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PERIODS OF MODULAR FORMS, TRACES OF HECKE OPERATORS, AND MULTIPLE ZETA VALUES

DON ZAGIER

The talk consisted of three somewhat separate parts, only loosely related to one another but all connected with the theory of periods of the modular group $\Gamma = PSL(2,\mathbb{Z})$. This group has the presentation $\Gamma = \langle S, U \mid S^2 = U^3 = 1\rangle$ with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $U = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and the element of infinite order $T = US = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. If V is a (right) representation of Γ (with the action denoted $v \mapsto v|\gamma$), then the group

$$Z_0^1(\Gamma, \mathbf{V}) = \{f : \Gamma \to \mathbf{V} \mid f(T) = 0, \quad f(\gamma_1\gamma_2) = f(\gamma_1)|\gamma_2 + f(\gamma_2) \quad (\forall \gamma_1, \gamma_2 \in \Gamma)\}$$

of parabolic 1-cocycles can be identified via $f \mapsto f(S)$ with the space

$$\mathbf{W} = \{v \in \mathbf{V} \mid v|(1+S) = v|(1+U+U^2) = 0\}$$

(we have extended the action of Γ to $\mathbb{Z}[\Gamma]$ by linearity), while the space of coboundaries

$$\{f : \Gamma \to \mathbf{V} \mid f(\gamma) = v|(\gamma - 1) \quad \forall\gamma \quad \text{for some } v \in \mathbf{V}\}$$

is identified with $\mathbf{W^0} = \{v|(1-S) \mid v \in \mathbf{V}^T\}$ ($\mathbf{V}^T = \mathrm{Ker}(1-T, \mathbf{V})$). We will see how this formalism, and in particular the characteristic equation $v + v|U + v|U^2 = 0$ or some variant of it, occur in several different problems in number theory.

1. A very elementary proof of the Eichler-Selberg trace formula

The best known example of the setup just described is given by the theory of periods of modular forms. Let $k > 2$ be an even integer ($k = 2$ can be treated similarly but a few details are different) and $\mathbf{V} = \mathbf{V}_k$ the space of polynomials of degree $\le k-2$, with the action $P \mapsto P|_{2-k}\gamma$, where $|_\nu$ has the usual meaning in the theory of modular forms, i.e., $P|_\nu\gamma(X) = (cX+d)^{-\nu}P(\frac{aX+b}{cX+d})$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Then $\mathbf{W} = \mathbf{W}_k$ is called the space of *period polynomials*, and the Eichler-Shimura-Manin theory of periods tells us that its quotient by the subspace $\mathbf{W^0} = \langle X^{k-2} - 1\rangle$ is isomorphic to the direct sum of two copies of $\mathbf{S} = \mathbf{S}_k$, the space of cusp forms of weight k on Γ, the two maps $\mathbf{S} \to \mathbf{W}$ being given by the even and odd parts of the polynomial

$$r_f(X) = \int_0^\infty f(z)\,(X-z)^{k-2}\,dz \qquad (f \in \mathbf{S}_k).$$

Alternatively, we can define the period mapping $f \mapsto r_f$ directly as a map from $\mathbf{S}$ to $Z_0^1(\Gamma, \mathbf{V})$ by using the so-called "Eichler integral": if $f(z) = \sum_{n=1}^{\infty} a_n\, q^n$ $(q = e^{2\pi i z})$ is in $\mathbf{S}_k$, then the $(k-1)$-fold integral $\tilde f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}}\, q^n$ has the property that

$$\left(\frac{1}{2\pi i}\frac{d}{dz}\right)^{k-1} \left((cz+d)^{k-2}\, \tilde f\big(\frac{az+b}{cz+d}\big) - \tilde f(z)\right) = (cz+d)^{-k}\, f\big(\frac{az+b}{cz+d}\big) - f(z) = 0$$

