Elementary Aspects of the Verlinde Formula and of the Harder-Narasimhan-Atiyah-Bott Formula

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Let $\mathcal{N}_{g,n,d}$ denote the moduli space of semistable n-dimensional vector bundles over a fixed Riemann surface of genus g and having as determinant bundle a fixed line bundle of degree d. This is a projective variety, smooth if $(d,n) = 1$ (so that all semistable bundles are stable). Its topology, which depends only on g, n and $d \pmod n$, and not on the specific Riemann surface or line bundle chosen, has been extensively studied. In particular, a recursive determination of the Betti numbers of $\mathcal{N}_{g,n,d}$ in the smooth case was proved by Harder and Narasimhan [HN] and by Atiyah and Bott [AB] (by totally different methods: the former by counting points of the moduli space over finite fields and using the Weil conjectures, and the latter via infinite-dimensional Morse theory with the Yang-Mills functional as Morse function).

Over $\mathcal{N}_{g,n,d}$ there is a canonically defined line bundle Θ, generalizing the classical theta bundle over the Jacobian of a curve (corresponding to the case when $n = 1$ and the determinant bundle is not fixed), and a basic tool in the study of the moduli spaces in question is the determination of the numbers $\dim H^0(\mathcal{N}_{g,n,d}, \Theta^k)$ for variable k. An explicit formula for these numbers was conjectured by the physicist E. Verlinde [Ve]. In the simplest case $n = 2$, it says that

$$\dim H^0(\mathcal{N}_{g,2,0}, \Theta^k) = D_+(g, k+2), \qquad \dim H^0(\mathcal{N}_{g,2,1}, \Theta^k) = D_-(g, 2k+2),$$

where

$$D_\varepsilon(g,k) = \left(\frac{k}{2}\right)^{g-1} \sum_{\substack{j \pmod k \\ j \not\equiv 0 \pmod k}} \frac{\varepsilon^{j-1}}{\sin^{2g-2} \frac{\pi j}{k}} \qquad (g,\, k \in \mathbb{N}, \quad \varepsilon = \pm 1, \quad \varepsilon^k = 1). \tag{1}$$

Verlinde's formula has now been proved by many people ([Sz], [BS], [Th], [Ki], [Do], [NR], [DW], [Z2] for $n = 2$; [Fa] and [BL], building on [TUY], and [Wi], [JK] for general n). In the present paper, in which all the algebraic geometry and topology are suppressed, we will not say anything about these proofs. Instead, we will discuss some of the many interesting number-theoretical and combinatorial aspects of the formula itself. We will look mostly at the case $n = 2$, where we collect together no fewer than ten formulas, some known and some new, for the numbers (1), but we also give in §2 a formula for $n = 3$ and in §3 a simple proof of a certain reciprocity property of the Verlinde numbers for general n.

In the final section of the paper, we solve the recurrences found by Harder-Narasimhan-Atiyah-Bott, obtaining a closed formula for the Poincaré polynomial of the moduli space $\mathcal{N}_{g,n,d}$ in all (smooth) cases. This calculation has several amusing aspects: to guess the formula we have to ignore the fact that polynomials in one variable commute, and to prove it we need an unusual descending induction over real numbers.

§1. The Verlinde formula for $n = 2$

In this section we discuss the numbers $D_\pm(g,k)$ defined by (1). We look mostly at $D_+(g,k)$, which we denote simply by $D(g,k)$, because of the relation

$$D_-(g,k) = D(g,k) - 2^g D(g,k/2) \qquad (k \text{ even}) \tag{2}$$

(for which, by the way, no geometric interpretation seems to be known). The first few values of $D(g,k)$ are given in the following table.

k :	1	2	3	4	5	6	7	8	9
$g = 0$	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8
2	0	1	4	10	20	35	56	84	120
3	0	1	8	36	120	329	784	1680	3312
4	0	1	16	136	800	3611	13328	42048	117072
5	0	1	32	528	5600	42065	241472	1122560	4411584
6	0	1	64	2080	40000	499955	4456256	30475264	168816960
7	0	1	128	8256	288000	5980889	82671232	831000576	6485090688

For fixed k, equation (1) already describes the function $g \mapsto D(g,k)$ as a sum of exponential functions, e.g. for $k \le 6$ we have

k	1	2	3	4	5	6
$D(g,k)$	0	1	2^g	$2^{g-1}(2^g+1)$	$2[(5+\sqrt{5})^{g-1}+(5-\sqrt{5})^{g-1}]$	$2^{2g-1}(3^{g-1}+1)+3^{g-1}$

This information can be put together into a generating function $\mathcal{D}_k(T) = \sum_{g=0}^{\infty} D(g,k)\, T^g$ which is then a rational function of T, e.g.,

k	1	2	3	4	5	6
$\mathcal{D}_k(T)$	1	$\dfrac{1}{1-T}$	$\dfrac{1}{1-2T}$	$\dfrac{1-3T}{1-6T+8T^2}$	$\dfrac{1-6T}{1-10T+20T^2}$	$\dfrac{1-14T+36T^2}{1-19T+96T^2-144T^3}$

The generating function $\mathcal{D}_k(T)$ is described in (ii) and (vi) of the Theorem below.

In the opposite direction, the function $k \mapsto D(g,k)$ for fixed g is a polynomial of degree 0, 1 or $3g-3$ for $g = 0$, 1, or ≥ 1, respectively. (Of course it must be, by the Hirzebruch-Riemann-Roch theorem, if Verlinde's formula is to be true.) Explicit formulas for this polynomial will be given in (iii), (iv) and (x) of Theorem 1 below; the polynomial nature of $D(g,k)$ is also clear from parts (ii), (v) and (ix) of that theorem. The first few values are given by

g	0	1	2	3	4
$D(g,k)$	1	$k-1$	$\dfrac{k^3-k}{6}$	$\dfrac{k^6+10k^4-11k^2}{180}$	$\dfrac{2k^9+21k^7+168k^5-191k^3}{7560}$

and a somewhat later value by

$$\begin{aligned} D(8,k) &= (60k^{21} + 1382k^{19} + 16380k^{17} + 133419k^{15} + 846560k^{13} \\ &\qquad + 4638816k^{11} + 31104000k^9 - 36740617k^7)/35026992000\,. \end{aligned}$$

Note the factor 691 (= numerator of the Bernoulli number B_{12}) in the coefficient of k^{19} here; it will be explained by part (iii) of Theorem 1. Again we can form a generating function $F_g(X) =$

$\sum_{k=0}^{\infty} D(g,k+2)\,X^k$ the first few values being given by

$$F_0(X)=\frac{1}{1-X},\quad F_1(X)=\frac{1}{(1-X)^2},\quad F_2(X)=\frac{1}{(1-X)^4},$$
$$F_3(X)=\frac{1+X+X^2+X^3}{(1-X)^7},\quad F_4(X)=\frac{1+6X+21X^2+40X^3+21X^4+6X^5+X^6}{(1-X)^{10}},$$
$$F_5(X)=\frac{1+19X+190X^2+946X^3+2012X^4+2012X^5+946X^6+190X^7+19X^8+X^9}{(1-X)^{13}}.$$

The fact that $D(g,k)$ for $g>1$ is a polynomial in k of degree $3g-3$, of parity $g-1$ and vanishing at $k=0$ and $k=1$, means that $F_g(X)$ has the form $(1+c_1X+\cdots+c_{3g-6}X^{3g-6})/(1-X)^{3g-2}$, where the coefficients c_i have the symmetry property $c_{3g-6-i}=c_i$. Moreover, in the examples we find that the c_i are always positive. These properties of symmetry and positivity correspond to the Gorenstein and Cohen-Macaulay properties, respectively, of the graded algebra $\bigoplus_k H^0(\mathcal{N}_{g,2,0},\Theta^k)$ of which $F_g(X)$ is the Hilbert series.

