m(m+1)/2}q^{A_{j,j}m(m+1)/2}t_i^m\prod_{i=1}^N\sigma_i^{mA_{i,j}}\Big(t^k\Omega_{A,m,k}^{\mathrm{FGI}}(t^m,q)\Big)\,,\end{aligned} \tag{136}$$

where $\sigma_j t := (t_1,\dots,qt_j,\dots,t_N)$.

Proof. From the definition of $I_{A,m,k}(t,x)$ and a simple change of coordinates in the Gaussian integration as done in [3] or [18], it is easy to see for $\ell \in (\mathbb{Z}/mm'\mathbb{Z})^N$ that $I_{A,\ell}(t,x)$ satisfies

$$I_{A,mm',\ell}(t,x) - I_{A,mm',\ell}(\sigma_j t,x) \;=\; (-1)^{A_{j,j}}t_j q^{A_{j,j}} I_{A,mm',\ell-\delta_j}\Big(\prod_{i=1}^N \sigma_i^{A_{i,j}}t,x\Big), \qquad j \;=\; 1,\dots,N\,, \tag{137}$$

where δ_j is the vector whose only non-zero entry is 1 in the j-th position. Therefore, from the basic properties of Gaussian polynomials we see that

$$\begin{aligned}&\sum_{i=0}^{m}(-1)^i q^{-i(i-1)/2}\binom{m}{i}_{q^{-1}} I_{A,mm',\ell}(\sigma_j^i t,x)\\ &\;=\;(-1)^{A_{j,j}m(m+1)/2}q^{A_{j,j}m(m+1)/2}t_i^m I_{A,mm',\ell-m\delta_i}\Big(\prod_{i=1}^N\sigma_i^{mA_{i,j}}t,q\Big)\,.\end{aligned} \tag{138}$$

The rest of the proof follows from the definition of $\Omega_{A,m,k}^{\mathrm{FGI}}$. □

2.6. **Algebraicity via WKB.** In this section we outline an independent proof of Lemma 2.12 of Section 2.5 using instead of formal Gaussian integration, the unique solution to the system of linear q-difference equations (33). This is essentially the WKB method [4] for linear q-difference equations [15], which writes the first derivative of a new function as a differential polynomial of known functions. In general, integrating will produce non-algebraic functions, but with some care one can overcome and solve this vanishing residue problem. Calculations similar to those of this section were done in unpublished work of Masha Vlasenko.

Rather than explain this method for the general case, we do so for the case of 1×1 integer matrices A. Consider the power series $z = z(t)$ defined by

$$1 - z \;=\; tz^A \tag{139}$$

and

$$X \;=\; \frac{z-1}{(1-A)z+A} \;=\; \begin{pmatrix} 1 & -1 \\ 1-A & A \end{pmatrix} \cdot z\,, \tag{140}$$

so that

$$z \;=\; \frac{AX+1}{(A-1)X+1} \;=\; \begin{pmatrix} A & 1 \\ A-1 & 1 \end{pmatrix} \cdot X\,. \tag{141}$$

Then notice that for $\Delta = X(AX+1)((A-1)X+1)$ we have

$$t\frac{d}{dt} \;=\; \Delta\frac{d}{dX}\,, \qquad X \;=\; \frac{t}{z}\frac{dz}{dt}\,. \tag{142}$$

This implies that

$$\exp\Big(ht\frac{d}{dt}\Big)\log(z(t)) - \log(z(t)) - Xh \;\in\; X(AX+1)((A-1)X+1)\mathbb{Q}[X][\![h]\!]\,. \tag{143}$$

Suppose that $F(t;q) \in \mathbb{Q}(\!(h)\!)[\![t]\!]$ is the unique solution to the equation

$$F(t;h) - F(e^h t;h) \;=\; tF(e^{Ah}t;h)\,, \tag{144}$$

and their exists $c_k(t) \in \mathbb{Q}[\![t]\!]$ so that $F(t;h)$ satisfies the following ansatz

$$F(t;h) \;=\; \exp\Big(\sum_{k=-1}^{\infty} c_k(t)h^k\Big)\,. \tag{145}$$

We will prove that such c_k exist and that for $k>0$ they are algebraic functions. In order to do this we consider the quotient

$$G(t;h) \;=\; \frac{F(e^h t;h)}{F(t;h)} \;=:\; z(t)\exp\Big(\sum_{k=1}^{\infty} b_k(t)h^k\Big)\,. \tag{146}$$

The functional equations of F and z imply that G satisfies the equation

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\Big(\frac{z(t)-G(t;h)}{1-z(t)}\Big)^k \;=\; \log\Big(1+\frac{z(t)-G(t;h)}{1-z(t)}\Big) \;=\; \sum_{j=0}^{A-1}\log\Big(\frac{G(e^{jh}t;h)}{z(t)}\Big)\,. \tag{147}$$

This implies that

$$-\frac{z(t)}{1-z(t)}b_1(t) \;=\; Ab_1(t) + \frac{1}{2}A(A-1)X \tag{148}$$

and

$$\frac{1}{2}\Big(\frac{z(t)}{1-z(t)}b_1(t)\Big)^2 - \frac{z(t)}{1-z(t)}\Big(b_2(t)+\frac{b_1(t)^2}{2}\Big) \;=\; Ab_2(t) + \sum_{j=0}^{A-1}\Big(jt\frac{d}{dt}\Big)\big(X(t)+b_1(t)\big)\,. \tag{149}$$

We have the following lemma.

Lemma 2.16. The coefficients $b_k(t)$ defined by equation (146) exists and satisfy the properties:

$$b_1(t) \;=\; \frac{A(A-1)}{2}X^2\,, \quad \text{and for } k\in\mathbb{Z}_{>1} \quad b_k(t) \;\in\; X\Delta\mathbb{Q}[X]\,. \tag{150}$$

Proof. Firstly, $b_1(t)$ can be explicitly computed using (141) and (148). Next we notice that

$$-\frac{zb_1(t)}{1-z} \;=\; \frac{(A-1)A}{2}X(AX+1) \;\in\; X\mathbb{Q}[X] \tag{151}$$

and

$$-\frac{zb_1(t)^2}{1-z} - \Big(-\frac{zb_1(t)}{1-z}\Big)^2 \;=\; \frac{(A-1)^2A^2}{4}X^2(AX+1)((A-1)X+1) \;\in\; X\Delta\mathbb{Q}[X]\,. \tag{152}$$

By induction on k, we see that for $k\in\mathbb{Z}_{>0}$ we have

$$-\frac{zb_1(t)^k}{1-z} - \Big(-\frac{zb_1(t)}{1-z}\Big)^k \;\in\; X\Delta\mathbb{Q}[X]\,. \tag{153}$$

Already this and Equation (149) implies that $b_2(t)\in X\Delta\mathbb{Q}[X]$. By induction, suppose that for some $K>2$ we have that $b_k(t)\in X\Delta\mathbb{Q}[X]$ for all $k<K$. Then we see that

$$\frac{1-G(t)}{1-z(t)} \quad\in\quad \frac{1-z(t)\exp(b_1(t)h)}{1-z(t)} + \frac{-z(t)b_K(t)h^K}{1-z(t)} + h^2X\Delta\mathbb{Q}[X,h] + \mathrm{O}(h^{K+1})\,, \tag{154}$$

and therefore

$$\log\Big(\frac{1-G(t)}{1-z(t)}\Big) \;\in\; \log\Big(\frac{1-z(t)\exp(b_1(t)h)}{1-z(t)}\Big) - \frac{z(t)b_K(t)h^K}{1-z(t)} + hX\Delta\mathbb{Q}[X,h] + \mathrm{O}(h^{K+1})\,. \tag{155}$$

From equation (153), we see that

$$\frac{1-z(t)\exp(b_1(t)h)}{1-z(t)} \;\in\; \exp\Big(\frac{-z(t)b_1(t)h}{1-z(t)}\Big) + hX\Delta\mathbb{Q}[X,h]\,, \tag{156}$$

and so

$$\log\Big(\frac{1-z(t)\exp(b_1(t)h)}{1-z(t)}\Big) \;\in\; \frac{-z(t)b_1(t)h}{1-z(t)} + hX\Delta\mathbb{Q}[X,h]\,. \tag{157}$$

Together with the fact that $zX^2(1-z)^{-1} \;=\; -X-AX^2 \;\in\; X\mathbb{Z}[X]$ and the functional equation (147) of G, this gives a relation for $b_K(t)$ of the form

$$\frac{-z(t)b_K(t)}{1-z(t)} - Ab_K(t) \;=\; X^{-1}b_K(t) \;\in\; \Delta\mathbb{Q}[X]\,. \tag{158}$$

This completes the proof. □

This lemma can now be used to determine the properties of the coefficients of $F(t;h)$.

Corollary 2.17. *The coefficients $c_k(t)$ in the function $F(t;h)$ exists and for $k \in \mathbb{Z}_{>0}$*

$$c_k(t) \;\in\; X\mathbb{Q}[X]\,. \tag{159}$$

Proof. Notice that

$$\frac{1}{\exp(ht\frac{d}{dt})-1}\log(G(t;h)) \;=\; \sum_{k=0}^{\infty}\frac{B_k}{k!}(ht\frac{d}{dt})^{k-1}\log(G(t;h)) \;=\; \log(F(t;h))\,. \tag{160}$$

Therefore, we see that to compute the coefficient $c_k(t)$ we need to integrate $b_{k+1}(t)$ with the initial condition $c_k(0)=0$. We see that

$$\int b_{k+1}(t)\frac{dt}{t} \;=\; \int b_{k+1}(t)\frac{dX}{\Delta} \;\in\; X\mathbb{Q}[X]\,, \tag{161}$$

since $\Delta^{-1}b_{k+1}(t)\in X\mathbb{Q}[X]$. □

2.7. **Synthesis.** This section combines complementary results about admissible series on the one hand and formal Gaussian integration on the other. It identifies the two collections $f_A(t,q)$ and $f_A^{\mathrm{FGI}}(t,q)$ and deduces a stronger property for their ring of coefficients. This will allow us to specialise to $t=1$ later and obtain elements of modules of the Habiro ring. Firstly, we will prove Theorem 3 of Section 1.6 in the following refined form.

Theorem 8. *For every symmetric matrix A with integer entries, positive integer m and residue class $k\in\{0,\dots,m-1\}^N$, we have:*

$$F_{A,m,k}(t^{1/m},q) \;=\; \Omega^{\mathrm{FGI}}_{A,m,k}(t,q), \qquad q \;=\; \zeta_{mm'}+x\,, \tag{162}$$

for any positive integer m', where these functions are defined by equations (32) and (125).

Proof. We will show that for every positive integer m', we have

$$F_{A,m,k}(t^{1/m},\zeta_{mm'}+x) \;=\; \Omega^{\mathrm{FGI}}_{A,m,k}(t,\zeta_{mm'}+x) \;\in\; \mathbb{Q}[\zeta_{mm'}](\!(x)\!)[\![t]\!]\,. \tag{163}$$

The proof of this equality uses crucially the fact that $F_{A,m,k}(t^{1/m},q)$ is the unique solution to a system of linear q-difference equations with initial condition $F(0,q)=1$. We discuss in detail the case where A is a 1×1 matrix, in which case the linear q-difference equation is

$$\begin{aligned}&\sum_{\ell=0}^{m}(-1)^{\ell}q^{-\ell(\ell-1)/2}\binom{m}{\ell}_{q^{-1}}\sigma_j^{\ell}\Big(t^kF_{A,m,k}(t,q)\Big)\\ &=\;(-1)^{Am(m+1)/2}q^{Am(m+1)/2}t^m\sigma^{Am}\Big(t^kF_{A,m,k}(t,q)\Big)\,.\end{aligned} \tag{164}$$

The first thing to note is that (164) has a unique solution in the set $t^k\mathbb{Q}[\zeta_{mm'}](\!(x)\!)[\![t^m]\!]$. Indeed, if $t^kF(t,q)=\sum_{j\ge 0}a_j(x)t^{k+mj}$ is a solution to (164) with $q=\zeta_{mm'}+x$, then $a_j(x)$ satisfies

$$\begin{aligned}&\sum_{\ell=0}^{m}(-1)^{\ell}q^{-\ell(\ell-1)/2}\binom{m}{\ell}_{q^{-1}}q^{k\ell+mj\ell}a_j(x)-(-1)^{Am(m+1)/2}q^{Am(m+1)/2+Am(j-1)}a_{j-1}(x)\\ &=\;(q^{k+mj+1-m};q)_m a_j(x)-(-1)^{Am(m+1)/2}q^{Am(m+1)/2+Am(j-1)}a_{j-1}(x)\;=\;0\end{aligned} \tag{165}$$

for all $j\ge 0$, which together with the initial condition $a_0(x)=1$ uniquely determines $a_j(x)$.

In fact, the above calculation and the definition of $F_{A,m,k}(t,q)$ prove that $F_{A,m,k}(t,q)$ satisfies (164). Lemma 2.15 of Section 2.5 shows that $\Omega^{\mathrm{FGI}}_{A,m,k}(t^m,q)$ is also a solution to (164). Moreover, the fact that $\det(\Lambda^{-1}) = O(t)$ dominates the t^{-1} appearing from the denominators of $\mathrm{Li}_{\le 0}(z)$ shows that $\Omega^{\mathrm{FGI}}_{A,m,k}(t,\zeta_{mm'}+x) \in \mathbb{Q}[\zeta_{mm'}]((x))[[t]]$.

This concludes the proof of the first part of the theorem when A is a 1×1 symmetric matrix. The proof in the general case is identical using equation (136), and is omitted. $\square$

A consequence of Theorem 8 is that the power series defined in Equations (64), (93), (65), and (66) for the admissible series and those defined in Equations (115), (119), and (120) for the FGI series coincide:

$$\delta \;=\; \delta^{\mathrm{FGI}}, \qquad V \;=\; V^{\mathrm{FGI}}, \qquad U_m \;=\; U^{\mathrm{FGI}}_m\,. \tag{166}$$

Proof of Lemma 2.6 of Section 2.3 and Lemma 2.9 of Section 2.4. These follow from Theorem 8 and the explicit formula of $\Omega^{\mathrm{FGI}}_{A,m,k}(t,q)$ given in equation (117). $\square$

Corollary 2.18. *If $F_A(t,q)$ is the series (31) and $V = V_A(t)$ is the associated potential, then V determines A and hence F_A.*

Proof. We have

$$t_j\partial_{t_j}V(t) \;=\; \log z_j(t), \qquad j \;=\; 1,\dots,N \tag{167}$$

Hence V determines z_j for $j=1,\dots,N$, and consequently the Hessian

$$\frac{\partial^2}{\partial z_i\partial z_j}V(t) \;=\; -A-\mathrm{diag}\Big(\frac{z(t)}{1-z(t)}\Big)\,. \tag{168}$$

The result follows. $\square$

Note that the proof of Theorem 8 uses the expansion of a solution of (164) into power series in t whose coefficients are power series in x, unlike the WKB method, which uses an expansion of a solution into power series in x whose coefficients are power series in t.

Proof of Theorem 4 of Section 1.6. Theorem 8 together with Lemmas 2.10 and 2.13 imply the coefficients of x in equation (39) are in $\mathbb{Z}[t^{1/m},\zeta_m][[t]] \cap S^{(m)}_p[\frac{1}{p},z^{1/m}]$. To complete the proof we will show that

$$\mathbb{Z}[t^{1/m},\zeta_m][[t]] \;\cap\; S^{(m)}_p[\tfrac{1}{p},z^{1/m}] \;\subseteq\; S^{(m)}_p[z^{1/m}]\,. \tag{169}$$

Elements in $S^{(m)}_p[\frac{1}{p},z^{1/m}]$ can be represented by polynomials in $t^{1/m}$ of degree less than m whose coefficients are polynomials in $z^{\pm 1/m}$, $t^{\pm 1}$ and δ^{-1}. Firstly, we find that $z(t)^{\pm 1/m} \in 1+t\mathbb{Z}[\frac{1}{m}][[t]]$ as the LHS of the equation

$$(-1)^{\mathrm{diag}(A)}z^{-A}(1-z) \;=\; t\,, \tag{170}$$

is integral and therefore inverting the series and solving for $1-z$ in terms of t leads to an integral power series in t. Secondly, as δ is a polynomial in z times a monomial in $z^{\pm 1}$ and has $\delta(0)=1$ we see that $\delta^{\pm 1} \in 1+t\mathbb{Z}[[t]]$. Thirdly, in any expression with t we can replace it with z using the functional Equations (34). Therefore, for any element of $S^{(m)}[\frac{1}{pm},z^{1/m}]$ we can remove all t from the coefficients of $t^{1/m}$ and factor out all denominators given that multiplying by $z^{\pm 1/m},\delta^{-1}$ preserve integrality of the power series. Therefore, elements in

$S^{(m)}[\frac{1}{pm}, z^{1/m}]$ can be represented by a unit in $\mathbb{Z}[\frac{1}{m}, t^{1/m}, \zeta_m][\![t]\!]$ times a polynomial in $(1-z)$. Assuming that this element is in the intersection in equation (169), we see that this implies that this polynomial is in $\mathbb{Z}[t^{1/m}, \zeta_m][\![1-z]\!]$. This implies that the polynomials in $1-z$ are p-integral and therefore implies the inclusion of equation (169). □

3. Integrality and gluing of p-adic series

In this section we will combine the previous sections to prove our main results. Firstly, we study the K-theory of local fields and its image under the p-adic regulator map. Then we provide the technical results needed in the proof of the main theorems. After that we prove our main results about the p-completed Habiro rings (Theorem 1 of the introduction and Theorem 10 of this section) by constructing explicit generators using the infinite Pochhammer symbol evaluated at roots of unity. We close by proving the remaining theorems (Theorems 2 and 5) of Section 1.6.

3.1. **Bloch group and p-adic dilogarithm.** We begin by recalling some basic facts about the dilogarithm function, the Bloch group, and their p-adic counterparts. The dilogarithm function, defined initially for $|x| < 1$ by

$$\mathrm{Li}_2(x) \;=\; \sum_{k=1}^{\infty} \frac{x^k}{k^2}\,, \tag{171}$$

satisfies the famous five-term functional equation. The five-term equation reflects the algebraic structure of the third algebraic K-group of a field, which is isomorphic, modulo well-understood torsion, with the Bloch group of the number field [8, 40]. Therefore, functions that satisfy the five-term relation give homomorphisms from the Bloch group, and hence from the third algebraic K-group. The Bloch-Wigner dilogarithm $D : \mathbb{C} \to \mathbb{R}$, given by $D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$, is one such function and satisfies the functional equation

$$D(x) + D(y) \;=\; D(xy) + D\Big(\frac{1-y}{1-x^{-1}}\Big) + D\Big(\frac{1-x}{1-y^{-1}}\Big)\,. \tag{172}$$

The usual dilogarithm is a holomorphic but multivalued function from $\mathbb{C} - \{0,1\}$. The Bloch-Wigner dilogarithm gives a well-defined continuous real valued function on $\mathbb{C}$. Taking a number field $\mathbb{K}$ and an infinite place σ (i.e. $\sigma : \mathbb{K} \to \mathbb{C}$ an embedding of the number field into $\mathbb{C}$) we can define a map $D_\sigma : B(\mathbb{K}) \to \mathbb{R}$ defined on the symbols in $B(\mathbb{K})$ by linearly extending $D_\sigma([z]) = D(\sigma(z))$. An analogous function valued in $\mathbb{C}_p$ was defined by Coleman [10] for finite places. The main point is to find an analytic extension of $\mathrm{Li}_2(z)$ in equation (171), which only converges for $z \in \mathbb{C}_p$ with $|z| < 1$. Coleman used an "analytic continuation along Frobenius" to define such a function after choosing a logarithm $\log : \mathbb{C}_p^\times \to \mathbb{C}_p$. For example, one could take the Iwasawa logarithm, which has $\log(p) = 0$. (The values of the dilogarithm only depend on this choice in the residue discs around 1 and ∞ and importantly the primes we consider will never require us to evaluate here.) Then Coleman obtains a well-defined function $\mathrm{Li}_2 : \mathbb{C}_p \smallsetminus \{1\} \to \mathbb{C}_p$ and defines

$$D_p : \mathbb{C}_p \smallsetminus \{0,1\} \to \mathbb{C}_p, \qquad z \mapsto \mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z)\,. \tag{173}$$

This function also satisfies equation (172). Therefore, this gives homomorphisms from the Bloch group in a completely analogous way to the Bloch-Wigner dilogarithm but for finite places (i.e. $\sigma : \mathbb{K} \to \mathbb{C}_p$ an embedding of the field). Conveniently, for unramified primes p of the field $\mathbb{K}$, we can combine these functions D_σ into one function $D_p : B(\mathbb{K}) \to \mathbb{K}_p$ such that the projection onto the various field factors of $\mathbb{K}_p$ corresponding to σ agree with the function D_σ. Let $R = \mathcal{O}_{\mathbb{K}}[\frac{1}{\Delta}]$ as in (11) and $R_p^\wedge$ denote its p-adic completion. The p-adic polylogarithm functions satisfy the following integrality properties:

Lemma 3.1. If $z \in R_p^\wedge$ and $|z|_p = 1$ and $|z-1|_p \geq 1$ then $D_p(z) \in p^2 R_p^\wedge$.