and hence that the function $\phi_f(\gamma)(z) = \big(\tilde f|_{2-k}(\gamma - 1)\big)(z)$ is a polynomial in z of degree $\le k-2$. The map $\gamma \mapsto \phi_f(\gamma)$ from Γ to $\mathbf{V}$ is automatically a cocyle (because it is formally a coboundary), sends T to 0 because $\tilde f$ is periodic, and maps S to a multiple of r_f, as one sees by writing $\tilde f(z)$ as $\frac{(2\pi i)^{k-1}}{(k-1)!} \int_z^{\infty} f(z')\,(z'-z)^{k-2}\, dz'$. Yet a third way to define the period map is in terms of the Hecke L-series $L(f,s) = \sum a_n n^{-s}$ of f, whose special values at the "critical points" $s = 1, 2, \ldots, k-1$ are up to simple multiples the coefficients of the polynomial $r_f(X)$. Finally, the isomorphism $\mathbf{W}/\mathbf{W}^0 \cong \mathbf{S} \oplus \mathbf{S}$ lifts naturally to an isomorphism $\mathbf{W} \cong \mathbf{S} \oplus \mathbf{M}$, where $\mathbf{M} = \mathbf{M}_k$ is the space of all modular forms of weight k on Γ.

The most important structure on the space $\mathbf{M}$ is the action of the Hecke algebra $\mathbb{T} = \langle T_n \rangle_{n\in\mathbb{N}}$. Here $T_n : \mathbf{S} \to \mathbf{S}$ is defined by $f \mapsto n^{k-1} \sum_{M \in \Gamma\backslash\mathcal{M}_n} f|_k M$, the sum being taken over the left Γ-cosets of $\mathcal{M}_n = \{M \in M_2(\mathbb{Z}) \mid \det M = n\}$ and $f|_k M$ defined by the same formula as given above. We can take coset representatives of $\Gamma\backslash\mathcal{M}_n$ which are upper triangular, i.e., we can define the action of T_n as (n^{k-1} times) the action of the element $T_n^\infty = \sum_{ad=n} \sum_{0\le b<d} \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ of $\mathbb{Z}[\mathcal{M}_n]$. On the other hand, the isomorphism $\mathbf{W} \cong \mathbf{S} \oplus \mathbf{M}$ lets us transfer the action of $\mathbb{T}$ to the space $\mathbf{W}$. This action and its consequences for the traces of Hecke operators are described in the following theorem.

Theorem. *Let n be a natural number.*
(a) Suppose that $\tilde T_n$ is an element of $\mathbb{Q}[\mathcal{M}_n]$ satisfying

$$(1-S)\,\tilde T_n = T_n^\infty\,(1-S) + (1-T)\,Y_n \tag{1}$$

for some $Y_n \in \mathbb{Q}[\mathcal{M}_n]$. Then for every k, $\mathbf{W}_k|\tilde T_n \subseteq \mathbf{W}_k$ and the action of $\tilde T_n$ on $\mathbf{W}_k$ corresponds to the action of the nth Hecke operator on $\mathbf{S}_k \oplus \mathbf{M}_k$.
(b) There exists an element $\tilde T_n \in \mathbb{Q}[\mathcal{M}_n]$ satisfying (1) and the two additional properties

$$(1-U)\,\tilde T_n\,(1+S) \;=\; 0 \qquad \big(\Leftrightarrow \tilde T_n \in (1+U+U^2)\,\mathcal{M}_n + \mathcal{M}_n\,(1-S)\big) \tag{2}$$

$$(1-S)\,\tilde T_n\,(1+U+U^2) \;=\; 0 \qquad \big(\Leftrightarrow \tilde T_n \in (1+S)\,\mathcal{M}_n + \mathcal{M}_n(1-U)\big)\,. \tag{3}$$

(c) For any choice of $\tilde T_n = \sum_{M\in\mathcal{M}_n} c(M)\,[M]$ as in (b), we have

$$tr(T_n, \mathbf{S}_k) + tr(T_n, \mathbf{M}_k) = \sum_{M\in\mathcal{M}_n} c(M)\, p_{k-2}(tr(M), n) \qquad (\forall k > 2)\,, \tag{4}$$

where $p_\nu(t,n) = \sum\limits_{0\le r\le \nu/2} (-1)^r \binom{\nu-r}{r} t^{\nu-r} n^r$ = *coefficient of* x^ν *in* $(1-tx+nx^2)^{-1}$.