We now state a theorem giving a variety of descriptions of the numbers $D(g,k)$. For some parts of the theorem it will be more convenient to use the rational numbers $V_h(k)$ defined by

$$D(g,k)=(k/2)^{g-1}\,V_{g-1}(k)\,. \tag{3}$$

Because of (2), we will (except in part (x)) give the results for $D=D_+$ only, mentioning one or two especially interesting cases for D_- after the theorem.

Theorem 1. *The same numbers $\{D(g,k)\}_{g,k\in\mathbb{N}}$ (resp. $\{V_h(k)\}_{h\ge0,\,k\in\mathbb{N}}$ related to $D(g,k)$ by (3)) are described by any of the following formulas:*

(i) (Trigonometric sum)

$$V_h(k)=\sum_{0<j<k}\Bigl(\sin\frac{\pi j}{k}\Bigr)^{-2h}. \tag{4}$$

(ii) (Generating Function)

$$1-\sum_{h=1}^{\infty}V_h(k)\,\sin^{2h}x\;=\;\frac{k\,\tan x}{\tan kx}\;\in\;\mathbb{Q}[k^2][[x^2]]\,. \tag{5}$$

(iii) (Explicit Polynomial)

$$V_h(k)=\sum_{s=0}^{h}\frac{(-1)^{s-1}2^{2s}B_{2s}}{(2s)!}\,c_{h,s}\,k^{2s}\qquad(h\ge1)\,, \tag{6}$$

where B_{2s} denotes the $(2s)$th Bernoulli number and $c_{h,s}$ the coefficient of x^{-2s} in the Laurent expansion of $(\sin x)^{-2h}$ at $x=0$.

(iv) (Divergent sum)

$$V_h(k)=(-4)^h\sum_{n=1}^{\infty}\left[k\binom{nk+h-1}{2h-1}-\binom{n+h-1}{2h-1}\right], \tag{7}$$

where $\sum_{n=1}^{\infty}P(n)$ for a polynomial P is to be interpreted via $\sum_{n=1}^{\infty}n^i=\zeta(-i)$.

(v) (Recursion)

$$V_h(k) = k - 1 + \frac{1}{k} \sum_{s=0}^{h-1} V_s(k)\, V_{h-s}(k-1) \qquad (h \geq 0). \tag{8}$$

(vi) (Continued Fraction)

$$\frac{1}{k} \sum_{h=0}^{\infty} V_h(k)\, T^h = \cfrac{1}{2 - 2T - \cfrac{1}{2 - \cfrac{1}{2 - 2T - \cfrac{1}{2 - \cfrac{1}{\ddots}}}}} \qquad \textit{(stop after } k-1 \textit{ terms).} \tag{9}$$

(vii) (Factorization/Fusion Rules)

$$D(g,k) = F_k(g,0)\,, \tag{10}$$

where F_k is the unique function on $\mathbb{Z}_{\geq 0} \times S_k$ ($S_k = \mathbb{Z}_{\geq 0} e_1 + \cdots + \mathbb{Z}_{\geq 0} e_k$) satisfying

$$F_k(g_1 + g_2, \mathbf{n}_1 + \mathbf{n}_2) = \sum_{0<a<k} F_k(g_1, \mathbf{n}_1 + e_a) F_k(g_2, \mathbf{n}_2 + e_a) \qquad (\forall g_1, g_2 \geq 0,\ \mathbf{n}_1, \mathbf{n}_2 \in S_k)\,,$$

$$F_k(g+1, \mathbf{n}) = \sum_{0<a<k} F_k(g, \mathbf{n} + 2e_a) \qquad (\forall g \geq 0,\ \mathbf{n} \in S_k)\,,$$

$F_k(0,0) = 1, \quad F_k(0, e_a) = 0, \quad F_k(0, e_a + e_b) = \delta_{ab},$ and

$$F_k(0, e_a + e_b + e_c) = \begin{cases} 1 & \textit{if } 2\max\{a,b,c\} < a+b+c < 2k,\ a+b+c \textit{ odd}, \\ 0 & \textit{otherwise.} \end{cases}$$

(viii) (Functional on Polynomials)

$$D(g,k) = \Phi_k[(U_0^2 + U_1^2 + \cdots + U_{k-1}^2)^g]\,, \tag{11}$$

where $U_n(x)$ denotes the nth Chebyshev polynomial of the second kind (defined by $U_n(\cos\theta) = \sin(n+1)\theta / \sin\theta$ or by $U_{-1} = 0$, $U_0 = 1$, $U_n = 2xU_{n-1} - U_{n-2}$) and $\Phi_k : \mathbb{C}[x] \to \mathbb{C}$ the functional defined by $\Phi_k[f] = 0$ for $f \in (U_{k-1})$, $\Phi_k[U_n] = \delta_{n0}$ for $0 \leq n \leq k-1$.

(ix) (Trace)

$$D(g,k) = \operatorname{tr}(M_k^{g-1})\,, \tag{12}$$

where M_k is the $(k-1) \times (k-1)$ matrix $\{M_k(a,b)\}_{0<a,b<k}$ with

$$M_k(a,b) = \begin{cases} \min(a,b) \cdot \min(k-a, k-b) & \textit{if } a \equiv b \pmod 2, \\ 0 & \textit{if } a \not\equiv b \pmod 2. \end{cases}$$

(x) (Determinant)

$$D_-(g, 2k) = \frac{2^g}{2!\,4! \cdots (2g)!} \det A\,, \tag{13}$$

where $A = \{a_{rs}\}_{0 \leq r < g, 0 < s \leq g}$ with $a_{0s} = 1$, $a_{rs} = (k+r)^{2s} - (k-r)^{2s}$ for $r \geq 1$.

Remarks. 1. Formula (5) can also be written in the form

$$\sum_{g=1}^{\infty} D(g,k) \left(\frac{2}{k} \sin^2 x\right)^{g-1} = \frac{k \sin(k-1)x}{\sin kx \cos x}$$

and formula (6) tells us that the polynomial $D(g,k)$ has the form

$$D(g,k) = \beta_{g-1}\,k^{3g-3} + \frac{g-1}{6}\,\beta_{g-2}\,k^{3g-5} + \frac{(g-1)(5g-4)}{360}\beta_{g-3}\,k^{3g-7} + \cdots + \\ + \frac{2^{g-3}(1+\frac{1}{4}+\cdots+\frac{1}{(g-2)^2})}{15(g-1)\binom{2g-2}{g-1}}\,k^{g+3} + \frac{2^{g-2}}{3(g-1)\binom{2g-2}{g-1}}\,k^{g+1} + C_g k^{g-1}$$

where β_r denotes $2^r|B_{2r}|/(2r)!$ (B_n = nth Bernoulli number) and C_g, the only negative coefficient in the polynomial, is fixed by $D_+(g,1)=0$.

2. There are similar results for D_-. We mention only the analogue of the generating function (ii) and the fusion rule description (vii), namely

$$\sum_{g=1}^{\infty} \frac{D_-(g,k)}{(k/2)^{g-1}} \sin^{2g-2} x = \frac{k\tan x}{\sin kx} \qquad \text{and} \qquad D_-(g,k) = F_k(g, e_{k-1})\,, \tag{14}$$

with F_k as in (vii). The first identity can be proved like (ii) or deduced from it using (2), and the second follows immediately from the general formula for $F_k(g,t)$ given in the proof of (vii) below.