Proof. From [7, Cor. 4.9] we know that

$$\mathrm{Li}_2(\zeta) \;\in\; p^2\mathbb{Z}_p[\zeta] \quad \text{and} \quad \mathrm{Li}_1(\zeta) \;\in\; p\mathbb{Z}_p[\zeta]\,. \tag{174}$$

This implies that $D_p(\zeta) \in p^2 R_p^\wedge$ if ζ is a root of unity. Then expanding these as Taylor series near ζ and noting that we assumed $p > 3$ is unramified proves the result on the unit discs near ζ. ☐

In what follows, we are mostly interested in evaluating $\mathrm{Li}_2(\zeta)$ when ζ is a root of unity of order prime to p. This case is important because these are rank 1 elements, i.e., they satisfy the equation

$$\varphi_p \zeta \;=\; \zeta^p\,. \tag{175}$$

Although the p-adic polylogarithm Li_n is a transcendental function for $n > 0$, its modulo p reduction can be described by a rational function, the finite polylogarithm function introduced by Kontsevich (unpublished note, reproduced as an appendix to [14])

$$\mathrm{li}_{n,p}(z) \;=\; \sum_{k=1}^{p-1} \frac{z^k}{k^n}\,. \tag{176}$$

Below, we denote by $\mathbb{Q}_{p^s}$ the unique unramified extension of $\mathbb{Q}_p$ of degree s (with residue field $\mathbb{F}_{p^s}$), and its ring of integers by $\mathbb{Z}_{p^s}$.

Proposition 3.2. [5, Cor. 2.2] *For a prime p and $\zeta \in \mu(\mathbb{Q}_{p^s})$ with $\zeta \neq 1$, we have*

$$p^{-2} D_p(\zeta^p) \;=\; \frac{1}{(\zeta-1)^p}\mathrm{li}_{2,p}(\zeta) \pmod{p}\,. \tag{177}$$

We can use this proposition to prove the following.

Proposition 3.3. *For every positive integer s, we have*

$$\mathrm{Span}_{\mathbb{Z}_p}\{p^{-2}D_p(\zeta) \,|\, \zeta \;\in\; \mu(\mathbb{Q}_{p^s})\} \;=\; \mathbb{Z}_{p^s}\,. \tag{178}$$

Proof. It suffices to prove (178) modulo p. Consider the rational function

$$f_p(x) \;=\; \frac{\mathrm{li}_{2,p}(x)}{(x-1)^p}\,. \tag{179}$$

Since $\mathrm{li}_{2,p}(1) \;\equiv\; 0 \pmod{p}$ and $\mathrm{li}'_{2,p}(1) \;\equiv\; 0 \pmod{p}$, there exists a polynomial $g_p(x)$ of degree $p-3$ such that $f_p(x) \;=\; \frac{g_p(x)}{(x-1)^{p-2}}$. We can view f_p as a map

$$f_p : \mathbb{F}_{p^s} \smallsetminus \{1\} \;\to\; \mathbb{F}_{p^s}\,. \tag{180}$$

Then we see that $f_p^{-1}(c) \subseteq \{x \in \mathbb{F}_{p^s} \smallsetminus \{1\} \,|\, c(x-1)^{p-2} - g_p(x)\}$ and therefore $\# f_p^{-1}(c) \leq p-2$. This gives a lower bound for the size of the image of f_p

$$\#\mathrm{image}(f_p) \;\geq\; \frac{p^s-1}{p-2} \;>\; p^{s-1}\,, \tag{181}$$

which implies that $p^{s-1} < p^r = \#\mathrm{Span}_{\mathbb{F}_p}(\mathrm{image}(f_p)) \leq p^s$. Therefore, we see that $\mathrm{Span}_{\mathbb{F}_p}(\mathrm{image}(f_p)) = \mathbb{F}_{p^s}$. Since $\mu(\mathbb{Q}_{p^s}) \simeq \mathbb{F}_{p^s}^\times$, Proposition 3.2 implies that $p^{-2}D_p(\zeta)$ span $\mathbb{F}_{p^s}$ as an $\mathbb{F}_p$-vector space. The result follows. ☐

Note that if $\mathbb{K}/\mathbb{Q}$ is a number field and p is unramified, so that $p = \prod_i \mathfrak{p}_i$, $N(\mathfrak{p}_i) = p^{s_i}$, then $\mathbb{K}_p \simeq \prod_i \mathbb{Q}_{p^{s_i}}$ and $K_n(\mathbb{K}_p) \simeq \prod_i K_n(\mathbb{Q}_{p^{s_i}})$ for every n.

Theorem 9. *Assume $p > 3$. We have a $\mathbb{Z}_p$-linear isomorphism*

$$D_p : K_3(\mathbb{K}_p; \mathbb{Z}_p) \to p^2 \mathcal{O}_{\mathbb{K}_p}\,, \tag{182}$$

and $K_3(\mathbb{K}_p;\mathbb{Z}_p)$ is generated as a $\mathbb{Z}_p$-module by $\{[\zeta] \,|\, \zeta \in \mu(\mathbb{K}_p)\}$.

For related work on p-adic regulators/dilogarithms, see Huber–Kings [24] and also Besser–de Jeu [6, Thm.1.6(2)].

Proof. It is known that $K_3(\mathbb{K}_p;\mathbb{Z}_p) \cong H^1(\mathbb{K}_p, \mathbb{Z}_p(2))$ is a free $\mathbb{Z}_p$-module of rank r, cf. [38, Thm. 7.4] (the torsion-freeness follows from the vanishing of $H^0(\mathbb{K}_p, \mathbb{F}_p(2))$, which follows from $p > 3$). Under the isomorphism between K_3 and the Bloch group (and noting that there is no contribution from $K_2 = K_2^M$ to $K_3(-;\mathbb{Z}_p)$) and using Lemma 3.1, one sees that the image is contained in $p^2\mathcal{O}_{\mathbb{K}_p}$. But the elements $\mathrm{ord}(\zeta) D_p(\zeta) \in \mathbb{K}_p$ generate the image. Therefore, this gives a surjective map between two free $\mathbb{Z}_p$-modules of the same rank and therefore the map has a trivial kernel. ☐

An example that illustrates this theorem is given in Example 4.3 in Section 4.4.

3.2. **Explicit sections of the p-completed Habiro ring.** In this section we consider the modules over the p-completed Habiro ring of Definition 1.3 of Section 1.4. We will prove Theorem 1 of Section 1.4 by explicitly constructing sections using Theorem 9 of Section 3.1. Throughout this section one can keep in mind $S = R_p^\wedge$, where $R = \mathcal{O}_\mathbb{K}$ and Δ are as in equation (11) and $S[\frac{1}{p}] = \mathbb{K}_p$. The next lemma is a variation of a well–known lemma of Dwork [29, IV.Lem.3] and will be the basic tool we use to prove integrality properties.

Lemma 3.4 (Dwork's lemma)**.** Fix a prime p and a positive integer m with $(m,p) = 1$ and $f(x) \in 1 + xS[\frac{1}{p}, \zeta_m][\![x]\!]$. Then $f(x) \in 1 + xS[\zeta_m][\![x]\!]$ if and only if

$$\frac{\varphi_p(f)((\zeta_m + x)^p - \zeta_m^p)}{f(x)^p} \;\in\; 1 + pxS[\zeta_m][\![x]\!]\,, \tag{183}$$

and $f(x) \in 1 + xS[\zeta_m] + \mathrm{O}(x^2)$, where φ_p is the Frobenius of S with $\varphi_p(\zeta_m) = \zeta_m^p$ and $\varphi_p(x) = x$.

Proof. Let

$$f(x) \;=\; 1 + \sum_{k=1}^\infty a_k x^k \quad \text{and} \quad \frac{\varphi_p(f)((\zeta_m + x)^p - \zeta_m^p)}{f(x)^p} \;=\; 1 + \sum_{k=1}^\infty b_k x^k\,. \tag{184}$$

Notice that if $a_k \in S[\zeta_m]$ then

$$\frac{\varphi_p(f)((\zeta_m + x)^p - \zeta_m^p)}{f(x)^p} \;=\; 1 \pmod{p}\,. \tag{185}$$

This completes half of the proof. Therefore, assume now that $b_k \in pS[\zeta_m]$. Let $\delta_{p,m}(x) = \frac{(\zeta_m+x)^p - x^p - \zeta_m^p}{px} \in \zeta_m^{p-1} + x\mathbb{Z}[\zeta_m, x]$. Then trivially

$$1 + \sum_{k=1}^{\infty} \varphi_p(a_k)(x^p + xp\delta_{p,m}(x))^k \;=\; \Big(1 + \sum_{k=1}^{\infty} a_k x^k\Big)^p \Big(1 + \sum_{k=1}^{\infty} b_k x^k\Big)\,. \tag{186}$$

Assume for induction that $a_i \in S[\zeta_m]$ for $i < n$. Then computing the n-th coefficient on both sides of the equation, we find that

$$\varphi_p(a_{n/p}) + \zeta_m^{p-1} p^n \varphi_p(a_n) + pS[\zeta_m] \;=\; \varphi_p(a_{n/p}) + pa_n + pS[\zeta_m]\,, \tag{187}$$

and so

$$a_n - \zeta_m^{p-1} p^{n-1} \varphi_p(a_n) \;\in\; S[\zeta_m]\,. \tag{188}$$

Therefore, if s is the order of the Frobenius endomorphism φ_p on $S[\zeta_m]$, then we get

$$(1 - \zeta_m^{s(p-1)} p^{s(n-1)}) a_n \;=\; \sum_{k=0}^{s-1} \zeta_m^{k(p-1)} p^{k(n-1)} (\varphi_p^k(a_n) - \zeta_m^{p-1} p^{n-1} \varphi_p^{k+1}(a_n)) \;\in\; S[\zeta_m]\,. \tag{189}$$

So assuming that $a_1 \in S[\zeta_m]$ we find that $a_n \in S[\zeta_m]$ by induction. □

When $D_p(\xi) = 0$ we can use the assumption of equation (21) in Definition 1.3 of Section 1.4 to prove integrality by applying Dwork's lemma. This will be important in the proof of the Theorem 1 of Section 1.4 that $\mathcal{H}_{R_p^\wedge,\xi}$ are free rank one bundles over $\mathcal{H}_{R_p^\wedge}$.

Corollary 3.5. *Fix a prime p and a p-complete torsion-free ring S. Suppose that*

$$g(q) \;=\; (g_m(x))_{m\ge 1,(m,p)=1}\,, \quad g_m(x) \;\in\; S[\tfrac{1}{p}, \zeta_m][\![x]\!] \tag{190}$$

is a collection of power series such that

$$\varphi_p g(q^p) - pg(q) \;\in\; \prod_{(m,p)=1} pS[\zeta_m][\![q - \zeta_m]\!]\,. \tag{191}$$

Then $f(q) := \exp(g(q))$ is well-defined and assuming $f(\zeta_m + x) \in (1 + pS[\zeta_m]) + xS[\zeta_m] + \mathrm{O}(x^2)$, it satisfies

$$f(q) \;\in\; \prod_{m\ge 1\,,(m,p)=1} S[\zeta_m][\![q - \zeta_m]\!]\,. \tag{192}$$

Proof. Notice that equation (191) implies that $g_m(0) \in pS[\zeta_m]$ and so the series $f_m(x)$ is well-defined. Moreover, equation (191) implies that

$$\exp(\varphi_p g_m((\zeta_m + x)^p - \zeta_m^p) - pg_m(x)) \;\in\; (1 + pS[\zeta_m]) + pxS[\zeta_m][\![x]\!]\,. \tag{193}$$

Therefore, the fact that $f_m(0) \in 1 + pS[\zeta_m]$ together with Dwork's lemma (Lemma 3.4) shows us that $f_m(0)^{-1} f_m(x) \in S[\zeta_m][\![x]\!]$ and hence $f_m(x) \in S[\zeta_m][\![x]\!]$. □

In view of Definition 1.3 of Section 1.4, we can focus on pairs consisting of a prime p and a positive integer m prime to p to define invertible sections. More generally, we can lift this to all roots of unity using the previous corollary and the following lemma.

Lemma 3.6. Fix a prime p, $\xi \in K_3(\mathbb{K})$, and a collection

$$f(q) \;=\; (f_m(x))_{m\geq 1}, \qquad f_m(x) \;\in\; \varepsilon_m(\xi)^{1/m}\,(R_p^\wedge[\zeta_m]^\times + xR_p^\wedge[\zeta_m] + x^2\mathbb{K}_p[\zeta_m][\![x]\!]) \tag{194}$$

that satisfies

$$\frac{\varphi_p\widehat{f}(q^p)}{\widehat{f}(q)^p} \in \prod_{(m,p)=1} \exp\left(\frac{p}{x}R_p^\wedge[\zeta_m][\![x]\!]\right), \tag{195}$$

with $\widehat{f}$ as in equation (22). Then for all positive integers m with $(m,p)=1$ there is a unique $\beta_m \in \mathbb{Z}_p$ such that the collection $\widetilde{f}_{mp^d}(x) = \zeta_{p^d}^{\beta_m} f_{mp^d}(x)$ satisfies

$$\frac{\widetilde{f}(q^\sigma)}{\widetilde{f}(q)^{\sigma^{-1}}} \in \prod_{(m,p)=1} R_p^\wedge[\zeta_m][\![x]\!], \qquad \text{for all } \sigma \in \mathbb{Z}_p^\times\,. \tag{196}$$

Proof. Firstly, for all $\sigma \in \mathbb{Z}_p^\times$, we have

$$\frac{\varphi_p f(q^{p\sigma})}{\varphi_p f(q^p)^{\sigma^{-1}}}\frac{f(q)^{p\sigma^{-1}}}{f(q^\sigma)^p} \;=\; \frac{\varphi_p \widehat{f}(q^{p\sigma})}{\varphi_p \widehat{f}(q^p)^{\sigma^{-1}}}\frac{\widehat{f}(q)^{p\sigma^{-1}}}{\widehat{f}(q^\sigma)^p} \in \prod_{(m,p)=1} \exp\left(pR_p^\wedge[\zeta_m][\![x]\!]\right). \tag{197}$$

This implies by Lemma 3.4, by induction on $d \in \mathbb{Z}_{\geq 0}$, that for $(m,p)=1$ there is a series $h_m(x) \in R_p^\wedge[\zeta_m][\![x]\!]$ and an $\alpha_\sigma \in \mathbb{Z}_p$ such that

$$f(q^\sigma)f(q)^{-\sigma^{-1}} \;=\; \zeta_{p^d}^{\alpha_\sigma} h_m(\zeta_m(\zeta_{p^d}-1)+x)\,, \qquad q \;=\; \zeta_m\zeta_{p^d}+x\,. \tag{198}$$

Notice that

$$\frac{f(q^{\sigma\sigma'})}{f(q)^{(\sigma\sigma')^{-1}}} \;=\; \frac{f(q^{\sigma\sigma'})}{f(q^{\sigma'})^{\sigma^{-1}}}\Big(\frac{f(q^{\sigma'})}{f(q)^{\sigma'^{-1}}}\Big)^{\sigma^{-1}} \;=\; \frac{f(q^{\sigma\sigma'})}{f(q^{\sigma})^{\sigma'^{-1}}}\Big(\frac{f(q^{\sigma})}{f(q)^{\sigma^{-1}}}\Big)^{\sigma'^{-1}} \tag{199}$$

and so

$$\alpha_{\sigma\sigma'} \;=\; \sigma'\alpha_\sigma + \sigma^{-1}\alpha_{\sigma'} \;=\; \sigma\alpha_{\sigma'} + \sigma'^{-1}\alpha_\sigma\,. \tag{200}$$

There is a unique such α_σ satisfying $\alpha_{\lim_{s\to\infty}\sigma^{p^s}} = \lim_{s\to\infty}\alpha_{\sigma^{p^s}}$, given by

$$\alpha_\sigma \;=\; (\sigma-\sigma^{-1})\frac{\alpha_\omega}{\omega-\omega^{-1}}\,, \tag{201}$$

where ω is a generator of the set of roots of unity in $\mathbb{Z}_p^\times$ for $p \neq 2,3$. Indeed, notice that for $p \neq 2,3$ we have $\omega-\omega^{-1} \in \mathbb{Z}_p^\times$. Letting $\beta_m = -\frac{\alpha_\omega}{\omega-\omega^{-1}}$ therefore completes the proof. □

Corollary 3.7. *If f is an invertible $L_p(\xi)$-section of Definition 1.3 of Section 1.4, then there exists a unique lift to collections*

$$f(q) \;=\; (f_m(x))_{m\geq 1}, \qquad f_m(x) \;\in\; (R_p^\wedge[\zeta_m]^{\times/m} + x\mathbb{K}_p[\zeta_m][\![x]\!]) \tag{202}$$

of power series satisfying the conditions of integrality and gluing, i.e.,

$$\frac{\varphi_p\widehat{f}(q^p)}{\widehat{f}(q)^p} \in \prod_{(m,p)=1} \exp\left(\frac{p}{x}R_p^\wedge[\zeta_m][\![x]\!]\right) \qquad \textit{and} \qquad \frac{f(q^\sigma)}{f(q)^{\sigma^{-1}}} \in \prod_{(m,p)=1} R_p^\wedge[\zeta_m][\![x]\!]\,, \tag{203}$$

where $\sigma \in \mathbb{Z}_p^\times$ *and* $\widehat{f}$ *is as in equation* (22).

Remark 3.8. Taking the logarithm of this extension of an invertible section $\widehat{f}(q)$ to all roots of unity (with some mild assumptions) can be shown to give a meromorphic function $\log(\widehat{f}(q))$ with simple poles at roots of unity with residues given by $m^{-2}D_p(\xi)$ and its images under powers of $p^{-2}\varphi_p$.

