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LINES ON THE DWORK QUINTIC PENCIL AND ITS HIGHER DEGREE ANALOGUES

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To the memory of my friend and teacher Friedrich Hirzebruch, who loved concrete examples in general and quintic hypersurfaces in particular.

Abstract

We give a reformulation of the recent results of Candelas et al. [**2**] describing pencils of lines on the quintic threefold

$$\left\{ (x_1 : \cdots : x_5) \in \mathbb{P}^4(\mathbb{C}) \;\middle|\; x_1^5 + \cdots + x_5^5 = 5\psi\, x_1 \cdots x_5 \right\}$$

in terms of the moduli space $M_{0,5}$ of curves of genus 0 with 5 marked points, and a generalization to pencils of lines on the degree n hypersurfaces

$$\left\{ (x_1 : \cdots : x_n) \in \mathbb{P}^{n-1}(\mathbb{C}) \;\middle|\; x_1^n + \cdots + x_n^n = n\psi\, x_1 \cdots x_n \right\}$$

in $\mathbb{P}^{n-1}(\mathbb{C})$ in terms of the moduli space $M_{0,n}$ for any odd integer $n \geq 5$.

This note is a small addendum to the beautiful recent paper by Philip Candelas, Xenia de la Ossa, Bert van Geemen, and Duco van Straten [**2**], in which the authors, extending earlier work by Anca Mustaţă [**5**], describe the lines on the Dwork quintic threefold

$$Q_\psi \;=\; \left\{ (x_1 : \cdots : x_5) \in \mathbb{P}^4(\mathbb{C}) \;\middle|\; x_1^5 + \cdots + x_5^5 = 5\psi\, x_1 \cdots x_5 \right\} \qquad (\psi \in \mathbb{C}), \tag{1}$$

showing that there are 375 isolated lines

$$\left\{(u : v : \alpha u : \beta v : 0) \mid (u:v) \in \mathbb{P}^1(\mathbb{C})\right\} \quad \text{and permutations,} \qquad \text{where } \alpha^5 = \beta^5 = -1, \tag{2}$$

and two continuous families, each of which is parametrized by a curve $\widetilde{C}_\varphi$ of genus 626 that is a 125-fold cover of a curve C_φ of genus 6. The two curves C_φ are given by the equations

$$G(\sigma,\tau) \;=\; \varphi\, H(\sigma,\tau) \tag{3}$$

in two variables σ and τ, where φ is one of the two roots of the equation

$$\frac{32}{\psi^5} \;=\; \varphi^2 + \frac{3}{4} \tag{4}$$

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and where $G = G(\sigma,\tau)$ and $H = H(\sigma,\tau)$ are the Laurent polynomials in σ and τ given by

$$(5)\ \ G \;=\; 3 - \frac{1}{2}\prod_{i=1}^{3}(\sigma_i+1) + \prod_{i=1}^{3}(\sigma_i^2-\sigma_i+1)\,, \qquad H \;=\; \prod_{i=1}^{3}(\sigma_i-1)\,,$$

with $(\sigma_1,\sigma_2,\sigma_3) = (\sigma,\tau,1/\sigma\tau)$. The obvious action of $\mathfrak{S}_5$ on Q_ψ corresponds in this description with the rather less obvious birational action of $\mathfrak{S}_5$ on $\mathbb{P}^2(\mathbb{C})$ generated by the two automorphisms

$$(6) \qquad (\sigma,\,\tau) \;\mapsto\; \Bigl(\frac{1}{\sigma},\,\frac{1}{\tau}\Bigr), \qquad (\sigma,\,\tau) \;\mapsto\; \Bigl(\tau,\,\frac{1-\sigma}{1-\sigma\tau}\Bigr)$$

of order 2 and 5, respectively, with the even permutations in $\mathfrak{S}_5$ preserving each of the curves C_φ and the odd ones interchanging them. The $\mathfrak{S}_5$-invariant curve $C_0 \subset \mathbb{P}^2(\mathbb{C})$ had been discovered already in 1897 by Wiman [**6**] and the family of $\mathfrak{A}_5$-invariant curves $C_\varphi \subset \mathbb{P}^2(\mathbb{C})$ by Edge [**3**] in 1981. (Here $\mathfrak{S}_n$ and $\mathfrak{A}_n$ as usual denote the symmetric and alternating groups on n letters.)