3. The formula (13) is a variant of a formula of Y. Laszlo (communicated to me by A. Beauville), who succeeded in evaluating $\dim H^0(\mathcal{N}_g, \Theta^{k-2})$ but obtained a result which looked quite different from Verlinde's formula. His result, after some simplification, can be written

$$D_-(g,2k) = \sum_{\varepsilon_1,\ldots,\varepsilon_{g-1}\in\{\pm1\}} \varepsilon_1\cdots\varepsilon_{g-1}\,\frac{\Delta(0,1,k+\varepsilon_1,k+2\varepsilon_2,\ldots,k+(g-1)\varepsilon_{g-1})}{\Delta(0,1,2,\ldots,g)}\,,$$

where $\Delta(\ell_0,\ell_1,\ldots,\ell_g) = \prod_{0\le j<i\le g}(\ell_i^2-\ell_j^2)$. Observing that $\Delta(0,1,\ldots,g) = 2^{-g}\,2!\,4!\cdots(2g)!$, writing each $\Delta(\ell_0,\ell_1,\ldots,\ell_g)$ appearing in the numerator as a Vandermonde determinant, and using the multilinearity of determinants, we easily see that this formula is equivalent to (x).

4. As mentioned in the introduction, several of the other formulas occurring in the theorem have occurred in the literature. In particular, I learned of (viii) (and of the whole question of moduli spaces of bundles and the Verlinde formula) from a lecture of R. Bott in Geneva in 1990, and both the generating function (ii) and the description (vii) in terms of "fusion rules" have appeared in various forms in papers of both mathematicians and physicists writing on the subject. Since I do not know the literature well enough to avoid gaffes, I will not try to give references or first discoverers. In any case, our aim is just to collect together in one convenient place some of the known expressions for the Verlinde numbers in the rank 2 case; the various proofs of equivalence, once the formulas have been found, are essentially exercises.

5. The formula (iii) exhibits the polynomiality of $V_h(k)$ and gives an explicit formula for its leading coefficient, and the same information can easily be read off from (ii) or (iv). The description (x), on the other hand, also show that $V_h(k)$ is a polynomial in k, but *a priori* of a higher degree than the true one.

Proof. (i) Here there is nothing to prove, as we are taking this as the definition.

(ii) The multiplication law ("distribution property") of the cotangent function gives

$$\frac{k\tan x}{\tan kx} = 1 + \frac{1}{2}\sum_{0<j<k}\left(\frac{\tan x}{\tan(x+\frac{\pi j}{k})} + \frac{\tan x}{\tan(x-\frac{\pi j}{k})}\right) = 1 - \frac{1}{2}\sum_{0<j<k}\frac{\sin^2 x}{\sin^2\frac{\pi j}{k} - \sin^2 x}\,,$$

and the result follows by expanding the summands on the right as geometric series in $\sin^2 x$.

(iii) Formula (ii) can be rewritten

$$V_h(k) = -\mathrm{Res}_{x=0}\left[\frac{k\tan x}{\tan kx}\,\frac{d(\sin x)}{\sin^{2h+1}x}\right] = -\mathrm{Res}_{x=0}\left[\frac{k}{\tan kx}\,\frac{dx}{\sin^{2h}x}\right].$$

The result follows since $\dfrac{k}{\tan kx} = \sum\limits_{s=0}^{\infty}\dfrac{(-4)^s B_{2s}}{(2s)!}\,k^{2s}x^{2s-1}$, $\dfrac{1}{\sin^{2h}x} = \sum\limits_{s=0}^{h} c_{h,s}\,x^{-2s} + \mathrm{O}(x^2)$.

We can also deduce (6) (except for the value of the constant term, which is determined by the requirement $V_h(1)=0$) directly from the definition (4). Indeed, for $s \ge 1$ denote by $f_s(x)$ the function $\dfrac{-1}{(2s-1)!}\,\dfrac{d^{2s-1}}{dx^{2s-1}}\cot x$. The standard partial fraction decomposition of $\cot x$ gives $f_s(x) = \sum_{n\in\mathbf{Z}}(x-\pi n)^{-2s}$, whence

$$\sum_{0<j<k} f_s\Big(\frac{\pi j}{k}\Big) = \frac{k^{2s}}{\pi^{2s}}\sum_{\substack{n\in\mathbf{Z}\\ n\not\equiv 0 \pmod k}}\frac{1}{n^{2s}} = \frac{2\zeta(2s)}{\pi^{2s}}\,(k^{2s}-1) = \frac{(-1)^{s-1}2^{2s}B_{2s}}{(2s)!}\,(k^{2s}-1).$$

On the other hand, induction on h shows that $\sin^{-2h}x$ is a linear combination of $f_1(x), \ldots, f_h(x)$, and the fact that $f_s(x) = x^{-2s} + \mathrm{O}(x)$ as $x\to 0$ implies that the coefficient of $f_s(x)$ in this representation is $c_{h,s}$.

Observe also that $c_{h,s}$ is p-integral for all primes $p > 2h-2s+1$, so any such prime dividing the numerator of $B_{2s}/(2s)!$ also divides the coefficient of k^{2s} in $V_h(k)$, e.g. the coefficient of k^{12} in the polynomial V_h is divisible by 691 for all $h \le 350$.

(iv) We can rewrite (iii) in the form $V_h(k) = \dfrac{(-4)^h}{(2h-1)!}\sum\limits_{s=1}^{h} c^*_{h,s}\zeta(1-2s)\,(k^{2s}-1)$ where $c^*_{h,s} = (-4)^{s-h}\dfrac{(2h-1)!}{(2s-1)!}\,c_{h,s}$. Differentiating the definition of $c_{h,s}$ twice, we obtain the recursion $c^*_{h+1,s} = c^*_{h,s-1} - h^2 c^*_{h,s}$, so $c^*_{h,s}$ is the coefficient of x^{2s-1} in $x(x^2-1^2)\cdots(x^2-(h-1)^2) = (2h-1)!\binom{x+h-1}{2h-1}$.

(v), (vi) These formulas follow more or less directly from (ii).

(vii) The uniqueness of a function $F_k(g,\mathbf{n})$ satisfying the given axioms is clear, since we can use the first two rules to reduce first g and then the weight $|\mathbf{n}|$ of $\mathbf{n}\in S_k$ (where $|n_1e_1+\cdots+n_{k-1}e_{k-1}| = n_1+\cdots+n_{k-1}$) until $g=0$ and $|\mathbf{n}|\le 3$, in which case the last rules give the value. To prove existence we write down an explicit formula:

$$F_k(g, n_1e_1+\cdots+n_{k-1}e_{k-1}) = \Big(\frac{k}{2}\Big)^{g-1}\sum_{0<j<k}\frac{1}{\sin^{2g-2}(\pi j/k)}\prod_{0<a<k}\left(\frac{\sin(\pi ja/k)}{\sin(\pi j/k)}\right)^{n_a}.$$

Clearly this formula includes (10), so we need only check that it satisfies the axioms.

We can write the proposed formula for $F_k(g,\mathbf{n})$ more succinctly as $\sum s(\zeta)^{g-1}r(\zeta)^{\mathbf{n}}$, where the sum is over all numbers $\zeta = e^{\pi ij/k}$ with $0<j<k$ (i.e. $\zeta^{2k}=1$, $0<\arg(\zeta)<\pi$),

$$s(\zeta) = \frac{-2k}{(\zeta-\zeta^{-1})^2} = \frac{k}{2\sin^2(\pi j/k)}, \qquad r_a(\zeta) = \frac{\zeta^a-\zeta^{-a}}{\zeta-\zeta^{-1}} = \frac{\sin(\pi ja/k)}{\sin(\pi j/k)},$$

and $r(\zeta)^{\mathbf{n}} = r_1(\zeta)^{n_1}\ldots r_{k-1}(\zeta)^{n_{k-1}}$ for $\mathbf{n} = n_1e_1+\ldots+n_{k-1}e_{k-1}\in S_k$. An easy calculation shows that $\sum\limits_{0<a<k} r_a(\zeta_1)r_a(\zeta_2)$ for two such numbers ζ_1 and ζ_2 equals $s(\zeta_1)$ if $\zeta_1=\zeta_2$ and 0 otherwise.

