AUTOMORPHIC PSEUDODIFFERENTIAL OPERATORS

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To the memory of Irene Dorfman

Introduction.

The theme of this paper is the correspondence between classical modular forms and pseudodifferential operators (ΨDO's) which have some kind of automorphic behaviour. In the simplest case, this correspondence is as follows. Let Γ be a discrete subgroup of $PSL_2(\mathbb{R})$, acting on the complex upper half-plane $\mathcal{H}$ in the usual way, and $f(z)$ a modular form of even weight k on Γ. Then there is a canonical lifting from f to a Γ-invariant ΨDO with leading term $f(z)\,\partial^{-k/2}$, where ∂ is the differential operator $\frac{d}{dz}$. This lifting and the fact that the product of two invariant ΨDO's is again an invariant ΨDO imply a non-commutative multiplicative structure on the space of all modular forms whose components are scalar multiples of the so-called Rankin-Cohen brackets (canonical bilinear maps on the space of modular forms on Γ defined by certain bilinear combinations of derivatives; the definition will be recalled later). This was already discussed briefly in the earlier paper [Z], where it was given as one of several "raisons d'être" for the Rankin-Cohen brackets.

The basic lifting from modular forms to invariant ΨDO's can be interpreted and developed in many ways. We shall discuss some of them in this paper. The two main generalizations are as follows:

(I) Just as one generalizes the notion of a modular function to the notion of a modular form, one can consider ΨDO's which are not invariant with respect to Γ but instead transform with some automorphy factor. Because of the non-commutativity of ΨDO's, however, we have new possibilities which do not occur in the classical case: one can consider "conjugate-automorphic" ΨDO's which under the action of a fractional linear transformation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ are multiplied by a $(cz+d)^{\kappa}$ on the left and by $(cz+d)^{-\kappa}$ on the right for some κ, or "automorphic ΨDO's of mixed weight" which transform by different automorphy factors on the left and right. The first way leads to a whole family of multiplications on the space of modular forms on Γ, each of which can be expressed in terms of the Rankin-Cohen brackets, but with coefficients which turn out to be intricate combinatorial expressions having beautiful and surprising properties. The second way gives even more structure on the space of modular forms and provides the clearest conceptual framework for the Rankin-Cohen brackets.

(II) The whole theory has a supersymmetric analogue. This is a natural generalization for the following reason. One of the disadvantages of the usual theory is that

The first author (PBC) was supported by the Institute for Advanced Study, Princeton, New Jersey, USA, NSF grant number DMS 9304580, by the Centre National de Recherche Scientifique, Lille, France and by the School MPCE, Macquarie University, NSW, Australia.

the derivative of the fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$ is $(cz+d)^{-2}$ and hence that there is no coupling between modular forms of even and odd weight: not only is the derivative of a modular form not quite modular (which is why the theory is so complicated), but its a weight is larger than the weight of the original form by 2 rather than by 1. But in the supersymmetric context, one has available a superdifferentiation operator D with square equal to d/dz and super-fractional linear transformations whose automorphy factor reduces modulo nilpotents to $(cz+d)^{-1}$ and hence effectively raises the weight of (super)modular forms by 1. Specifically, in the supercomplex plane $\mathbb{C}^{1|1}$ one has one even coordinate z and one odd one ζ, with $z\zeta = \zeta z$, $\zeta^2 = 0$, so a superanalytic function has the form $F(z,\zeta) = f(z) + g(z)\zeta$ with f and g holomorphic functions of z. The differential operator $D = \frac{\partial}{\partial \zeta} + \zeta \frac{\partial}{\partial z}$ sends F to $g(z) + f'(z)\zeta$, so that $D^2 = \partial$ as claimed; and we get the desired theory by working with ΨDO's based on powers of D rather than of ∂.

The structure of the paper is as follows. In §1 we define ΨDO's and give the basic result about lifting modular forms to invariant ΨDO's. In §2 we describe other proofs and interpretations of that result and a generalization to ΨDO's with non-integral powers of ∂. The next few sections treat point (I) above: in §§3–5 we define canonical liftings of modular forms to various kinds of automorphic ΨDO's and describe the induced multiplications on the space of modular forms explicitly in terms of Rankin-Cohen brackets, and in §6 we give a conceptual proof (in terms of the non-commutative residue map and the duality between modular forms of weights k and $2-k$) of the surprising symmetries exhibited by the numerical coefficients appearing in these formulas. Point (II), the supersymmetric generalization of the theory, is treated in §7. We explain the superanalogues of modular forms and of ΨDO's and state and prove the superanalogue of the basic lifting property.

The last section contains some scattered remarks and questions. Whereas in the main body of the paper we described our constructions in the context of classical automorphic forms, here we try to put them in the framework of the theory of completely integrable Hamiltonian systems to which Irene Dorfman made a significant contribution (see e. g. [GD]).

§1. Lifting modular forms to pseudodifferential operators.

Let z be a local coordinate for $\mathbb{C}$. We have the associated differential operator $\partial = \frac{d}{dz}$ which transforms under a coordinate change $z \mapsto \tilde{z}$ as $\partial = \partial(\tilde{z}) \cdot \tilde{\partial}$. Let R be a ring of functions on $\mathbb{C}$ on which ∂ acts, so that the pair (R, ∂) is a ring with derivation. By a *pseudodifferential operator* (ΨDO) over R we will mean a formal Laurent series in the formal inverse ∂^{-1} of ∂ with coefficients in R, i.e. an element of the vector space

$$\Psi\mathrm{DO}(R) = \left\{ \sum_{n \in \mathbb{Z}} h_n \partial^{-n} : h_n \in R, \quad h_n = 0 \text{ if } n \ll 0 \right\}. \tag{1.1}$$

The subspace $\mathrm{DO}(R)$ of *differential operators* over R, consisting of sums as in (1.1) but with $n \geq 0$, is a ring under composition, and the formula for the multiplication of differential operators implied by Leibniz's rule, viz.,

$$\left(\sum_n g_n(z)\, \partial^n \right) \left(\sum_m h_m(z)\, \partial^m \right) = \sum_{n,m} \sum_{r \geq 0} \binom{n}{r} g_n\, \partial^r(h_m)\, \partial^{n+m-r}, \tag{1.2}$$

can be extended to the full space $\Psi\mathrm{DO}(R)$ if we remember that for $l \in \mathbb{Z}_{\geq 0}$ the binomial coefficient $\binom{w}{l} = w(w-1)\cdots(w-l+1)/l!$ is a polynomial in w and hence is defined for any integral (or even complex) value of w. We have an increasing filtration of $\Psi\mathrm{DO}(R)$ by the subspaces

$$\Psi\mathrm{DO}(R)_w = \left\{ \sum_{n=0}^{\infty} f_n \partial^{w-n}, \quad f_n \in R \right\} \tag{1.3}$$

with $w \in \mathbb{Z}$. It follows from formula (1.2) that this filtration is compatible with the ring structure in the sense that

$$\Psi\mathrm{DO}(R)_{w_1} \cdot \Psi\mathrm{DO}(R)_{w_2} \subseteq \Psi\mathrm{DO}(R)_{w_1+w_2} \qquad \forall w_1,\, w_2\,. \tag{1.4}$$

In particular, the subspace $\Psi\mathrm{DO}(R)_0$ of pure ΨDO's is a subring of $\Psi\mathrm{DO}(R)$, and $\Psi\mathrm{DO}(R)$ has an (additive) direct sum decomposition as $\Psi\mathrm{DO}(R)_{-1} \oplus \mathrm{DO}(R)$. We have a short exact sequence

$$0 \to \Psi\mathrm{DO}(R)_{w-1} \to \Psi\mathrm{DO}(R)_w \to R \to 0 \tag{1.5}$$

for every w, where the final map sends $\sum_{m\geq 0} f_m \partial^{w-m}$ to f_0 (symbol map).

We shall be interested in the behavior of ΨDO's under (groups of) transformations of the coordinate z. Under a coordinate change $z \mapsto \tilde{z}$ the differentiation operator ∂ is transformed to $\tilde{\partial} = j^{-1}\,\partial$, where $j = d\tilde{z}/dz$ is the Jacobian of the transformation, and there is a corresponding action on ΨDO's (cf. [KZ1])

$$\tilde{\partial}^w = j^{-w}\partial^w - \tbinom{w}{2} j' j^{-w-1}\partial^{w-1} + [3\tbinom{w+1}{4} j'^2 + \tbinom{w}{3} j j''] j^{-w-2}\partial^{w-2} + \cdots \tag{1.6}$$

(prove this by induction on w for $w \in \mathbb{Z}_{\geq 0}$, and then extend to all w). In particular, the exact sequence (1.5) is equivariant with respect to coordinate transforms if we define the action on the last term by $f(z) \mapsto j^{-w} f(\tilde{z})$.

If the coordinate change is a fractional linear transformation $\tilde{z} = g(z) = \frac{az+b}{cz+d}$ with $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(\mathbb{C})$, then $j = (cz+d)^{-2}$, all the terms multiplying ∂^{w-n} in (1.6) become proportional, and the equation simplifies to

$$\tilde{\partial}^w = [(cz+d)^2\,\partial]^w = \sum_{n=0}^{\infty} n!\, \tbinom{w}{n} \tbinom{w-1}{n}\, c^n (cz+d)^{2w-n}\, \partial^{w-n}\,. \tag{1.7}$$

(Again one proves this by induction for $w \in \mathbb{Z}_{\geq 0}$ and then extends to other values of w.) The action on the symbol, and hence on the last term in the sequence (1.5), is the classical action $f \mapsto f|_{-2w} g$, where $f|_k$ is defined for $k \in \mathbb{Z}$ by $(f|_k g)(z) := (cz+d)^{-k} f\big(\frac{az+b}{cz+d}\big)$. If we have a group $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$, acting on R via its fractional linear action on $\mathbb{C}$, then we will denote by $M_k(R,\Gamma)$ or simply by $M_k(\Gamma)$ the space of invariants of R under the action $f \mapsto f|_k$ of Γ. If we take for R the ring $\mathcal{F}$ of all holomorphic functions in the complex upper half-plane $\mathcal{H}$ which are bounded by a power of $(|z|^2+1)/\Im(z)$, and Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ of finite covolume, then $M_k(\Gamma)$ is the usual space of holomorphic modular forms on Γ and is finite-dimensional for all $k \in \mathbb{Z}$ and zero for $k < 0$, but we can also take larger rings of functions (like the ring of all holomorphic functions in $\mathcal{H}$, or all those of at most exponential growth at the cusps) to allow modular forms of negative weight. By taking Γ-invariants in (1.5) we get (with $k = -w$) a sequence

$$0 \to \Psi\mathrm{DO}(R)^{\Gamma}_{-k-1} \to \Psi\mathrm{DO}(R)^{\Gamma}_{-k} \to M_{2k}(\Gamma) \to 0 \tag{1.8}$$

which is exact except perhaps for the last arrow. The basic fact studied in this paper is the following proposition, which says that (1.5) has a canonical equivariant splitting and hence that the sequence (1.8) is exact and splits canonically.

Proposition 1. *For $k \geq 1$ define an operator $\mathcal{L}_k : R \to \Psi\mathrm{DO}(R)_{-k}$ by*

$$\mathcal{L}_k(f) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+k)!\,(n+k-1)!}{n!\,(n+2k-1)!}\, f^{(n)} \partial^{-k-n}$$

and an operator $\mathcal{L}_{-k} : R \to \mathrm{DO}(R)_k$ by

$$\mathcal{L}_{-k}(f) = \sum_{n=0}^{k-1} \frac{(2k-n)!}{n!\,(k-n)!\,(k-n-1)!}\, f^{(n)}\, \partial^{k-n}\,,$$

and set $\mathcal{L}_0(f) = f$. Then $\mathcal{L}_k(f|_{2k}g) = \mathcal{L}_k(f) \circ g$ for all $g \in \mathrm{PSL}(2,\mathbb{C})$ and all $k \in \mathbb{Z}$. In particular, if $f \in M_{2k}(\Gamma)$ for some subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ then $\mathcal{L}_k(f) \in \Psi\mathrm{DO}(R)^{\Gamma}_{-k}$.