Now that we have understood the basic analytic properties of $L_p(\xi)$-sections, we can give our explicit construction. Locally our modules are indexed by an element

$$\widehat{\xi} \;\in\; \mathrm{Im}(K_3(\mathbb{K}) \to K_3(\mathbb{K}_p;\mathbb{Z}_p)\otimes\mathbb{Z}_p) \tag{204}$$

and our explicit invertible $L_p(\xi)$-sections will depend on a presentation of $\widehat{\xi}$ of the form

$$\widehat{\xi} \;=\; \sum_{\zeta\in\mu(\mathbb{K}_p)} a_\zeta[\zeta] \;\in\; K_3(\mathbb{K}_p)\otimes\mathbb{Z}_p \tag{205}$$

using Theorem 9 of Section 3.1. Using this data alone, we define a collection of power series over local fields that, unlike its global field counterpart, does not require formal Gaussian integration or admissible series, but just the infinite Pochhammer symbol. What's more, the constant term is simply the unit ε, and is obviously non-vanishing.

Definition 3.9. Fix a prime p and $\zeta \in \mu(\mathbb{K}_p)$. Then we define the collection $\Psi_{[\zeta],p}(q) = (\Psi_{[\zeta],p,m}(x))_{m\geq 1\,(m,p)=1}$ by

$$\begin{aligned}\Psi_{[\zeta],p,m}(x) \;&=\; \exp\Big(-\frac{\mathrm{Li}_2(\zeta)}{m^2\log(q)}\Big)\,\varepsilon_m([\zeta])^{1/m}(q^{m/2}\zeta;q^m)_\infty^{1/m}\,, \qquad q \;=\; \zeta_m + x\\ &\in \varepsilon_m([\zeta])^{1/m}(1+x\mathbb{K}_p[\zeta_m][\![x]\!])\,.\end{aligned} \tag{206}$$

Moreover, for every $\widehat{\xi}$ given in equation (205) we define

$$\Psi_{\widehat{\xi},p} \;=\; \prod_\zeta (\Psi_{[\zeta],p})^{a_\zeta}\,. \tag{207}$$

A slight variation of Proposition 2.2 of Section 2.1 and equation (55), to include the factor of $q^{1/2}$, implies the following theorem.

Theorem 10. *For all* $\widehat{\xi}$ *as in equation* (204) *and unramified* $p > 3$, $\Psi_{\widehat{\xi}}$ *is an* $L_p(\widehat{\xi})$*-section.*

We can use this to prove Theorem 1 of Section 1.4.

Proof of Theorem 1 of Section 1.4. Combining Theorem 10 with Corollary 3.5 immediately implies Theorem 1 of Section 1.4.

Alternatively, we can use replace the use of Theorem 10 by a general construction from telescoping sums. For any

$$\begin{aligned} h(q) \;&\in\; \prod_{m\geq 1\,,(m,p)=1} \frac{1}{x}R_p^\wedge[\zeta_m][\![x]\!]\,, \qquad \text{such that}\\ h(\zeta_m+x) \;&=\; \frac{\varphi_p D_p(\xi) - p^2 D_p(\xi)}{m^2p^2\log(q)} + \frac{1}{mp}\log\Big(\frac{(\sigma_p\varepsilon(\xi))^p}{\varepsilon(\xi)}\Big) + \mathrm{O}(x^2)\,,\end{aligned} \tag{208}$$

we can define an $L_p(\xi)$-section by

$$\log(f(q)) \;=\; \frac{1}{m}\log(\varepsilon(\xi)) + \sum_{k=0}^{\infty}\frac{\varphi_p^k}{p^k}\Big(\frac{\varphi_p D_p(\xi) - p^2 D_p(\xi)}{m^2 p^{2+k}\log(q)} + \frac{1}{mp}\log\Big(\frac{\varphi_p\varepsilon(\xi)}{\varepsilon(\xi)}\Big) - h(q^{p^k})\Big)\,, \tag{209}$$

which is a convergent sum. □

This then has the following corollary, which can be interpreted as saying that the Ψ of definition 3.9 is a local lift of the unit computed in [9].

Corollary 3.10. *The collection $\Psi_{\widehat{\xi},p} \in \mathcal{H}^\times_{R_p^\wedge,\widehat{\xi}}/\mathcal{H}^\times_{R_p^\wedge}$ only depends on $\widehat{\xi}$ from Equation (204) and not on the presentation of equation (205).*

Proof. If we have two sets of a_ζ that represent $\widehat{\xi}$ as in equation (205), then the quotient is in $\mathcal{H}^\times_{R_p^\wedge}$. □

3.3. **Proof of the main theorems.** We are now ready to give proofs of our main theorems.

Proof of Theorem 2 of Section 1.4. Firstly, we see that $\mathcal{H}_R \subseteq \mathcal{H}_{R,0}$ as $f(q) = 1$ is a global invertible $L(0)$-section. For all p prime to Δ, the constant collection 1 is an invertible $L_p(0)$-section. Therefore, from Corollary 3.7 of Section 3.2 and the gluing condition (24) of Definition 1.4, if $f \in \mathcal{H}_{R,0}$, then $f \in \mathcal{H}_{R_p^\wedge}$. Varying over all primes shows that $f \in \mathcal{H}_R$. This completes the proof that $\mathcal{H}_{R,0} = \mathcal{H}_R$. The second statement follows from the fact that ε_m is multiplicative while D_p is additive and that the conditions equation (24) and equation (21) are multiplicative as a consequence. The final statement follows from the combination of the previous two. □

Proof of Theorem 5 of Section 1.6. Firstly, we show that the constant term of $f_{A,z,m}(x)$ is $\varepsilon_m(\xi)^{1/m}$ times an element of $\mathbb{K}[\zeta_m]$. For m prime to Δ (where Δ includes the primes 2 and 3 and finitely many other primes that depend only on the number field $\mathbb{K}$), this follows from Theorem 1.6 and Equation (14) of [9, Thm.1.6] combined with Hutchinson [25].

Secondly, we will prove that for a system of q-difference equations associated to the combination in equation (24) there is a unique solution in formal power series in t. We will explicitly describe the case when $N = 1$. (We again omit the case when $N > 1$, since it completely analogous but notationally heavier.) To do this, we will use standard methods in the study of q-holonomic modules. Recall the q-difference equation (33) and define

$$\psi_{A,\mu}(t,q) \;=\; F_A(q^\mu t, q)\,. \tag{210}$$

Fix $\gamma \in \mathbb{Z}_{>0}$ and consider the function

$$(\mu,\nu,t) \mapsto \psi^{(\gamma)}_{A,\mu,\nu}(t,q) \;=\; \psi_{A,\mu_1}(t,q^\gamma)\cdots\psi_{A,\mu_\gamma}(t,q^\gamma)\psi_{A,\nu}(t,q^{-1})\,, \tag{211}$$

where $\mu = (\mu_1,\dots,\mu_\gamma) \in \mathbb{Z}^\gamma$ and $\nu \in \mathbb{Z}$. It satisfies $\gamma+2$ equations, corresponding to the number of variables:

$$\begin{aligned} \psi^{(\gamma)}_{A,\mu,\nu}(t,q) - \psi^{(\gamma)}_{A,\mu+\delta_i,\nu}(t,q) \;&=\; (-1)^A q^{A\gamma} t\,\psi^{(\gamma)}_{A,\mu+A\delta_i,\nu}(t,q) \qquad (i = 1,\cdots,\gamma)\,,\\ \psi^{(\gamma)}_{A,\mu,\nu}(t,q) - \psi^{(\gamma)}_{A,\mu,\nu+1}(t,q) \;&=\; (-1)^A q^{-A} t\,\psi^{(\gamma)}_{A,\mu,\nu+A}(t,q)\,,\\ \psi^{(\gamma)}_{A,\mu}(qt,q) \;&=\; \psi^{(\gamma)}_{A,\mu+1,\nu-1}(t,q)\,. \end{aligned} \tag{212}$$

These equations have a unique solution for power series in t of the form $1 + \mathrm{O}(t)$. This follows from the fact that the first two equations imply that the coefficient of t^k in $\psi^{(\gamma)}$ is constant in μ, ν modulo coefficients of smaller powers of t. Assuming this, the last equation implies that this coefficient multiplied by $q^k - 1$ is given entirely by combinations of lower order terms. Therefore, everything is determined by the coefficient of t^0.

From the explicit formulas of equation (116) and equation (117), Theorem 8 of Section 2.7, Theorem 4 of Section 1.6, and Galois invariance under $z^{1/m} \mapsto \zeta_m z^{1/m}$; we see that $\psi^{(\gamma)}_{A,\mu,\nu}(t^{1/m}, \zeta_m + x) \in S^{(m)}_p[\![x]\!]$, for all primes p prime to Δ and m prime to γ. Therefore, $\psi^{(\gamma)}_{A,\mu,\nu}(t^{1/m}, \zeta_m + x) \in S^{(m)}[\![x]\!]$. Moreover, since there is a unique solution to the Equations (212), if we p-complete and re-expand by $x \mapsto x + \zeta_{pm} - \zeta_m$ then we must find an equality after applying a Frobenius, which maps $t \mapsto t^p$. Therefore, these equalities persist when we specialise $t = 1$. This proves that $f_{A,z}$ satisfies equation (24).

For the rest of the proof, we fix a positive integer m, a prime p with $(m, p) = 1$ and a congruence class $k \in (\mathbb{Z}/m\mathbb{Z})^N$. Theorems 4 of Section 1.6 and 8 of Section 2.7 imply that each term $\Omega^{\mathrm{FGI}}_{A,m,k}(t, x)$ of equation (125), specialised to $t = 1$, satisfies equation (39), which is an essentially equation (21) restricted to a disc. Summing up using (116) and (125), we obtain that

$$f^{\mathrm{FGI}}_{A,m}(1, x) \;=\; \sum_{k\in(\mathbb{Z}/m\mathbb{Z})^N} \frac{(-1)^{\mathrm{diag}(A)\cdot k} q^{\frac{1}{2}(k^t A k + \mathrm{diag}(A)\cdot k)}}{(q;q)_{k_1}\cdots(q;q)_{k_N}} \Omega^{\mathrm{FGI}}_{A,m,k}(1, q)\,, \quad q \;=\; \zeta_m + x\,. \tag{213}$$

It follows that $f^{\mathrm{FGI}}_A(1, q)$ gives an element of the module $\mathcal{H}_{R[\delta^{-1/2}]^\wedge_p[\zeta_m, z^{1/m}],\xi}$ for q-restricted to the disc around ζ_m. Finally, from the invariance under $z^{1/m} \mapsto \zeta_m z^{1/m}$ from Lemma 2.12 of Section 2.5, we see that $f_{A,z}(q)$ descends to an element of $\mathcal{H}_{R[\delta^{-1/2}]^\wedge_p,\xi}$ restricted to the disc around ζ_m.

Varying over all m prime to Δ and combining this with the previous result that equation (24) is satisfied, we conclude that $f_{A,z}(q) \in \mathcal{H}_{R[\delta^{-1/2}],\xi}|_\Delta$. □

Proof of Corollary 1.11 of Section 1.6. This would immediately follow from Theorem 5 of Section 1.6, Theorem 2 of Section 1.4, and Proposition 1.5 of Section 1.4; if we did not have the technical assumption on the order of roots of unity being prime to Δ.

However, equation (44) does immediately follow from the proof of Theorem 5 of Section 1.6 that showed $f_{A,z}$ satisfies equation (24), which in particular holds when $\gamma = 1$.

For part (b), a weaker statement asserting that $f_{A,z}(q)^r \in \mathcal{H}_{R[1/\sqrt{\delta}]}|_\Delta$ follows from Theorem 5 of Section 1.6 and Theorem 2 of Section 1.4. To replace $\mathcal{H}_{R[1/\sqrt{\delta}]}|_\Delta$ by $\mathcal{H}_{R[1/\sqrt{\delta}]}$, note the surjective map $K_3(\mathbb{K}) \to B(\mathbb{K})$ of Suslin (see e.g., [42, Eqn.(1.1)]). Hence, if $r\xi = 0$, then the image of ξ in the Bloch group can be written as a sum of 5-term relations in the Bloch group. Then the identity of [28] that was used in [9] shows that the constant term at ζ_m is in $S^{(m)}$. □

Remark 3.11. In fact, the specialisation $t = q^\nu$ for $\nu \in \mathbb{Z}^N$ (i.e., $t_j = q^{\nu_j}$ for integers ν_j) defines an element $f_{A,z,\nu}(q) \in \mathcal{H}_{R[1/\sqrt{\delta}],\xi}|_\Delta$. The proof is identical to the proof of Theorem 5 of Section 1.6 and is omitted.

Proof of Proposition 1.5 of Section 1.4. The proofs of the proposition are all elementary and outlined as follows:

- Property (a) and (d) follow from the χ^{-1} equivariance of ε.
- Property (b) follows from Theorem 2 of Section 1.4.
- Property (c) is trivial and follows from the definition of γ^* just prior to the proposition.
- Property (e) this property follows from the fact that, varying over γ, γ', the collections $f(q^{\gamma_1}) \cdots f(q^{\gamma_n}) f(q^{-\gamma'_1}) \cdots f(q^{-\gamma'_{n'}})$, where $\sum_k \gamma_k^{-1} - \sum_k \gamma_k'^{-1} = 0$ determine $f(q)$. Alternatively, one could use the remark of Section 3.2 to connect various roots of unity.
- Property (f) follows from the fact that the constants $f_m(0) \in R_p^\wedge[\zeta_m, \varepsilon_m^{1/m}]$ for all m and p with $(mp, \Delta) = 1$.

□

An immediate corollary of part (f) is the following result, which even in very simple instances like the 4_1-knot seems very difficult to prove directly.

Corollary 3.12. *For m prime to Δ, the constants $U_m^{\mathrm{FGI}}(1)$ from Equation (120) are Δ-integral.*

4. Examples and computations

4.1. **Symmetrisation and a residue formula.** Elements of the usual Habiro ring $\mathcal{H}_{\mathbb{Z}}$ are extremely easy to write down. For example, the ring $\mathbb{Z}[q]$ is a subring of $\mathcal{H}_{\mathbb{Z}}$. This is not the case for the Habiro ring of a number field, where $\mathcal{O}[q]$ is no longer a subring of $\mathcal{H}_{\mathcal{O}[1/\Delta]}$. One can construct elements of these rings for certain presentations of a number field. In this section we explain how to get formulas for such elements using combinatorial data.

Fix a symmetric integer matrix A, and consider the following expression

$$J_A(t, w, q) \;=\; \sum_{n \in \mathbb{Z}_{\geq 0}^N} \frac{(-q^{\frac{1}{2}})^{n^{\mathrm{T}} A n} q^{\frac{1}{2}\mathrm{diag}(A)\cdot n} w^{An} t^n}{(qw;q)_n}\,, \tag{214}$$

where $t^n = t_1^{n_1} \cdots t_N^{n_N}$ and $(qw;q)_n = (qw_1;q)_{n_1} \cdots (qw_N;q)_{n_N}$.

We can expand this sum when $q = \zeta_m + x$ is near a root of unity ζ_m and obtain an element of $\mathbb{Z}[\zeta_m][w^{\pm 1}, (1-w)^{-1}][\![t]\!][\![x]\!]$. In fact, it is not hard to see that

$$J_A(t, w, \zeta_m + x) \;\in\; \mathbb{Z}[\zeta_m][t, w_i^{\pm 1}, (1 - t_i^m P_i(w^m))^{-1} \,|\, i \;=\; 1, \dots, N][\![x]\!]\,, \tag{215}$$

where

$$P_i(z) \;=\; (-1)^{A_{ii}} (1 - z_i)^{-1} \prod_{j=1}^{N} z_j^{A_{ij}}, \qquad i = 1, \dots, N\,. \tag{216}$$

We define the collection $\Psi_A(t, q) \;=\; (\Psi_{A,m}(t, x))_{m \geq 1}$ by

$$\Psi_{A,m}(t, x) \;=\; \operatorname*{Res}_{w^m = z} t^{-1/m} J_A(t^{1/m}, w, \zeta_m + x)\, \frac{dw}{w}\,, \tag{217}$$

where the residue is taken over all $w^m = z$ and z satisfies the equations

$$t_i P_i(z) \ = \ 1, \qquad i \ = \ 1, \dots, N\,. \tag{218}$$

The value at ζ_m is given by the manifestly integral formula

$$\Psi_{A,m}(t,0) \ = \ \frac{1}{\delta_A(t) m^N} \sum_{w^m=z} \sum_{n \in \mathbb{Z}^N/m\mathbb{Z}^N} \frac{(-\zeta^{\frac{1}{2}})^{n^{\mathrm{T}} A n} \zeta^{\frac{1}{2}\mathrm{diag}(A)\cdot n} w^{An} t^n}{(qw;q)_n} \tag{219}$$

with $\delta_A(t)$ as in (119). The next theorem says in particular that this number is the same as the symmetrised value $f_A^{\mathrm{sym}}(t,\zeta_m) = f_A(t,\zeta_m) f_A(t,\zeta_m^{-1})$, which therefore is also integral.

Theorem 11. *For every positive integer m and f_A of Theorem 3 in Section 1.6, we have*

$$\Psi_A(t,q) \ = \ f_A(t,q) f_A(t,q^{-1}) \ \in \ \mathbb{Z}[\zeta_m][\![x]\!][\![t^{1/m}]\!], \qquad q \ = \ \zeta_m + x\,. \tag{220}$$

Proof. The series $\Psi_A(t,q)$ and $f_A(t,q) f_A(t,q^{-1})$ are the specialisations to $\mu = \nu = 0$ of two families indexed by two integer vectors $\mu, \nu \in \mathbb{Z}^N$

$$\begin{aligned} f_{A,\mu,\nu}(t,q) \ &= \ f_A(q^{m\mu} t, q) f_A(q^{-m\nu} t, q^{-1})\,, \\ \Psi_{A,\mu,\nu}(t,q) \ &= \ \operatorname*{Res}_{w^m=z} q^{\nu} w^{\mu+\nu} t_1^{-1/m} \cdots t_N^{-1/m} J_A(q^{\mu} t^{1/m}, w, \zeta_m + x) \, \frac{dw}{w}\,. \end{aligned} \tag{221}$$

Both families satisfy the same system of q-difference equations. In the one-dimensional case (i.e. $N = 1$), for $q = \zeta_m + x$ we have

$$\begin{aligned} f_{A,\mu,\nu}(t,q) - f_{A,\mu+1,\nu}(t,q) \ &= \ (-1)^A q^A t^{1/m} f_{A,\mu+A,\nu}(t,q)\,, \\ f_{A,\mu,\nu}(t,q) - f_{A,\mu,\nu+1}(t,q) \ &= \ (-1)^A q^{-A} t^{1/m} f_{A,\mu,\nu+A}(t,q)\,, \\ f_{A,\mu,\nu}(q^m t, q) \ &= \ f_{A,\mu+1,\nu-1}(t,q)\,. \end{aligned} \tag{222}$$

To see that $\Psi_{A,\mu,\nu}(t,q)$ also satisfies these equations note that

$$\begin{aligned} J_A(t,w,q) - w J_A(qt,w,q) \ &= \ -(1-w) + (-1)^A q^A t w^A J_A(q^A t, w, q)\,, \\ (1-qw) J_A(t,w,q) \ &= \ -(1-qw) + (-1)^A q^{-A} t q^A (qw)^A J_A(t, qw, q)\,. \end{aligned} \tag{223}$$

Therefore after taking the residues with a change of variables in the RHS of the second equation $qw \mapsto w$ we find that $\Psi_{A,\mu,\nu}(t,q)$ satisfies the Equations (222). In both cases the solutions can be taken of the form

$$\sum_{k=0}^{\infty} a_{\mu,\nu,k}(q)\, t^{k/m} \ \in \ \mathbb{Z}[\zeta_m][\![x]\!][\![t^{1/m}]\!]\,. \tag{224}$$

The q-difference equations imply that for

$$\begin{aligned} a_{\mu,\nu,k} - a_{\mu+1,\nu,k} \ &= \ (-1)^A q^A a_{\mu+A,\nu,k-1}\,, \\ a_{\mu,\nu,k} - a_{\mu,\nu+1,k} \ &= \ (-1)^A q^{-A} a_{\mu,\nu+A,k-1}\,, \\ q^k a_{\mu,\nu,k} \ &= \ a_{\mu+1,\nu-1,k}\,, \end{aligned} \tag{225}$$

where we set $a_{\mu,\nu,k} = 0$ for $k < 0$. This completely determines all $a_{\mu,\nu,k}$ from the value of $a_{0,0,0}$. Notice that $f_{A,m}(0,q) f_{A,m}(0,q^{-1}) = 1$. To see that $\Psi_{A,\mu,\nu}(0,q)$ also equals 1, notice that only the coefficient of x^0 contributes to the coefficient of t^0, because the residues of the factors $(1-t_i^m P_i(w^m))^{-\ell}$ contribute at least one factor of t when $\ell > 0$. Therefore an explicit computation leads to $\Psi_{A,m}(0,x) = 1$. □

To get elements of the Habiro ring of a number field, we specialise to $t = 1$ and assume that the equations (218) define a reduced zero-dimensional scheme over $\mathbb{Q}$. Fix a solution z of these equations and denote the corresponding collection of power series by $\Psi_{A,\mu,\nu,z}(q)$. A solution z generates a number field $\mathbb{K}$. Combining Theorem 3 of Section 1.6, Theorem 11 and Theorem 5 of Section 1.6, we obtain that $\Psi_{A,\mu,\nu,z}(q) \in \mathcal{H}_R$.