Hence

$$\sum_{0<a<k}\Big(\sum_{\zeta_1} s(\zeta_1)^{g_1-1}\, r(\zeta_1)^{\mathbf{n}_1}\, r_a(\zeta_1)\Big)\Big(\sum_{\zeta_2} s(\zeta_2)^{g_2-1}\, r(\zeta_2)^{\mathbf{n}_2}\, r_a(\zeta_2)\Big) = \sum_{\zeta} s(\zeta)^{g_1+g_2-1}\, r(\zeta)^{\mathbf{n}_1+\mathbf{n}_2},$$

$$\sum_{0<a<k}\Big(\sum_{\zeta} s(\zeta)^{g-1}\, r(\zeta)^{\mathbf{n}}\, r_a(\zeta)^2\Big) = \sum_{\zeta} s(\zeta)^{g}\, r(\zeta)^{\mathbf{n}},$$

which are the first two axioms. For the last, in the non-trivial case $|\mathbf{n}| = 3$, we have

$$\sum_{\zeta} s(\zeta)^{-1}\, r_a(\zeta)\, r_b(\zeta)\, r_c(\zeta) = \frac{-1}{2k}\sum_{\zeta}\frac{(\zeta^a-\zeta^{-a})(\zeta^b-\zeta^{-b})(\zeta^c-\zeta^{-c})}{\zeta-\zeta^{-1}}.$$

The sum is 0 if $a+b+c+1$ is odd (replace ζ by $-\zeta^{-1}$) and otherwise is one-half of the corresponding sum over all $(2k)$th roots of unity, so we find

$$\begin{aligned} F_k(0, e_a+e_b+e_c) &= \frac{1}{2k}\sum_{\zeta^{2k}=1}(\zeta^{a-b}-\zeta^{a+b})(\zeta^{c-1}+\zeta^{c-3}+\cdots+\zeta^{3-c}+\zeta^{1-c}) \\ &= \begin{cases} 1 & \text{if } |a-b|<c \\ 0 & \text{otherwise} \end{cases} - \begin{cases} 1 & \text{if } a+b<c \text{ or } a+b+c>2k, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

which agrees with the given "fusion formula."

(viii) Define a functional $\Psi_k : \mathbb{C}[x] \to \mathbb{C}$ by

$$\Psi_k[f] = \frac{1}{k}\sum_{j \pmod{2k}} f\big(\cos\frac{\pi j}{k}\big)\sin^2\frac{\pi j}{k} = -\frac{1}{4k}\sum_{\zeta^{2k}=1}(\zeta-\zeta^{-1})^2 f\big(\frac{\zeta+\zeta^{-1}}{2}\big).$$

Then $\Psi_k[f]=0$ for f in the ideal (U_{k-1}) because $U_{k-1}(\cos\frac{\pi j}{k})=0$ for all j, and

$$\Psi_k[U_n] = -\frac{1}{4k}\sum_{\substack{\zeta^{2k}=1\\ \zeta^2\neq 1}}(\zeta-\zeta^{-1})^2\,\frac{\zeta^{n+1}-\zeta^{-n-1}}{\zeta-\zeta^{-1}} = -\frac{1}{4k}\sum_{\zeta^{2k}=1}(\zeta^{n+2}-\zeta^{n}-\zeta^{-n}+\zeta^{-n-2})$$

equals 1 for $n=0$ and 0 for $1\le n\le k-1$. Hence $\Psi_k=\Phi_k$. On the other hand, if $H(x)$ denotes the polynomial $U_0(x)^2+\cdots+U_{k-1}(x)^2$, then

$$H\big(\frac{\zeta+\zeta^{-1}}{2}\big) = \sum_{j=0}^{k-1}\Big(\frac{\zeta^{j+1}-\zeta^{-j-1}}{\zeta-\zeta^{-1}}\Big)^2 = \frac{-2k}{(\zeta-\zeta^{-1})^2}$$

for $\zeta^{2k}=1$, $\zeta^2\neq 1$. Hence

$$\Phi_k(H^g) = \Psi_k(H^g) = \frac{1}{2}\sum_{\substack{\zeta^{2k}=1\\ \zeta^2\neq1}}\Big(\frac{-2k}{(\zeta-\zeta^{-1})^2}\Big)^{g-1} = \big(\frac{k}{2}\big)^{g-1}\sum_{0<j<k}\big(\sin\frac{\pi j}{k}\big)^{-2g+2} = D(g,k).$$

(ix) It is an exercise in the summation of geometric series to check that for each $0<j<k$ the vector v_j with components $(v_j)_a = \sin(\pi j/k)$ $(0<a<k)$ is an eigenvector of M_k with eigenvalue $k/2\sin^2(\pi j/k)$. The equivalence of (4) and (12) is obvious from this. The relation of (ix) to (vii)

is that M_k is the sum of the matrices N_a^2 $(0 < a < k)$, where N_a is the $(k-1)\times(k-1)$ matrix whose (b,c) entry is the number $F_k(0, e_a+e_b+e_c)$ given in the last formula of (vii).

(x) Define numbers λ_{pr} $(p,\, r \ge 0)$ by $\lambda_{0r} = \delta_{r,0}$ and $\lambda_{pr} = (-1)^{p+r}\frac{r}{p}\binom{2p}{p+r}$ for $p \ge 1$, and set $b_{ps} = \sum_r \lambda_{pr} a_{rs}$. Then the matrix $B = (b_{ps})_{0\le p<g,\,1\le s\le g}$ is the product of A with a $g\times g$ unipotent matrix, so has the same determinant. On the other hand, we have

$$\sum_{s=1}^{\infty} a_{rs}\frac{(-1)^{s-1}\,x^{2s}}{(2s)!} = \begin{cases} -\cos[(k+r)x] + \cos[(k-r)x] \;=\; 2\sin kx \sin rx & (r\ge 1) \\ 1-\cos x \;=\; 2\,\sin^2(x/2) & (r=0) \end{cases}$$

and

$$\sum_{r=0}^{\infty}\lambda_{pr}\sin rx = -\frac{1}{2p}\,\frac{d}{dx}\left[\sum_{n=0}^{2p}(-1)^n\binom{2p}{n}\,e^{i(p-n)x}\right] = \left(-4\,\sin^2(x/2)\right)^{p-1}\,\sin x$$

for $p \ge 1$, so

$$\beta_p(x) := \sum_{s=1}^{\infty} b_{ps}\frac{(-1)^{s-1}\,x^{2s}}{(2s)!} = \begin{cases} 2\sin kx\,(-4\sin^2 x/2)^{p-1}\,\sin x & (p\ge 1)\,, \\ 2\,\sin^2(x/2) & (p=0)\,. \end{cases}$$

In particular, $\beta_0(x) = x^2/2 + \cdots \in \mathbb{Q}[[x^2]]$ and $\beta_p(x) = 2k(-1)^{p-1}x^{2p} + \cdots \in \mathbb{Q}[[x^2]]$, so we can write $\beta_0(x) = \sum_{p=1}^{\infty} v_p\beta_p(x)$. Then $v_p = (-1)^{p-1}D_-(p,k)/(4k)^p$ by (14). Replacing $(\beta_0, \beta_1, \ldots, \beta_{g-1})$ by $(\beta_1, \ldots, \beta_{g-1}, \beta_0 - \sum_{p=1}^{g-1} v_p\beta_p)$ corresponds to subtracting from the first row of B a linear combination of the other rows and then moving it to the bottom row, which changes the determinant by a factor $(-1)^{g-1}$. But this new matrix, acting on the basis $\{(-1)^{s-1}x^{2s}/(2s)!\}_{1\le s\le g}$ of the g-dimensional vector space $x^2\mathbb{Q}[[x^2]]/x^{2g+2}\mathbb{Q}[[x^2]]$, is triangular with diagonal entries $2k\cdot 2!\,, \ldots, 2k\cdot(2g-2)!\,, 2k\cdot(2g)!\,v_g$. The assertion follows.