Proof. Write $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By induction on n we have the formula

$$\frac{d^n}{dz^n}\big(f|_k g(z)\big) = \sum_{r=0}^{n} \frac{n!}{r!} \binom{k+n-1}{n-r} \frac{(-c)^{n-r}}{(cz+d)^{k+n+r}}\, f^{(r)}\Big(\frac{az+b}{cz+d}\Big) \tag{1.9}$$

for any $k \in \mathbb{Z}$ and any $n \geq 0$, where $f^{(r)}$ denotes $\partial^r f$ as usual. From this and (1.7) we find that for $k > 0$ both $\mathcal{L}_k(f|_{2k}g)(z)$ and $(\mathcal{L}_k(f) \circ g)(z)$ are equal to

$$\sum_{r,\,m \geq 0} \frac{(m+r+k)!\,(m+r+k-1)!}{m!\,r!\,(2k+r-1)!} \frac{(-1)^r\, c^m}{(cz+d)^{2k+m}}\, f^{(r)}\Big(\frac{az+b}{cz+d}\Big)\, \partial^{-k-m-r}\,.$$

The proof for $k < 0$, is similar, and the proof for $k = 0$ is of course trivial. ■

§2. Interpretations and extensions of the basic lifting.

In this section we discuss some further aspects of the proposition just proved. In particular we describe the relationship between modular forms, invariant ΨDO's, and "Jacobi-like forms" (this was the point of view taken in [Z1]), give a different and more conceptual proof of Proposition 1 in terms of the Casimir operator for $\mathrm{sl}(2,\mathbb{C})$, and describe an extension to generalized ΨDO's where one allows non-integral powers of ∂.

Jacobi-like forms. One interpretation of the lifting from modular forms with respect to Γ to Γ-invariant ΨDO's is to identify both spaces with the space $\mathcal{J}(\Gamma)$ of *Jacobi-like forms*, namely power series $\Phi(z,X) \in R[[X]]$ satisfying the transformation law

$$\Phi\Big(\frac{az+b}{cz+d}, \frac{X}{(cz+d)^2}\Big) = e^{cX/(cz+d)}\, \Phi(z,X) \qquad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\,. \tag{2.1}$$

(Here Γ is a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and R a Γ-invariant ring of functions in $\mathcal{H}$, e.g. the ring $\mathcal{F}$ defined in §1.) This space is filtered by the subspaces $\mathcal{J}(\Gamma)_k = \mathcal{J}(\Gamma) \cap X^k R[[X]]$. Clearly, if $\Phi(z,X)$ belongs to $\mathcal{J}(\Gamma)_k$ and has leading term $f(z)X^k$, then $f|_{2k}\gamma = f$ for all $\gamma \in \Gamma$, so we have a sequence

$$0 \to \mathcal{J}(\Gamma)_{k+1} \to \mathcal{J}(\Gamma)_k \to M_{2k}(\Gamma) \to 0 \tag{2.2}$$

which is exact except possibly at the last place. The following proposition, which is a sharpening of Prop. 1 for the case of positive weights, says that this sequence splits and is canonically isomorphic to the split short exact sequence (1.8).

Proposition 2. *Let* $\phi_k = \phi_k(z)$ $(k = 1, 2, \dots)$ *be elements of* R. *Then the following are equivalent:*

(1) $\Phi(z, X) := \sum_{k=1}^{\infty} \phi_k(z)\, X^k \;\in\; \mathcal{J}(\Gamma)$;

(2) $\psi(z) := \sum_{k=1}^{\infty} (-1)^k\, k!\, (k-1)!\, \phi_k(z)\, \partial^{-k} \;\in\; \Psi\mathrm{DO}(R)^\Gamma$;

(3) $\phi_k|_{2k}\gamma(z) = \sum_{n=0}^{k-1} \frac{1}{n!} \big(\frac{c}{cz+d}\big)^n \phi_{k-n}(z)$ *for all* $k \geq 1$ *and all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$;

(4) $\sum_{r=0}^{k-1} (-1)^r \frac{(2k-2-r)!}{r!}\, \phi_{k-r}^{(r)}(z) \;\in\; M_{2k}(\Gamma)$ *for all* $k \geq 1$;

(5) $\phi_n(z) = \sum_{r=0}^{n-1} \frac{1}{r!\,(2n-r-1)!}\, f_{n-r}^{(r)}(z)$ *where* $f_k \in M_{2k}(\Gamma) \quad (\forall k \geq 1)$.

Proof. One checks that each of the properties in question is equivalent to the transformation law (3). For (1) this is obvious from the definition (2.1), for (2) it follows directly from (1.7), and for (5) it follows from (1.9). Property (4) can be checked the same way or we can note that by a simple binomial coefficient identity it is equivalent to (5) if we define f_k to be $2k-1$ times the sum in (4). ∎

We can restate the result of Proposition 2 in the following way. Denote by $\mathcal{M}(\Gamma)_+ = \prod_{n>0} M_{2n}(\Gamma)$ the space of sequences of modular forms of positive weights, with the trivial filtration by the subspaces $\mathcal{M}(\Gamma)_k = \prod_{n\geq k} M_{2n}(\Gamma)$, with successive quotients $\mathcal{M}(\Gamma)_k/\mathcal{M}(\Gamma)_{k-1} = M_{2k}(\Gamma)$. Proposition 2 says that $\mathcal{M}(\Gamma)_1$ is canonically isomorphic as a filtered vector space to both the space $\mathcal{J}(\Gamma)_1$ of Jacobi-like forms with no constant term and the space $\Psi\mathrm{DO}(R)_1^\Gamma$, the correspondence sending the sequence $(f_1, f_2, \dots)$ $(f_k \in M_{2k}(\Gamma))$ to the elements $\Phi \in \mathcal{J}(\Gamma)_1$ and $\psi \in \Psi\mathrm{DO}(R)_1^\Gamma$ defined by (1) and (2), respectively. Note also that, by linearity, the Jacobi-like property of Φ and the Γ-invariance of ψ need only be checked in the case when there is only a single non-zero f_k. In this case, writing f for f_k, we find that the Φ is simply the *Cohen-Kuznetsov lifting*

$$\tilde{f}(z, X) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!(n+2k-1)!}\, X^{n+k} \tag{2.3}$$

of f whose Jacobi-like property was discovered in [Ku] and [Co], while ψ is precisely the lifting $\mathcal{L}_k(f)$ of Proposition 1.

The reason for the name "Jacobi-like," by the way, is that the space $\mathcal{J}(\Gamma)_k$ can be identified via $\Phi(z, 2\pi i m u^2) = u^{2k}\phi(z, u)$ with the set of all $\phi(z, u) \in R[[u^2]]$ satisfying $\phi(\frac{az+b}{cz+d}, \frac{u}{cz+d}) = (cz+d)^{2k}\, e^{2\pi i c m u^2/(cz+d)}\, \phi(z, X)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and this is one of the two transformation laws characterizing Jacobi forms of weight k and index m in the sense of [EZ].

The Casimir operator. The proof of Proposition 1 by direct computation as given in §1 is very short, but not particularly enlightening. We now describe another way to see the existence (and uniqueness) of the equivariant splitting map $\mathcal{L}_k$ which was pointed out to us by Beilinson. Let $\mathrm{SL}(2, \mathbb{C})$ act by fractional linear transformations as usual. The action of its Lie algebra $\mathrm{sl}(2, \mathbb{C})$ is then given by the three vector fields $L_j = z^{j+1}\partial$ $(j = -1, 0, 1)$, with Lie bracket given by commutation. There is an induced operation of $\mathrm{sl}(2, \mathbb{C})$ on ΨDO's by commutation (adjoint representation). Explicitly, we have $L_{-1}(f\partial^w) = [\partial, f\partial^w] = f'\partial^w$ and similarly $L_0(f\partial^w) = (zf' - wf)\partial^w$, $L_1(f\partial^w) = (z^2f' - 2wzf)\partial^w - w(w+1)zf\partial^{w-1}$, so a short computation shows that the Casimir operator

$$C = L_0^2 - \frac{1}{2}(L_1L_{-1} + L_{-1}L_1)\,,$$

which acts trivially on functions, acts on ΨDO's by

$$C(f\partial^w) = w(w+1)f\partial^w + w(w-1)f'\partial^{w-1}\,.$$

In particular, the induced action of C on the quotient $\Psi\mathrm{DO}(R)_w/\Psi\mathrm{DO}(R)_{w-1} \cong R$ in (1.5) is multiplication by $w(w+1)$, so if there is any equivariant splitting of this sequence then the lift ψ of $f \in R$ to $\Psi\mathrm{DO}(R)_w$ must be an eigenvector of C with eigenvalue $w(w+1)$. Writing $w = -k$ and $\psi(z) = \sum_{n=0}^{\infty} f_n(z)\,\partial^{-k-n}$, we find

$$\big[C - k(k-1)\big]\,\psi = \sum_{n=1}^{\infty}\big[n(n+2k-1)\,f_n + (n+k)(n+k-1)\,f'_{n-1}\big]\,\partial^{-k-n}\,,$$

and equating all coefficients of this to 0 we find by induction that each f_n is a multiple of the nth derivative $f^{(n)}$ with coefficients as given in Proposition 1. (To get exactly the lift $\mathcal{L}_k$ we must normalize by taking $f_0 = \lambda_k f$ with $\lambda_k = \binom{2k-1}{k}^{-1}$ if $k \ge 0$ and $\lambda_k = \frac{(2|k|)!}{|k|!(|k|-1)!}$ if $k < 0$.)

Generalized pseudodifferential operators. Since a ΨDO is defined as a formal expression anyway, one can allow symbols ∂^w with arbitrary complex powers w. Both the transformation property (1.6) of ΨDO's under changes of variables and the rule (1.2) for multiplying ΨDO's involve together with each power ∂^w all lower powers ∂^{w-n} with n a positive integer, so we again define $\Psi\mathrm{DO}(R)_w$ for any $w \in \mathbb{C}$ by equation (1.3) and define a generalized ΨDO as an element of any such space or a finite sum of such elements [KZ1]. Because formula (1.2) involves only binomial coefficients whose lower index is a nonnegative integer, and hence makes (formal) sense even for non-integral m and n, the space $\Psi\mathrm{DO}(R)_{\mathbb{C}}$ of generalized ΨDO's is a ring just as before, formula (1.4) still holds, and there is a direct sum decomposition

$$\Psi\mathrm{DO}(R)_{\mathbb{C}} = \bigoplus_{w\in\mathbb{C}/\mathbb{Z}} \Psi\mathrm{DO}(R)_{w+\mathbb{Z}}\,, \qquad \Psi\mathrm{DO}(R)_{w+\mathbb{Z}} := \bigcup_{k\in\mathbb{Z}} \Psi\mathrm{DO}(R)_{w+k}\,.$$

The summand $\Psi\mathrm{DO}(R)_{\mathbb{Z}}$ is the ring $\Psi\mathrm{DO}(R)$ previously considered and each other summand $\Psi\mathrm{DO}(R)_{w+\mathbb{Z}}$ is a module over this ring and is filtered by the subspaces $\Psi\mathrm{DO}(R)_{w+n}$ ($n \in \mathbb{Z}$), and we again have the exact sequence (1.5).

Formula (1.6) defines the behavior of the generalized ΨDO's under coordinate changes (again the binomial coefficients make sense even for w non-integral), and formula (1.7) their behavior under the action of $SL(2,\mathbb{C})$. Of course there is now a problem because the quantity j^{-w} or $(cz+d)^{2w}$ is not uniquely defined for w non-integral. This can be overcome in several ways. In the case when R is a space of functions on the upper half-plane $\mathcal{H}$, we replace the group $\mathrm{SL}(2,\mathbb{R})$ by its universal covering, consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ together with a choice of logarithm of $cz+d$ in $\mathcal{H}$, and take for Γ a subgroup of this covering which maps isomorphically onto a discrete co-finite volume subgroup of $\mathrm{SL}(2,\mathbb{R})$. In this case the elements of $M_{-2w}(\Gamma)$ are essentially what are classically known as modular forms with multiplier systems. This does not work for $\mathrm{SL}(2,\mathbb{C})$ acting on $\mathbb{P}^1$ because then there is no global logarithm of $cz+d$. But actually, as we could have pointed out even when looking at the case of integral weight, there is no reason that we have to work with a ring R of functions defined globally on all of $\mathcal{H}$ or all of $\mathbb{P}^1$: all of the considerations in §1 were local, so in the formulas of that section we could always have considered ΨDO's $\psi(z)=\sum f_n(z)\partial^{w-n}$ defined for z in some open subset of $\mathbb{C}$, and coordinate changes $z\mapsto\tilde z$ mapping this set to some possibly different open subset. On a simply connected open set on which j or $cz+d$ has no zeros or poles, we can choose a branch of j^{-w} or $(cz+d)^{2w}$ and make sense of all the formulas we have been writing. The correct language to describe all of this would be that of sheaves of D-modules over Riemann surfaces with a projective structure (i.e. having an atlas such that the coordinate transformation maps between charts are fractional linear), as will be discussed in §8. For now we will ignore this issue and use the same terminology as before, with the understanding that the results have to be interpreted in one of the ways just indicated. Proposition 1 then generalizes to the following result.