Theorem 12. *For all $\mu, \nu \in \mathbb{Z}^N$, we have $\Psi_{A,\mu,\nu,z}(q) \in \mathcal{H}_R$.*

Some cases of the above theorem were first proven in the thesis of Ferdinand Wagner. Here is a concrete example for the cubic field of discriminant -23.

Example 4.1. Consider the sum

$$J(t,w,q) \;=\; \sum_{k=0}^{\infty} (-1)^k \frac{q^{3k(k+1)/2} w^{3k} t^k}{(qw;q)_k}\,. \tag{226}$$

Expanding $J(t,w,1+x)$ as a power series in x and observing that each coefficient is a sum of derivatives of geometric series in t, we find that

$$\begin{aligned} J(t,w,1+x) \;=\; & \frac{-1+w}{-tw^3+w-1} + \frac{3tw^3-5tw^4+2tw^5}{(-tw^3+w-1)^3}x \\ & + \frac{1}{(-tw^3+w-1)^5}\Big(3tw^3-9tw^4+10tw^5+(-21t^2-5t)w^6+(50t^2+t)w^7 \\ & - 39t^2w^8+(3t^3+10t^2)w^9-4t^3w^{10}+t^3w^{11}\Big)x^2+O(x^3) \end{aligned} \tag{227}$$

and

$$\begin{aligned} \Psi_1(t,x) \;=\; & \frac{2tz^2+3tz-9t}{27t-4} + \frac{x^2}{(27t-4)^4}\big((-4374t^4-2106t^3+3t^2)z^2 \\ & + (-2187t^4-2997t^3-129t^2+2t)z + (2916t^3+1404t^2-2t)\big) + \mathrm{O}(x^3)\,, \end{aligned} \tag{228}$$

where z satisfies the equation

$$1-z \;=\; -tz^3\,. \tag{229}$$

Specialising to $t=1$ we find the expansion

$$\Psi_1(x) \;=\; \frac{2z^2+3z-9}{23} + \frac{-6477z^2-5311z+4318}{23^4}x^2 + O(x^3) \tag{230}$$

of the element of the Habiro ring $R = \mathbb{Z}[z, \frac{1}{23}]$, where $z^3-z+1=0$ generates the cubic field of discriminant -23. Using

$$\delta \;=\; -2t - \frac{t}{1-z} \;=\; -tz^2-tz+(-3t+1) \tag{231}$$

one can compare these equations to equation (247) and equation (248).

The next remark is for knot-theorists.

Remark 4.2. Use the matrices

$$A_{4_1} \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_{5_2} \;=\; \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad A_{(-2,3,7)} \;=\; \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} \tag{232}$$

in Theorem 5 of Section 1.6 to compute the asymptotic series $f^{(K)}(h)$ of the three simplest hyperbolic knots (for the $(-2,3,7)$ pretzel knot, the matrix was given in [20, Rem.A.5]). To get the asymptotic series of [11] for any knot, one can use a triangulation of it from `SnapPy`, choose quad types with Neumann–Zagier matrices $(\mathbf{A}|\mathbf{B})$ satisfying that $\mathbf{B}^{-1}\mathbf{A}$ is integral (if this is possible) and apply Theorem 5 of Section 1.6 with $A = \mathrm{I} - \mathbf{B}^{-1}\mathbf{A}$.

4.2. **A rank one admissible series.** q-hypergeometric series give admissible series that are easy to analyse. We illustrate this with the example of the 1×1 matrix $A = (3)$ in (31)

$$F(t,q) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k q^{3k(k+1)/2}}{(q;q)_k} t^k \;\in\; \mathbb{Q}(q)[\![t]\!] \,. \tag{233}$$

It satisfies the linear q-difference equation

$$F(t,q) - F(qt,q) + q^3 t F(q^3 t, q) \;=\; 0 \,. \tag{234}$$

The DT invariant of (27) $c_{n,i}$ is non-zero only for $3n+1 \leq i \leq n^2+n+1$ (and exceptionally, for $c_{1,3} = 1$) and satisfy the positivity $c_{n,i} \in \mathbb{N}$. The first few values of $c_{n,i}$ are given by

n	$c_{n,i},\ i = 3n+1, \dots, n^2+n+1$
2	1
3	1, 1, 0, 1
4	1, 1, 2, 1, 2, 1, 1, 0, 1
5	1, 2, 3, 4, 4, 5, 4, 4, 3, 3, 2, 2, 1, 1, 0, 1
6	1, 2, 5, 7, 11, 11, 15, 13, 15, 13, 14, 10, 12, 8, 8, 6, 6, 3, 4, 2, 2, 1, 1, 0, 1

(235)

Despite appearances in low degrees, the DT invariants grow fast (in fact they grow like $e^{C\sqrt{n}}$ for a constant C). For example,

$$c_{20,142} \;=\; 44549701024 \,. \tag{236}$$

The WKB expansion (see e.g., [4]) of the solution $F(t,q)$ of (234) allows one to compute the leading asymptotics of $f_1(t,x)$ in terms of the power series $z(t)$ and $V(t)$ defined by

$$1 - z \;=\; -tz^3, \qquad z(0) \;=\; 1 \,, \tag{237}$$

where

$$z(t) \;=\; 1+t+3t^2+12t^3+55t^4+273t^5+1428t^6+7752t^7+43263t^8+246675t^9+O(t^{10}) \tag{238}$$

and

$$\begin{aligned} V(t) \;&=\; -\,\mathrm{Li}_2(1-z(t)) - \frac{3}{2}(\log(z(t)))^2 \\ &= t + \frac{5}{4}t^2 + \frac{28}{9}t^3 + \frac{165}{16}t^4 + \frac{1001}{25}t^5 + \frac{1547}{9}t^6 + \frac{38760}{49}t^7 + \frac{245157}{64}t^8 + O(t^9) \,. \end{aligned} \tag{239}$$

Using the auxiliary function

$$\begin{aligned}\delta(t) \;&=\; -2t-\frac{t}{1-z(t)}\\ &=\; 1-5t-3t^2-10t^3-42t^4-198t^5-1001t^6-5304t^7-29070t^8+\dots \;\in\; \mathbb{Z}[\![t]\!]\end{aligned} \tag{240}$$

and setting $q=1+x$ and abbreviating $V(t)$, $\delta(t)$ and $z(t)$ by V, δ and z, respectively, we obtain the first few coefficients of $f_1(t,x)$ as follows:

$$\begin{aligned}\widehat{f}_1(t,x) \;&=\; e^{\frac{V}{x}}\frac{1}{\sqrt{\delta}}\Big(1+\frac{1}{24\delta^3}((308t^3-74t^2)z^2+(234t^3-74t^2)z+(216t^3-382t^2+74t))x\\ &+\frac{1}{1152\delta^6}((748116t^6-893688t^5+281084t^4-19924t^3-1104t^2)z^2\\ &+(397872t^6-685624t^5+257848t^4-21028t^3-1104t^2)z\\ &+(186624t^6-1252224t^5+1127196t^4-303216t^3+18820t^2+1104t))x^2+O(x^3)\Big)\,.\end{aligned} \tag{241}$$

In general, the coefficient of x^k in $f_1(t,x)e^{-\frac{V(t)}{x}}\delta^{\frac{1}{2}+3k}$ is in $\mathbb{Q}[t,z]$, where z satisfies (237).

Recall the symmetrisation $G(t,q)$ and $F^{\rm sym}(t,q)$ from equation (61). equation (234) implies that G satisfies a non-linear (Ricatti type) q-difference equation

$$1-G(t,q)+q^3t\,G(t,q)\,G(qt,q)\,G(q^2t,q) \;=\; 0\,. \tag{242}$$

It follows from this that the limit $\lim_{q\to1}G(tq,q)=z(t)$ exists and satisfies the algebraic equation (237). Equation (63) gives the factorisation

$$z(t) \;=\; \prod_{n\geq1}(1-t^n)^{-n\sum_{i\in\mathbb{Z}}c_{n,i}} \tag{243}$$

which proves that the exponent of $1-t^n$ in the above product expansion is divisible by n. This is an application of [30], where one may think of the polynomials $L_n(q)$ as a categorification of the integers $n\,c_{n,i}$.

The symmetrised series $F^{\rm sym}(t,q)$ was not considered previously in the literature. It is easy to show that it satisfies a sixth order linear q-difference equation, which we omit. Below, we will treat the series $G(t,q)$ and $F^{\rm sym}(t,q)$ on the same footing.

We now illustrate a remarkable aspect of the series $G(t,q)$ and $F^{\rm sym}(t,q)$, namely their $(q-1)$-expansion. It is clear that they both lie in the completed ring $\mathbb{Z}[\![t]\!][\![q-1]\!]$ but more is true. The expansion (241) contains a volume prefactor that cancels, as well as universal denominators for each power of x, which also cancel, so that what remains are series in $\mathbb{Z}[t^{\pm1},z,1/\delta]$. Explicitly, we can write $G(t,q)$

$$G(t,1+x) \;=\; \sum_{k\geq0}g_k(t)x^k \tag{244}$$

and then it follows from (242) and induction that $g_0=z$ and $\delta^{3k}g_k\in\mathbb{Z}[t^{\pm1},z]$, e.g.,

$$\begin{aligned}
g_0 &= z\,,\\
\delta^3 g_1 &= (15t^3-3t^2)z^2+(18t^3-3t^2)z+(-18t^2+3t)\,,\\
\delta^6 g_2 &= (711t^6-708t^5+107t^4+17t^3-3t^2)z^2+(405t^6-586t^5+115t^4+14t^3-3t^2)z\\
&\quad+(-1176t^5+828t^4-96t^3-20t^2+3t)\,,\\
\delta^9 g_3 &= (26325t^9-69399t^8+32035t^7+6234t^6-6470t^5+1259t^4-69t^3-t^2)z^2\\
&\quad+(9720t^9-47322t^8+29899t^7+2658t^6-5430t^5+1187t^4-70t^3-t^2)z\\
&\quad+(-56187t^8+95787t^7-32141t^6-10699t^5+7584t^4-1330t^3+68t^2+t)\,.
\end{aligned} \tag{245}$$

Likewise, we have

$$F^{\mathrm{sym}}(t,1+x) \;=\; \sum_{k\ge 0} f_k^{\mathrm{sym}}(t)x^k\,, \tag{246}$$

where $\delta^{3k+1} f_k^{\mathrm{sym}} \in \mathbb{Z}[t^{\pm 1}, z]$, with the first few values given by

$$\begin{aligned}
\delta f_0^{\mathrm{sym}} &= 1\,,\\
\delta^4 f_1^{\mathrm{sym}} &= 0\,,\\
\delta^7 f_2^{\mathrm{sym}} &= (39t^6-109t^5-18t^4+34t^3-5t^2)z^2+(9t^6-85t^5+t^4+29t^3-5t^2)z\\
&\quad+(-96t^5+124t^4+42t^3-39t^2+5t)\,,\\
\delta^{10} f_3^{\mathrm{sym}} &= (1296t^9-9183t^8+7230t^7+1604t^6-2730t^5+858t^4-109t^3+5t^2)z^2\\
&\quad+(243t^9-5328t^8+6310t^7+415t^6-2139t^5+764t^4-104t^3+5t^2)z\\
&\quad+(-3969t^8+14304t^7-7635t^6-3231t^5+3405t^4-957t^3+114t^2-5t)\,.
\end{aligned} \tag{247}$$

But a further surprise is awaiting us when we expand $G(t,q)$ and $F^{\mathrm{sym}}(t,q)$ at $q=\zeta_m+x$. As expected, we now get series in $\mathbb{Z}[\zeta_m][t^{\pm 1}, z, 1/\delta][\![x]\!]$. But these series glue, after applying p-Frobenius. Concretely, if we specialise $t=1$, then we obtain

$$F^{\mathrm{sym}}(1,1+x) \;=\; \frac{1}{\delta}+(-59z^2-51z+36)\frac{x^2}{\delta^7}+(-1029z^2+166z+2026)\frac{x^3}{\delta^{10}}+O(x^4)\,, \tag{248}$$

where z satisfies the cubic equation $1-z+z^3=0$ and $\delta=-2-1/(1-z)=-z^2-z-2$, an algebraic integer of norm -23. z generates a cubic field $F=\mathbb{Q}(z)$ of discriminant -23. Let $R=\mathbb{Z}[z,1/23]$.

The specialisation $F^{\mathrm{sym}}(1,q)\in\mathbb{Z}[z,\frac{1}{23}][\![q-1]\!]$ and $G(1,q)$ lie in $\mathcal{H}_{\mathbb{Z}[z,1/23]}$ and its field of fractions, respectively. This illustrates part (a) of Corollary 1.11 of Section 1.6.

These are elements of the Habiro ring of the étale map

$$\mathbb{Z}[t^{\pm 1},1/\delta]\to\mathbb{Z}[t^{\pm 1},1/\delta][z]/(1-z+tz^3)\,. \tag{249}$$

When $m=1$ and A is a 1×1 matrix, the power series $z(t)=z_A(t)$, $V(t)=V_A(t)$ and $\delta(t)=\delta_A(t)$ have coefficients polynomials in A. Indeed, the unique power series $z(t)=z_A(t)$

$$1-z \;=\; (-1)^A t z^A, \qquad z(0) \;=\; 1\,. \tag{250}$$

has coefficients integer-valued polynomials of A (for integer A), with the first few given by

$$z(t) \;=\; 1-(-1)^A t + At^2 - \frac{1}{2}(-1)^A A(3A-1)t^3 + O(t^4)\,. \tag{251}$$

This, together with

$$\begin{aligned} z(t) \;&=\; \lim_{q\to 1}\frac{F(tq,q)}{F(t,q)} \;=\; \lim_{q\to 1}\exp\Big(-\sum_{n,\ell\geq 1}\sum_{i\in\mathbb{Z}}\frac{L_n(q^\ell)}{\ell(1-q^\ell)}(q^{\ell n}-1)t^{\ell n}\Big)\\ &=\; \exp\big(\sum_{n\geq 1} nL_n(1)\mathrm{Li}_1(t^n)\big) \;=\; \exp\big(t\partial_t\sum_{n\geq 1}L_n(1)\mathrm{Li}_2(t^n)\big) \;=\; \exp\big(t\partial_t V(t)\big)\,, \end{aligned}$$

implies that $V(t)$ satisfies

$$\begin{aligned} V(t) \;&=\; -\mathrm{Li}_2(1-z(t)) - \frac{A}{2}(\log(z(t)))^2 \;\in\; \mathbb{Q}[\![t]\!], \qquad V(0) \;=\; 0\\ &=\; -(-1)^A t - \frac{2A-1}{4}t^2 + (-1)^A\frac{(3A-1)(3A-2)}{18}t^3 + O(t^4)\,. \end{aligned} \tag{252}$$

Finally,

$$\begin{aligned} \delta(t) \;&=\; (-1)^A\Big((A-1)t + \frac{t}{1-z(t)}\Big) \;\in\; \mathbb{Z}[\![t]\!], \qquad \delta(0) \;=\; 1\\ &=\; 1-(-1)^A(2A-1)t + \frac{A(A-1)}{2}t^2 + (-1)^A\frac{(2A-1)A(A-1)}{3}t^3 + O(t^4)\,. \end{aligned} \tag{253}$$

The rest of the coefficients of the unique zero-slope solution $F_A(t,q)$ of (164) can be computed inductively as was illustrated in the beginning of this section with the example of $A=3$.

4.3. **Torsion in the Bloch group from admissible series.** In this section we illustrate part (b) of Corollary 1.11 of Section 1.6 with the matrix

$$A \;=\; \begin{pmatrix} 8 & 5\\ 5 & 4\end{pmatrix} \tag{254}$$

(symmetric and positive definite) taken from the survey article [40].

The Nahm equations for $z=(z_1,z_2)$

$$1-z_1 \;=\; z_1^8 z_2^5, \qquad 1-z_2 \;=\; z_1^5 z_2^4\,, \tag{255}$$

have eight solutions in two Galois orbits defined over two quartic fields, one given by

$$z_1^4+z_1^3+3z_1^2-3z_1-1 \;=\; 0\,, \qquad z_2 \;= ?? \tag{256}$$

and the other by

$$z_1^4-z_1^3+3z_1^2-3z_1+1 \;=\; 0\,, \qquad z_2 \;= ??\,. \tag{257}$$

Since A is positive definite equation (255) has a unique solution in $(0,1)^2$, given to a few decimals by $(0.88483\cdots, 0.78939\cdots)$, belonging to the real embedding of the quartic number field F of type $(2,1)$ and discriminant $-5^2\cdot 19$ defined by (256). This solution of the Nahm equation is non-degenerate and defines an element $\xi=[z_1]+[z_2]$ of the Bloch group $B(F)$. It turns out to be 60-torsion, and the corresponding series $(f_{A,z,\nu})^{60}$ belongs to $\mathcal{H}_{\mathcal{O}_F[1/(5\cdot 19)]}$ for all $\nu\in\mathbb{Z}^2$ (conjecturally, but provably if we invert 6). The q-holonomic module of the

Nahm sum associated to A has rank 8, spanned by $\mathbb{Q}(q)$-linear combinations of the Nahm sums $F_\nu(q)$ defined by

$$F_\nu(q) \;=\; \sum_{n=(n_1,n_2)\in\mathbb{N}^2} \frac{q^{\frac{1}{2}n^tAn+n^t\nu}}{(q;q)_{n_1}(q;q)_{n_2}}, \qquad \nu \;\in\; \mathbb{Z}^2\,. \tag{258}$$

The radial asymptotics of these q-hypergeometric functions as q approaches a root of unity allow one to compute the asymptotic series $f_{A,z,\nu}$ (abbreviated by f for $\nu = (0,0)$ below). This is done using a numerical computation of the function F_ν, followed by an acceleration that improves the precision of the found numbers, and their eventual recognition as exact algebraic numbers (explained in detail in [21, 20]). Applying this method, we find that the asymptotics of $F_0(1+x)$ for $x \in \mathbb{R}$ as $x \to 0$ have the form

$$F_0(1+x) \;\sim\; \widehat{f}_1((1+x)^{-1}+1)\,, \tag{259}$$

where

$$\widehat{f}_1(x) \;=\; e^{\frac{\pi^2}{15}\frac{1}{\log(1+x)}}\frac{1}{\sqrt{\delta}}\big(1+a_1x+a_2x^2+a_3x^3+O(x^4)\big) \tag{260}$$

with δ and a_i in F given by

$$\begin{aligned}
\delta \;&=\; \frac{753-505z_1-124z_1^2-186z_1^3}{5}\,,\\
a_1 \;&=\; \frac{-1284z_1^3+384z_1^2-5520z_1+2047}{2^2\cdot 3\cdot 5^2\cdot 19^2}\,,\\
a_2 \;&=\; \frac{-3084024z_1^3-11262336z_1^2-1073760z_1+17201653}{2^5\cdot 3^2\cdot 5^2\cdot 19^4}\,,\\
a_3 \;&=\; \frac{-1185017476284z_1^3+1129707725184z_1^2-5869777630320z_1+1818824190547}{2^7\cdot 3^4\cdot 5^4\cdot 19^6}\,.
\end{aligned} \tag{261}$$

We computed the series $f_1(x)$ up to $\mathrm{O}(x)^{28}$. The denominator of the series to order 27 a priori should include all primes less than or equal to 29 as found in [21, Sec.9.2, Thm.9.1]. However, the denominator is actually $2^{77}\cdot 3^{40}\cdot 5^{34}\cdot 19^{52}$, so the series is Δ-integral. Moreover, if we take the 60-th power of $f_{0,1}(x)$, we find that the denominator improves to $5^{33}\cdot 19^{54}$, where the remaining primes 5 and 19 are the prime factors of the discriminant $-5^4\cdot 19$ of the number field.