§2. The Verlinde formula for $n = 3$

The Verlinde formula for bundles of rank 3 and trivial determinant is

$$\dim H^0(\mathcal{N}_{g,3,0}\,, \Theta^k) = 3^{g-1}\left(\frac{k+3}{8}\right)^{2g-2} V_{g-1,g-1,g-1}(k+3)\,,$$

where

$$V_{l,m,n}(k) = \sum_{\substack{a,b,c \pmod k \\ a+b+c\equiv 0 \pmod k}} \left(\sin\frac{\pi a}{k}\right)^{-2l}\left(\sin\frac{\pi b}{k}\right)^{-2m}\left(\sin\frac{\pi c}{k}\right)^{-2n} \qquad (k,\, l,\, m,\, n \in \mathbb{N})\,.$$

(Here the meaningless terms with $a\equiv 0$, $b \equiv 0$ or $c\equiv 0 \pmod k$ are to be omitted.) With methods similar to those used in proving part (ii) of the Theorem above, one obtains

$$V_{l,m,n}(k) = \text{ coefficient of } \sin^{2l}x\cdot\sin^{2m}y\cdot\sin^{2n}z \text{ in } \frac{f_k(x+y+z)}{f_k(x)f_k(y)\,f_k(z)}\,, \tag{15}$$

where $f_k(x) = \dfrac{\sin kx}{k\,\sin x}$. In particular, $V_{l,m,n}(k)$ is a polynomial of degree $l+m+n$ in k^2 with rational coefficients, the first few values being

$$\begin{aligned}
V_{1,1,1}(k) &= \frac{2^5}{3\cdot 7!}\,(k^2-1)\,(k^2-4)\,(k^2+47)\,,\\
V_{1,1,2}(k) &= \frac{2^8}{5\cdot 9!}\,(k^2-1)\,(k^2-4)\,(k^4+40k^2+679)\\
V_{1,2,2}(k) &= \frac{2^{10}}{3\cdot 11!}\,(k^2-1)\,(k^2-4)\,(k^2+19)\,(k^4+19k^2+628)\,,\\
V_{1,1,3}(k) &= \frac{2^9}{3\cdot 11!}\,(k^2-1)\,(k^2-4)\,(3k^6+125k^4+1757k^2+21155)\,,\\
V_{2,2,2}(k) &= \frac{2^{11}}{15\cdot 13!}\,(k^2-1)\,(k^2-4)\,(19k^8+875k^6+22317k^4+505625k^2+5691964)\,.
\end{aligned}$$

The leading coefficient $v_{l,m,n}$ of this polynomial is given by

$$v_{l,m,n} = \frac{1}{\pi^{2l+2m+2n}} \sum_{\substack{a,b,c\in\mathbf{Z}\smallsetminus\{0\}\\ a+b+c=0}} \frac{1}{a^{2l}b^{2m}c^{2n}} = \text{ coefficient of } x^{2l}y^{2m}z^{2n} \text{ in } \frac{f(x+y+z)}{f(x)f(y)f(z)}$$

with $f(x)=\dfrac{\sin x}{x}$ (as one sees by dividing x, y and z by k in the preceding generating function and letting k tend to infinity). In the Verlinde case $l=m=n=h$ $(=g-1)$ this number can be expressed in terms of Bernoulli numbers as

$$v_{h,h,h} = (-1)^h 2^{6h+1} \sum_{r=0}^{h} \binom{4h-2r-1}{2h-1} \frac{B_{2r}}{(2r)!}\frac{B_{6h-2r}}{(6h-2r)!}\,, \tag{16}$$

a formula also obtained by Stavros Garoufalidis and Leonard Weinstein (see also [Z1]).

The method of calculation also works for the Verlinde formula of higher rank n (given in §3) and expresses the Verlinde number in this case as the coefficient of $\prod(\sin x_i)^{2g-2}$ in a certain trigonometric function of $n(n-1)/2$ variables x_i. However, already for $n=4$ the form I obtained for this trigonometric function (of 6 variables) was as a sum of about 80 terms, and the algebra of putting these terms over a common denominator and simplifying defeated both me and the computer.

§3. A reciprocity property of the Verlinde numbers

The Verlinde formula for general rank n and trivial determinant bundle says that

$$\dim H^0(\mathcal{N}_{g,n,0}\,,\Theta^k) = D(g,n,k+n)\,,$$

where

$$D(g,n,N) = n^{g-1}\left(\frac{N}{2^n}\right)^{(n-1)(g-1)} \sum_{\substack{j_1,\dots,j_{n-1}>0\\ j_1+\dots+j_{n-1}<N}} \prod_{1\le\mu<\nu\le n} \frac{1}{\left(\sin \pi\frac{j_\mu+\dots+j_{\nu-1}}{N}\right)^{2g-2}}\,.$$

In November 1992, A. Beauville wrote to me that "according to the physicists" one should have

$$k^g\, D(g,n,k+n) = n^g\, D(g,k,k+n) \tag{17}$$

and that he and Y. Laszlo had a "long and ugly" proof of this for $k = 2$ (the case $k = 1$ is easy). Here is a simple proof for arbitrary k and n.

Write G for $\mathbb{Z}/N\mathbb{Z}$ and s_ν for $j_1 + \cdots j_{\nu-1}$. Then

$$\frac{N^g}{n^g} D(g,n,N) = \frac{N}{n} N^{n(g-1)} \sum_{0=s_1<s_2<\cdots<s_n<N} \prod_{1\le\mu<\nu\le n} \frac{1}{\left(2\sin\pi\frac{s_\nu-s_\mu}{N}\right)^{2g-2}}$$
$$= \frac{N}{n} N^{n(g-1)} \sum_{\substack{S\subset G\\ |S|=n,\,0\in S}} \prod_{\substack{s,\,t\in S\\ s\ne t}} \frac{1}{\left|2\sin\pi\frac{s-t}{N}\right|^{g-1}} .$$

Since the product is invariant under translations $S \mapsto S + j$, and the probability that an n-subset of G contains 0 is $\frac{n}{N}$, we can drop the condition "$0 \in S$" and omit the factor $\frac{N}{n}$, so

$$\frac{N^g}{n^g} D(g,n,k) = \sum_{\substack{S\subset G\\ |S|=n}} \prod_{s\in S} \left(N \prod_{\substack{t\in S\\ t\ne s}} \frac{1}{\left|2\sin\pi\frac{s-t}{N}\right|}\right)^{g-1} = \sum_{\substack{S\subset G\\ |S|=n}} \prod_{\substack{s\in S\\ s'\notin S}} \left|2\sin\pi\frac{s-s'}{N}\right|^{g-1} ,$$

where we have used that $N = \prod_{t\in G,\, t\ne s} \left|2\sin\pi\frac{s-t}{N}\right|$ for any $s \in G$. The last expression is seen by interchanging S and $G \setminus S$ to be symmetric under $n \leftrightarrow k := N - n$.