Proposition 3. *Let $w\in\mathbb{C}$, $2w$ not a nonnegative integer. Then the map*

$$\mathcal{D}_w : R\to \Psi\mathrm{DO}(R)_w\,,\qquad \mathcal{D}_w(f)=\sum_{n=0}^{\infty}\frac{\binom{w}{n}\binom{w-1}{n}}{\binom{2w}{n}}\,f^{(n)}\,\partial^{w-n} \tag{2.4}$$

satisfies $\mathcal{D}_w(f|_{-2w}g)=\mathcal{D}_w(f)\circ g$ for all $g\in\mathrm{SL}(2,\mathbb{C})$, so $\mathcal{D}_w$ gives an equivariant splitting of the exact sequence (1.5)*. If w is a nonnegative integer, then the same assertion remains true if the sum in the definition of $\mathcal{D}_w$ is replaced by a sum from $n=0$ to $n=w$. If w is a positive half-integer, then there is no equivariant splitting of the sequence* (1.5)*.*

Proof. For $w\in\mathbb{Z}$ this is the same as the statement of Proposition 1, since one easily checks that $\mathcal{D}_{-k}=\lambda_k\mathcal{L}_k$ for $k\in\mathbb{Z}$, with λ_k defined as at the end of the last subsection. Both the proof by direct computation given in §1 and the proof given above using the Casimir operator given above apply unchanged for general w (with the change of notation that we again use w instead of $k=-w$, which was more convenient before because classical modular forms have positive weight). The proof using the Casimir operator showed the uniqueness of the lift and hence also its non-existence in the case when w is a positive half-integer (corresponding to modular forms of odd negative weight), since the recursive relation $n(n-2w-1)\,f_n=-(n-w)(n-w-1)\,f'_{n-1}$ cannot be solved in general for $n=2w+1$. In this case there is a lifting if and only if f is a polynomial of weight $\le 2w$. ■

§3. A non-commutative multiplication of modular forms.

Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. As explained in the last section, one interpretation of Proposition 1 is that $\Psi\mathrm{DO}(R)^\Gamma$ is canonically isomorphic to the space $\mathcal{M}(\Gamma)=\prod_{k\gg-\infty}M_{2k}(\Gamma)$ of semi-infinite sequences of modular forms on Γ (i.e. sequences $f_k\in M_{2k}(\Gamma)$ with $f_k=0$ for all but finitely many negative k; in the first subsection of §2 we looked only at the subspace $\mathcal{M}(\Gamma)_+$ of sequences of forms of positive weight). On the other hand, the product of Γ-invariant ΨDO's is again Γ-invariant, so there is an induced non-commutative ring structure on $\mathcal{M}(\Gamma)$. In this section we describe it explicitly in terms of the "Rankin-Cohen brackets." These are the bilinear maps

$$[\ ,\ ]_n=[\ ,\ ]_n^{(k,l)}:M_{2k}\otimes M_{2l}\to M_{2k+2l+2n}\qquad (k,\,l\in\mathbb{Z}\,,\quad n\in\mathbb{Z}_{\ge0}\,)$$

defined by the formula

$$[f,g]_n^{(k,l)}(z) = \sum_{m=0}^{n} (-1)^m \binom{2k+n-1}{n-m} \binom{2l+n-1}{m} f^{(m)}(z)\, g^{(n-m)}(z)\,. \qquad (3.1)$$

(Here $\phi^{(m)} = \partial^m \phi$ as usual, and we have dropped the Γ in the notation for spaces of modular forms; we will also usually omit the superscripts "(k,l)" on the brackets except when necessary for clarity, since we will always apply them with superscripts equal to half the weights of the arguments.) They were introduced and shown to be modular in 1974 by H. Cohen [Co], this result being a special case of a general theorem of Rankin [Ra] describing all multilinear differential operators which send modular forms to other modular forms. The easiest proof of the modularity of $[f,g]_n$ is to use the Cohen-Kuznetsov lifting (2.3) from modular forms to Jacobi-like forms: the transformation law (2.1) shows that the product $\tilde{f}(z,-X)$ and $\tilde{g}(z,X)$ is invariant under $(z,X) \mapsto \left(\frac{az+b}{cz+d}, \frac{X}{(cz+d)^2}\right)$, which means that the coefficient of X^{k+l+n} in this product is modular of weight $2k+2l+2n$, and this coefficient is just a scalar multiple of $[f,g]_n$. It is also easy to see that the combination (3.1) is the only universal bilinear combination of derivatives of f and g which goes from $M_{2k} \otimes M_{2l}$ to $M_{2k+2l+2n}$.

Proposition 4. *For integers n, k, $l \geq 0$ define coefficients $t_n(k,l)$ by*

$$t_n(k,l) = \frac{1}{\binom{-2l}{n}} \sum_{r+s=n} \frac{\binom{-k}{r}\binom{-k-1}{r}}{\binom{-2k}{r}} \frac{\binom{n+k+l}{s}\binom{n+k+l-1}{s}}{\binom{2n+2k+2l-2}{s}}\,. \qquad (3.2)$$

Then the multiplication μ on $\mathcal{M}(\Gamma)$ defined by

$$\mu(f,g) = \sum_{n=0}^{\infty} t_n(k,l)\, [f,g]_n^{(k,l)} \qquad \big(f \in M_{2k}(\Gamma),\ g \in M_{2l}(\Gamma)\big)$$

is associative and the lifting map $\mathcal{D} = \prod_w \mathcal{D}_w : \mathcal{M}(\Gamma) \to \Psi\mathrm{DO}(R)^\Gamma$ is a ring homomorphism with respect to this multiplication.

Proof. As already mentioned, the isomorphism between $\mathcal{M}(\Gamma)$ and $\Psi\mathrm{DO}(R)^\Gamma$ permits us to transfer the non-commutative structure on the latter space to the former one, i.e., to associate to $f \in M_{2k}$ and $g \in M_{2l}$ a unique sequence of elements $h_n \in M_{2k+2l+2n}$ $(n = 0, 1, \dots)$ such that $\mathcal{D}_{-k}(f)\, \mathcal{D}_{-l}(g) = \sum_{n=0}^{\infty} \mathcal{D}_{-k-l-n}(h_n)$. The map $(f,g) \mapsto h_n$ from $M_{2k} \otimes M_{2l}$ to $M_{2k+2l+2n}$ is expressed by a universal formula as a linear combination of products of the first n derivatives of f and g, so by the uniqueness mentioned above it must be a multiple of the Rankin-Cohen bracket, i.e. we have $h_n = t_n\, [f,g]_n$ for all $n \geq 0$, where the coefficient t_n depends only on n and on the weights k and l. Substituting the definitions of the Rankin-Cohen brackets and of $\mathcal{D}$ and multiplying everything out, we obtain a rather complicated identity which overdetermines the coefficients t_n: for each $m \geq 0$ the comparison of the coefficients of $f^{(n)} g^{(m)} \partial^{-k-l-n-m}$ on the two sides of the equation for $n = 0, 1, 2, \dots$ gives an infinite sequence of equations which inductively determine the coefficients t_n. For $m = 0$ these equations are

$$\frac{\binom{n+k}{n}\binom{n+k-1}{n}}{\binom{n+2k-1}{n}} = \sum_{r+s=n} \frac{\binom{2l+r-1}{r}\binom{n+k+l}{s}\binom{n+k+l-1}{s}}{\binom{n+r+2k+2l-1}{s}}\, t_r\,,$$

and this can easily be inverted to yield the formula for $t_n = t_n(k,l)$ given in the proposition. ■

Computing the first three coefficients t_n from (3.2), we find

$$t_0 = 1, \qquad t_1 = -\frac{1}{4}, \qquad t_2 = \frac{1}{16}\left(1 + \frac{3}{(2k+1)(2l+1)(2k+2l+1)}\right), \tag{3.3}$$

and computing a few more coefficients we are led to conjecture the formula

$$t_n(k,l) = (-\frac{1}{4})^n \sum_{j\geq 0} \binom{n}{2j} \frac{\binom{-\frac{3}{2}}{j}\binom{-\frac{1}{2}}{j}\binom{\frac{1}{2}}{j}}{\binom{-k-\frac{1}{2}}{j}\binom{-l-\frac{1}{2}}{j}\binom{n+k+l-\frac{3}{2}}{j}} \,. \tag{3.4}$$

(Note that the sum on the right is finite since $\binom{n}{2j}$ vanishes for $j > n/2$.) The equivalence of (3.2) and (3.4) is a special case of the following result, whose fairly complicated proof will be given in a separate paper [Z2].

IDENTITY. *For an integer $n \geq 0$ and variables X, Y, Z satisfying $X+Y+Z = n-1$, we have*

$$\frac{(-4)^n}{\binom{2X}{n}} \sum_{r+s=n} \frac{\binom{Y}{r}\binom{Y-1}{r}}{\binom{2Y}{r}} \frac{\binom{Z}{s}\binom{Z+1}{s}}{\binom{2Z}{s}} = \sum_{j\geq 0} \binom{n}{2j} \frac{\binom{-\frac{3}{2}}{j}\binom{-\frac{1}{2}}{j}\binom{\frac{1}{2}}{j}}{\binom{X-\frac{1}{2}}{j}\binom{Y-\frac{1}{2}}{j}\binom{Z-\frac{1}{2}}{j}} \,. \tag{3.5}$$

A first corollary of the identity (3.4) is that the coefficient $t_n(k,l)$ is symmetric in k and l, a property which is not immediately obvious from the definition (the product of ΨDO's is neither commutative nor anti-commutative) and is not at all obvious from the closed formula (3.2). But in fact there is an even less obvious three-fold symmetry which is seen best in the formulation (3.5): even though the expression on the left apparently has a slightly different dependence on Y and Z, and a totally different dependence on X, the identity shows that it is in fact symmetric in all three variables. Going back to the special case (3.4), we see that this is equivalent to the symmetries

$$t_n(k,l) = t_n(l,k) = t_n(k, 1-n-k-l) \qquad \forall k,\, l \in \mathbb{Z}, \quad n \in \mathbb{Z}_{\geq 0} \,. \tag{3.6}$$

(This makes sense because the denominators in (3.4) do not vanish for any integral values of k and l and the sum is finite, so that $t_n(k,l)$ is defined for all $k,\, l \in \mathbb{Z}$.) An explanation of this symmetry in terms of residues will be given in §6.

§4. Conjugate-automorphic ΨDO's and new multiplications on $\mathcal{M}(\Gamma)$.

In this section we will show how to generalise the above discussion to produce a whole family of new associative multiplications. The starting point for this was an observation by W. Eholzer, who discovered (and verified for the first few terms of the expansion) that the anti-commutative bracket

$$[f,g]_E \;:=\; \sum_{n \text{ odd}} [f,g]_n \tag{4.1}$$

satisfies the Jacobi identity and hence equips $\mathcal{M}(\Gamma)$ with the structure of a Lie algebra. Since the nth Rankin-Cohen bracket is $(-1)^n$-symmetric, the bracket $[f,g]_E$ is just the odd part $\frac{1}{2}(f*g - g*f)$ of the *Eholzer product*

$$f * g \;:=\; \sum_{n=0}^{\infty} [f,g]_n \,. \tag{4.2}$$

so Eholzer's observation suggested the following result:

Proposition 5. *The multiplication $*$ on $\mathcal{M}(\Gamma)$ defined by* (4.2) *is associative.*

Comparing this statement with Proposition 4, we see that both have the same form, except that the complicated coefficients $t_n = t_n(k,l)$ defined by (3.2) or (3.4) are replaced simply by 1. On the other hand, from the special cases in (3.3) we see that the coefficients $(-4)^n t_n$ (where the factor $(-4)^n$ of course does not affect the associativity of the product $\sum t_n [f,g]_n$) are a kind of "small deformation" of 1. This suggested that there might be a whole family of multiplications of $\mathcal{M}(\Gamma)$ of which both Propositions 4 and 5 are specializations, and after a fair amount of experimentation a formula which worked empirically was discovered:

Theorem 1. *For $\kappa \in \mathbb{C}$ define coefficients $t_n^\kappa(k,l)$ $(n = 0, 1, 2, \ldots)$ by*

$$t_n^\kappa(k,l) = \left(-\frac{1}{4}\right)^n \sum_{j \geq 0} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j}\binom{\kappa-\frac{3}{2}}{j}\binom{\frac{1}{2}-\kappa}{j}}{\binom{-k-\frac{1}{2}}{j}\binom{-l-\frac{1}{2}}{j}\binom{n+k+l-\frac{3}{2}}{j}} . \tag{4.3}$$

Then the multiplication μ^κ on $\mathcal{M}(\Gamma)$ defined by

$$\mu^\kappa(f,g) = \sum_{n=0}^{\infty} t_n^\kappa(k,l)\, [f,g]_n^{(k,l)} \qquad \big(f \in M_{2k}(\Gamma),\ g \in M_{2l}(\Gamma)\big) \tag{4.4}$$

is associative.