The same experiment can be performed for other values of $\nu \in \mathbb{Z}^2$. Actually one needs only 8 values since the holonomic rank of the 2-parameter q-holonomic function $\nu \mapsto F_\nu(q)$ is 8. Doing so, we find the same Δ-integrality results.

Incidentally, the element corresponding to the second quartic field E is not a torsion element of $B(E)$, and its corresponding series, computed from the asymptotics of $F_0(\mathbf{e}(-1/\tau))$ when τ approaches infinity near the real line, does not exhibit any Δ-integrality properties.

4.4. **Some p-adic computations.** In this section we discuss p-adic computations, starting with the absolute basics. Hensel's lemma states that if a polynomial factors into irreducible polynomials over $\mathbb{Z}/p\mathbb{Z}$, then this factorisation lifts to a unique factorisation over $\mathbb{Z}/p^n\mathbb{Z}$. Therefore, if ξ is a generator of the field $\mathbb{K}$ with minimal polynomial $P(x)$ and p is an unramified prime, applying Hensel's lemma to $P(x)$ lifts the Frobenius automorphism R/pR

to $R_p^\wedge \cong R \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where $R = \mathcal{O}_{\mathbb{K}}[1/\Delta]$. Hensel's lemma is constructive and we can easily use it to compute the Frobenius automorphism. Firstly, notice that

$$P(\xi^p) \;\equiv\; 0 \pmod{p}\,. \tag{262}$$

Suppose by induction that for some d there is a unique $\alpha \in R/p^{d-1}R$ such that

$$P(\xi^p + \alpha p) \;\equiv\; 0 \pmod{p^d}\,. \tag{263}$$

Choose a lift of α to R/p^dR. Let $\beta \in R/pR$. Then we notice that

$$\frac{P(\xi^p + \alpha\, p + \beta\, p^d) - P(\xi^p + \alpha\, p)}{p^d} \;\equiv\; P'(\xi^p)\,\beta \pmod{p}\,. \tag{264}$$

As p is an unramified prime $P'(\xi^p) \in (R/pR)^\times$ and hence we can set

$$\beta \;=\; -\frac{P(\xi^p + \alpha\, p)}{p^d P'(\xi^p)} \;\in\; R/pR\,. \tag{265}$$

With this choice we find that

$$P(\xi^p + \alpha p + \beta p^d) \;\equiv\; 0 \pmod{p^{d+1}}\,. \tag{266}$$

Therefore, by induction, we can lift the element $\xi^p \in R/pR$ to an element $\varphi_p(\xi) \in R_p^\wedge$ such that $P(\varphi_p(\xi)) = 0$ and $\varphi_p(\xi) = \xi^p \in R/pR$.

Next we will explain how to evaluate the p-adic polylogarithm. We can evaluate this function on an element t of a p-adic field using the algorithm of Besser–de Jeu described in [7]. Let us recall how this is done.

- When $|t| < 1$, we can use the power series definition of $\mathrm{Li}_n(t)$ at $t = 0$.
- When $t = \zeta \neq 1$, a root of unity of unity, we use the series expansion of $\mathrm{Li}_n^{(p)}(t/(1-t))$, which converges for $|t| < p^{\frac{1}{p-1}}$ for $t = t_0 = \zeta/(1-\zeta)$ (satisfying $|t_0| = 1$) to compute $\mathrm{Li}_n^{(p)}(\zeta)$, together with equation [7, Prop. 4.2]

$$\mathrm{Li}_n(\zeta) \;=\; (p^{ns} - 1)^{-1} \sum_{r=0}^{s-1} p^{n(s-r)} \mathrm{Li}_n^{(p)}(\zeta_{p^s-1}^{p^r}) \tag{267}$$

  to compute $\mathrm{Li}_n(\zeta)$.
- When $|t - \zeta| < 1$, for ζ as above, use the Taylor series expansion of $\mathrm{Li}_n(t)$ at $t = \zeta$, together with the fact that $(t\frac{\partial}{\partial t})^s|_{t=\zeta}\mathrm{Li}_n(t) = \mathrm{Li}_{n-s}(\zeta)$ to compute $\mathrm{Li}_n(t)$.
- When $|t| > 1$, use the inversion formula

$$\mathrm{Li}_n(z) + (-1)^n\, \mathrm{Li}_n(z^{-1}) \;=\; -\frac{1}{n!} \log(z)^n \tag{268}$$

  to compute $\mathrm{Li}_n(t)$.
- Finally, one can also compute $\mathrm{Li}_n(t)$ when $0 < |t-1| < 1$ by considering the function $\mathrm{Li}_n(t) - \log(t)\mathrm{Li}_{n-1}(t)/(n-1)$, which is analytic in that domain.

The next example illustrates Theorem 9 of Section 3.1.

Example 4.3. Suppose that $\mathbb{K} = \mathbb{Q}(\alpha)/(\alpha^3 - \alpha^2 + 1)$. Let

$$z_1 \;=\; 1 - \alpha^2\,, \qquad z_2 \;=\; z_1^2 - z_1 + 2\,, \qquad z_3 \;=\; z_1\,. \tag{269}$$

Then $\xi = [z_1] + [z_2] + [z_3] \in B(\mathbb{K})$ represents the element of the Bloch group of 5_2. Consider the prime $p = 5$, where $\mathbb{K}_5 \cong \mathbb{Q}_{5^2} \times \mathbb{Q}_5$ is a product of two unramified extension of $\mathbb{Q}_5$, one of degree two and one of degree one. We have

$$\begin{aligned} D_5(\xi) \;&=\; D_5(z_1) + D_5(z_2) + D_5(z_3) \\ &=\; (3 \cdot 5^2 + 5^3 + 2 \cdot 5^4 + \cdots)\alpha^2 + (5^2 + 3 \cdot 5^3 + \cdots)\alpha + (2 \cdot 5^2 + 3 \cdot 5^3 + \cdots)\,. \end{aligned} \tag{270}$$

The roots of unity $\mu(\mathbb{K}_5)$ is a product of finite cyclic groups of order $5^2 - 1 = 24$ and $5 - 1 = 4$. There is an order 24 subgroup generated by ζ_{24}, where

$$\zeta_{24} \;=\; \lim_{s\to\infty} \alpha^{5^{2s}} \;=\; (4 \cdot 5^2 + \cdots)\alpha^2 + (1 + \cdots)\alpha + (3 \cdot 5 + \cdots)\,. \tag{271}$$

Then we have

$$\begin{aligned} D_5(\zeta_{24}) \;&=\; (4\cdot 5^2 + 4\cdot 5^3 + \cdots)\alpha^2 + (4\cdot 5^2 + 2\cdot 5^3 + \cdots)\alpha + (2\cdot 5^2 + 5^3 + 2\cdot 5^4 + \cdots), \\ D_5(\zeta_{24}^5) \;&=\; (2\cdot 5^2 + 2\cdot 5^3 + 5^4 + \cdots)\alpha^2 + (3\cdot 5^2 + 3\cdot 5^3 + 5^4 + \cdots)\alpha + (5^2 + 3\cdot 5^4 + \cdots), \\ D_5(\zeta_{24}^6) \;&=\; (3\cdot 5^2 + 3\cdot 5^3 + 2\cdot 5^4 + \cdots)\alpha^2 + (3\cdot 5^2 + 2\cdot 5^3 + 3\cdot 5^4 + \cdots)\alpha + (2\cdot 5^2 + 4\cdot 5^4 + \cdots). \end{aligned} \tag{272}$$

This implies that

$$D_5(\xi) \;=\; (1+4\cdot 5+3\cdot 5^2+\cdots)D_5(\zeta_{24})+(3+5+\cdots)D_5(\zeta_{24}^2)+(1+5+4\cdot 5^2+\cdots)D_5(\zeta_{24}^6) \tag{273}$$

which illustrates Theorem 9 of Section 3.1. This implies that

$$\xi \;=\; (1+4\cdot 5+3\cdot 5^2+\cdots)[\zeta_{24}]+(3+5+\cdots)[\zeta_{24}^2]+(1+5+4\cdot 5^2+\cdots)[\zeta_{24}^6] \;\in\; B(\mathbb{K}_5)\otimes\mathbb{Q}_5\,. \tag{274}$$

4.5. **The 4_1 knot.** Every result presented in the paper has been numerically verified. We will describe a select few of the computations that were carried out for two examples associated to the knots for 4_1 and 5_2.

Firstly, there are a variety of methods available, both numerical and exact, to compute these collections of power series at roots of unity. Using the exact formulas of formal Gaussian integration, we can easily compute the series around $q = 1$. For example in [18, Sec. 7] the computation was detailed for 4_1. With ψ as in equation (113), this gives the series

$$f_1^{4_1}(x) \;=\; (1+x)^{\frac{1}{8}} \langle \psi_{0,\zeta_6,1}(w_1,x)\psi_{0,\zeta_6,1}(w_2,x)\rangle_{\Lambda_{4_1}} \;=\; (1+x)^{\frac{1}{6}} \Big\langle \exp\Big(\frac{w}{2}\Big)\psi_{0,\zeta_6,1}(w,x)^2 \Big\rangle_{2\zeta_6-1}\,, \tag{275}$$

where

$$\Lambda_{4_1} \;=\; \begin{pmatrix} \zeta_6 - 1 & -1 \\ -1 & \zeta_6 - 1 \end{pmatrix}. \tag{276}$$

This simple expression allows the computation of this series to high precision. We have computed of order 600 in this example. The first few terms are given in the Introduction 1.1 in equation (1) for $q = e^h = 1 + x$, with $\Phi_1^{4_1}(h) = f_1^{4_1}(x)$. The integrality of the symmetrisation was then given in equation (2). The Ohtsuki property of the series was checked with this data. This field is $\mathbb{Q}(\sqrt{-3})$ and has non-trivial roots of unity. In fact, the symbols of these roots of unity generate the Bloch group. In this case we can construct a generator of the Habiro module globally using the infinite Pochhammer symbol with a root of unity as argument. Indeed, we find

$$\begin{aligned}&\sqrt[4]{-3}\,\widehat{f}_1^{4_1}(x)\,(1+x)^{-\frac{1}{24}}\,(\zeta_6^{-2};1+x)_\infty^3(1-\zeta_6)^{-\frac{3}{2}}\\&=\;1+\frac{1}{3^3}(5\zeta_6-7)x+\frac{1}{3^5}(-30\zeta_6+31)x^2+\frac{1}{3^9}(1565\zeta_6-1444)x^3+\cdots\,.\end{aligned} \tag{277}$$

This whole series is integral away from 3. This observation was a key clue to finding a definition of the Habiro modules.

4.6. **The pair of the 5_2 and the $(-2,3,7)$-pretzel knots.** In this section we give a pair of elements of the same module over the Habiro ring of the cubic field $\mathbb{K}$ of discriminant -23 that come from the asymptotic series associated to a pair of hyperbolic knots, namely the 5_2 and the $(-2,3,7)$-pretzel knot. This pair of knots was studied in detail in [21] and was instrumental in formulating the results and conjectures of that paper. Recall that this pair of knots has common trace field $\mathbb{K}$ generated by a solution to the cubic equation $\xi^3-\xi^2+1=0$. In fact, these knots are scissors congruent; their complements can be decomposed in three ideal tetrahedra with shapes in $\mathbb{K}$, but assembled differently for each knot. It follows that the corresponding elements of the Bloch group are equal, modulo 6-torsion. Although several terms of their series $f_m^{(5_2)}(x)$ and $f_m^{(-2,3,7)}(x)$ were computed at various roots of unity of small order m, no relation between the two series was found.

There are several methods to compute the series $f_m^{(5_2)}(x)$ and $f_m^{(-2,3,7)}(x)$:

- numerically compute the Kashaev invariant, its asymptotic expansion at roots of unity of order m as in [21, Eqn.(1.5)] using high precision and extrapolation, and then lifting to elements of $\mathbb{K}$,
- numerically compute the asymptotics of q-series associated to these knots using high precision and extrapolation, and then lifting to elements of $\mathbb{K}$, as in [20]
- compute using exact arithmetic the formal Gaussian integral associated to these knots.

The state-integrals of these knots a priori are 3 and 4-dimensional, but they reduce to explicit 1-dimensional state integrals for both (see [2, Eqn.(39)] for the 5_2 knot and [16, Eqn.(58)] for the $(-2,3,7)$-pretzel knot). When $m=1$, the third method applied to these 1-dimensional integrals gives an efficient way to compute the series $f_1^{(5_2)}(x)+\mathrm{O}(x)^{401}$ and $f_1^{(-2,3,7)}(x)+\mathrm{O}(x)^{401}$; see [1, Sec.4]. Although the coefficients of both series have a universal denominator defined in [21, Thm.9.1] the product (keeping in mind that $q=1+x$, $q^{-1}=1-x/(x+1)$)

$$\begin{aligned}f_1^{(5_2)}(x)\,f_1^{(-2,3,7)}\Big(-\frac{x}{1+x}\Big)\;=c\Big(1&+\frac{-7\alpha^2+20\alpha+33}{2^4\cdot 23}x\\&+\frac{226541\alpha^2-275879\alpha-218336}{2^9\cdot 23^3}x^2\\&+\frac{-95096039\alpha^2+85905420\alpha+49207882}{2^{13}\cdot 23^4}x^3+\dots\Big)\end{aligned} \tag{278}$$

has denominators given by powers of 2 and 23, e.g. the denominator of x^{400} is $2^{1997}\cdot 23^{581}$. Here α satisfies $\alpha^3-\alpha^2+1=0$. The constant term is the product of the square roots of the

δ-invariant of the two knots

$$c \;=\; \frac{1}{\sqrt{-6\alpha^2+10\alpha-4}}\cdot\frac{1}{\sqrt{-24\alpha^2+32\alpha-26}} \;=\; \frac{1}{\sqrt{2}\cdot(2\alpha^2-2\alpha+3)} \tag{279}$$

Despite the above similarity, the series $f_1^{(5_2)}(x)$ and $f_1^{(-2,3,7)}(x)$ are quite different from each other: for instance the coefficients of the series of 5_2 "see" only the cubic trace field of discriminant -23, whereas those of $(-2,3,7)$ see in addition the abelian field $\mathbb{Q}(2\cos(2\pi/7))$ of discriminant 49. Thus, even the rank of the étale algebras is different.

We can explore 5_2 in some more detail. A Gaussian integral in this case can be given

$$f_1^{5_2}(x) \;=\; \frac{1}{\sqrt{\delta_{5_2}}}\Big\langle e^w\psi_{0,z,1}(w,x)^3\Big\rangle_{3\alpha-2}\,, \tag{280}$$

where $\alpha^3-\alpha^2+1=0$ and $\delta=3\alpha-2$. The first few terms of this series are well documented[5] but we will give them again here:

$$\begin{aligned}\Phi_1^{5_2}(h) \;=\; \frac{1}{\sqrt{3\alpha-2}}\Big(1&+\frac{765\alpha^2-1086\alpha+1043}{24\,(3\alpha-2)^3}h+\frac{1757583\alpha^2-2956029\alpha+2241964}{1152\,(3\alpha-2)^6}h^2\\ &+\frac{21285784611\alpha^2-37166037066\alpha+27969826252}{414720\,(3\alpha-2)^9}h^3+\cdots\Big)\end{aligned} \tag{281}$$

Since $q=1+x$ and $q^{-1}=1-x/(1+x)$, it follows that the symmetrisation is given by

$$\begin{aligned}\delta f_1^{5_2}(x)\,f_1^{5_2}\Big(-\frac{x}{1+x}\Big) \;=\; 1&+\frac{1}{23^3}(465\alpha^2-465\alpha+54)x^2+\frac{1}{23^3}(-465\alpha^2+465\alpha-54)x^3\\ &+\frac{1}{23^6}(4934541\alpha^2-4934541\alpha+462834)x^4+\cdots\,.\end{aligned} \tag{282}$$

This data given is enough to compute the first digit of the constant of the symmetrised series 5-adically at ζ_5. Indeed, we find that

$$f_1^{5_2}(\zeta_5-1)\,f_1^{5_2}(\zeta_5^{-1}-1) \;\equiv\; (\alpha^2+3\alpha+2)\zeta_5^3+(\alpha^2+3\alpha+2)\zeta_5^2 \pmod 5\,. \tag{283}$$

while

$$f_5^{5_2}(0)^2 \;\equiv\; (\alpha^2+4\alpha)x^3+(\alpha^2+4\alpha)x^2 \pmod 5\,. \tag{284}$$

This agrees with the gluing in the Habiro ring as

$$(\alpha^2+4\alpha)^5 \;\equiv\; \alpha^2+3\alpha+2 \pmod 5\,. \tag{285}$$

These computations can be carried out to any desired order and for any roots of unity by replacing the p-power map with the Frobenius automorphism described in Section 4.4.