§4. On the Harder-Narasimhan-Atiyah-Bott dimension formula

As stated in the introduction, a recursive formula for the Betti numbers of $\mathcal{N}_{g,n,d}$ in the smooth case $(n,d) = 1$ was given by Harder-Narasimhan (implicitly, but made explicit by Desale-Ramanan [DR]) and by Atiyah-Bott [AB]. The formula, expressed via a Poincaré polynomial, is

$$B_{g,n,d}(t) := \sum_{i=0}^{\infty} \dim H^i(\mathcal{N}_{g,n,d})\, t^i = \frac{1-t^2}{(1+t)^{2g}} P_{n,d} \qquad \text{if } (n,d)=1 \tag{18}$$

where $P_{n,d} = P_{n,d}(g,t)$ is the rational function of t defined inductively by the requirement that

$$\sum_{k=1}^{n} \sum_{\substack{n_1,\dots,n_k>0\\ n_1+\cdots+n_k=n}} \sum_{\substack{d_1/n_1>\dots>d_k/n_k\\ d_1+\cdots+d_k=d}} t^{2N_g(n_1,\dots,n_k;d_1,\dots,d_k)} P_{n_1,d_1}\cdots P_{n_k,d_k} = P_n \tag{19}$$

for all $n \in \mathbb{N}$ and $d \in \mathbb{Z}$, where

$$P_n = P_n(g,t) := \frac{(1+t)^{2g}(1+t^3)^{2g}\cdots(1+t^{2n-1})^{2g}}{(1-t^2)^2(1-t^4)^2\cdots(1-t^{2n-2})^2(1-t^{2n})} \qquad (n\in\mathbb{N}) \tag{20}$$

and

$$N_g(n_1,\dots,n_k;d_1,\dots,d_k) = \sum_{1\le i<j\le k} \left(d_i n_j - d_j n_i + (g-1)n_i n_j\right) .$$

(The factor $(1+t)^{2g}$ in (18) is the Poincaré polynomial of the Jacobian of a curve of genus g and would be absent if we had defined $\mathcal{N}_{g,n,d}$ as the moduli space of all vector bundles of rank n and degree d, rather than fixing the determinant line bundle.) Notice that the periodicity property $P_{n,d+ln} = P_{n,d}$ ($l \in \mathbb{Z}$) follows immediately from (19) since we can replace each d_i by $d_i + ln_i$ without changing the inequalities $d_1/n_1 > \dots > d_k/n_k$ or the value of $N_g(n_1,\dots,n_k;d_1,\dots,d_k)$.

Before giving the solution of the recursion (19) in the general case, let us look in detail at the cases $n = 1$, 2, 3 and 4 (already computed in [DR], Theorem 2, p. 241). For $n = 1$ we have immediately

$$P_1 = P_{1,d}\,, \qquad P_{1,d} = P_1 = \frac{(1+t)^{2g}}{1-t^2}\,, \qquad B_{g,1,d}(t) = 1$$

as it should be. For $n = 2$ we find

$$P_2 = P_{2,d} + \sum_{d_1+d_2=d,\ d_1>d_2} t^{2(d_1-d_2)+2g-2}\, P_{1,d_1} P_{1,d_2} = P_{2,d} + \frac{t^{2g-2+a}}{1-t^4}\, P_1^2$$

where a is 2 if d is odd and 4 if d is even. For $d = 1$ this gives the well-known formula

$$B_{g,2,1}(t) = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{2g}}{1-t^4}\,\frac{(1+t)^{2g}}{(1-t^2)} = \frac{(1+t^3)^{2g}-(t+t^2)^{2g}}{(1+t^3)^2-(t+t^2)^2} \tag{21}$$

(the second form of the result makes it clear that this expression is indeed a polynomial in t). For $n = 3$ the calculation is already much messier, with several geometric series to be summed and various case distinctions according to the values of the various d_i modulo 2 or 3. The result for $d = 1$ is

$$P_{3,1} = P_3 - \frac{t^{4g-2}(1+t^2)}{1-t^6}\, P_1 P_2 + \frac{t^{6g-2}}{(1-t^4)^2}\, P_1^3\,. \tag{22}$$

By the time we reach $n = 4$ the calculation becomes really horrible. The result (for $d = 1$) is

$$P_{4,1} = P_4 - \frac{t^{6g-4}}{1-t^4}\, P_1 P_3 - \frac{t^{8g-4}}{1-t^8}\, P_2^2 + \frac{t^{10g-6}(1+t^2)^2}{(1-t^6)^2}\, P_1^2 P_2 - \frac{t^{12g-6}}{(1-t^4)^3}\, P_1^4\,. \tag{23}$$

That the right-hand sides of (22) and (23), divided by P_1, are polynomials in t, as they must be if (18) is to hold, is not at all obvious. (A direct proof will be given at the end of the section, when we rewrite these expressions in terms of a generating function.) More serious is that one cannot see—or at least, *I* could not see—what the pattern in the coefficients in these two equations is. The problem is that $P_{n,d}$ ends up getting written as a linear combination of all products $P_{n_1}\cdots P_{n_k}$ with $(n_1,\ldots,n_k)$ running over all partitions of n, but the coefficient of each such product arises as a sum of many terms coming from different parts of the recursive procedure. The surprising solution turned out to be to ignore the fact that the expressions P_{n_i,d_i} commute! That is to say, we go back to the recursion (19) and calculate inductively in n, substituting at each stage for $P_{n',d'}$ $(n' < n)$ the formulas already obtained as polynomials in P_m's, but in doing so we *do not permit ourselves to change the order of multiplication.* For $n = 3$ this procedure gives

$$P_{3,1} = P_3 - \frac{t^{4g}}{1-t^6}\, P_1 P_2 - \frac{t^{4g-2}}{1-t^6}\, P_2 P_1 + \frac{t^{6g-2}}{(1-t^4)^2}\, P_1^3$$

which is hardly any simpler than (22), but for $n = 4$ we find

$$\begin{aligned} P_{4,1} = P_4 &- \frac{t^{6g}}{1-t^8}\, P_1 P_3 - \frac{t^{6g-4}}{1-t^8}\, P_3 P_1 - \frac{t^{8g-4}}{1-t^8}\, P_2^2 + \frac{t^{10g-4}}{(1-t^4)(1-t^6)}\, P_1^2 P_2 \\ &+ \frac{t^{10g-4}}{(1-t^6)^2}\, P_1 P_2 P_1 + \frac{t^{10g-6}}{(1-t^4)(1-t^6)}\, P_2 P_1^2 - \frac{t^{12g-6}}{(1-t^4)^3}\, P_1^4 \end{aligned}$$

and now it already is becoming much easier to discern a pattern. Studying these formulas and the corresponding results for $P_{3,0}$, $P_{4,0}$ and $P_{4,2}$ (which are the only other cases of interest for $n \le 4$, since $P_{n,d}$ as a function of d is even and periodic of period n), one finally discovers the general formula given in the theorem below.

Before stating the result, we make one preliminary transformation to simplify the recursion. Define new rational functions of t by $Q_n(t) = t^{-n^2(g-1)}P_n(t)$ and $Q_{n,d}(t) = t^{-n^2(g-1)}P_{n,d}(t)$. Then, because $(\sum n_i)^2 - \sum n_i^2 = 2\sum_{i<j} n_i n_j$, the explicit dependence on g in the recursion (19) is eliminated, i.e., this recursion becomes

$$Q_n = \sum_{k=1}^{n} \sum_{\substack{n_1,\ldots,n_k>0 \\ n_1+\cdots+n_k=n}} \sum_{\substack{d_1/n_1>\ldots>d_k/n_k \\ d_1+\cdots+d_k=d}} x^{N(n_1,\ldots,n_k;d_1,\ldots,d_k)}\, Q_{n_1,d_1}\cdots Q_{n_k,d_k} \qquad (\forall n,\, d) \tag{24}$$

where $x = t^2$ and

$$N(n_1,\ldots,n_k;d_1,\ldots,d_k) = \sum_{1\le i<j\le k} (d_i n_j - d_j n_i)$$

($= N_1$ in the previous notation). The solution of this recursive system is given by:

Theorem 2. *Let Q_n and $Q_{n,d}$ ($n \in \mathbb{N}$, $d \in \mathbb{Z}/n\mathbb{Z}$) be elements of a not necessarily commutative algebra over the field of formal power series $\mathbb{Q}((x))$ which are related by (24). Then for any n and d, we have*

$$Q_{n,d} = \sum_{k=1}^{n} \sum_{\substack{n_1,\ldots,n_k>0 \\ n_1+\cdots+n_k=n}} \frac{(-1)^{k-1}\, x^{M(n_1,\ldots,n_k;d/n)}}{\left(1-x^{n_1+n_2}\right)\cdots\left(1-x^{n_{k-1}+n_k}\right)}\, Q_{n_1}\cdots Q_{n_k}\,, \tag{25}$$

where $M(n_1,\ldots,n_k;\lambda)$ for $n_i \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ is defined by

$$M(n_1,\ldots,n_k;\,\lambda) = \sum_{j=1}^{k-1} (n_j + n_{j+1})\,\langle\,(n_1+\cdots+n_j)\,\lambda\,\rangle\,.$$

Here $\langle x\rangle = 1 + [x] - x$ for a real number x denotes the unique $t \in (0,1]$ with $x + t \in \mathbb{Z}$.