The first few coefficients $t_n = t_n^{(s)}(k,l)$ are

$$t_0 = 1, \qquad t_1 = -\frac{1}{4}, \qquad t_2 = \frac{1}{16}\left(1 + \frac{(1-2\kappa)(3-2\kappa)}{(2k+1)(2l+1)(2k+2l+1)}\right).$$

¿From these special cases or from the formula (4.3) we again see non-trivial symmetries, namely

$$t_n^\kappa(k,l) = t_n^\kappa(l,k) = t_n^\kappa(k, 1-n-k-l) \tag{4.5}$$

(generalizing (3.6)) and

$$t_n^\kappa(k,l) = t_n^{2-\kappa}(k,l) \tag{4.6}$$

(which says that the multiplications μ^κ and $\mu^{2-\kappa}$ coincide). We will discuss both of these equations in §6 in terms of the residue map and the duality between automorphic forms of weights κ and $2-\kappa$. We note that Proposition 4 is the special case $\kappa = 0$ (or $\kappa = 2$) of Theorem 1 and Proposition 5 (up to a harmless rescaling of t_n by $(-4)^n$) is the special case $\kappa = 1/2$ or $\kappa = 3/2$. Another interesting special case is given by taking $\kappa = 1/\varepsilon$, multiplying t_n by a factor $(-4\varepsilon)^n$ (again, this does not affect the statement about associativity), and letting ε tend to 0. The resulting coefficient $t_n^{(\infty)}(k,l)$ is simpler than the general coefficient t_n^κ, since in the limit all terms in (4.3) except the one (if any) with $2j = n$ vanish and we have

$$t_{2j}^{(\infty)}(k,l) = \binom{-\frac{1}{2}}{j} \Big/ j!^2 \binom{k+j-\frac{1}{2}}{j}\binom{l+j-\frac{1}{2}}{j}\binom{k+l+2j-\frac{3}{2}}{j}, \qquad t_{2j+1}^{(\infty)}(k,l) = 0\,.$$

The vanishing of $t_n^{(\infty)}(k,l)$ for n odd means that the corresponding multiplication $\mu^{(\infty)}$, unlike the multiplications μ^κ for κ finite, is commutative.

Problem. Find a natural interpretation for this ring structure $\mu^{(\infty)}$ on $\mathcal{M}(\Gamma)$.

We now turn to the proof of Theorem 1. One can prove it by direct combinatorial manipulation of the sums of binomial coefficients involved. However, this proof is not only very laborious, but also does not explain where the new multiplications come from. Instead we give a proof using pseudodifferential operators with a new invariance property. Namely, we can use the non-commutativity of the ring of ΨDO's to define a "twisted" action of $\mathrm{SL}(2,\mathbb{C})$ by

$$\bigl(\psi|_\kappa g\bigr)(z) = (cz+d)^{-\kappa}\,\psi\bigl(\frac{az+b}{cz+d}\bigr)\,(cz+d)^{\kappa} \qquad \bigl(\kappa\in\mathbb{C}, \quad g=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\bigr)\,. \tag{4.7}$$

Note that this makes sense even for non-integral κ since any two determinations of the factor $(cz+d)^\kappa$ differ by a scalar factor and scalars commute with ΨDO's. If $\Gamma\subset \mathrm{PSL}(2,\mathbb{C})$ is a group acting on the ring R as usual, then we call an element of $\Psi\mathrm{DO}(R)$ which is Γ-invariant with respect to the action (4.7), i.e., which satisfies

$$\psi\bigl(\frac{az+b}{cz+d}\bigr) = (cz+d)^{\kappa}\psi(z)(cz+d)^{-\kappa} \qquad \text{for all } \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma\,, \tag{4.8}$$

a *conjugate-automorphic pseudodifferential operator of weight* κ with respect to Γ. We denote the space of such elements by $\Psi\mathrm{DO}(\Gamma)^\kappa$ and write $\Psi\mathrm{DO}(\Gamma)^\kappa_w$ for its intersection with $\Psi\mathrm{DO}(R)_w$. (We omit R from the notation; usually we think of the case when Γ is a discrete subgrop of $\mathrm{PSL}(2,\mathbb{R})$ and $R=\mathcal{F}$.) Since conjugation of a ΨDO by a function does not change the leading term (symbol), we see that the exact sequence (1.5) is equivariant with respect to the action of Γ on the first two terms by (4.7) and on the last term by $|_{-2w}$, so taking invariants we get a sequence

$$0\to \Psi\mathrm{DO}(\Gamma)^\kappa_{w-1}\to \Psi\mathrm{DO}(\Gamma)^\kappa_{w}\to M_{-2w}(\Gamma)\to 0 \tag{4.9}$$

which is exact except possibly for the final term. We then have the following generalization of Proposition 3:

Proposition 6. *The map* $\mathcal{D}^\kappa_w : R\to \Psi\mathrm{DO}(R)_w$ *defined by*

$$\mathcal{D}^\kappa_w(f) = \sum_{n=0}^{\infty} \frac{\binom{w}{n}\binom{w+\kappa-1}{n}}{\binom{2w}{n}}\, f^{(n)}\,\partial^{w-n} \tag{4.10}$$

(where the sum must be replaced by $\sum_{n=0}^{w}$ *if* w *is a nonnegative integer and is not defined if* w *is a positive half-integer) satisfies* $\mathcal{D}^\kappa_w(f|_{-2w}g) = \mathcal{D}^\kappa_w(f)|_\kappa g$ *for all* $g\in\mathrm{SL}(2,\mathbb{C})$. *In particular, the sequence* (4.9) *is exact and splits canonically.*

Proof. The proof, either by direct calculation, via Jacobi-like forms, or using the Casimir operator, is exactly the same as before. ■

Now we proceed just as in §3. The lifting $\mathcal{D}^\kappa = \prod_k \mathcal{D}^\kappa_w$ gives an isomorphism from $\mathcal{M}(\Gamma)$ to $\Psi\mathrm{DO}(R,\Gamma)^\kappa$, the inverse map being given explicitly by

$$\sum_{n\ll\infty} g_n\,\partial^n \mapsto \{f_k\}_{k\gg-\infty}\,, \qquad f_k = \sum_{r=0}^{\infty}\frac{\binom{k-1}{r}\binom{k-\kappa}{r}}{\binom{2k-2}{r}}\, g^{(r)}_{r-k}\ \in M_{2k}\,, \tag{4.11}$$

generalizing (4) of Proposition 2, §2. On the other hand, it is clear that the product of two conjugate-automorphic ΨDO's of weight κ is again conjugate-automorphic of the same weight, so by transporting the multiplication of ΨDO's to $\mathcal{M}(\Gamma)$ by $\mathcal{D}^\kappa$ we get a new ring structure μ^κ on $\mathcal{M}(\Gamma)$. Again the uniqueness of the Rankin-Cohen brackets says that we must have $\mathcal{D}^\kappa(f)\mathcal{D}^\kappa(g) = \sum_n t_n[f,g]_n$ for all $f\in M_{2k}$, $g\in M_{2l}$ for some universal coefficients $t_n = t^\kappa_n(k,l)$, and by substituting all definitions and multiplying out what this says we get an infinite sequence of equations for the t_n of which the simplest is

$$\frac{\binom{n+k-\kappa}{n}\binom{n+k-1}{n}}{\binom{n+2k-1}{n}} = \sum_{r+s=n} \frac{\binom{2l+r-1}{r}\binom{n+k+l-\kappa}{s}\binom{n+k+l-1}{s}}{\binom{n+r+2k+2l-1}{s}}\, t_r\,.$$

Inverting this as in the previous case we find the closed formula

$$t_n^\kappa(k,l) = \frac{1}{\binom{-2l}{n}} \sum_{r+s=n} \frac{\binom{-k}{r}\binom{-k-1+\kappa}{r}}{\binom{-2k}{r}} \frac{\binom{n+k+l-\kappa}{s}\binom{n+k+l-1}{s}}{\binom{2n+2k+2l-2}{s}}\,. \tag{4.12}$$

That this is equivalent to (4.3) follows from the following generalization of the identity given in §3, and whose proof again will be postponed to the paper [Z2]:

IDENTITY. *For an integer $n \ge 0$ and variables a, x, y, z satisfying $x+y+z = n-1$, we have*

$$\frac{(-4)^n}{\binom{2x}{n}} \sum_{r+s=n} \frac{\binom{y}{r}\binom{y-a}{r}}{\binom{2y}{r}} \frac{\binom{z}{s}\binom{z+a}{s}}{\binom{2z}{s}} = \sum_{j\ge 0} \binom{n}{2j} \frac{\binom{-\frac12}{j}\binom{a-\frac12}{j}\binom{-a-\frac12}{j}}{\binom{x-\frac12}{j}\binom{y-\frac12}{j}\binom{z-\frac12}{j}}\,. \tag{4.13}$$

(In our case $x=-l$, $y=-k$, $z=n+k+l-1$, and $a=1-\kappa$.) Again this identity reveals surprising "hidden symmetries": the left-hand side is symmetric under interchanging y and z and simultaneously replacing a by $-a$ and has no other evident symmetries, but the identity shows that it is in fact symmetric in all three variables x, y, z and at the same time an even function of a. In terms of the coefficients $t_n^\kappa(k,l)$, these symmetries become the equations (4.5) and (4.6) mentioned above.

As a final remark, we observe that in the special case $\kappa = 1/2$ corresponding to the Eholzer multiplication (4.2), not only the multiplication but also the formula for the lifting map $\mathcal{D}^\kappa$ simplifies, since (4.10) becomes simply

$$\mathcal{D}_w^{1/2}(f) = \sum_{n=0}^{\infty} 4^n \binom{2w-n}{n} f^{(n)}\, \partial^{w-n}\,. \tag{4.14}$$

§5. Automorphic ΨDO's of mixed weight.