The final experiment to describe here is the integrality of the series $f_1^{5_2}(x)$ after division by the generator $\Psi_{\xi,p}(x)$. We can use Example 4.3 of Section 4.4 to give the description of

[5] Here we again use $\Phi(h)$ instead of $f(x)$ where $q=e^h=1+x$. Also, there are factors of $q=e^h$ and ζ_8 that differ here from the formulas in [21, Equ. 4].

our series. For some constant C we find

$$\begin{aligned}
&C\, f_5^{5_2}(x)\Psi_{[\zeta_{24}],5}(x)^{-(1+4\cdot5+3\cdot5^2+\cdots)}\Psi_{[\zeta_{24}^2],5}(x)^{-(3+5+\cdots)}\Psi_{[\zeta_{24}^6],5}(x)^{-(1+5+4\cdot5^2+\cdots)}\\
&= 1+((2+4\cdot5+4\cdot5^3+\cdots)\alpha^2+(2+3\cdot5+3\cdot5^2+3\cdot5^3+\cdots)\alpha+(3+\cdots))x\\
&+((3+2\cdot5+2\cdot5^2+3\cdot5^3+\cdots)\alpha^2+(3+4\cdot5^2+3\cdot5^3+\cdots)\alpha+(3+3\cdot5+2\cdot5^2+3\cdot5^3+\cdots))x^2\\
&+((2+4\cdot5+5^2+4\cdot5^3+\cdots)\alpha^2+(3+4\cdot5^2+5^3+\cdots)\alpha+(2+3\cdot5^2+3\cdot5^3+\cdots))x^3+\cdots\,.
\end{aligned} \tag{286}$$

The coefficient of x^{200} is given by

$$(1+5^2+5^3+\cdots)\xi^2+(3+2\cdot5+2\cdot5^2+4\cdot5^3+\cdots)\xi+(4+3\cdot5+2\cdot5^2+5^3+\cdots), \tag{287}$$

which is clearly 5-integral.

The perturbative series associated to the three boundary-parabolic representations of the $(-2,3,7)$-pretzel knot with values in the abelian number field $\mathbb{K}=\mathbb{Q}(2\cos(2\pi/7))$ give an element of $\mathcal{H}_{\mathcal{O}_{\mathbb{K}}[1/7]}$. The corresponding element of the Bloch group is 3-torsion. We have computed 400 terms of the series at $m=1$ and have checked their 7-integrality. For instance, the coefficient of x^{400} of this series is $2^{1997}\cdot3^{596}\cdot7^{466}$, which improves to $2^{1997}\cdot7^{466}$ upon taking the third power of the series, illustrating part (b) of Corollary 1.11 of Section 1.6.

4.7. **A note on modularity.** Modular forms at roots of unity give rise to special elements associated to torsion classes in K_3. We will consider two simple examples: one coming from a quadratic Gauss sum (and hence related to the asymptotics of Jacobi θ-series) and the other related to the Roger-Ramanujan functions, which are well known to be modular.

Example 4.4 (A quadratic Gauss sum)**.** Consider

$$\begin{aligned}
F_m(x) \;&=\; \frac{1}{m}\Big(\sum_{k\in\mathbb{Z}/m\mathbb{Z}}\zeta_m^{k^2}\Big)\Big(\sum_{k\in\mathbb{Z}/m\mathbb{Z}}\zeta_m^{-k^2}\Big) \;=\; \frac{1}{2}(1+\zeta_4^m)(1+\zeta_4^{-m})\\
&=\; \begin{cases} 1 & \text{if } m\equiv 1 \pmod 4\,,\\ 0 & \text{if } m\equiv 2 \pmod 4\,,\\ 1 & \text{if } m\equiv 3 \pmod 4\,,\\ 2 & \text{if } m\equiv 4 \pmod 4\,.\end{cases}
\end{aligned} \tag{288}$$

We see that this gives an almost trivial element of the Habiro ring $\mathcal{H}_{\mathbb{Z}[1/2]}$. We can also consider

$$G_m(x) \;=\; \frac{\zeta_4}{m}\Big(\sum_{k\in\mathbb{Z}/m\mathbb{Z}}\zeta_m^{k^2}\Big)^2 \;=\; \frac{1}{2}(1+\zeta_4^m)^2 \;=\; \begin{cases} i & \text{if } m\equiv 1 \pmod 4\,,\\ 0 & \text{if } m\equiv 2 \pmod 4\,,\\ -i & \text{if } m\equiv 3 \pmod 4\,,\\ 2 & \text{if } m\equiv 4 \pmod 4\,,\end{cases} \tag{289}$$

giving a slightly less trivial element of $\mathcal{H}_{\mathbb{Z}[i,1/2]}$.

Example 4.5 (Rogers-Ramanujan symmetrised)**.** The Rogers-Ramanujan function has an associated field $\mathbb{Q}(\sqrt{5})$, which has an abelian Galois group over $\mathbb{Q}$. Consider

$$J(z,q) \;=\; \sum_{k=0}^{\infty}\frac{q^{k(k+1)}z^{2k}}{(qz;q)_k} \;\in\; \mathcal{H}_{\mathbb{Q}(z)}\,. \tag{290}$$

This J has the special property that

$$J(z,\zeta_m(1-u)) \in \mathbb{Z}[z,(1-z^m-z^{2m})^{-1}][\zeta_m][\![u]\!]\,. \tag{291}$$

We define for $\xi^2+\xi-1=0$

$$F_m(u) = \sum_{\theta^m=\xi} \operatorname*{Res}_{z=\theta} J(z,\zeta_m(1-u))\frac{dz}{z}\,. \tag{292}$$

Each $F_m(u)$ in $\mathbb{Q}(\sqrt{5},\zeta_m)$ is constant, the first five values being:

$$\begin{aligned} F_1(u) &= -\frac{1}{2}-\frac{\sqrt{5}}{10}\,, & F_2(u) &= -\frac{1}{2}+\frac{\sqrt{5}}{10}\,,\\ F_3(u) &= -\frac{1}{2}+\frac{\sqrt{5}}{10}\,, & F_4(u) &= -\frac{1}{2}-\frac{\sqrt{5}}{10}\,,\\ F_5(u) &= \frac{\sqrt{5}}{5}\zeta_5^3+\frac{\sqrt{5}}{5}\zeta_5^2-\frac{1}{2}+\frac{\sqrt{5}}{10}\,. \end{aligned} \tag{293}$$

Note that, depending on the embeddings, $F_5(u)$ is equal to either 0 or -1 in $\mathbb{C}$. In fact, using the modularity of the Rogers-Ramanujan functions, one can show that

$$-5F_m^2(u)-5F_m(u) = \begin{cases} 1 & \text{if } (m,5) = 1\,,\\ 0 & \text{otherwise}\,, \end{cases} \tag{294}$$

the expressions on either side being the value at $\zeta_m(1-u)$ of an element of $\mathcal{H}_{\mathbb{Z}[1/5]}$. More generally, $\mathcal{H}_{\mathbb{Z}[1/5]}$ contains an element χ_N sending ζ_m to $\delta_{v_5(m),N}$. These elements can be constructed using the operation $f(q)\mapsto f(q^5)$ — which does act on the usual Habiro ring of $\mathbb{Z}$. Note that the operation $f(q)\mapsto f(q^k)$ is in general only defined on the Habiro ring of a ring R if it contains $1/k$. For example for $k=2$ we see that $f((1-u)^2)=f((-(1-u))^2)$ and hence the re-expansion will not require a Frobenius φ_2, which it should.)

This example illustrates how the different roots of unity can be disconnected by denominators. This viewpoint will be pushed further in the next section; where it will be used to give an alternative approach to the Habiro ring of a number field.

5. Alternative approach: Habiro rings via congruences

In this final section we study the basic properties of the original Habiro ring $\mathcal{H}$ and its generalisation to the Habiro $\mathcal{H}_R$ of a ring R. The most important point is how to identify $\mathcal{H}$ explicitly within the ring of "Galois invariant functions near roots of unity" by means of a sequence of congruences. The Habiro ring $\mathcal{H}_R$ when R is the ring of integers (or S-integers) of a number field is then given by the *same* collection of congruences, but twisted by the application of a certain Frobenius automorphism at each root of unity. We will consider filtrations with finite quotients of both the "naive" Habiro ring and the space of functions near roots of unity. On these finite quotients we will explicitly describe the embedding of the Habiro ring. This will be a simple map between lattices and we will give an explicit formula for the index, which will illustrate the strength of the constraints for a function near roots of unity to be in a Habiro ring. We will illustrate the procedure on several examples. Finally, we will explore some exotic functions near roots of unity that exhibit some Habiro-like properties.

Throughout this section, the ground rings R we will consider will be associative, commutative, unital, and torsion-free rings R. The original Habiro ring will be when $R = \mathbb{Z}$ but more generally R can be thought of as the ring of integers of a number field with a finite set of primes inverted.

5.1. **Functions near roots of unity and the naive Habiro ring.** In Habiro's original work [22] the Habiro rings of $\mathbb{Z}$ and $\mathbb{Q}$ were defined as $\mathbf{H} := \mathbf{H}_{\mathbb{Z}}$ and $\mathbf{H}_{\mathbb{Q}}$, where $\mathbf{H}_R$ is defined for any ground ring R by

$$\mathbf{H}_R \;=\; \varprojlim_n R[q]/(q;q)_n R[q]\,. \tag{295}$$

(Here we use the notation $\mathbf{H}_R$, defined as an inverse limit, as opposed to $\mathcal{H}_R$, which is defined as a subset of $\mathcal{P}_R$ in Definition 1.1 of Section 1.3.) (The two rings $\mathbf{H} = \mathbf{H}_{\mathbb{Z}}$ and $\mathcal{H} = \mathcal{H}_{\mathbb{Z}}$ can be identified canonically, but in general $\mathbf{H}_R$, which we will call the *naive Habiro ring*, and $\mathcal{H}_R$ are different.) Note that $\mathbf{H}_{\mathbb{Q}} \neq \mathbf{H}_{\mathbb{Z}} \otimes \mathbb{Q}$. Indeed, the ring $\mathbf{H}_{\mathbb{Z}}$ is an integral domain while $\mathbf{H}_{\mathbb{Q}}$ has a large number of zero divisors [22, Sec. 7.5].

The definition of $\mathbf{H}_R$ leads immediately to expressions of the form (4) with $P_n(q) \in R[q]$. These expressions are not unique but can be made unique by assuming $\deg P_n < n$ so that $P_n(q) = \sum_{k=0}^{n-1} a_{n,k} q^k$ for some $a_{n,k} \in R$. (The example from equation (5) has $a_{n,k}$ given simply by $\delta_{k,0}$.) The coefficients $a_{n,k}$ give an explicit isomorphism of R-modules

$$\begin{array}{ccccc} \mathbf{H}_R & = & \prod_{n=1}^{\infty}(R \oplus qR \oplus \cdots \oplus q^{n-1}R)(q;q)_{n-1} & \cong & \prod_{n>k\geq 0} R\,, \\ \sum_{n=1}^{\infty} P_n(q)(q;q)_{n-1} & \mapsto & \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} a_{n,k} q^k (q;q)_{n-1} & \mapsto & \{a_{n,k}\}_{n>k\geq 0}\,. \end{array} \tag{296}$$

As in Section 1.2, we have a map $\iota : \mathbf{H}_R \to \mathcal{P}_R$, where $\mathcal{P}_R$ was defined in equation (8):

$$\begin{array}{ccccc} \mathcal{P}_R & := & \Big(\prod_{\zeta\in\mu_\infty} R[\zeta][\![u]\!]\Big)^{\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})} & \cong & \prod_{m\geq 1} R[\zeta_m][\![u]\!]\,, \\ f(q) & \mapsto & (f_\zeta(u) = F(\zeta(1-u)))_\zeta & \mapsto & (f_m(u) = f_{\zeta_m}(u))_m\,. \end{array} \tag{297}$$

Here, for each m, we use the local variable u defined by $q = \zeta_m(1-u)$. (In the Introduction, we used $q = \zeta_m + x$ instead, so u and x are related by $x = -\zeta_m u$ and f_m differs here from that of equation 6 by the change of variable.) The map $\iota : \mathbf{H}_R \to \mathcal{P}_R$ splits as a product of maps $\iota = \prod_{m\geq 1} \iota_m$, where for $F \in \mathbf{H}_R$, we define the map ι_m by the expansion at $q = \zeta_m(1-u)$, i.e. $\iota_m(F) = F(\zeta_m(1-u)) \in R[\zeta_m][\![u]\!]$.

Similar to the coordinates $a_{n,k}$ coming from the isomorphism (296), the R-module $\mathcal{P}_R$ has a natural basis over R given by $\{\gamma_{m,\ell,j}\}_{m,\,\ell\geq 1,\,\phi(m)>j\geq 0}$, where

$$f_m(u) \;=\; \sum_{\ell=1}^{\infty} C_\ell(\zeta_m)\, u^{\ell-1}\,, \qquad C_\ell(\zeta_m) \;=\; \sum_{j=0}^{\phi(m)-1} \gamma_{m,\ell,j}\, \zeta_m^j \;\in\; \mathbb{Z}[\zeta_m]\,. \tag{298}$$

(Again, the shift by 1 — making $C_\ell(\zeta_m)$ the coefficient of $u^{\ell-1}$ rather than u^ℓ — makes later formulas more natural, because its properties depend on the multiplicative nature of ℓ as opposed to $\ell - 1$.) We then have explicit isomorphisms of abelian groups,

$$\begin{array}{ccccc} \mathcal{P}_R & \cong & \prod_{n>0}\prod_{m|n} R[\zeta_m] & \cong & \prod_{n>0}\prod_{\substack{m\ell=n \\ \phi(m)>j\geq 0}} R\,, \\ \{f_m(u)\}_{m>0} & \mapsto & \{C_\ell(\zeta_m)\}_{m,\ell>0} & \mapsto & \{\gamma_{m,\ell,j}\}_{\substack{m,\ell>0 \\ \phi(m)>j\geq 0}}\,. \end{array} \tag{299}$$

All of these R-modules come with natural filtrations and these are compatible. From equation (296) we can naturally consider

$$\begin{aligned} \mathbf{H}_{R,N} \;:=\; (q;q)_{N-1}\mathbf{H}_R \;&=\; \left\{ \sum_{n=N}^{\infty}\sum_{k=0}^{n-1} a_{n,k}q^k(q;q)_{n-1} \;\middle|\; a_{n,k} \;\in\; R \right\} \\ &=\; \prod_{n=N}^{\infty}(R\oplus qR\oplus\cdots\oplus q^{n-1}R)(q;q)_{n-1} \;\cong\; \prod_{n\geq N,\, n>k\geq 0} R\,, \end{aligned} \tag{300}$$

i.e. $\mathbf{H}_{R,N}$ consists of sums $\sum_{n=N}^{\infty} P_n(q)(q;q)_{n-1}$, which in coordinates is described by a collection $\{a_{n,k}\}_{n>k\geq 0}$ with $a_{n,k}=0$ for $n<N$. Denote the quotient $\mathbf{H}_R/\mathbf{H}_{R,N}$ by $\mathbf{H}_R^N$. We thus have an increasing sequence $\mathbf{H}_{R,N}$ of submodules of $\mathbf{H}_R$ and a decreasing sequence $\mathbf{H}_R^N$ of quotients, i.e.

$$0 \;\subset\; \cdots \;\subset\; \mathbf{H}_{R,2} \;\subset\; \mathbf{H}_{R,1} \;=\; \mathbf{H}_R \;\twoheadrightarrow\; \cdots \;\twoheadrightarrow\; \mathbf{H}_R^2 \;\twoheadrightarrow\; \mathbf{H}_R^1 \;=\; 0\,. \tag{301}$$

Note that as R-modules,

$$\mathbf{H}_R^N \;\cong\; R[q]/(q;q)_{N-1}R[q] \;\cong\; \bigoplus_{n=1}^{N-1}(R\oplus qR\oplus\cdots\oplus q^{n-1}R)(q;q)_{n-1}\,, \tag{302}$$

where the first isomorphism is induced by the canonical inclusion $R[q]\hookrightarrow \mathbf{H}_R$. It follows that $\mathbf{H}_R = \varprojlim_N \mathbf{H}_R^N$ (or equivalently $\bigcap_N \mathbf{H}_{R,N}=\{0\}$). Note also that the exact sequence of R-modules $0\to\mathbf{H}_{R,N}\to\mathbf{H}_R\to\mathbf{H}_R^N\to 0$ splits so that $\mathbf{H}_R\cong\mathbf{H}_{R,N}\oplus\mathbf{H}_R^N$ as R-modules for every N.

We now look at how expansions near roots of unity interact with the filtration $\{\mathbf{H}_{R,N}\}_{N>0}$ on $\mathbf{H}_R$. If we note that $(q;q)_{N-1}$ vanishes at a primitive m-th root of unity ζ_m to order $\lceil N/m\rceil-1$, then we see that ι maps this filtration to the filtration $\{\mathcal{P}_{R,N}\}_{N>0}$ on $\mathcal{P}_R$ given by

$$\mathcal{P}_{R,N} \;:=\; \big\{f \;\in\; \mathcal{P}_R \mid f_m(u) \;=\; O(u^{\lceil N/m\rceil-1})\big\} \;\cong\; \prod_{n=N}^{\infty}\prod_{m|n} R[\zeta_m] \;\cong\; \prod_{n=N}^{\infty}\prod_{\substack{m\ell=n\\ \phi(m)>j\geq 0}} R\,, \tag{303}$$

where the isomorphisms are of R-modules. In coordinates, $\mathcal{P}_{R,N}$ consists of $\{C_\ell(\zeta_m)\}_{m,\ell>0}$ in $\mathcal{P}_R$ satisfying $C_\ell(\zeta_m)=0$ for $m\ell<N$. As before, we denote the quotient $\mathcal{P}/\mathcal{P}_{R,N}$ by $\mathcal{P}_R^N$. We thus have an increasing sequence $\mathcal{P}_{R,N}$ of submodules of $\mathcal{P}_R$ and a decreasing sequence $\mathcal{P}_R^N$ of quotients, just as in equation (301). Moreover, we have a splitting of R-modules so that $\mathcal{P}_R\cong\mathcal{P}_{R,N}\oplus\mathcal{P}_R^N$. Note also that we have the canonical isomorphism

$$\mathcal{P}_R^N \;\cong\; \bigoplus_{m=1}^{N}\big(R[\zeta_m][u] \;+\; \mathrm{O}(u^{\lceil N/m\rceil-1})\big)\,, \tag{304}$$

which implies that $\mathcal{P}_R=\varprojlim_N \mathcal{P}_R^N$ (or equivalently $\bigcap_N \mathcal{P}_{R,N}=\{0\}$), just as for $\mathbf{H}$. We see that $\iota(\mathbf{H}_{R,N})\subseteq\mathcal{P}_{R,N}$ and therefore we have well-defined maps

$$\iota^N : \mathbf{H}_R^N \to \mathcal{P}_R^N\,, \tag{305}$$

each factoring as $\prod_{m\geq 1}\iota_m^N$. This is an injective map between free R-modules of the same rank (equal $N(N-1)/2$), invertible after tensoring with $\mathbb{Q}$. This implies in turn that ι is an injection over R and an isomorphism over $R\otimes\mathbb{Q}$, e.g.,

$$\begin{array}{ccc} \mathbf{H}_{\mathbb{Z}} & \hookrightarrow & \mathcal{P}_{\mathbb{Z}} \\ \cap & & \cap \\ \mathbf{H}_{\mathbb{Q}} & \cong & \mathcal{P}_{\mathbb{Q}} \end{array} \tag{306}$$

for the original Habiro rings.