Corollary. *For n and d coprime the Poincaré polynomial of $\mathcal{N}_{g,n,d}$ is given by*

$$B_{g,n,d}(t) = P_1^{-1} \sum_{k=1}^{n} \sum_{\substack{n_1,\ldots,n_k>0 \\ n_1+\cdots+n_k=n}} \frac{(-1)^{k-1}\, t^{2M_g(n_1,\ldots,n_k;d/n)}}{\left(1-t^{2n_1+2n_2}\right)\cdots\left(1-t^{2n_{k-1}+2n_k}\right)}\, P_{n_1}\cdots P_{n_k}\,, \tag{26}$$

with $P_n = P_n(t)$ as in (20) and $M_{g-1}(n_1,\ldots,n_k;\lambda) = M(n_1,\ldots,n_k;\lambda) + (g-1)\sum_{i<j} n_i n_j$.

Remarks. 1. The exponent $M(n_1,\ldots,n_k;d/n)$ in (25) is always an integer, since $\langle x\rangle \equiv -x$ (mod 1) and $\sum (n_j + n_{j+1})(n_1+\cdots+n_j)(d/n) = d(n - n_k)$ is an integer.

2. The periodicity of $Q_{n,d}$ in d is obvious from (25), since $\langle x\rangle$ is periodic in x of period 1.

3. Observe that, surprisingly, formula (25) is *simpler* than the recursion (24) of which it is the solution: we have the same sum over (ordered) partitions of n, but no longer have the infinite sum over partitions of d.

Proof. The proof of Theorem 2, not surprisingly, uses induction. However, to make this work we have to generalize the theorem in a rather odd way, by introducing a *real* parameter with respect to which we can perform an induction!

Theorem 3. *Let the hypotheses be as in the previous theorem. Then the two quantities*

$$Q^\lambda_{n,d} = \sum_{k=1}^{n} \sum_{\substack{n_1,\dots,n_k>0 \\ n_1+\dots+n_k=n}} \sum_{\substack{\lambda \ge d_1/n_1 > \dots > d_k/n_k \\ d_1+\dots+d_k=d}} x^{N(n_1,\dots,n_k;d_1,\dots,d_k)} Q_{n_1,d_1} \cdots Q_{n_k,d_k} \tag{27}$$

$$R^\lambda_{n,d} = \sum_{k=1}^{n} \sum_{\substack{n_1,\dots,n_k>0 \\ n_1+\dots+n_k=n}} \frac{(-1)^{k-1}\, x^{(n-n_k)(\lambda n-d)+M(n_1,\dots,n_k;\lambda)}}{(1-x^{n_1+n_2})\cdots(1-x^{n_{k-1}+n_k})} Q_{n_1}\cdots Q_{n_k}\,, \tag{28}$$

agree for every real number $\lambda \ge d/n$.

Theorem 2 follows because for $\lambda = d/n$ the inner sum in (27) is empty unless $k=1$, so $Q^\lambda_{n,d} = Q_{n,d}$. Note also that the periodicity property now reads $Q^{\lambda+1}_{n,d+n} = Q^\lambda_{n,d}$ (and similarly for $R^\lambda_{n,d}$).

Proof. We assume by induction that $Q^\lambda_{n',d} = R^\lambda_{n',d}$ for all $n' < n$ (and all d and λ). This is normal enough. But then for a given n and d we will do a "downward induction over real numbers," showing that

(i) the theorem is true for the "initial value" $\lambda = \infty$

(ii) if the theorem is true for λ, then it is true for $\lambda - \varepsilon$ for sufficiently small $\varepsilon > 0$.

This peculiar induction works for the following reason. First, both $Q^\lambda_{n,d}$ and $R^\lambda_{n,d}$ are step functions of λ, jumping only at rational values with denominator strictly less than n. Indeed, $Q^\lambda_{n,d}$ jumps at λ only if $\lambda = d_1/n_1$ for some k-tuples $(n_1,\dots,n_k) \in \mathbb{N}^k$, $(d_1,\dots,d_k) \in \mathbb{Z}^k$ with $\sum n_i = n$, $\sum d_i = d$, and k cannot be be 1 for $\lambda > d/n$. Similarly, $R^\lambda_{n,d}$ jumps only if $(n_1+\dots+n_j)\lambda$ is an integer for some decomposition $n = n_1+\dots+n_k$ and some $1 \le j < k$. At a jump point $\lambda = c/m$ $(0 < m < n)$, both $Q^\lambda_{n,d}$ and $R^\lambda_{n,d}$ take on the same value as they do slightly to the right. To prove (ii), we will show (using the inductive assumption) that the differences $\Delta Q^\lambda_{n,d} = Q^\lambda_{n,d} - Q^{\lambda^-}_{n,d}$ and $\Delta R^\lambda_{n,d} = R^\lambda_{n,d} - R^{\lambda^-}_{n,d}$ agree (here λ^- denotes any value $\lambda - \varepsilon$ with $\varepsilon > 0$ sufficiently small). This shows that the difference $Q^\lambda_{n,d} - R^\lambda_{n,d}$ is independent of λ. The meaning of (i) is that $Q^\lambda_{n,d}$ and $R^\lambda_{n,d}$ agree modulo x^N for all N and $\lambda \ge \lambda_0(N)$ sufficiently large. Combining these statements shows that the two expressions agree (as power series in x) for all λ.

We first prove (i). This is easy. As $\lambda \to \infty$, there is no restriction in (27), so $Q^\lambda_{n,d}$ tends x-adically to Q_n by virtue of the relation (24). On the other hand, the exponent in the numerator of the fraction in (28) tends to zero x-adically as $\lambda \to \infty$ in all cases except $k=1$, so $R^\lambda_{n,d}$ also has the limit Q_n.

We now compute the jumps of $Q^\lambda_{n,d}$ and $R^\lambda_{n,d}$ at rational values of λ with denominator $< n$. As already stated, the only terms of $Q^\lambda_{n,d}$ which differ at λ and at λ^- are those with $d_1/n_1 = \lambda$. Write m for n_1, so $0 < m < n$, $m\lambda \in \mathbb{Z}$. We have

$$\begin{aligned} N(n_1,\dots,n_k;d_1,\dots,d_k) &= \sum_{j=2}^{k}(\lambda m n_i - m d_i) + \sum_{2\le i<j\le k}(d_i n_j - d_j n_i) \\ &= \lambda m\,(n-m) - m\,(d-\lambda m) + N(n_2,\dots,n_k;d_2,\dots,d_k)\,. \end{aligned}$$

Hence, renaming $k-1$ and n_{i+1}, d_{i+1} $(i \ge 1)$ as k and n_i, d_i, we find

$$\Delta Q^\lambda_{n,d} = \sum_{0<m<n,\ m\lambda\in\mathbb{Z}} x^{m(\lambda n-d)}\, Q_{m,\lambda m}\, Q^\lambda_{n-m,d-\lambda m}\,. \tag{29}$$

Now turn to $R^\lambda_{n,d}$ and look at a single summand $(n_1,\dots,n_k)$ in (28). Denote generically by J those indices $j \in \{1,\dots,k\}$ for which $m_J := n_1+\dots+n_J$ is a multiple of the denominator of λ.