We can generalize still further by considering the action of Γ defined by

$$\bigl(\psi|_{\kappa_1,\kappa_2}\gamma\bigr)(z) = (cz+d)^{-\kappa_1}\,\psi\Bigl(\frac{az+b}{cz+d}\Bigr)\,(cz+d)^{\kappa_2} \qquad \Bigl(\gamma = \bigl(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\bigr) \in \Gamma\Bigr)$$

where κ_1 and κ_2 are complex constants. If κ_1 and κ_2 differ by an integer, then this makes sense independently of the branch of $\log(cz+d)$ chosen; if not, then we either have to pick a lifting from Γ to the universal cover of $\mathrm{SL}(2,\mathbb{R})$ or else work with locally defined functions, as discussed at the end of §2. We call the elements of $\Psi\mathrm{DO}(R)$ which are Γ-invariant with respect to this action, i.e., which satisfy the transformation law

$$\psi\Bigl(\frac{az+b}{cz+d}\Bigr) = (cz+d)^{\kappa_1}\,\psi(z)\,(cz+d)^{-\kappa_2} \qquad \forall \bigl(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\bigr) \in \Gamma\,, \tag{5.1}$$

automorphic pseudodifferential operators of mixed weight (κ_1,κ_2) with respect to Γ. We denote the space of such operators by $\Psi\mathrm{DO}(\Gamma)^{\kappa_1,\kappa_2}$ and its intersection with $\Psi\mathrm{DO}(R)_w$ by $\Psi\mathrm{DO}(\Gamma)_w^{\kappa_1,\kappa_2}$. If $\psi(z) = \sum_{n\ge 0} f_n(z)\partial^{w-n}$ belongs to this latter space, then its leading coefficient f_0 is Γ-invariant with respect to the action $|_{\kappa_1-\kappa_2-2w}$, so the sequence (4.9) generalizes to

$$0 \to \Psi\mathrm{DO}(\Gamma)_{w-1}^{\kappa_1,\kappa_2} \to \Psi\mathrm{DO}(\Gamma)_w^{\kappa_1,\kappa_2} \to M_{\kappa_1-\kappa_2-2w}(\Gamma) \to 0 \tag{5.2}$$

and the liftings described in the previous sections to the following proposition:

Proposition 7. *The map* $\mathcal{D}_w^{\kappa_1,\kappa_2} : R \to \Psi\mathrm{DO}(R)_w$ *defined by*

$$\mathcal{D}_w^{\kappa_1,\kappa_2}(f) = \sum_{n=0}^{\infty} \frac{\binom{w}{n}\binom{w+\kappa_2-1}{n}}{\binom{\kappa_2-\kappa_1+2w}{n}}\, f^{(n)}\,\partial^{w-n}\,, \tag{5.3}$$

where the upper index in the sum must be replaced by $\sum_{n=0}^{w}$ *if* w *is a non-negative integer and values of* w *for which the denominator of any of the coefficients vanishes must be excluded, satisfies*

$$(\mathcal{D}_w^{\kappa_1,\kappa_2} f)|_{\kappa_1,\kappa_2}\, g = \mathcal{D}_w^{\kappa_1,\kappa_2}(f|_{\kappa_1-\kappa_2-2w}\, g) \qquad \forall g \in \mathrm{SL}(2,\mathbb{C})\,. \tag{5.4}$$

In particular, the sequence (5.2) *is exact and splits canonically.*

Just as before, we could prove the proposition by direct computations as in §1 or else by an argument using Jacobi-like forms or the Casimir operator as in §2. Now, however, there is a new argument which is perhaps the simplest of all. In the special case when $w = n$ is a non-negative integer, the lifting $\mathcal{D}_w^{\kappa_1,\kappa_2}(f)$ of an element $f \in M_{\kappa_1-\kappa_2-2n}$ is a differential rather than a pseudodifferential operator, and hence acts on functions. Moreover, it is clear from the transformation law (5.1) that if $g \in M_{\kappa_2}(\Gamma)$ and $\psi \in \mathrm{DO}(\Gamma)^{\kappa_1,\kappa_2}$, then the image $\psi(z)g(z)$ belongs to $M_{\kappa_1}(\Gamma)$. Hence, changing notation from κ_1, κ_2 to $2k = \kappa_1 - \kappa_2 - 2n$, $2l = \kappa_2$, we see that the map $f \otimes g \mapsto (\mathcal{D}_n^{2k+2l+2n,2l} f)(g)$ goes from $M_{2k}(\Gamma) \otimes M_{2l}(\Gamma)$ to $M_{2k+2l+2n}(\Gamma)$, and comparing the definition (5.3) with the definition (3.1), we see that this map is (up to a scalar) nothing else than the Rankin-Cohen bracket, as indeed it must be by the uniqueness of the latter. Turning this around, the fact that the Rankin-Cohen bracket is given in terms of derivatives means that, for a fixed $f \in M_{2k}(\Gamma)$, the operator $[f,\cdot]_n^{(k,l)}$ is a differential operator which sends $M_{2l}(\Gamma)$ to $M_{2k+2l+2n}(\Gamma)$ and hence satisfies the transformation law (5.2) (with $\kappa_1 = 2k+2l+2n$, $\kappa_2 = 2l$), so that the modularity property of the bracket implies the equivariance property of the lifting (5.3) in this case. Since this equivariance property is at each level equivalent to a finite number of binomial coefficient identities, and since this argument shows that these identities (which are polynomial in κ_1 and κ_2) are true for infinitely many values κ_1, κ_2, this special case is enough to prove the proposition.

Now just as in the previous cases $\kappa_1 = \kappa_2 = 0$ and $\kappa_1 = \kappa_2 = \kappa$, this proposition induces an isomorphism between $\mathcal{M}(\Gamma)$ and $\Psi\mathrm{DO}(\Gamma)^{\kappa_1,\kappa_2}$. However, the latter space is no longer a ring, so this does not directly induce a single multiplication on the space of modular forms. Instead, we clearly have

$$\Psi\mathrm{DO}(\Gamma)_{w_1}^{\kappa_1,\kappa_2}\ \Psi\mathrm{DO}(\Gamma)_{w_2}^{\kappa_2,\kappa_3}\ \subseteq\ \Psi\mathrm{DO}(\Gamma)_{w_1+w_2}^{\kappa_1,\kappa_3} \tag{5.5}$$

(if we restrict this to differential operators rather than ΨDO's then $\mathrm{DO}(R)^{\kappa_1,\kappa_2}$ can be thought of as giving homomorphisms from M_{κ_2} to M_{κ_1} as just explained, and this is just the composition of homomorphisms) and combining this with the lifting of Proposition 7 we get a corresponding collection of multiplication maps $\mu^{\kappa_1,\kappa_2,\kappa_3}$ on $\mathcal{M}(\Gamma)$ which satisfy the evident associativity property (groupoid structure). These multiplications must again be expressible in terms of Rankin-Cohen brackets, i.e., we must have

$$\mathcal{D}_{w_1}^{\kappa_1,\kappa_2}(f)\,\mathcal{D}_{w_2}^{\kappa_2,\kappa_3}(g) = \sum_{n=0}^{\infty} t_n^{\kappa_1,\kappa_2,\kappa_3}(k,l)\,\mathcal{D}_{w_1+w_2-n}^{\kappa_1,\kappa_3}([f,g]_n) \tag{5.6}$$

for some numerical coefficients $t_n^{\kappa_1,\kappa_2,\kappa_3}(k,l)$, where $2k$ and $2l$ are the weights of f and g and $w_1 = \frac{1}{2}(\kappa_1-\kappa_2)-k$, $w_2=\frac{1}{2}(\kappa_2-\kappa_3)-l$. These coefficients can be evaluated as before to give the formula

$$t_n^{\kappa_1,\kappa_2,\kappa_3}(k,l) = \bigl(-\frac{1}{4}\bigr)^n\, T_n\bigl(1-\kappa_1, 1-\kappa_3, 1-\kappa_2;\, -l, -k, k+l+n-1\bigr)\,, \qquad (5.7)$$

where $T_n(a,b,c;x,y,z)$ is defined for a non-negative integer n and variables a, b, c, x, y, z with $x+y+z=n-1$ by the formula

$$T_n(a,b,c;x,y,z) = \frac{(-4)^n}{\binom{2x}{n}} \sum_{r+s=n} \frac{\binom{y-\frac{a+c}{2}}{r}\binom{y+\frac{c-a}{2}}{r}}{\binom{2y}{r}}\, \frac{\binom{z+\frac{a-b}{2}}{s}\binom{z+\frac{a+b}{2}}{s}}{\binom{2z}{s}}\,, \qquad (5.8)$$

which reduces to the left-hand side of (4.13) in case $a=b=c$.

In §6 we will use the interpretation (5.7) of the numbers T_n to prove the following purely combinatorial result.

Theorem 2. *The coefficient $T_n(a,b,c;x,y,z)$ is symmetric in the three pairs of variables (a,x), (b,y), and (c,z) and is an even function of a, b and c.*

As examples of the theorem, we found (with effort!) the symmetric expressions

$$\begin{aligned}
T_0 &= 1\,,\\
T_1 &= 1 + \frac{1}{4}\left(\frac{a^2}{yz}+\frac{b^2}{xz}+\frac{c^2}{xy}\right) \qquad\qquad (x+y+z=0),\\
T_2 &= 1 + \frac{1}{2}\left(\frac{a^2}{yz}+\frac{b^2}{xz}+\frac{c^2}{xy}\right) - \frac{1}{(2x-1)(2y-1)(2z-1)}\\
&\quad + \frac{1}{4}\left(\frac{a^2(a^2-2)}{yz(2y-1)(2z-1)}+\frac{b^2(b^2-2)}{xz(2x-1)(2z-1)}+\frac{c^2(c^2-2)}{xy(2x-1)(2y-1)}\right)\\
&\quad + \frac{1}{4xyz}\left(\frac{b^2c^2}{2x-1}+\frac{a^2c^2}{2y-1}+\frac{a^2b^2}{2z-1}\right) \qquad\qquad (x+y+z=1)\,.
\end{aligned}$$

The expression for T_1 clearly simplifies to 1 if $a=b=c$, but already for $n=2$ the verification that T_2 reduces to $1+(4a^2-1)/(2x-1)(2y-1)(2z-1)$ when $a=b=c$ (as it must by (4.13)) requires the non-obvious identities

$$\begin{aligned}
\frac{1}{zx}+\frac{1}{xy}+\frac{1}{yz}-\frac{8}{(2x-1)(2y-1)(2z-1)} &= \frac{-1}{xyz}\left(\frac{1}{2x-1}+\frac{1}{2y-1}+\frac{1}{2z-1}\right)\\
= \frac{1}{zx(2z-1)(2x-1)} &+ \frac{1}{xy(2x-1)(2y-1)} + \frac{1}{yz(2y-1)(2z-1)}
\end{aligned}$$

for variables x, y, z with $x+y+z=1$. It would be nice to find a direct combinatorial proof of Theorem 2 or, even better, a closed formula for $T_n(a,b,c;x,y,z)$ which

(a) makes the symmetries stated in Theorem 2 evident, and
(b) reduces to the right-hand side of (4.13) when $a=b=c$,

but so far we could not find a formula having either one of these properties.

§6. Residues, duality, and symmetry.

In the present section we found a striking symmetry among the *three* weights k, l, and $m := 1-k-l-n$ in the formulas giving the coefficients of the nth bracket $[f,g]_n$ ($f \in M_{2k}$, $g \in M_{2l}$) in the various multiplications on $\mathcal{M}(\Gamma)$. To explain it, we use the *non-commutative residue map*

$$\mathrm{Res}_\partial : \sum_m h_m(z)\partial^m \mapsto h_{-1}(z)dz \;\in\; H(R) := \Omega^1(R)/d\Omega^0(R)\,,$$

where $\Omega^1(R) = R\,dz$ denotes the space of formal differentials $f(z)\,dz$ ($f \in R$) and $d\Omega^0(R) = dR$ the subspace of exact differentials $f'(z)dz$, $f \in R$. This residue map was introduced in [Ma2] and shown to have the properties

$$\mathrm{Res}_\partial(\psi \circ g) = \mathrm{Res}_\partial(\psi) \circ g \tag{6.1}$$

for any ψ and any holomorphic map $z \mapsto g(z)$ (invariance under holomorphic change of coordinates) and

$$\mathrm{Res}_\partial(\psi_1(z)\psi_2(z)) = \mathrm{Res}_\partial(\psi_2(z)\psi_1(z)) \tag{6.2}$$

for any two ΨDO's ψ_1 and $\psi_2(z)$ (trace property). The invariance under changes of variables implies in particular that Res_∂ maps $\Psi\mathrm{DO}(R)^\Gamma$ to $H(R)^\Gamma$ if Γ is a group of fractional linear transformations acting on R, and the conjugacy-invariance property (6.2) implies that the same is true for the space $\Psi\mathrm{DO}(R,\Gamma)^\kappa$ of conjugate-automorphic ΨDO's of arbitrary (complex) weight κ. The space $H(R)^\Gamma$ is isomorphic via $f(z)\,dz \mapsto f(z)$ to the space $H(R,\Gamma) = M_2(\Gamma)/\partial(M_0(\Gamma))$, and by abuse of notation we will simply identify these spaces and write Res_∂ for the corresponding map $\Psi\mathrm{DO}(R,\Gamma)^\kappa \to H(R,\Gamma)$. We must choose R large enough that there are plenty of modular forms of positive and negative weight, so that we can test an identity in $M_{2k}(\Gamma)$ by checking whether its product with an arbitrary element of $M_{2-2k}(R,\Gamma)$ is 0 in $H(R,\Gamma)$. For instance, we could take R to be the set of all functions which are meromorphic in the upper half-plane including the cusps, or the subspace of those with poles at most at some specified non-empty Γ-invariant set S.