We will sketch the proofs of parts (a) and (c) of the theorem in a moment. Part (b) is proved by giving an explicit formula (which we omit), e.g. for $n=1$ or 2 we can take

$$\tilde{T}_1 = \frac{1}{6}\begin{pmatrix}1&0\\0&1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0&-1\\1&0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1&-1\\1&0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}0&1\\-1&1\end{pmatrix} = \frac{1}{6} - \frac{1}{2}S - \frac{1}{3}(U+U^2),$$
$$\tilde{T}_2 = \begin{pmatrix}2&0\\0&1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix} - \begin{pmatrix}1&-1\\2&0\end{pmatrix} - \begin{pmatrix}0&2\\-1&1\end{pmatrix} - \begin{pmatrix}0&-2\\1&0\end{pmatrix}.$$

In any given case one can check directly that properties (1), (2), and (3) are satisfied. For instance, for the above $\tilde{T}_1$ we can check that (1) holds with $Y_1 = -\frac{1}{3}(1+U+U^2)$, while (2) and (3) follow by writing $\tilde{T}_1$ as $-\frac{1}{3}(1+U+U^2)+\frac{1}{2}(1-S)$ or $-\frac{1}{2}(1+S)+\frac{1}{3}(2+U)(1-U)$, respectively, and similarly for $\tilde{T}_2$ with $Y_2 = \begin{pmatrix}1&-1\\0&2\end{pmatrix} + \begin{pmatrix}2&0\\-1&1\end{pmatrix} + \begin{pmatrix}1&1\\-1&1\end{pmatrix}$. From the explicit formula for $\tilde{T}_n$ one finds that

$$\sum_{\mathrm{tr}(M)=t} c(M) + \sum_{\mathrm{tr}(M)=-t} c(M) = -2\,H(4n-t^2)$$

for any $t\in\mathbb{Z}$, where $H(n)$ for $n>0$ is the Hurwitz-Kronecker class number (= number of Γ-equivalence classes of positive definite binary quadratic forms of discriminant $-n$, the forms with a stabilizer of order 2 or 3 in Γ being counted with multiplicity 1/2 or 1/3), while $H(n)$ for $n\le 0$ is defined as $-1/12$ if $n=0$, $-u/2$ if $n=-u^2$ with $u\in\mathbb{N}$, and 0 if $-n$ is not a perfect square. We thus recover the trace formula in its classical form

$$\mathrm{tr}(T_n,\mathbf{S}_k) + \mathrm{tr}(T_n,\mathbf{M}_k) = -\sum_{t\in\mathbb{Z}} H(4n-t^2)\,p_{k-2}(t,n)\,.$$

(The sum is finite since t^2-4n is a positive non-square for $|t|>n+1$.)

We now indicate briefly the proof of parts (a) and (c) of the theorem. The period polynomial r_f of $f\in\mathbf{S}$ is, as we saw, equal (up to a constant depending only on k) to $\tilde{f}|(1-S)$, and mapping f to $f|_k T_n$ corresponds to $\tilde{f}\mapsto \tilde{f}|_{2-k}T_n^\infty$. But $\tilde{f}$ is T-invariant, so (1) implies $(\tilde{f}|T_n^\infty)|(1-S) = (\tilde{f}|(1-S))|\tilde{T}_n$ or $r_{f|T_n} = r_f|\tilde{T}_n$, and this is exactly the assertion of part (a). For part (c), we argue as follows. Let $\mathbf{A}=\mathbf{V}^S$ and $\mathbf{B}=\mathbf{V}^U$ be the fixed point sets of S and U on $\mathbf{V}$. They intersect transversally since S and U generate Γ and $\mathbf{V}^\Gamma=\{0\}$ (because $k>2$). On the other hand, $\mathbf{V}$ has a non-degenerate Γ-invariant scalar product (given by $(X^n,X^m)=(-1)^n n!m!\delta_{m+n,k-2}$), and using this gives

$$\mathbf{A}^\perp = \big(\mathrm{Ker}(1-S)\big)^\perp = \mathrm{Ker}(1+S)\,,\qquad \mathbf{B}^\perp = \big(\mathrm{Ker}(1-U)\big)^\perp = \mathrm{Ker}(1+U+U^2)$$

and hence $\mathbf{W}=\mathbf{A}^\perp\cap\mathbf{B}^\perp = (\mathbf{A}\oplus\mathbf{B})^\perp$. Also, by equations (2) and (3), $\mathbf{A}^\perp|\tilde{T}_n\subseteq\mathbf{B}^\perp$, $\mathbf{B}^\perp|\tilde{T}_n\subseteq\mathbf{A}^\perp$, while we already know that $\tilde{T}_n$ maps $\mathbf{W}$ to itself. These three facts and simple linear algebra show that the trace of $\tilde{T}_n$ on $\mathbf{W}$ is the same as its trace on the whole space $\mathbf{V}$. (Choose a complement $\mathbf{C}$ to $\mathbf{A}\oplus\mathbf{B}$ in $\mathbf{V}$ and look at the block decomposition of the matrix representing $\tilde{T}_n$ with respect to the dual direct sum decomposition of $\mathbf{V}$.) This gives statement (c) because it is easily checked that the trace of the action of $M \in \mathcal{M}_n$ on $\mathbf{V}_k$ is equal to $p_{k-2}(\mathrm{tr}(M), n)$.