To prove the last statements, we consider the associated graded groups,

$$\begin{aligned} \mathcal{P}_{R,N}/\mathcal{P}_{R,N+1} \;&\cong\; \bigoplus_{m\ell=N}\big(u^{\ell-1}R[\zeta_m][\![u]\!] \,+\, \mathrm{O}(u^\ell)\big) \;\cong\; \bigoplus_{m\ell=N} R[\zeta_m]\,, \\ \mathbf{H}_{R,N}/\mathbf{H}_{R,N+1} \;&\cong\; R[q]/(1-q^N)R[q]\,, \end{aligned} \tag{307}$$

where the last isomorphism is induced from the first isomorphism of equation (302). Notice that

$$\mathrm{rk}(\mathcal{P}_{R,N}/\mathcal{P}_{R,N+1}) \;=\; \sum_{m|N}\mathrm{rk}(R[z_m]) \;=\; \sum_{m|N}\phi(m) \;=\; N \;=\; \mathrm{rk}(\mathbf{H}_{R,N}/\mathbf{H}_{R,N+1})\,, \tag{308}$$

which implies the values given in the above formulas for the ranks of $\mathbf{H}_R^N$. The map ι descends to these graded quotients. For $P(q)\in R[q]/(1-q^N)$, we multiply by $(q;q)_{N-1}$ and expand at ζ_m for $m|N$. For $q=\zeta_m(1-u)$

$$(q;q)_{m\ell-1} \;=\; D_{m,\ell}\,u^{\ell-1} + O(u^\ell), \qquad D_{m,\ell} \;:=\; m^{2\ell-1}(\ell-1)!\,, \tag{309}$$

by a simple calculation, which is left to the reader. Therefore, ι is represented on these quotients by the map

$$P(q) \;\mapsto\; \bigoplus_{m\ell=N}\big(D_{m,\ell}\,u^{\ell-1}\,P(\zeta_m) \,+\, \mathrm{O}(u^\ell)\big)\,. \tag{310}$$

Since $D_{m,\ell}\in\mathbb{Z}_{>0}$ is a non-zero integer, it follows that ι is an injection on these quotients and an isomorphism after tensoring with $\mathbb{Q}$. (Indeed, here we have used that $P(q)\in(1-q^N)\mathbb{Z}[q]$ if and only if $P(\zeta_m)=0$ for all $m|N$.) Given that ι respects the grading and is injective on the associated graded pieces, it is injective. This fact implies that understanding $\mathbf{H}_R$ amounts to understanding the image of ι, which was the philosophy used in Definition 1.1 of Section 1.3.

We now restate the above considerations in terms of the R-bases $\{q^k(q;q)_{n-1}\}_{k\leq 0,\,n>0}$ and $\{\zeta_m^j u^{\ell-1}\}$ and the corresponding coordinates $\{a_{n,k}\}$ and $\{\gamma_{m,\ell,j}\}$ for $\mathbf{H}_R$ and $\mathcal{P}_R$ as defined in Equations (296) and (299), respectively. These bases give rise to a square matrix $\mathbf{M}_N$ representing the map ι^N. This matrix has integers entries and is independent of the ring R. (This is equivalent to noticing that the choice of basis and coordinates imply isomorphisms $\mathbf{H}_R\cong\mathbf{H}_{\mathbb{Z}}\otimes_{\mathbb{Z}}R$ and $\mathcal{P}_R\cong\mathcal{P}_{\mathbb{Z}}\otimes_{\mathbb{Z}}R$, with $\iota_R=\iota_{\mathbb{Z}}\otimes_{\mathbb{Z}}1$, with $\mathbf{M}_N$ represents $\iota_{\mathbb{Z}}^N$.) Also, since ι^N respects the filtrations, the matrix $\mathbf{M}_N$ is in block lower triangular form, as illustrated for

$N = 5$ by the following equation.

$$\begin{pmatrix} \gamma_{1,1,0} \\ \hline \gamma_{2,1,0} \\ \gamma_{1,2,0} \\ \hline \gamma_{3,1,0} \\ \gamma_{3,1,1} \\ \gamma_{1,3,0} \\ \hline \gamma_{4,1,0} \\ \gamma_{4,1,1} \\ \gamma_{2,2,0} \\ \gamma_{1,4,0} \end{pmatrix} = \left(\begin{array}{c|ccc|ccc|cccc} 1 & & & & & & & & & & \\ \hline 1 & 2 & -2 & & & & & & & & \\ 0 & 1 & 1 & & & & & & & & \\ \hline 1 & 1 & 1 & 3 & 0 & -3 & & & & & \\ 0 & -1 & 2 & 0 & 3 & -3 & & & & & \\ 0 & 0 & -1 & 2 & 2 & 2 & & & & & \\ \hline 1 & 1 & 1 & 2 & 2 & -2 & 4 & 0 & -4 & 0 \\ 0 & -1 & 1 & -2 & 2 & 2 & 0 & 4 & 0 & -4 \\ 0 & -1 & 3 & 4 & -4 & 4 & 8 & -8 & 8 & -8 \\ 0 & 0 & 0 & -1 & -3 & -5 & 6 & 6 & 6 & 6 \end{array}\right) \begin{pmatrix} a_{1,0} \\ \hline a_{2,0} \\ a_{2,1} \\ \hline a_{3,0} \\ a_{3,1} \\ a_{3,2} \\ \hline a_{4,0} \\ a_{4,1} \\ a_{4,2} \\ a_{4,3} \end{pmatrix}. \tag{311}$$

Finally, we can give an explict formula for the absolute value of determinant of $\mathbf{M}_N$.

Proposition 5.1. *The number $D(N) := |\det(\mathbf{M}_N)|$ is given by*

$$D(N) \;=\; \prod_{n=1}^{N} D_1(n) D_2(n)\,, \tag{312}$$

where

$$D_1(n) \;=\; \prod_{m\ell=n} D_{m,\ell}^{\phi(m)} \;=\; \prod_{m\ell=n} m^{(2\ell-1)\phi(m)}(\ell-1)!^{\phi(m)} \tag{313}$$

with $D_{m,\ell}$ defined as in equation (309) and

$$D_2(n) \;=\; n^{\frac{n}{2}} \prod_{m|n} |\mathrm{disc}(\Phi_m)|^{-\frac{1}{2}} \;=\; \prod_{m|n} \Big(\frac{n}{m} \prod_{p|m\,,\,p \text{ prime}} p^{\frac{1}{p-1}}\Big)^{\frac{\phi(m)}{2}}. \tag{314}$$

Note that $D_2(n)$ is an integer since if $p|m$ then $\phi(m)$ is divisible by $2(p-1)$ except for $m = p^k, 2p^k$, where $\phi(p^k) = \phi(2p^k)$, which can be seen to be integral for both even and odd n.

We have tabulated the first few of these numbers below. One can see their extremely rapid growth. This illustrates how small the Habiro ring is inside the set of functions near roots of unity with integral Taylor expansions.

N	1	2	3	4	5	6	7	8	9
$D_1(N)$	1	2	18	768	15000	2.0×10^8	8.5×10^7	6.5×10^{13}	5.1×10^{15}
$D_2(N)$	1	2	3	8	5	72	7	128	81
$D(N)$	1	4	216	1327104	99532800000	1.4×10^{21}	8.6×10^{29}	7.1×10^{45}	3.0×10^{63}

These numbers are not only very large, but also highly factored. Here are two bigger examples illustrating both properties for $D(N)$:

$$\begin{aligned} D(25) \;&=\; 2^{1050} \cdot 3^{469} \cdot 5^{255} \cdot 7^{118} \cdot 11^{71} \cdot 13^{25} \cdot 17^{25} \cdot 19^{25} \cdot 23^{25} \;\approx\; 2.35 \times 10^{1016}\,, \\ D(100) \;&=\; 2^{26102} \cdot 3^{12800} \cdot 5^{6404} \cdot 7^{3965} \cdot 11^{2158} \cdot 13^{1514} \cdot 17^{961} \cdot 19^{983} \cdot 23^{763} \cdot 29^{497} \\ &\quad \times 31^{503} \cdot 37^{285} \cdot 41^{289} \cdot 43^{291} \cdot 47^{295} \cdot 53^{100} \cdot 59^{100} \cdot 61^{100} \cdot 67^{100} \cdot 71^{100} \\ &\quad \times 73^{100} \cdot 79^{100} \cdot 83^{100} \cdot 89^{100} \cdot 97^{100} \;\approx\; 9.33 \times 10^{34419}\,. \end{aligned} \tag{315}$$

Since the nuymber D(N) is the cardinality of the cokernel of $\iota_{\mathbb{Z}}^N$, we see that begin in the image of ι^N is equivalent to a set of congruences on the coordinates $\gamma_{m,\ell,j}$ to a total modulus $D(N)$. In particular, we have a well-defined map

$$(\iota^N)^{-1} : \mathcal{P}_R/\mathcal{P}_R^N \;\to\; D(N)^{-1}R[q]/(q;q)_{N-1}R[q]\,, \tag{316}$$

and the condition for a given element $H \in \mathcal{P}_R$ to be the image of ι is that

$$(\iota^N)^{-1}(H) \;\in\; R[q]/(q;q)_{N-1}R[q] \tag{317}$$

for all $N \in \mathbb{Z}_{>0}$. We can express this in terms of matrices as follows:

Proposition 5.2. *If $H \in \mathcal{P}_R$ with coordinates $\gamma_{m,\ell,j} \in R$ $(m, \ell \geq 1, 0 \leq j < \phi(m))$, then $H \in \iota(\mathbf{H}_R)$ if and only if*

$$\mathbf{M}_N^*\Big((\gamma_{m,\ell,j})_{\substack{m\ell<N\\ 0\leq j<\phi(m)}}\Big) \;\equiv\; 0 \pmod{D(N)} \tag{318}$$

for all $N \in \mathbb{Z}_{>0}$, where $\mathbf{M}_N^* := D(N)\,\mathbf{M}_N^{-1} \in M_{N(N-1)/2}(\mathbb{Z})$.

Note that $D(N)$ and $\mathbf{M}_N^*$ have many common factors, which simplifies $\mathbf{M}_N^{-1}$. This means that in practice the congruences that one constructs can be simplified. For example, using the previous coordinates in equation (296) and equation (299), we find that for $N = 5$ these conditions become

$$\left(\begin{array}{c|cc|ccc|cccc}
1 & & & & & & & & & \\
\hline
-1 & 1 & 2 & & & & & & & \\
1 & -1 & 2 & & & & & & & \\
\hline
-7 & -9 & -6 & 16 & -8 & 12 & & & & \\
-1 & 9 & 6 & -8 & 16 & 12 & & & & \\
17 & -9 & 18 & -8 & -8 & 12 & & & & \\
\hline
-13 & 9 & -13 & -32 & 0 & 0 & 36 & 0 & 9 & 12 \\
-14 & -18 & -1 & 32 & -32 & 12 & 0 & 36 & -9 & 12 \\
9 & 27 & 23 & 0 & 32 & 24 & -36 & 0 & 9 & 12 \\
68 & -36 & 59 & -32 & 0 & 36 & 0 & -36 & -9 & 12
\end{array}\right)
\left(\begin{array}{c}
\gamma_{1,1,0} \\ \hline \gamma_{2,1,0} \\ \gamma_{1,2,0} \\ \hline \gamma_{3,1,0} \\ \gamma_{3,1,1} \\ \gamma_{1,3,0} \\ \hline \gamma_{4,1,0} \\ \gamma_{4,1,1} \\ \gamma_{2,2,0} \\ \gamma_{1,4,0}
\end{array}\right)
\in
\left(\begin{array}{c}
R \\ \hline 4\,R \\ 4\,R \\ \hline 72\,R \\ 72\,R \\ 72\,R \\ \hline 288\,R \\ 288\,R \\ 288\,R \\ 288\,R
\end{array}\right). \tag{319}$$

Summarising, we have found an explicit inductive set of congruences that the coefficients of an element of $\mathcal{P}_R$ must satisfy to be an element of the image of ι of the naive Habiro ring of R, the first few of these congruences (corresponding to $N = 4$) being

$$\begin{aligned}
-\gamma_{1,1,0} + \gamma_{2,1,0} + 2\gamma_{1,2,0} &\equiv 0 \pmod{4}\,,\\
\gamma_{1,1,0} - \gamma_{2,1,0} + 2\gamma_{1,2,0} &\equiv 0 \pmod{4}\,,\\
-7\gamma_{1,1,0} - 9\gamma_{2,1,0} - 6\gamma_{1,2,0} + 16\gamma_{3,1,0} - 8\gamma_{3,1,1} + 12\gamma_{1,3,0} &\equiv 0 \pmod{72}\,,\\
-\gamma_{1,1,0} + 9\gamma_{2,1,0} + 6\gamma_{1,2,0} - 8\gamma_{3,1,0} + 16\gamma_{3,1,1} + 12\gamma_{1,3,0} &\equiv 0 \pmod{72}\,,\\
17\gamma_{1,1,0} - 9\gamma_{2,1,0} + 18\gamma_{1,2,0} - 8\gamma_{3,1,0} - 8\gamma_{3,1,1} + 12\gamma_{1,3,0} &\equiv 0 \pmod{72}\,.
\end{aligned} \tag{320}$$

These equations are not independent and we can reduce them to

$$\begin{aligned}
\gamma_{1,1,0} - \gamma_{2,1,0} + 2\gamma_{1,2,0} + 4\gamma_{1,3,0} &\equiv 0 \pmod{8}\,,\\
\gamma_{1,1,0} - \gamma_{3,1,0} - \gamma_{3,1,1} - 3\gamma_{1,3,0} &\equiv 0 \pmod{9}\,,\\
\gamma_{1,2,0} + \gamma_{3,1,1} &\equiv 0 \pmod{3}\,.
\end{aligned} \tag{321}$$

Those familiar with the Habiro ring will easily recognise these congruences as those discovered by Ohtsuki [33]. Indeed, we find that they are equivalent to the beginning of the expansions

$$\gamma_{2,1,0} \;=\; \sum_{k=0}^{\infty} \gamma_{1,k,0}\, 2^k \in R_2^{\wedge}\,, \quad \text{and} \quad \gamma_{3,1,0} \,+\, \zeta_3\, \gamma_{3,1,1} \;=\; \sum_{k=0}^{\infty} \gamma_{1,k,0}\, (1 \,-\, \zeta_3)^k \in R_3^{\wedge}[\zeta_3]\,. \quad (322)$$

5.2. **The Habiro ring of a number field.** In the previous subsection, although everything was done over an arbitrary ground ring R, the considerations were really based on the case when $R = \mathbb{Z}$ (the original Habiro ring), because the congruences needed to recognise whether an element of $\mathcal{P}_R$ comes from the naive Habiro ring of R were independent of R and came from this special case. In this subsection, we will consider more general ground rings and describe the relation between the naive Habiro ring and the Habiro ring studied in Sections 1–4. We will explain that the natural definition involves twisting the compatibility conditions by suitable Frobenius automorphisms similar to those that appeared in Definition 1.1 of Section 1.3. We will focus on ground rings given by rings of integers (or S-integers, meaning that we adjoin the reciprocal of a non-zero integer) of a number field, or various completions of this.

Let R be the ring $\mathcal{O}_{\mathbb{K}}[1/\Delta]$, where $\mathcal{O}_{\mathbb{K}}$ is the ring of integers of a number field $\mathbb{K}$ and Δ any non-zero integer divisible by the discriminant of $\mathbb{K}$. Let p be a prime that does not divide Δ and is therefore unramified, so that

$$p \;=\; \mathfrak{p}_1 \cdots \mathfrak{p}_n\,, \qquad (323)$$

with the ideals $\mathfrak{p}_i$ distinct. This gives rise to an isomorphism

$$R/pR \;\cong\; R/\mathfrak{p}_1 R \times \cdots \times R/\mathfrak{p}_n R\,, \qquad (324)$$

where each $R/\mathfrak{p}_i R \cong \mathbb{F}_{q_i}$ with $q_i = |R/\mathfrak{p}_i R|$ is a finite field with cyclic Galois group over $\mathbb{F}_p$ generated by the Frobenius automorphism

$$x \mapsto x^p\,. \qquad (325)$$

The Frobenius automorphisms on each factor $R/\mathfrak{p}_i R$ give rise to a Frobenius automorphism $\varphi_p : R/pR \cong R/pR$ given by the same formula. Hensel's lemma states that if a polynomial factors into irreducible polynomials over $\mathbb{Z}/p\mathbb{Z}$, then this factorisation lifts to a unique factorisation over $\mathbb{Z}/p^n\mathbb{Z}$. Therefore, if ξ is a generator of the field $\mathbb{K}$ with minimal polynomial $P(x)$, then applying Hensel's lemma to $P(x)$ lifts the Frobenius automorphism of R/pR to an automorphism of $\varphi_p : R/p^nR \cong R/p^nR$ for all $n \in \mathbb{Z}_{>0}$. Hensel's lemma is completely constructive, as was explained in Section 4.4. These Frobenius lifts can be used to define automorphisms for any R/MR for $M \in \mathbb{Z}$ prime to the discriminant of $\mathbb{K}$. To do this, we write $M = \prod_{p \text{ prime}} p^{d_p}$. The Chinese remainder theorem gives a canonical isomorphism

$$R/MR \;\cong\; \prod_{p \text{ prime}} R/p^{d_p}R\,. \qquad (326)$$

We can define $\varphi_p : R/MR \to R/MR$ to be the Frobenius automorphism on the p-th factor and the identity on the others. Then for $m \in \mathbb{Z}_{>0}$ (or even $m \in \mathbb{Q}^{\times}$) we define

$$\varphi_m \;=\; \prod_{p} \varphi_p^{v_p(m)} : R/MR \to R/MR\,, \qquad (327)$$

where φ_p on R/MR is simply the identity if p does not divide M. This can then be used to give Frobenius automorphisms on the completion of the integers over all unramified primes at once. Consider the completion

$$\widehat{R} \;:=\; \varprojlim_{M} R/MR \;\cong\; \prod_{p \text{ prime}} R_p\,, \tag{328}$$

where $R_p := \varprojlim_n R/p^nR \cong R \otimes \mathbb{Z}_p$. Notice that if $p|\Delta$ then $R_p = \{0\}$ is the zero ring. This completion then has Frobenius automorphisms for each $m \in \mathbb{Q}^\times$

$$\varphi_m : \widehat{R} \to \widehat{R}\,. \tag{329}$$

We can lift φ_m uniquely to an automorphism $\varphi_m : \widehat{R}[\zeta_n][\![x]\!] \to \widehat{R}[\zeta_n][\![x]\!]$ acting trivially on ζ_n and x. These then combine to an automorphism $\varphi : \mathcal{P}_{\widehat{R}} \to \mathcal{P}_{\widehat{R}}$ defined by

$$(\varphi\, f)_m \;=\; \varphi_m(f_m(x)) \;\in\; \widehat{R}[\zeta_m][\![x]\!]\,. \tag{330}$$

We will also use φ to denote the composite map $\mathcal{P}_R \hookrightarrow \mathcal{P}_{\widehat{R}} \xrightarrow{\varphi} \mathcal{P}_{\widehat{R}}$. We can now restate the definition of the Habiro ring (Definition 1.1 of Section 1.3) as follows.

Proposition 5.3. *The Habiro $\mathcal{H}_R$ of $R = \mathcal{O}_{\mathbb{K}}[1/\Delta]$ is given by*

$$\mathcal{H}_R \;=\; \left\{ H \in \mathcal{P}_R \;\middle|\; \varphi(H) \in \iota(\mathbf{H}_{\widehat{R}}) \right\}, \tag{331}$$

where $\mathbf{H}_{\widehat{R}}$ is defined as in (295). Equivalently, $\mathcal{H}_R = \mathcal{P}_R \cap \mathcal{H}_{\widehat{R}}$ with $\mathcal{H}_{\widehat{R}} = \varphi^{-1}(\iota(\mathbf{H}_{\widehat{R}}))$.

Using Proposition 5.2 of Section 5.1, we can reformulate this concretely as follows:

Proposition 5.4. *Let $H \in \mathcal{P}_R$ with coordinates $\gamma_{m,\ell,j} \in R$ $(m, \ell \geq 1, 0 \leq j < \phi(m))$. Then $H \in \mathcal{H}_R$ if and only if*

$$\mathbf{M}_N^*\left((\varphi_m \gamma_{m,\ell,j})_{\substack{m\ell<N \\ 0\leq j<\phi(m)}} \right) \;\equiv\; 0 \pmod{D(N)} \tag{332}$$

for all $N \in \mathbb{Z}_{>0}$, where $\mathbf{M}_N^*$ is the matrix defined in Proposition 5.2 of Section 5.1.