We also define a projection map P from $M_*(\Gamma) = \oplus_k M_{2k}(\Gamma)$ to $H(R,\Gamma)$ by sending $f \in M_{2k}(\Gamma)$ to 0 if $k \neq 1$ and to its natural image in $H(R,\Gamma)$ if $k=1$.

Proposition 8. (i) *The map* P *annihilates all higher Rankin-Cohen brackets, i.e.* $\mathrm{P}([f,g]_n) = 0$ *for all* $f,\, g \in M_*(\Gamma)$ *and all* $n > 0$.

(ii) *The "triple bracket"* $\{f,g,h\}_n := \mathrm{P}([f,g]_n h)$ $(f,\, g,\, h \in \mathcal{M}_*(\Gamma),\; n \geq 0)$ *is invariant under cyclical permutation of its three arguments.*

Proof. Suppose $f \in M_{2k}(\Gamma)$ and $g \in M_{2l}(\Gamma)$. If $k+l+n \neq 1$, then $\mathrm{P}([f,g]_n)$ vanishes by definition. If $k+l+n = 1$ then a one-line computation shows that

$$n[f,g]_n \;=\; (k-l)\,\partial\big([f,g]_{n-1}\big)$$

and hence that $[f,g]_n$ vanishes in $H(R,\Gamma)$ if $n \neq 0$. This proves (i). To prove (ii), let $h \in M_{2m}(\Gamma)$ be a third modular form, and suppose that $k+l+m+n = 1$ (otherwise $\{f,g,h\}_n$ and $\{g,h,f\}_n$ are zero by definition). Let $\equiv$ denote congruence modulo dR. From $f'g \equiv -fg'$ we get $(-1)^p f^{(p)} g \equiv g^{(p)} f$ by induction and hence

$$(-1)p\, f^{(p)}\, g^{(q)}\, h \equiv (g^{(q)}h)^{(p)}\, f \equiv \sum_{r+s=p} \tbinom{p}{s} g^{(q+r)}\, h^{(s)}\, f$$

by Leibniz's rule, so

$$\begin{aligned}
[f,g]_n h &= \sum_{p+q=n} (-1)^p \binom{2k+n-1}{q}\binom{2l+n-1}{p} f^{(p)}\, g^{(q)}\, h \\
&\equiv \sum_{q+r+s=n} \binom{2k+n-1}{q}\binom{2l+n-1}{r+s}\binom{r+s}{s} g^{(q+r)}\, h^{(s)}\, f \\
&= \sum_{s=0}^{n} \binom{2l+n-1}{s}\Big\{ \sum_{q+r=n-s} \binom{2k+n-1}{q}\binom{2l+n-s-1}{r}\Big\}\, g^{(n-s)}\, h^{(s)}\, f\,.
\end{aligned}$$

But the term in braces is given by

$$\{\cdots\} = \binom{2k+2l+2n-s-2}{n-s} = \binom{-2m-s-2}{n-s} = (-1)^{n-s}\binom{n+2m-1}{n-s}\,,$$

so this last expression equals $[g,h]_n f$, proving the claim. We also note that (i) is a special case of (ii), since $\mathrm{P}([f,g]_n) = \{f,g,1\}_n = \{g,1,f\}_n = 0$ if $n>0$. ■

We can now give the promised explanation of the cyclic symmetry property (3.6) of the coefficients $t_n(k,l)$. Let k, l, m, n be integers with $n \ge 0$ and $k+l+m+n=1$ and let f, g, and h be modular forms of weight $2k$, $2l$, and $2m$, respectively. (For the application to (3.6) we imagine that k and l are positive and hence that m is negative, but the signs play no role.) Write $\mathcal{D}$ for the lifting map from $\mathcal{M}(\Gamma)$ to $\Psi\mathrm{DO}(R)^\Gamma$ (so $\mathcal{D}(F) = \mathcal{D}_{-K}(F)$ for F modular of weight K) and μ for the multiplication on $\mathcal{M}(\Gamma)$ defined in §3, Proposition 4, so that $\mathcal{D}(F)\mathcal{D}(G) = \mathcal{D}(\mu(F,G))$ for any F and G in $\mathcal{M}(\Gamma)$. Also $\mathrm{Res}_\partial(\mathcal{D}(F)) = \mathrm{P}(F)$ for any modular form F, because the coefficient of ∂^{-1} in $\mathcal{D}(F)$ is F if F has weight 2 and is either 0 or else a higher derivative of F if F has any other weight. Hence for any two modular forms F and G we have

$$\mathrm{Res}_\partial\big(\mathcal{D}(F)\mathcal{D}(G)\big) = \mathrm{Res}_\partial\big(\mathcal{D}\big(\mu(F,G)\big)\big) = \mathrm{P}\big(\mu(F,G)\big) = \mathrm{P}(FG)\,,$$

where the last line follows from part (i) of Proposition 8 and the fact that $\mu(F,G)$ is the sum of FG plus a linear combination of higher Rankin-Cohen brackets. Applying this with $F=\mu(f,g)$ and $G=h$ we find

$$\mathrm{Res}_\partial\big(\mathcal{D}(f)\mathcal{D}(g)\mathcal{D}(h)\big) = \mathrm{Res}_\partial\big(\mathcal{D}(\mu(f,g))\mathcal{D}(h)\big) = \mathrm{P}\big(\mu(f,g)h\big) = t_n(k,l)\,\{f,g,h\}_n\,.$$

The expression on the left is invariant under cyclic permutation of f, g and h by the trace property (6.1), and the triple bracket $\{f,g,h\}_n$ is invariant under cyclic permutations by part (ii) of Proposition 8, so the coefficient $t_n(k,l)$ must have the same symmetry, i.e., $t_n(k,l) = t_n(l,m) = t_n(l, 1-n-k-l)$.

The same argument works unchanged if we replace $\Psi\mathrm{DO}(R)^\Gamma$ by the group $\Psi\mathrm{DO}(R,\Gamma)^\kappa$ of conjugate-invariant ΨDO's of weight κ and $\mathcal{D}$ by the lifting map $\mathcal{M}(\Gamma) \to \Psi\mathrm{DO}(R,\Gamma)^\kappa$ constructed in §4, so we also get an explanation of the analogous cyclic symmetry property of the more general coefficients $t_n^\kappa(k,l)$.

Everything also goes through in the case of mixed weights introduced in the last section. Choose three complex numbers κ_1, κ_2, and κ_3, and consider equation (5.6). Multiplying this equation on the right by $\mathcal{D}_{w_3}^{\kappa_3,\kappa_1}(h)$, where h is a modular form of weight $2m$ with $k+l+m=1-n$ for some integer $n\ge 0$ and w_3 is defined as $\frac{1}{2}(\kappa_3-\kappa_1)-m$, we find by a second application of the same equation that

$$\begin{aligned}
&\mathcal{D}_{w_1}^{\kappa_1,\kappa_2}(f)\,\mathcal{D}_{w_2}^{\kappa_2,\kappa_3}(g)\,\mathcal{D}_{w_3}^{\kappa_3,\kappa_1}(h) \\
&\quad = \sum_{r,\,s\ge 0} t_r^{\kappa_1,\kappa_2,\kappa_3}(k,l)\, t_s^{\kappa_1,\kappa_3,\kappa_1}(k+l+r,m)\,\mathcal{D}_{w_1+w_2+w_3-r-s}^{\kappa_1}([[f,g]_r,h]_s)\,.
\end{aligned}$$

Now applying Res_∂ to both sides and arguing as before we find

$$\mathrm{Res}_\partial\big(\mathcal{D}_{w_1}^{\kappa_1,\kappa_2}(f)\,\mathcal{D}_{w_2}^{\kappa_2,\kappa_3}(g)\,\mathcal{D}_{w_3}^{\kappa_3,\kappa_1}(h)\big) = t_n^{\kappa_1,\kappa_2,\kappa_3}(k,l)\cdot\{f,g,h\}_n\,. \tag{6.3}$$

This implies just as before the invariance of $t_n^{\kappa_1,\kappa_2,\kappa_3}(k,l)$ with respect to simultaneous cyclic permutation of $(\kappa_1,\kappa_2,\kappa_3)$ and of $(k,l,-k-l-n+1)$.

Proof of Theorem 2. Formula (5.7) together with the cyclic symmetry just proved implies that $T_n(a,b,c;x,y,z)$ is invariant with respect to cyclic permutations of the three pairs of variables (a,x), (b,y), and (c,z). On the other hand, it is clear from the defining formula (5.8) that $T_n(a,b,c;x,y,z)$ is an even function of b and of c. From the cyclic invariance it follows that the three variables a, b, and c play equal roles, so it is also an even function of a. On the other hand, by interchanging the roles of r and s in (5.8) we see that T_n is unchanged if we interchange (b,y) and (c,z) and simultaneously replace a by $-a$, so we obtain also the invariance of T_n under odd permutations of (a,x), (b,y), and (c,z). ■

In terms of the coefficients t_n of the multiplications of ΨDO's of mixed weights, Theorem 2 says that these coefficients are invariant not only under cyclic permutations of the indices, but also under interchange of k and l (and simultaneously of κ_1 and κ_3), as well as under each of the three involutions $\kappa_i\mapsto 2-\kappa_i$. We have given an intrinsic explanation of the first symmetry in terms of the residue map, but this is only a subgroup of order 3 out of a total symmetry group $\mathfrak{S}_3\ltimes(\mathbb{Z}/2Z)^3$ of order 48. We now explain where the other symmetries come from. For this we will use both a duality and an isomorphism between the (abstract) spaces of modular forms of weight κ and weight $2-\kappa$.

We first give an argument which shows that

$$t_n^{\kappa_1,\kappa_2,\kappa_3}(k,l) = t_n^{2-\kappa_3,2-\kappa_2,2-\kappa_1}(l,k) \tag{6.4}$$

and hence that the coefficients t_n are invariant if we subject $(k,l,m=1-n-k-l)$ to any odd permutation, apply the corresponding permutation to the κ_i's, and simultaneously replace each κ_i by $2-\kappa_i$.

There is a canonical involution $A\mapsto A^*$ on the ring $\mathrm{DO}(R)$ of differential operators over R defined by the property that $A(f)\,g\equiv f\,A^*(g)\pmod{dR}$ for all $f,\,g\in R$. This involution is the identity on functions, sends ∂ to $-\partial$ (formula for integration by parts!), and satisfies $(AB)^*=B^*A^*$, so it must be given by

$$\Big(\sum_n f_n\,\partial^n\Big)^* = \sum_n(-1)^n\,\partial^n\,f_n\,. \tag{6.5}$$

We can now use this formula to extend $*$ to all of $\Psi\mathrm{DO}(R)$, and all its formal properties (like being a ring anti-automorphism) must remain true, since all such properties are equivalent to binomial coefficient identities which hold identically if they hold for positive integers. We also find the further property

$$\mathrm{Res}_\partial\big(\psi^*\big) = -\mathrm{Res}_\partial\big(\psi\big)\qquad\forall\psi\in\Psi\mathrm{DO}(R)\,. \tag{6.6}$$

Indeed, any ψ can be decomposed as $\psi_1+\psi_2$ with $\psi_1\in\mathrm{DO}(R)$ and $\psi_2=\sum_{n=1}^\infty h_n\partial^{-n}$; then ψ_1 and ψ_1^* are differential operators and hence map to 0 under Res_∂ while $\psi_2=h_1\,\partial^{-1}+\mathrm{O}(\partial^{-2})$ and $\psi_2=-\partial^{-1}\,h_1+\mathrm{O}(\partial^{-2})=-h_1\,\partial^{-1}+\mathrm{O}(\partial^{-2})$

have opposite images under Res_∂ . Finally, one can check either from the defining property of $*$ or by direct computation that

$$\big(\psi\circ\gamma\big)^*(z) = (cz+d)^{-2}\,\psi^*(\gamma z)(cz+d)^2 \qquad \forall\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$$

In particular, if ψ belongs to $\Psi\mathrm{DO}(\Gamma)^{\kappa_1,\kappa_2}$ then ψ^* lies in $\Psi\mathrm{DO}(\Gamma)^{2-\kappa_2,2-\kappa_1}$. We also have:

$$\mathcal{D}_w^{\kappa_1,\kappa_2}(f) = (-1)^w\,\mathcal{D}_w^{2-\kappa_2,2-\kappa_1}(f)^* \qquad \text{if } f\in M_{\kappa_1-\kappa_2-2w}(\Gamma)\,. \tag{6.7}$$

Indeed, the map $f\mapsto \mathcal{D}_w^{2-\kappa_2,2-\kappa_1}(f)^*$ is an equivariant splitting of (5.2) and hence by uniqueness is a multiple of $\mathcal{D}_w^{\kappa_1,\kappa_2}(f)$, and the multiple is $(-1)^w$ because of (6.5) and the fact that the leading term of $\mathcal{D}_w^*(f)$ is $f\,\partial^w$.