A complete proof of the theorem discussed in this section will be published later. For a proof of part (a) (including explicit constructions of $\tilde{T}_n$ satisfying (1)), as well as a more detailed review of the classical theory of periods, see [1] or [6].

2. Characterization of Maass wave forms by a simple functional equation

Recall that a *Maass wave form* of eigenvalue λ on Γ is a Γ-invariant function on the upper half-plane which is small at infinity and is an eigenfunction of the hyperbolic Laplace operator $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ $(z = x+iy \in \mathfrak{H})$. If we write the eigenvalue as $s(1-s)$ with $s \in \mathbb{C}$, then an equivalent condition is that $u(z)$ is invariant under $z \mapsto -1/z$ and has a convergent Fourier expansion of the form

$$u(x+iy) = \frac{\sqrt{y}}{2} \sum_{n\in\mathbb{Z},\, n\neq 0} A_n\, (2\pi|n|)^{s-\frac{1}{2}}\, K_{s-1/2}(2\pi|n|y)\, e^{2\pi i n x} \tag{5}$$

($K_\nu(t)$ = K-Bessel function). We call u *even* if $A_n = A_{-n}$ for all n, so that $u(z)$ has a Fourier cosine expansion. The eigenvalue $s(1-s)$ is necessarily real and positive, and one knows that the set of values which occur is discrete (the smallest is about 90), but no methods except numerical ones are known to describe the spectrum. Recently, John Lewis (then a student of S. Helgason) made the very surprising discovery that *if an (even) Maass wave form of eigenvalue $s(1-s)$ on Γ exists, then there is a holomorphic (!) function ψ on $\mathbb{C} \setminus (-\infty, 0]$, vanishing at 1 and satisfying the functional equation*

$$\psi(z) = \psi(z+1) + z^{-2s}\, \psi(1+z^{-1}) \qquad (\forall z \in \mathbb{C} \setminus (-\infty, 0])\,. \tag{6}$$

(Notice that the expression z^{-2s} on the right-hand side makes sense by writing z as $e^{\log z}$ with $|\Im(\log z)| < \pi$.) Moreover, the converse is true under some restrictions on ψ. Even more surprisingly, essentially the same fact emerges from apparently unrelated work of Dieter Mayer [3] expressing the Selberg zeta function as the Ruelle zeta-function of a dynamical system! We first describe Lewis's result in more detail and then describe Mayer's result and the relationship of the two results to each other and to periods.

Lewis [2] actually establishes an analytic correspondence between Maass wave forms of eigenvalue $s(1-s)$ and solutions of (6). His method goes via a series of integral (Hankel and Laplace) transforms, but the final result can be stated simply in terms of Fourier and Taylor expansions: if $u(z)$ has the expansion (5), then the Taylor expansion of ψ at 1 is given by $\psi(1+z) = \sum_{j=2}^{\infty} C_j z^j$ with

$$C_j = \sum_{2\le r\le j} \frac{(-1)^{j+1-r/2}\, \Gamma(j+2s)}{(2\pi)^r\, (j-r+1)!} \sum_{n\neq 0} \frac{A_n}{n^r}\,, \tag{7}$$

and conversely, if C_j are the Taylor coefficients at 1 of a holomorphic solution of (6), then the Fourier coefficients of the corresponding Maass wave form are given by

$$A_n = \sum_{j=2}^{\infty} \frac{C_j}{\Gamma(j+2s)}\, (2\pi i n)^j \tag{8}$$

and automatically—although not obviously—satisfy $A_n = A_{-n}$.