Then $M(n_1,\cdots,n_k;\lambda)=M(n_1,\cdots,n_k;\lambda^-)+\sum_J(n_J+n_{J+1})$, so

$$x^{M(n_1,\cdots,n_k;\lambda^-)}-x^{M(n_1,\cdots,n_k;\lambda)}=x^{M(n_1,\cdots,n_k;\lambda^-)}\left(1-x^{\sum_J(n_J+n_{J+1})}\right)$$
$$=\sum_J x^{M(n_1,\cdots,n_k;\lambda^-)+\sum_{J'<J}(n_{J'}+n_{J'+1})}\left(1-x^{(n_J+n_{J+1})}\right)$$
$$=\sum_J x^{M(n_1,\cdots,n_J;\lambda)}\left(1-x^{(n_J+n_{J+1})}\right)\cdot x^{M(n_{J+1},\cdots,k;\lambda)},$$

$$\Delta\left(\frac{x^{(n-n_k)(\lambda n-d)+M(n_1,\dots,n_k;\lambda)}}{(1-x^{n_1+n_2})\cdots(1-x^{n_{k-1}+n_k})}\right)=\sum_J x^{m_J(\lambda n-d)}\frac{x^{M(n_1,\dots,n_J;\lambda)}}{(1-x^{n_1+n_2})\cdots(1-x^{n_{J-1}+n_J})}\cdot\frac{x^{(n-m_J-n_k)(\lambda n-d)+M(n_{J+1},\dots,n_k;\lambda)}}{(1-x^{n_{J+1}+n_{J+2}})\cdots(1-x^{n_{k-1}+n_k})}$$

and hence, grouping together all terms by values of $m=m_J$,

$$\Delta R^\lambda_{n,d}=\sum_{0<m<n,\ m\lambda\in\mathbf{Z}} x^{m(\lambda n-d)}\,R^\lambda_{m,\lambda m}\,R^\lambda_{n-m,d-\lambda m}\,.$$

But $R^\lambda_{m,\lambda m}=Q^\lambda_{m,\lambda m}=Q_{m,\lambda m}$ and $R^\lambda_{n-m,d-\lambda m}=Q^\lambda_{n-m,d-\lambda m}$ by induction (since m and $n-m$ are less than n), so the right-hand side of this agrees with (29). This completes the proof.

We end the section with some remarks on the polynomial nature of the functions occurring on the right-hand side of (18). From (20) and (26) we see that the generating function of the Poincaré polynomials $B_{g,n,d}(t)$ for fixed n and d has the form

$$\sum_{g\ge1}B_{g,n,d}(t)\,X^{g-1}=\Big(\sum_{0\le j<p(n)}A_j\,X^j\Big)\Big/\prod_\pi(1-C_\pi X)\tag{30}$$

where $p(n)$ is the number of partitions of n, $A_j=A_j(t)$ is a rational function of t, and the product is over the partition $\pi=\{n_1\le\ldots\le n_k,\ n_1+\cdots+n_k=n\}$, with

$$C_{n_1,\dots,n_k}=t^{n^2}(1+t)^{-2}\prod_{j=1}^k\Big(t^{-n_j^2}(1+t)^2(1+t^3)^2\cdots(1+t^{2n_j-1})^2\Big).$$

(This already follows from the general result on the shape of $B_{g,n,d}(t)$ given in [DR], Theorem 1.) For any given n and d the finitely many coefficients $A_j(t)$ are calculable, and if they are all polynomials then this exhibits the polynomial nature of all the $B_{g,n,d}(t)$. In particular, for $n\le4$ and $d=1$ we find from (21) – (23)

$$\sum_{g\ge1}B_{g,2,1}(t)\,X^{g-1}=\frac{1}{\big(1-(1+t^3)^2X\big)\big(1-t^2(1+t)^2X\big)},$$

$$\sum_{g\ge1}B_{g,3,1}(t)\,X^{g-1}=\frac{1+t^2(1+t)(1+t^3)^3(1+t+t^2)\,X}{\big(1-(1+t^3)^2(1+t^5)^2X\big)\big(1-t^4(1+t)^2(1+t^3)^2X\big)\big(1-t^6(1+t)^4X\big)},$$

$$\sum_{g\ge1}B_{g,4,1}(t)\,X^{g-1}=\frac{1+A_1X+A_2X^2+A_3X^3}{(1-C_4X)(1-C_{1,3}X)(1-C_{2,2}X)(1-C_{1,1,2}X)(1-C_{1,1,1,1}X)}$$

with

$$
\begin{aligned}
A_1 &= t^2\,(1+t)^3\,(1+t^3)^3\,(1-t+3t^2-4t^3+7t^4-5t^5+10t^6 \\
&\qquad -5t^7+10t^8-5t^9+7t^{10}-4t^{11}+3t^{12}-t^{13}+t^{14})\,, \\
A_2 &= -\,t^{10}\,(1+t)^6\,(1+t^3)^5\,(1+t^5)\,(1-4t+7t^2-11t^3+16t^4-20t^5 \\
&\qquad +24t^6-21t^7+24t^8-20t^9+16t^{10}-11t^{11}+7t^{12}-4t^{13}+t^{14})\,, \\
A_3 &= -\,t^{18}\,(1+t)^{10}\,(1+t^3)^6\,(1+t^5)^2\,(1-2t+5t^2-6t^3+10t^4-12t^5+17t^6 \\
&\qquad -16t^7+19t^8-16t^9+17t^{10}-12t^{11}+10t^{12}-6t^{13}+5t^{14}-2t^{15}+t^{16})\,.
\end{aligned}
$$

Thus the $B_{g,n,d}(t)$ are polynomials for these values of n. However, we do not know an elementary way to see this in general. Perhaps there is a "nice" closed formula for the rational function in (30) as a sum of rational functions each of which has a simple numerator and a denominator consisting of only a few (say, at most n) of the $p(n)$ factors $1-C_\pi X$, but we have not been able to guess the general form of such an expression.

On the other hand, from $Q_n(1/t) = -Q_n(t)$ and the recursions one easily deduces that $Q_{n,d}(1/t) = -Q_{n,d}(t)$. It follows that $t^{(n^2-1)(2g-2)}B_{g,n,d}(1/t) = B_{g,n,d}(t)$, so once we know that $B_{g,n,d}$ is a polynomial we also know that it is monic of degree $(n^2-1)(2g-2)$ and has symmetric coefficients, corresponding to Poincaré duality for the smooth $(n^2-1)(g-1)$-dimensional manifold $\mathcal{N}_{g,n,d}$. But even the special case $B_{1,n,d}(t) \equiv 1$ (or $A_0 = 1$) of this, which is geometrically obvious, is not at all obvious from the "explicit" formulas.

Finally, we remark that the expression (26) makes is easy to compute the Poincaré polynomials on a computer. For instance, the Betti numbers of the 48-dimensional varieties $\mathcal{N}_{2,5,1}$ and $\mathcal{N}_{2,5,2}$ (the first cases not covered by (21) – (23)) are

$$
\begin{aligned}
&[1,0,1,4,3,8,11,20,32,44,70,100,151,200,281,392,511,668, \\
&\quad 841,1064,1283,1496,1680,1828,1918,1828,1680,\ldots,4,1,0,1]
\end{aligned}
$$

and

$$
\begin{aligned}
&[1,0,1,4,3,8,11,20,32,44,71,104,158,212,305,432,574,764, \\
&\quad 977,1256,1532,1792,2031,2212,2304,2212,2031,\ldots,4,1,0,1]\,,
\end{aligned}
$$

respectively.

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