Combining (6.3), (6.6) and (6.7) and noting that $w_1+w_2+w_3=n-1$, we find

$$\begin{aligned} t_n^{\kappa_1,\kappa_2,\kappa_3}(k,l)\,\{f,g,h\}_n &= -\mathrm{Res}_\partial\Big(\big(\mathcal{D}_{w_1}^{\kappa_1,\kappa_2}(f)\,\mathcal{D}_{w_2}^{\kappa_2,\kappa_3}(g)\,\mathcal{D}_{w_3}^{\kappa_3,\kappa_1}(h)\big)^*\Big)\\ &= -\mathrm{Res}_\partial\big(\mathcal{D}_{w_3}^{\kappa_3,\kappa_1}(h)^*\,\mathcal{D}_{w_2}^{\kappa_2,\kappa_3}(g)^*\,\mathcal{D}_{w_1}^{\kappa_1,\kappa_2}(f)^*\big)\\ &= (-1)^n\,\mathrm{Res}_\partial\big(\mathcal{D}_{w_3}^{2-\kappa_1,2-\kappa_3}(h)\,\mathcal{D}_{w_2}^{2-\kappa_3,2-\kappa_2}(g)\,\mathcal{D}_{w_1}^{2-\kappa_2,2-\kappa_1}(f)\big)\\ &= (-1)^n\,t_n^{2-\kappa_3,2-\kappa_2,2-\kappa_1}(m,l)\,\{h,g,f\}_n\,. \end{aligned}$$

But $(-1)^n\{h,g,f\}_n=\{f,g,h\}_n$ by Proposition 8 and the $(-1)^n$-symmetry of the nth Rankin-Cohen bracket, so this equation (after one more cyclic permutation of its arguments) implies (6.4).

Finally, we have to see why each κ_i can be replaced by $2-\kappa$; this will give the rest of our symmetry group (so far we have explained only 6 out of a total of 48 symmetries) and in particular show why the original coefficients $t_n(k,l)$ of §3 are symmetric in k and l. Consider the case when the w of $\mathcal{D}_w^{\kappa_1,\kappa_2}$ is a positive integer, so that $\mathcal{D}_w^{\kappa_1,\kappa_2}(f)$ is a differential rather than a pseudo-differential operator. Then, as discussed in §5, it maps the space of modular forms of weight κ_2 on Γ to the space of modular forms of weight κ_1. Suppose that $\kappa_1=2-h$ for some positive even integer h. (As usual, these restrictions on w and κ_1 are not important since in proving formal identities it is enough to prove them for infinitely many special cases.) It is a classical and basic fact of the theory of modular forms that one has

$$\frac{d^{h-1}}{dz^{h-1}}\big(f|_{2-h}g\big) = \frac{d^{h-1}}{dz^{h-1}}\big(f\big)|_h g \qquad \forall g\in \mathrm{SL}(2,\mathbb{C}),\quad h\in\mathbb{N}\,.$$

(This identity, sometimes known as Van der Pol's formula, is the basis of Eichler cohomology and the theory of periods of modular forms.) Hence ∂^{h-1} maps $M_{2-h}(\Gamma)$ to $M_h(\Gamma)$, so if $\psi\in\mathrm{DO}(R,\Gamma)^{2-h,\kappa_2}$ and $f\in M_{\kappa_2}(\Gamma)$ then $\partial^{h-1}(\psi(f))\in M_h$. This says that the product in $\Psi\mathrm{DO}(R)$ of ∂^{h-1} and ψ belongs to $\Psi\mathrm{DO}(R,\Gamma)^{h,\kappa_2}$. Replacing $\kappa_1=2-h$ by an arbitrary value of κ_1, we see that we have proved that $\Psi\mathrm{DO}(R,\Gamma)^{\kappa_1,\kappa_2}$ is canonically isomorphic to $\Psi\mathrm{DO}(R,\Gamma)^{2-\kappa_1,\kappa_2}$ by left multiplication with $\partial^{1-\kappa_1}$. The same argument shows that it is also canonically isomorphic to $\Psi\mathrm{DO}(R,\Gamma)^{\kappa_1,2-\kappa_2}$ by right multiplication with ∂^{κ_2-1}. It follows that the equations

$$\partial^{1-\kappa_1}\circ\mathcal{D}_w^{\kappa_1,\kappa_2}(f) = \mathcal{D}_{w+1-\kappa_1}^{2-\kappa_1,\kappa_2}(f)\,, \qquad \mathcal{D}_w^{\kappa_1,\kappa_2}(f)\circ\partial^{\kappa_2-1} = \mathcal{D}_{w+\kappa_2-1}^{\kappa_1,2-\kappa_2}(f) \tag{6.8}$$

must be true up to scalar factors, and by looking at the leading term one sees that these factors equal 1. (As a check, note that the second of these equations implies that the coefficient of $f^(n)\partial^{-n}$ in (5.3) must be invariant under $(\kappa_1,\kappa_2,w)\mapsto(\kappa_1,2-\kappa_2,w+\kappa_2-1)$, and this is indeed true.) These identities let one replace κ by $2-\kappa$ wherever they occur as superscripts, which was the observed symmetry.

§7. Supermodular forms and superpseudodifferential operators.

We work on the supercomplex plane $\mathbb{C}^{1|1}$ with local coordinate (z,ζ) where $\zeta^2=0$ and canonical supersymmetric (SUSY) structure given by the maximal non-integrable structure distribution of rank $0|1$ generated by the vector field $D=\frac{\partial}{\partial\zeta}+\zeta\frac{\partial}{\partial z}$ satisfying $D^2=\frac{\partial}{\partial z}$. This is (up to isomorphism) the unique such SUSY structure extendible to $\mathbb{P}^{1|1}$. (For an exposition of those aspects of the theory of supersymmetry needed for the present paper see [Ma2].) If $(\tilde z,\tilde\zeta)$ is another local coordinate defining the same SUSY structure, then $D=J\cdot\tilde D$ where $J=D(\tilde\zeta)$ is the superanalogue of the usual Jacobian. We let R be a $\mathbb{Z}/2$-graded ring of functions on $\mathbb{C}^{1|1}$ on which D acts; these will have the form $F(z,\zeta)=f(z)+\zeta g(z)$ where the functions $f(z)$ and $g(z)$ can themselves have coefficients in a superring (or $\mathbb{Z}/2$-graded ring) of constants Λ. By convention, even coordinates or constants will always be denoted by Latin letters $a,b,c,d\ldots$ and odd coordinates or constants by Greek letters $\alpha,\beta,\gamma,\delta\ldots$. Even constants and variables commute with even and odd constants and variables, while odd constants and variables anti-commute with odd constants and variables (and in particular have square zero).

The superanalogue of the group $\mathrm{PSL}(2,\mathbb{C})$ is the group $\mathrm{PC}(2,\mathbb{C}^{1|1})$ whose elements are matrices

$$A=\begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix} \tag{7.1}$$

satisfying

$$ad-bc-\alpha\beta=1,\qquad e^2+2\gamma\delta=1,\quad \alpha e=a\delta-c\gamma,\qquad \beta e=b\delta-d\gamma$$

together with the condition that e reduces to 1 modulo nilpotent elements. (The last condition prevents both A and $-A$ from belonging to the group.) The matrix (7.1) acts on $\mathbb{C}^{1|1}$ by the "fractional linear SUSY-compatible transformation"

$$(z,\zeta)\ \mapsto\ (\tilde z,\,\tilde\zeta)\ =\ \Bigl(\frac{az+b+\gamma\zeta}{cz+d+\delta\zeta},\,\frac{\alpha z+\beta+e\zeta}{cz+d+\delta\zeta}\Bigr) \tag{7.2}$$

and on R by sending $F(z,\zeta)=f(z)+\zeta g(z)$ to $F^A(z,\zeta)=f(\tilde z)+\tilde\zeta g(\tilde z)$. A calculation shows that the superjacobian $J(A)=D(\tilde\zeta)$ of the transformation (7.2) is equal to $(cz+d+\delta\zeta)^{-1}$. We will use this as the automorphy factor to define supermodular forms (notice that it becomes the square root of the classical automorphy factor $d\tilde z/dz$ when $\delta=0$). For an integer k and a (discrete) subgroup $\Gamma\subset\mathrm{PC}(2,\mathbb{C}^{1|1})$), we denote by $\mathrm{SM}_k(\Gamma,R)$ the space of supermodular forms of weight k, i.e., elements of R satisfying, for $A\in\Gamma$ as in (7.1),

$$F\Bigl(\frac{az+b+\gamma\zeta}{cz+d+\delta\zeta},\frac{\alpha z+\beta+e\zeta}{cz+d+\delta\zeta}\Bigr)=(cz+d+\delta\zeta)^kF(z,\zeta)\,.$$

By direct calculation we find that this is equivalent to the two equations

$$(1-k\alpha\beta)\,f(z)-(cz+d)^{-k}f\Bigl(\frac{az+b}{cz+d}\Bigr)=e\,(\alpha z+\beta)\,g(z)$$

$$e\,g(z)-(cz+d)^{-k-1}\,g\Bigl(\frac{az+b}{cz+d}\Bigr)=(\alpha z+\beta)\,f'(z)+k\,\frac{c(\alpha z+\beta)+\delta e}{cz+d}\,f(z)\,.$$

Notice that when $\alpha = \beta = \gamma = \delta = 0$ and $e = 1$ the element A corresponds to an element of $\mathrm{PSL}(2, \mathbb{C})$ and these two equations give the separate transformation laws

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$$
$$g(\frac{az+b}{cz+d}) = (cz+d)^{k+1} g(z)$$

corresponding to the transformation law for M_k and M_{k+1} respectively, so the new theory automatically combines the cases of modular forms of even and odd weight.

We next turn to the definition of *superpseudodifferential operators*, see also [MR]. We first need the analogue of the Leibniz formula. The usual Leibniz formula $\partial(fg) = \partial(f)g + f\partial g$ is replaced in the supercase by

$$D(FG) = D(F)\, G + \sigma(F)\, D(G), \qquad F,\, G \in R$$

where the involution σ is the *grading automorphism* of R, equal to 1 on the even part and to -1 on the odd part of R (in other words, D is a superderivation). This formula generalises by induction on m to the graded Leibniz formula

$$D^m(FG) = \sum_{k=0}^{\infty} \binom{m}{r}_s D^r(\sigma^{m-r}(F))\, D^{m-r}(G) \tag{7.3}$$

for all integers $m \geq 0$, where the *supersymmetric binomial coefficients* $\binom{m}{r}_s$ are defined by

$$\binom{m}{r}_s = \begin{cases} \binom{[m/2]}{[r/2]} & \text{if } r \text{ is even or } m \text{ is odd,} \\ 0 & \text{if } r \text{ is odd and } m \text{ is even,} \end{cases}$$

with $[x]$ as usual denoting the integral part of a real number x, so we can define a multiplication on the space $\mathrm{S\Psi DO}(R)$ of super-ΨDO's (Laurent series in D^{-1}) by

$$FD^m \cdot GD^n = \sum_{r \geq 0} \binom{m}{r}_s FD^r(\sigma^{m-r}G)\, D^{m+n-r} \qquad (m,\, n \in \mathbb{Z})\,,$$

and with respect to multiplication the subspace $\mathrm{SDO}(R) = R[D] \subset \mathrm{S\Psi DO}(R)$ of superdifferential operators is a subring. As before, we have a filtration of $\mathrm{S\Psi DO}(R)$ by the subspaces

$$\mathrm{S\Psi DO}(R)_w = \left\{ \sum_{m=0}^{\infty} F_m D^{w-m}, \quad F_m \in R \right\}$$

and this filtration is compatible with the ring structure.