Before explaining the connection of this with periods, we briefly describe the work of Mayer which leads to essentially the same correspondence from a completely different point of view. The relationship comes from the Selberg zeta function, which has two entirely different definitions, one in terms of the spectral theory of the Laplace operator in $\Sigma = \mathfrak{H}/\Gamma$ and one in terms of the closed geodesics on Σ. (The equality of the two results from the Selberg trace formula and is, so far as I know, the only reason for making either definition in the first place.) Lewis's work connects with the first point of view, Mayer's with the second. Specifically, the relationship between the closed geodesics and periodic continued fractions relates the Selberg zeta function (with the second definition) to the dynamics of the "continued fraction map" $F : [0, 1) \to [0, 1)$ which maps x to the fractional part of $1/x$ (and, say, to 0 if $x = 0$), and this in turn leads to the functional equation (6). In more detail:

To a dynamical system $F : X \to X$ and a weight function $h : X \to \mathbb{C}$ one associates for each integer $n \geq 1$ a *partition function*

$$Z_n(F, h) = \sum_{x \in X,\ F^n x = x} h(x)\, h(Fx)\, h(F^2 x) \cdots h(F^{n-1} x)$$

(sum over n-periodic points). In our case, $X = [0, 1)$, F is the continued fraction map, and we take $h(x) = x^{2s}$ for some fixed $s \in \mathbb{C}$ with $\Re(s) > \frac{1}{2}$ (to make the series converge); we then write $Z_n(s)$ rather than $Z_n(F, h)$. Using the technique of "transfer operators" and Grothendieck's theory of nuclear operators, Mayer shows that $Z_n(s)$ is given by

$$Z_n(s) = \operatorname{tr}(\mathcal{L}_s^n) - (-1)^n \operatorname{tr}(\mathcal{L}_{s+1}^n) \qquad (\forall n \geq 0),$$

where $\mathcal{L}_s$ is the operator on the space of holomorphic functions in the disc $|z - 2| < 3/2$ defined by

$$(\mathcal{L}_s \psi)(z) = \sum_{m=0}^{\infty} \Bigl(\frac{1}{m+z}\Bigr)^s \psi\Bigl(1 + \frac{1}{m+z}\Bigr).$$

On the other hand, the Selberg zeta function is defined by $Z_{\text{Selberg}}(s) = \prod_{k=0}^{\infty} \zeta_{SR}(s+k)^{-1}$, where $\zeta_{SR}(s)$ (the letters "SR" stand for Smale-Ruelle) is defined as the product over all closed primitive geodesics in Σ of $(1 - e^{-Ls})$, L being the length of the geodesic. The connection between closed geodesics and periodic continued fractions shows that $\zeta_{SR}(s)$ equals $\exp\bigl(\sum_{n=1}^{\infty} \frac{1}{n} Z_{2n}(s)\bigr)$. (Only even indices occur because the map $x \mapsto 1/x - m$ implicit in the definition of F corresponds to a matrix in $PGL(2, \mathbb{Z})$ of determinant -1, so only even iterates of F correspond to the action of Γ.) Putting all of this together and using (the infinite-dimensional analogue of) the formula $\exp\bigl(\sum_{n=1}^{\infty} \operatorname{tr}(L^n)\bigr) = \det(1 - L)$, we find that $\zeta_{SR}(s) = \dfrac{\det(1 - \mathcal{L}_{s+1}^2)}{\det(1 - \mathcal{L}_s^2)}$ and hence finally $Z_{\text{Selberg}} = \det(1 - \mathcal{L}_s^2)$. Therefore—if we again ignore the difficulties connected with the fact that our operators are acting on infinite-dimensional spaces—the zeros of the Selberg zeta function, which are the eigenvalues of Δ acting on $L^2(\Sigma)$, should correspond to the values of s for which $\mathcal{L}_s$ has the eigenvalue ± 1.

One should also expect (and in fact it can be proved) that the eigenvalues $+1$ and -1 for $\mathcal{L}_s$ correspond to even and odd Maass wave forms. Hence finally the eigenvalues of even Maass forms should correspond to solutions of $\mathcal{L}_s\psi = \psi$, but since $\mathcal{L}_s\psi(z) - \mathcal{L}_s\psi(z+1) = z^{-2s}\psi(1+z^{-1})$, this exactly corresponds to Lewis's equation (6), the condition $\psi(1) = 0$ being needed to make the series defining $\mathcal{L}_s\psi$ convergent. (Of course, all of this is formal and there are many analytic details to be checked.)