This gives an easy numerical check of whether a function near roots of unity is an element of the Habiro ring of R.

Remark 5.5. In this section, we have only treated the Habiro ring $\mathcal{H}_R$ and not its modules $\mathcal{H}_{R,\xi}$ indexed by $\xi \in K_3(\mathbb{K})$. Given that Theorem 1 implies the completion $\mathcal{H}_{\widehat{R},\xi}$ is a free $\mathcal{H}_{\widehat{R}}$-module, we can use similar descriptions to that of Proposition 5.4 to describe the modules by simply dividing elements by a chosen generator of $\mathcal{H}_{\widehat{R},\xi}$.

5.3. **Examples.** We now explore how the approach described in last two subsections works for some different ground rings. We will illustrate how one can test whether a Galois invariant function near roots of unity is actually an element of a Habiro ring.

Example 5.6. Recall the Kashaev invariant of the trefoil from equation (5) (often called the Kontsevich-Zagier series). We can compute the first few terms in the expansions at

$\zeta_1, \zeta_2, \zeta_3, \zeta_4$ (with $q = \zeta_m(1-u)$)

$$\begin{aligned} F_{3_1,1}(u) &= 1 + u + 2u^2 + 5u^3 + \mathrm{O}(u^4)\,, \\ F_{3_1,2}(u) &= 3 + 11u + \mathrm{O}(u^2)\,, \\ F_{3_1,3}(u) &= 5 - \zeta_3 + \mathrm{O}(u)\,, \\ F_{3_1,4}(u) &= 8 - 3\zeta_4 + \mathrm{O}(u)\,. \end{aligned} \tag{333}$$

These give rise to the vector

$$\begin{array}{rcccccccccc} (& \gamma_{1,1,0} \mid & \gamma_{2,1,0} & \gamma_{1,2,0} \mid & \gamma_{3,1,0} & \gamma_{3,1,1} & \gamma_{1,3,0} \mid & \gamma_{4,1,0} & \gamma_{4,1,1} & \gamma_{2,2,0} & \gamma_{1,4,0}\ ) \\ = (& 1 \mid & 3 & 1 \mid & 5 & -1 & 2 \mid & 8 & -3 & 11 & 5\ )\,, \end{array} \tag{334}$$

which when multiplied by $\mathbf{M}_5^{-1}$ gives the expected

$$\begin{array}{rcccccccccc} (& a_{1,0} \mid & a_{2,0} & a_{2,1} \mid & a_{3,0} & a_{3,1} & a_{3,2} \mid & a_{4,0} & a_{4,1} & a_{4,2} & a_{4,3}) \\ = (& 1 \mid & 1 & 0 \mid & 1 & 0 & 0 \mid & 1 & 0 & 0 & 0)\,. \end{array} \tag{335}$$

To see just how delicate the property of being in the Habiro ring is, notice that simply changing the number $\gamma_{1,1,0}$ from 1 to 2 by would give the sequence

$$\Big(2 \;\Big|\; \frac{3}{4} \quad \frac{1}{4} \;\Big|\; \frac{65}{72} \quad \frac{-1}{72} \quad \frac{17}{72} \;\Big|\; \frac{275}{288} \quad \frac{-7}{144} \quad \frac{1}{32} \quad \frac{17}{72} \Big)\,, \tag{336}$$

where the integrality has been completely ruined.

Example 5.7. In this example we will see what the introduction of denominators does to the Habiro ring. The Habiro function $F(q) \in \mathbf{H}_{\mathbb{Q}}$ whose image under ι is the collection of (constant) power series

$$F(\zeta_m(1-u)) \;=\; \begin{cases} 1 & \text{if } m \text{ is odd}\,, \\ 0 & \text{if } m \text{ is even} \end{cases} \tag{337}$$

does not belong to $\mathbf{H}_{\mathbb{Z}}$, but is an element of $\mathbf{H}_{\mathbb{Z}[1/2]}$. Indeed, it can be written

$$F(q) \;=\; 1 + \frac{1}{4}(-1+q)\,(q;q)_1 + \frac{1}{8}(1-q+q^2)\,(q;q)_2 + \frac{1}{32}(-5+2q+q^2+4q^3)\,(q;q)_3 + \cdots\,. \tag{338}$$

Similarly, the element $G(q) = F(q^2) - F(q)$, whose image under ι is the collection of (constant) power series

$$G(\zeta_m(1-u)) \;=\; \begin{cases} 1 & \text{if } m \equiv 2 \pmod 4 \\ 0 & \text{otherwise}\,, \end{cases} \tag{339}$$

is also an element of $\mathbf{H}_{\mathbb{Z}[1/2]}$, and can be written

$$G(q) \;=\; \frac{1}{4}(1-q)\,(q;q)_1 + \frac{1}{8}(-1+q-q^2)\,(q;q)_2 + \frac{1}{32}(1-2q+3q^2-4q^3)\,(q;q)_3 + \cdots\,. \tag{340}$$

Example 5.8. Our final example is an element of the Habiro ring of the ring of integers, with 1/23 adjoined, of the field $\mathbb{K} = \mathbb{Q}(\alpha)/(\alpha^3 - \alpha^2 + 1)$, the cubic field of discriminant

-23, associated to the Kashaev invariant of the knot 5_2. We define a collection of series $(\Psi_{5_2,m})_{m\geq 1}$ using the residue formulas given in Section 4.1. Specifically, we define

$$\Psi_{5_2,m}(u) \;:=\; \operatorname*{Res}_{w^m=z} J_{5_2}(1,w,\zeta_m(1-u))\frac{dw}{w}\,, \tag{341}$$

where

$$J_{5_2}(t,w,q) \;:=\; \sum_{k=0}^{\infty}\frac{q^{k(k+1)}\,w^{2k}}{(qw;q)_k^3}t^k \tag{342}$$

and set $z=1-\alpha^2$ so that $z^2(1-z)^{-3}=1$. The collection $\Psi_{5_2,m}$ is related to the symmetrised series $f^{(5_2)}(q)\,f^{(5_2)}(q^{-1})$ discussed in Section 4.6. For example, equation (282) is equal to $(3\alpha-2)\Psi_{5_2,1}(-x)$. One can compute the first few coordinates via the expansions

$$\begin{aligned}
\Psi_{5_2,1}(u) \;&=\; \frac{1}{23}(-9\alpha^2+3\alpha+2)+\frac{1}{23^4}(444\alpha^2+3417\alpha-1287)u^2\\
&\qquad +\frac{1}{23^4}(444\alpha^2+3417\alpha-1287)u^3+\mathrm{O}(u^4)\,,\\
\Psi_{5_2,2}(u) \;&=\; \frac{1}{23}(-43\alpha^2+22\alpha+7)+\mathrm{O}(u^2)\,,\\
\Psi_{5_2,3}(u) \;&=\; \frac{1}{23}(-111\alpha^2+60\alpha+17)+\mathrm{O}(u)\,,\\
\Psi_{5_2,4}(u) \;&=\; \frac{1}{23}(-242\alpha^2+119\alpha+41)+\mathrm{O}(u)\,,
\end{aligned} \tag{343}$$

giving rise to the vector

$$\begin{pmatrix} \gamma_{1,1,0} \\ \hline \gamma_{2,1,0} \\ \gamma_{1,2,0} \\ \hline \gamma_{3,1,0} \\ \gamma_{3,1,1} \\ \gamma_{1,3,0} \\ \hline \gamma_{4,1,0} \\ \gamma_{4,1,1} \\ \gamma_{2,2,0} \\ \gamma_{1,4,0} \end{pmatrix} = \begin{pmatrix} \frac{1}{23}(-9\alpha^2+3\alpha+2) \\ \hline \frac{1}{23}(-43\alpha^2+22\alpha+7) \\ 0 \\ \hline \frac{1}{23}(-111\alpha^2+60\alpha+17) \\ 0 \\ \frac{1}{23^4}(444\alpha^2+3417\alpha-1287) \\ \hline \frac{1}{23}(-242\alpha^2+119\alpha+41) \\ 0 \\ 0 \\ \frac{1}{23^4}(444\alpha^2+3417\alpha-1287) \end{pmatrix}. \tag{344}$$

(That the coordinates here are in $\mathbb{K}$ as opposed to $\mathbb{K}[\zeta_m]$ is because $\phi(m)\leq 2$ for $m<5$ and Ψ_{5_2} is a symmetrisation; in general, the coordinates are in $\mathbb{K}[\zeta_m+\zeta_m^{-1}]$.) This vector when multiplied by $\mathbf{M}_5^{-1}$ gives the vector

$$\begin{pmatrix} \frac{1}{23}(-9\alpha^2+3\alpha+2) \\ \hline \frac{1}{92}(-34\alpha^2+19\alpha+5) \\ \frac{1}{92}(34\alpha^2-19\alpha-5) \\ \hline \frac{1}{6716184}(-5376038\alpha^2+3018917\alpha+785707) \\ \frac{1}{6716184}(2070166\alpha^2-1142197\alpha-309323) \\ \frac{1}{6716184}(4552234\alpha^2-2529235\alpha-674333) \\ \hline \frac{1}{26864736}(-22020494\alpha^2+10246115\alpha+3924793) \\ \frac{1}{13432368}(-5376038\alpha^2+3018917\alpha+785707) \\ \frac{1}{8954912}(10100386\alpha^2-4938301\alpha-1720695) \\ \frac{1}{6716184}(4552234\alpha^2-2529235\alpha-674333) \end{pmatrix}, \tag{345}$$

which has denominator $26864736 = 2^5 \cdot 3 \cdot 23^4$. If we instead apply the correct Frobenius automorphism and then multiply by $\mathbf{M}_5^{-1}$ we find

$$\begin{aligned} \begin{pmatrix} \gamma_{1,1,0} \\ \hline \varphi_2(\gamma_{2,1,0}) \\ \hline \gamma_{1,2,0} \\ \hline \varphi_3(\gamma_{3,1,0}) \\ \varphi_3(\gamma_{3,1,1}) \\ \hline \gamma_{1,3,0} \\ \hline \varphi_4(\gamma_{4,1,0}) \\ \varphi_4(\gamma_{4,1,1}) \\ \varphi_2(\gamma_{2,2,0}) \\ \gamma_{1,4,0} \end{pmatrix} &= \frac{1}{23^4} \begin{pmatrix} 225\alpha^2+213\alpha+142 \\ 241\alpha^2+113\alpha+74 \\ 0 \\ 279\alpha^2+132\alpha+151 \\ 0 \\ 156\alpha^2+249\alpha+153 \\ 161\alpha^2+115\alpha+4 \\ 0 \\ 0 \\ 156\alpha^2+249\alpha+153 \end{pmatrix} \pmod{288}, \\ \begin{pmatrix} a_{1,0} \\ \hline a_{2,0} \\ a_{2,1} \\ \hline a_{3,0} \\ a_{3,1} \\ a_{3,2} \\ \hline a_{4,0} \\ a_{4,1} \\ a_{4,2} \\ a_{4,3} \end{pmatrix} &= \frac{1}{23^4} \begin{pmatrix} 225\alpha^2+213\alpha+142 \pmod{288} \\ 4\alpha^2+47\alpha+55 \pmod{72} \\ 68\alpha^2+25\alpha+17 \pmod{72} \\ 0 \pmod{4} \\ 2\alpha^2+2\alpha \pmod{4} \\ 2\alpha^2+3\alpha+1 \pmod{4} \\ 0 \pmod{1} \\ 0 \pmod{1} \\ 0 \pmod{1} \\ 0 \pmod{1} \end{pmatrix}, \end{aligned} \tag{346}$$

where now everything is integral away from the prime 23.

5.4. **Some wilder Habiro-like elements.** In this final subsection we describe some exotic functions that behave like elements of Habiro rings that involve different fields at each root of unity. On the face of it, it could seem that this would make patching together series via p-adic re-expansion impossible. The reason that it nevertheless works is that the fields in question become canonically isomorphic after completion and the gluing of equation (13) can take place. We will explain this in a simple example, but it could be generalised easily by increasing the dimensions of the sums involved. These elements will generalise the construction given in Section 4.1.

Take a polynomial $P(X) \in \mathbb{Z}[X]$ and set

$$f_P(w,t;q) \;=\; \sum_{k=0}^{\infty}\biggl(\prod_{j=0}^{k-1} P(q^j w)\biggr)\, t^k\,. \tag{347}$$

We would like to take $t = 1$. However, the function does not converge there, just as in the case of admissible series studied in Section 2. As in Theorem 4 of Section 1.6, the x-expansion of $f_P(w, t; \zeta + x)$ for a root of unity ζ has coefficients contained in rational functions in t and w. Therefore, we can specialise to $t = 1$. For example, near $\zeta = 1$ we find that

$$\begin{aligned} f_P(w,t;1+x) \;&=\; \sum_{k=0}^{\infty}\biggl(\prod_{j=0}^{k-1} P(q^j w)\biggr) t^k \\ &=\; \sum_{k=0}^{\infty} t^k\Bigl(P(w)^k + \binom{k}{2} P(w)^{k-1}\, P'(w) x + \mathrm{O}(x^2)\Bigr) \\ &=\; \frac{1}{1-tP(w)} \;+\; \frac{tP'(w)}{(1-tP(w))^3}x \;+\; \mathrm{O}(x^2)\,, \end{aligned} \tag{348}$$

in which we can set $t = 1$ to find

$$f_P(w,1;1+x) \;=\; \frac{1}{1-P(w)} \;+\; \frac{P'(w)}{(1-P(w))^3}x \;+\; \mathrm{O}(x^2)\,. \tag{349}$$

Similarly, at any primitive m-th root of unity ζ_m, one can show that there are polynomials $A_{k,m}(w) \in \mathbb{Z}[\zeta_m, w]$ so that

$$f_P(w,1;\zeta_m + x) \;=\; \sum_{k=0}^{\infty} \frac{A_{k,m}(w)}{(1-P_m(w^m))^{2k+1}} x^k\,, \tag{350}$$

where

$$P_m(w) \;=\; \prod_{j=1}^{m} P(\zeta_m^j w^{1/m}) \;=\; \prod_{u^m = w} P(u)\,. \tag{351}$$

Notice that if (and only if) $P(w)$ has the special form

$$P(w) \;=\; w^a (1-w)^b\,, \tag{352}$$

for some $a, b \in \mathbb{Z}$, then

$$P_m(w) \;=\; P(w)\,, \tag{353}$$

for all m, giving functions similar to those considered in Section 4.1. Going back to the general case, if we expand $f_P(w, 1; \zeta_m + x)$ near $w = \zeta_m^j \alpha_m^{1/m}$, where $P_m(\alpha_m) = 1$, we get a

Laurent series. Therefore, we can take the residue of these Laurent expansions, just as was done in Section 4.1, to define series

$$f_{P,m}(x) \;=\; \sum_{z^m=\alpha_m} \operatorname*{Res}_{w=z} f_P(w,1;\zeta+x)\frac{dw}{w} \;\in\; \mathcal{O}_{P,m}[\zeta][\![x]\!]\,, \tag{354}$$

where $\mathcal{O}_{P,m} = \mathcal{O}_{\mathbb{Q}(\alpha_m)}\big[\frac{1}{\mathrm{disc}(1-P_m)}\big]$. The most important property is that for p not dividing Δ, sending α_{pm} to α_m modulo p gives a canonical isomorphism

$$\mathcal{O}_{P,pm}/p\mathcal{O}_{P,pm} \;\cong\; \mathcal{O}_{P,m}/p\mathcal{O}_{P,m}\,. \tag{355}$$

Hensel's lemma implies that there exists a unique lift to an isomorphism from $\mathcal{O}_{P,pm}/p^n\mathcal{O}_{P,pm}$ to $\mathcal{O}_{P,m}/p^n\mathcal{O}_{P,m}$ for all n, which is exactly what we need for the gluing of equation (13).

Example 5.9. Take $P(X) = -X^3+8$. Then

$$P_m(X) \;=\; \begin{cases} -X^{3m}+8^m & \text{if } m\equiv 1,2 \pmod 3\,,\\ -X^{3m}+3\cdot 2^m X^{2m} - 3\cdot 4^m X^m + 8^m & \text{if } m \equiv 0 \pmod 3\,. \end{cases} \tag{356}$$

Therefore, we see that $\mathcal{O}_{P,m}$ is a cubic ring generated by ξ satisfying

$$\begin{aligned} \xi^3-(8^m-1) \;&=\; 0 && \text{if } m\equiv 1,2 \pmod 3\,,\\ \xi^3 - 3\cdot 2^m\xi^2 + 3\cdot 4^m\xi - (8^m-1) \;&=\; 0 && \text{if } m\equiv 0 \pmod 3\,. \end{aligned} \tag{357}$$

Note that

$$\xi^3 - 3\cdot 2^m\xi^2 + 3\cdot 4^m\xi - (8^m-1) \;=\; (\xi+1-2^m)(\xi^2-(2^{m+1}-1)\xi + 4^m + 2^m + 1)\,. \tag{358}$$

This makes obvious the isomorphism of equation (355), since for example $\xi^3-(8^{pm}-1) \equiv \xi^3-(8^m-1)$ modulo p. In this case we can compute and surprisingly (though for the moment only conjecturally) find that

$$f_{P,m}(x) \;=\; \begin{cases} \frac{1}{21} & \text{if } m\equiv 1,2 \pmod 3\,,\\ 0 & \text{if } m\equiv 0 \pmod 3\,, \end{cases} \tag{359}$$

at least for small values of m. If this is true, then $7f_P$ belongs to $\mathbf{H}_{\mathbb{Z}[1/3]}$ and its image under ι to $\mathcal{H}_{\mathbb{Z}[1/3]}$.

Example 5.10. In the previous case, our "exotic" construction gave (at least conjecturally) an element of an ordinary Habiro ring. We take another example to see some more interesting behaviour. Choose $P(X) = -X^3+7X^2-14$. Then we find that

$$\begin{aligned} f_{P,1}(x) \;&=\; \frac{1}{14505}(21\xi_1^2 - 49\xi_1 - 551)\\ &\quad + \frac{1}{983690367661125}(7160238883\xi_1^2 - 22515890236\xi_1 - 64413491205)x^2 + \cdots\,,\\ f_{P,2}(x) \;&=\; \frac{1}{15179085}(11711\xi_2^2 - 534814\xi_2 + 555516) + \cdots\,, \end{aligned} \tag{360}$$

where ξ_1 and ξ_2 are cubic irrationalities defined by

$$\xi_1^3 - 7\xi_1^2 + 15 \;=\; 0\,, \qquad \xi_2^3 - 49\xi_2^2 + 196\xi_2 - 195 \;=\; 0 \tag{361}$$

Again we expect Ohtsuki congruences modulo p^n between the series around roots of unity whose orders are related by multiplication by the prime p. In this example we can check the first Ohtsuki congruence between the expansions around $q = 1$ and -1. This will be a congruence modulo 2. Specifically, this first Ohtsuki congruence is equivalent to the equality

$$f_{P,1}(-2) \equiv \xi^2 + \xi + 1 \equiv f_{P,2}(0)^2 \equiv \varphi_2(f_{P,2}(0)) \pmod{2}\,, \tag{362}$$

where $\xi^3 + \xi^2 + 1 \equiv 0$ modulo 2.

To prove identities of this kind one can use the simple q-holonomic methods used to prove Theorem 5 of Section 1.6.

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