In the supercase, the group Γ acts on $\mathrm{S\Psi DO}(R)$ via its action on R and on D. The element A of Γ as in (7.1) transforms D into $(cz + d + \delta\zeta)D$. The ring $\mathrm{S\Psi DO}(R)^\Gamma$ denotes the Γ-invariant elements of $\mathrm{S\Psi DO}(R)$. We have the filtration $\mathrm{S\Psi DO}(R)^\Gamma_k, k \in \mathbb{Z}$, inherited from the filtration of $\mathrm{S\Psi DO}(R)$. The analogue of (1.8) for the supercase is the sequence, which is split short exact by Theorem 3 below, involving supermodular forms of weight k (for all parities of k)

$$0 \to \mathrm{S\Psi DO}(R)^\Gamma_{-k-1} \to \mathrm{S\Psi DO}(R)^\Gamma_{-k} \to SM_k(\Gamma) \to 0\,. \tag{7.4}$$

The analogue of Proposition 1 for supermodular forms is as follows.

Theorem 3. *For $k>0$ define an operator $S\mathcal{L}_k : R \to \mathrm{S\Psi DO}(R)_{-k}$ by*

$$S\mathcal{L}_k(F) = \sum_{n=0}^{\infty} (-1)^{[n/2]} \frac{[\frac{n+k}{2}]!\,[\frac{n+k-1}{2}]!}{[\frac{n}{2}]!\,[\frac{n+2k-1}{2}]!}\, D^n(F)\, D^{-k-n}$$

and an operator $S\mathcal{L}_{-k} : R \to \mathrm{SDO}(R)_k$ by

$$S\mathcal{L}_{-k}(F) = \sum_{n=0}^{k-1} \frac{[\frac{2k-n}{2}]!}{[\frac{n}{2}]!\,[\frac{k-n}{2}]!\,[\frac{k-n-1}{2}]!}\, D^n(F)\, D^{k-n}\,,$$

and set $S\mathcal{L}_0(F)=F$. Then $S\mathcal{L}_k(F^A J(A)^k) = S\mathcal{L}_k(F)\circ A$ for any $A \in \mathrm{PC}(2,\mathbb{C}^{1|1})$ and any $k\in\mathbb{Z}$. In particular, if $F\in SM_k(\Gamma)$ for any $k\in\mathbb{Z}$ then $S\mathcal{L}_k(F)$ is a Γ-invariant superpseudodifferential operator .

Remark. Just as in §2, if we denote by $S\mathcal{D}_w : R \to \mathrm{S\Psi DO}(R)_w$ the lifting map renormalized to have leading coefficient FD^w then we can write the formulas for positive and negative w uniformly using binomial coefficients as

$$S\mathcal{D}_w(F) = \sum_{n\ge 0} \frac{\binom{[\frac{w}{2}]}{[\frac{n+1}{2}]}\binom{[\frac{w-1}{2}]}{[\frac{n}{2}]}}{\binom{w}{[\frac{n+1}{2}]}}\, D^n(F)\, D^{w-n} \qquad (w\in\mathbb{Z})\,, \tag{7.5}$$

where the sum goes only up to $n=w$ if $w\ge 0$.

Proof of Theorem 3. We imitate the proof of Proposition 1 in §1. The analogues of (1.7) and (1.9), both proved by induction, are

$$\big[(cz+d+\delta\zeta)\circ D\big]^w = (cz+d+\delta\zeta)^w \sum_{r=0}^{\infty} \alpha_r(w)\, \Phi_r(A)\, D^{w-r} \tag{7.6}$$

and

$$D^n\big(F^A\,(cz+d+\delta\zeta)^w\big) = \sum_{r=0}^{n} \beta_r(w,n)\,(D^{n-r}F)^A\,\Phi_r(A)\,(cz+d+\delta\zeta)^{w-n+r}$$

where $\Phi_r(A)$ is defined for A as in (7.1) by

$$\Phi_r(A) = \left(\frac{c}{cz+d+\delta\zeta}\right)^{[\frac{r}{2}]} \left(\frac{c\zeta-\delta}{cz+d+\delta\zeta}\right)^{[\frac{r+1}{2}]-[\frac{r}{2}]}$$

and the numerical coefficients $\alpha_r(w)$ and $\beta_r(w,n)$ are given by

$$\alpha_r(w) = \frac{[\frac{w}{2}]!\,[\frac{w-1}{2}]!}{[\frac{r}{2}]!\,[\frac{w-r}{2}]!\,[\frac{w-r-1}{2}]!}\,, \qquad \beta_r(n,w) = \frac{[\frac{n}{2}]!\,[w-\frac{n-r}{2}]!}{[\frac{r}{2}]!\,[\frac{n-r}{2}]!\,[w-\frac{n}{2}]!}\,.$$

(The last two formulas are written for $w>0$; there are similar formula for $w=-k<0$ and again a uniform formula using binomial coefficients). Now letting $\gamma_n(w)$ denote the coefficient of $D^n(F)\,D^{w-n}$ in the definition of $\mathcal{L}_{-w}(F)$ (or of

$S\mathcal{D}_w(F)$), we find that the desired equality is equivalent to the trivially verified identity $\gamma_s(w)\,\alpha_r(w-s) = \gamma_n(w)\,\beta_r(w,n)$. ■

One can also give proof of Theorem 3 along the lines of the one in §2 using the superanalogue of the Casimir operator. The other results of this paper can also all be generalized to the SUSY case, but we will not do this here. We say a few words about the super-version of the generalized ΨDO's mentioned in §2. The obvious idea of taking complex powers of D does not work. Instead, we must take linear combinations over R of formal symbols ∂^u and $\partial^u D$ with $u \in \mathbb{C}$, the multiplication being defined by $D^2 = \partial$ and by Leibniz's rule and its superextension (7.3). The transformation behavior under changes of coordinates (7.2) is given by the same formula (7.6) except that when one replaces D^w by $\partial^u D^p$ with $u \in \mathbb{C}$ and $p \in \{0,1\}$ one must reinterpret the formula

$$\alpha_r(w) = \left[\tfrac{r+1}{2}\right]! \binom{\left[\frac{w}{2}\right]}{\left[\frac{r+1}{2}\right]} \binom{\left[\frac{w-1}{2}\right]}{\left[\frac{r}{2}\right]}$$

which was valid for $w \in \mathbb{Z}$ by replacing $\left[\frac{w}{2}\right]$ and $\left[\frac{w-1}{2}\right]$ by u and $u+p-1$, respectively, i.e., by the unique expression which is correct when $u \in \mathbb{Z}$ and $w = 2u+p$, and similarly for the lifting formula (7.5). The considerations of §§3–6 about the multiplications of modular forms induced by the multiplications of various kinds of automorphic ΨDO's can be generalized in the more or less obvious way (thus an automorphic super-ΨDO of mixed weight is just a super-ΨDO which is multiplied on the left and on the right by some powers of $J(A)$ under the action of $A \in \Gamma$), and the arguments given in the last section can also be generalized using the superversion of the non-commutative residue map given in [MR]. Some of these things may be carried out in more detail in a later paper.

§8. Concluding remarks.

The study of formal ΨDO's in the last two decades was primarily motivated by the needs of the theory of completely integrable systems of non-linear differential equations like the Korteweg-de Vries equation and the Kadomtsev-Petviashvili hierarchy: see e.g. [KZ2] for some recent developments and extensive references. A few remarks added here may help the interested reader to put our constructions in this framework.

A sheaf-theoretic version. Let X be a complex Riemannian surface, not necessarily compact. For any open subset $U \subset X$ let $\mathcal{O}_X(U)$ be the ring of holomorphic functions in U. If U admits a local coordinate z, put $\partial_z = \partial/\partial z$ and form the ring $\mathcal{E}_{X,z}(U) = \{\sum_m h_m \partial_z^{-m} | h_m \in \mathcal{O}_X(U)\}$. A change of local coordinate induces a canonical isomorphism of the respective rings compatible with restriction to smaller sets, so that we get a sheaf of rings $\mathcal{E}_X$. It is naturally filtered by the subsheaves $\mathcal{E}_X^{(-m)}$, and the associated sheaf of graded algebras is $\oplus_m \omega_X^{\otimes m}$ where ω_X is the sheaf of holomorphic differentials. Assume now that X is additionally endowed with a projective structure p i. e. with a maximal atlas (U_α, z_α) whose transition functions $z_\alpha = f_{\alpha,\beta}(z_\beta)$ are fractional linear. Define the local lifting maps $\Lambda_{m,z_\alpha} : \omega_{X,z_\alpha}^{\otimes m} \to \mathcal{E}_X^{(-m)}$ by the same formulas as in §1. They will be automatically compatible on the intersections and therefore define a sheafified lifting map $\Lambda_m(p) : \omega_X^{\otimes m} \to \mathcal{E}_X^{(-m)}$ depending only on the flat structure p. This must be evident from the Beilinson construction of the lifting using Casimir operators discussed in §2. In fact, p determines a sheaf of $sl(2)$-algebras on X consisting of projectively flat tangent fields, and the local Casimirs in the relevant sheaf of universal enveloping algebras glue to form a global section $C(p)$. Then $\Lambda_m(p)$ is a differential operator (of infinite order for $m \geq 1$) identifying $\omega_X^{\otimes m}$ with a subsheaf of $\mathcal{E}_X^{(-m)}$ of operators with the same top symbol consisting of the eigenvectors of $C(p)$ with eigenvalue $m(m-1)$.

In the context of automorphic forms we considered essentially a modular curve $X_\Gamma = \mathcal{H}/\Gamma$ with a fixed projective structure coming from H. Now we can vary p and ask how $\Lambda(p) = \oplus\Lambda_m(p)$ varies with p. Formally, $C(p)$ varies isospectrally so that for any pair of flat structures p, p' we have

$$C(p') = T(p',p)C(p)T(p',p)^{-1},\ \Lambda(p') = T(p',p)\Lambda(p)$$

for some $T(p',p)$ (acting e.g. upon $\Gamma(X, \mathcal{E}_X^{(-1)})$ for compact X of genus ≥ 2).

Now, all p's on X form an affine space associated with the vector space of quadratic holomorphic differentials on X: locally we have $p' - p = S_z^{z'}(dz)^2$ where p (resp. p') corresponds to a local flat coordinate z (resp. z'), and $S_z^{z'}$ is the Schwarz derivative (see e. g. A. Tyurin's report [Ty]).

Question. Is it true that $T(p',p)$ depends only on $p'-p$?

For example, a direct calculation (essentially made in the main text) shows that $\Lambda_m(p) = \Lambda_m(p')$ for $m = 1, 0, -1, -2$, whereas

$$(\Lambda_m(p) - \Lambda_m(p'))(f(dz)^{-3}) = \frac{1}{5} f\left(3\frac{(\partial_z j)^2}{j^2} - 2\frac{\partial_z^2 j}{j}\right)\partial_z$$

where $j = \partial z'/\partial z$. This means that $\Lambda_m(p) - \Lambda_m(p')$ is essentially multiplication by $p' - p$ (if one writes $(dz)^{-1}$ instead of ∂_z at the last place of the right hand side).

Question. Can $T(p',p)$ be described in terms of derivations and multiplication in $\Gamma(X, \mathcal{E}_X)$?

All of this has a straightforward supersymmetric version.

Complex powers and $\mathcal{D}$-modules. The complex powers of ∂ were treated in [KZ1] in the Hamiltonian context. If one attempts to sheafify them, then one has to make some sense of complex powers of holomorphic functions because they appear already on the level of coordinate change for principal symbols. A well known way to interpret f^w for complex w is to treat it as a section of a $\mathcal{D}_X$-module. This of course incorporates the formal rule of derivation which we used to define $\Psi\mathrm{DO}_w$. This problem deserves further investigation. Let us mention in addition that the complex eigenvalues of the Casimir operator were recently used to define so-called "matrices of complex size" which are infinite-dimensional algebras $U(sl_2)/(C - w(w-1))$ where C is the Casimir (cf. [KM].)

ΨDO as a Lie algebra and its central extension. In the context of the automorphic forms, we related via liftings the multiplication in ΨDO with Cohen-Kuznetsov brackets. We could have looked at the Lie bracket in ΨDO instead. The point is that this Lie algebra admits a nontrivial central extension which can be suggestively described by introducing the formal expression $\log\partial$ and the commutator

$$[\log\partial, \sum h_m\partial^{-m}] := \sum_m\sum_{k\geq 1}\frac{(-1)^{k+1}}{k}\partial^k h_m \partial^{-m-k}$$

which is then used to define a cocycle $c(A,B) = tr\,([\log\partial, A]\circ B)$ for an appropriate trace functional tr. This construction is important for clarifying the Poisson-Lie structure of ΨDO. Does it admit a sensible descent to modular forms?

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