We now turn to the relation with periods. Note that equation (7) expresses each Taylor coefficient of $\psi(z)$ at $z = 1$ as a finite linear combination of values of the Hecke L-function attached to u at integral arguments. This is like the period r_f of a cusp form $f \in \mathbf{S}$ in the Eichler -Shimura-Manin theory, since the coefficients of the period are just the special values of the L-function of f at the arguments $1, 2, \ldots, k-1$. In fact this analogy goes further and there is even an actual connection. Namely, suppose that $2s = 2-k$ where $k > 2$ is an even integer. Then $\Gamma(j+2s)^{-1} = 0$ for $j = 2, 3, \ldots, k-2$, so the right-hand side of equation (8) vanishes for every n if $C_j = 0$ for $j > k-2$. In other words, (8) implies that any function $\psi \in \mathbf{V}_k$ satisfying (6) with $2s = 2-k$ and vanishing at $z = 1$ is in the kernel of the Lewis correspondence $\psi \mapsto u$. Suppose that ψ is such a function. We can write (6) as $\psi = \psi|T|(1+\varepsilon)$ where $\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_{-1}$. In particular, $\psi = \psi|\varepsilon$, so $\psi = \psi|T + \psi|\varepsilon T\varepsilon = \psi|(U+U^2)S$. But then it follows that $\psi|(1+S) = \psi|(1+U+U^2)$ and hence that this element vanishes (it is invariant under both S and U, and $\mathbf{V}^\Gamma = \{0\}$), so that $\psi \in \mathbf{W}$. Conversely, by reversing the steps we see that any ψ in the $(+1)$-eigenspace of the action of ε on $\mathbf{W}_k$ (ε acts on $\mathbf{W}$ because it commutes with S and $U+U^2$) is a solution of (6) with $2s = 2-k$. Such a ψ is automatically an odd polynomial, since $\psi|S = -\psi = -\psi|\varepsilon$ and $S\varepsilon = \varepsilon S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and automatically vanishes doubly at $z = 1$ (take $z = -1$ in (6) to see that ψ vanishes at -1 and hence also at $+1$, while the equation $\psi|\varepsilon = \psi$ shows that the order of vanishing is even). Therefore the kernel of the map $\psi \mapsto u$ when $2s = 2-k$ is exactly the set of odd polynomials in $\mathbf{W}_k$, and is isomorphic via the Eichler-Shimura-Manin correspondence to the space of cusp forms of weight k on Γ.

We thus have two classes of solutions of (6) with the auxiliary condition $\psi(1) = 0$: the ones coming from (cuspidal) Maass wave forms, for which the number s lies on the line $\Re(s) = 1/2$, and the ones coming from holomorphic cusp forms, for which s is a negative integer. There is a third class, noticed by Lewis, coming from the zeros of the Riemann zeta function. Specifically, for $s \in \mathbb{C}$ with $\Re(s) > 1$ set $\psi_s(z) = \sum_{m,\,n\geq 0}^{*} \frac{1}{(mz+n)^{2s}}$, where the asterisk means that the terms where one inequality is an equality are to be counted with multiplicity $1/2$ and the term $m = n = 0$ omitted. This series, a special case of the Barnes double zeta function, can be thought of as a sort of partial Eisenstein series of complex weight. It makes sense for $z \in \mathbb{C} \setminus (-\infty, 0]$, since each term $mz+n$ then lies in the same domain and hence has a well-defined logarithm. Moreover, we have

$$\begin{aligned}
\psi_s(z+1) + z^{-2s}\,\psi_s(1+z^{-1}) &= \sum_{m,\,n\geq 0}^{*}\left(\frac{1}{(mz+(m+n))^{2s}} + \frac{1}{((m+n)z+m)^{2s}}\right) \\
&= \left(\sum_{n\geq m\geq 0}^{*} + \sum_{m\geq n\geq 0}^{*}\right)\frac{1}{(mz+n)^{2s}} = \psi_s(z)\,,
\end{aligned}$$

so that ψ_s is a solution of (6). The function $\psi_s(z)$ can be meromorphically continued to all s, with a pole only at $s = 1$, and equation (6) remains true for all s by analytic continuation. But it is easily seen that $\psi_s(1) = \zeta(2s-1)$ for $\Re(s) > 1$, so a further class of solutions of (6) vanishing at $z = 1$ is given by the functions $\psi_{(1+\rho)/2}$ where ρ ranges over the non-trivial zeros of the Riemann zeta function.

3. Riemann zeta values, Eisenstein series, and multiple zeta values

It has been known since Euler that $\zeta(k)$ for every positive even integer k is a rational multiple of π^k. Here is a completely elementary way to show that $\zeta(k)$ is a rational multiple of P^k for some P and all k, without knowing what P is. Suppose we are given for some even $k > 2$ a homogeneous polynomial $f(m,n) \in \mathbb{Q}[m^{-1}, n^{-1}]$ of degree k satisfying

$$f(m,n) - f(m+n,n) - f(m,m+n) = \sum_{0<j<k,\ j \text{ even}} c_j\, m^{-j} n^{-k+j} \tag{9}$$

for some numbers $c_j \in \mathbb{Q}$, and that $f(1,1) = \lambda \neq 0$ (for example, $f(m,n) = m^{-1}n^{-k+1} + \frac{1}{2}\sum_{r=2}^{k-2} m^{-r}n^{-k+r} + m^{-k+1}n^{-1}$, with $c_j = 1$, $\lambda = (k+1)/2$). Then

$$\sum_{j=2}^{k-2} c_j\,\zeta(j)\,\zeta(k-j) = \Big(\sum_{m,n>0} - \sum_{m>n>0} - \sum_{n>m>0} \Big) f(m,n) = \sum_{n>0} f(n,n) = \lambda\,\zeta(k)\,, \tag{10}$$

so the inductive assumption $\zeta(j) \in \mathbb{Q}\,P^j$ ($j < k$ even) implies that $\zeta(k) \in \mathbb{Q}\,P^k$.

The attentive reader will have noticed that the left-hand side of (9) has the form $f|(1 - T - \varepsilon T \varepsilon)$, where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\varepsilon T \varepsilon = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and that $1 - T - \varepsilon T\varepsilon$ is the very operator whose kernel on the space $\mathbf{V}_k$ of polynomials of degree $k-2$ was seen in Section 2 to coincide with the period subspace $\mathbf{W}$. This suggests that there is a relation between the above proof and the theory of periods. Indeed, there is. The Riemann zeta-value $\zeta(k)$ is the limiting value at the cusp of the Eisenstein series

$$G_k(z) \;=\; \frac{1}{2} \sum_{(a,b)\neq(0,0)} \frac{1}{(az+b)^k} \;=\; \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \Big(\sum_{d|n} d^{k-1} \Big)\, e^{2\pi i n z}\,,$$

and the collections of numbers $\{c_j\}$ occurring in identities of the form (9) are exactly those where the sum $\sum c_j G_j(z) G_{k-j}(z)$ is a multiple of (more precisely, λ times) the Eisenstein series $G_k(z)$. (This statement must be modified slightly if c_2 or c_{k-2} is non-zero, because $G_2(z)$ is not quite a modular form; in this case, one must modify $G_2 G_{k-2}$ by adding an appropriate multiple of the derivative of G_{k-2}.) A proof of this using only combinatorial manipulations and the Fourier coefficients of G_k is given in the very nice article [4], but one can also give a proof in terms of the definition of $G_k(z)$ as $\sum (az+b)^{-k}$ by simply interpreting m and n in the proof above as elements of the lattice $\mathbb{Z}z + \mathbb{Z}$, with "$m > 0$" interpreted to mean "$m = az + b$ with $a > 0$ or with $b > a = 0$"; then the same formal manipulation as before immediately gives $\sum c_j G_j G_{k-j} = \lambda G_k$. (There are problems of non-absolute convergence if c_2 or c_{k-2} are non-zero which force the slight modification mentioned above.) The connection with the theory of periods now arises because the Petersson scalar product of a cusp form $f \in \mathbf{S}_k$ with the product of Eisenstein series $G_j G_{k-2}$ (made modular by adding a multiple of G'_{k-2} if j equals 2 or $k-2$) is, by virtue of an identity of Rankin, essentially the $(j-1)$st period of f (i.e., the coefficient of X^{j-1} in $r_f(X)$). Therefore, since G_k spans the orthogonal complement of $\mathbf{S}_k \subset \mathbf{M}_k$, the relations of the form $\sum c_j G_j G_{k-j} = \lambda G_k$ correspond exactly to the relations among the coefficients of the (odd part of the) period polynomials of cusp forms of weight k.

In a different direction, the above can be thought of as a way of finding linear relations over $\mathbb{Q}$ satisfied by the "double zeta values" $\zeta(j, k-j)$ $(1 \le j \le k-2)$. These are the special case $r = 2$ of the "multiple zeta values"

$$\zeta(k_1, \ldots, k_r) = \sum_{0<n_1<\ldots<n_r} \frac{1}{n_1^{k_1} \ldots n_r^{k_r}} \qquad (k_i \ge 1,\ k_r \ge 2)$$

which seem to be very fascinating numbers and whose systematic study is only now beginning. A brief discussion of this connection is given in [7]. Roughly, it is as follows. The numbers $\zeta(j, k-j)$ satisfy the basic relation

$$\sum_{s=2}^{k-1} \left[\binom{s-1}{j-1} + \binom{s-1}{k-j-1} \right] \zeta(k-s, s) = \zeta(j)\, \zeta(k-j) \qquad (2 \le j \le k/2) \tag{11}$$

(coming from a partial fraction expansion) as well as the more or less obvious relation

$$\zeta(j, k-j) + \zeta(k-j, j) = \zeta(j)\, \zeta(k-j) - \zeta(k) \qquad (2 \le j \le k/2)\,. \tag{12}$$

But for k even there are approximately $k/6$ linear dependences among these relations, forcing the same number of $\mathbb{Q}$-linear relations among the numbers $\zeta(j)\zeta(k-j)$ and $\zeta(k)$, and these are precisely the same as the relations (10) obtained above. There is also a connection with the identity discussed at the end of Section 2, as follows. Define a function F_k on $[0, \infty)$ by

$$F_k(x) = \sum_{p=1}^{\infty} \frac{\psi(px)}{p^{k-1}}, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \lim_{Q\to\infty} \left(\log Q - \sum_{q=0}^{Q} \frac{1}{x+q} \right).$$

Then

$$\frac{(-1)^k}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} F_k(x) = \sum_{p\ge1,\, q\ge0} \frac{1}{(px+q)^k}\,,$$

which up to trivial modifications is the function $\psi_s(x)$ considered at the end of the last section, with $s = k/2$. Integrating $k-1$ times the basic identities satisfied by $\psi_s(z)$, we find

$$\begin{aligned} F_k(x) + x^{k-2} F_k(\tfrac{1}{x}) &= A_k(x) - \zeta(k)(x^{-1} + x^{k-1})\,, \\ F_k(x) - F_k(x-1) + x^{k-2} F_k(1 - \tfrac{1}{x}) &= B_k(x) \end{aligned} \tag{13}$$

with polynomials A_k, B_k of degree $k-2$. On the other hand, by looking at the Taylor expansion of F_k near $x=1$ we find that the coefficients of A_k and B_k can be expressed in terms of the numbers $\zeta(j,k-j)$. The relations (11) and (12) satisfied by the $\zeta(j,k-j)$ say that A_k and B_k define a cocycle in $\mathbf{V}_k$, and they are "explained" by (13), which expresses these polynomials as a coboundary.

Finally, we mention that there is an interesting connection between the formulas discussed in this section and the main theorem of [5], which is an identity expressing all periods of all modular forms on $SL(2,\mathbb{Z})$ in terms of a multiplicative combination of Jacobi theta functions which satisfies the basic period relation $v|(1+U+U^2)=0$ as a consequence of the Riemann theta relations. However, we do not elaborate on this connection here.

References

[1] YJ. Choie and D. Zagier, Rational period functions. In *A Tribute to Emil Grosswald: Number Theory and Related Analysis*, Cont. Math. **143**, A.M.S., Providence 1993, 89–108.

[2] J. Lewis, A space of entire functions equivalent to the even Maass cusp forms. Preprint, Framingham State College 1991.

[3] D. Mayer, The thermodynamic formalism approach to Selberg's zeta function for $PSL(2,\mathbb{Z})$. Bull. AMS **25** (1991), 55–60.

[4] N. Skoruppa, A quick combinatorial proof of Eisenstein series identities. J. Number Theory **43** (1993) 68–73.

[5] D. Zagier, Periods of modular forms and Jacobi theta functions. Invent. math. **104** (1991), 449–465.

[6] D. Zagier, Hecke operators and periods of modular forms. Israel Math. Conference Proc. **3** (1990), 321–336.

[7] D. Zagier, Values of zeta functions and their applications. To appear in the proceedings of the first European Mathematical Congress (Paris, 1992).

Department of Mathematics, Faculty of Science, Kyushu University, Hakozaki 6-10-1, Higashi-ku, Fukuoka 812, Japan

(permanent address: Max-Planck-Institut für Mathematik, Gottfried-Claren-Straße 26, 5300 Bonn 3, Germany)