The starting point for this note is the observation that the space $\mathbb{P}^2(\mathbb{C})$, with the (birational) coordinates σ and τ and the action of $\mathfrak{S}_5$ described by (6), can be interpreted naturally as the moduli space of $M_{0,5}$ of 5-tuples of points on a curve of genus 0: this moduli space can be parametrized by choosing the curve of genus 0 to be $\mathbb{P}^1(\mathbb{C})$ and the points to be 0, ∞, $1/\tau$, 1, and σ (or, in the slightly more symmetrical σ_i-coordinates, by choosing the points to be 0, ∞, a, b, and c in $\mathbb{P}^1(\mathbb{C})$ where $(cb^{-1},ba^{-1},ac^{-1}) = (\sigma_1,\sigma_2,\sigma_3)$). The non-linear action (6) of $\mathfrak{S}_5$ on (σ,τ) then corresponds simply to permuting the 5 points and applying an element of $\mathrm{PGL}(2,\mathbb{C})$ to put the image back into the form $(0,\infty,1/\tau',1,\sigma')$. It turns out that by using this interpretation one can rewrite the results of Candelas *et al* in a more natural and less computational way, intimately related to the classical invariant theory of binary quintic forms. This is done in §§1–2, while §3 is devoted to describing the equation of the hypersurface of degree 250 in $\mathbb{P}^4(\mathbb{C})$ in which all of the lines lie. Finally, in §4 we give the surprisingly easy generalization to the higher-degree and higher-dimensional hypersurfaces

$$Q_\psi^{(n)} \;=\; \bigl\{\,(x_1:\cdots:x_n)\in\mathbb{P}^{n-1}(\mathbb{C}) \bigm| x_1^n+\cdots+x_n^n \;=\; n\psi\,x_1\cdots x_n\,\bigr\} \qquad (7) \qquad (\psi\in\mathbb{C}),$$

with odd $n \ge 5$, which contain $(n-4)$-dimensional pencils of lines related in a natural way to $M_{0,n}$.

1. The moduli space $M_{0,5}$ and invariants of binary quintics

The moduli space $M_{0,5}$ of curves of genus 0 with 5 (distinct) marked points is given by

$$\begin{aligned} M_{0,5} \;&=\; \mathrm{PGL}(2,\mathbb{C}) \setminus \bigl\{\mathbf{z} = (z_1,\dots,z_5) \in \mathbb{P}^1(\mathbb{C})^5 \bigm| z_i \neq z_j \text{ for } i \neq j\bigr\} \\ &=\; \mathrm{GL}(2,\mathbb{C}) \setminus \Bigl(\mathbb{C}^{2\times 5} \smallsetminus \bigcup_{i<j}\{\Delta_{ij} = 0\}\Bigr) / (\mathbb{C}^*)^5_{\mathrm{diag}} \end{aligned} \tag{8}$$

where $\mathbb{C}^{2\times 5}$ denotes the space of 2×5 matrices $Z = (Z_1 \cdots Z_5) = \begin{pmatrix}\xi_1 & \cdots & \xi_5 \\ \eta_1 & \cdots & \eta_5\end{pmatrix}$ and $\Delta_{ij} = |Z_i\ Z_j| = \xi_i\eta_j - \xi_j\eta_i$ $(1 \le i,\, j \le 5,\ i \neq j)$, with $z_i = \xi_i/\eta_i$. As already stated in the introduction, we can coordinatize it explicitly, though at the expense of obscuring the $\mathfrak{S}_5$-symmetry, by

$$\begin{aligned} \bigl\{(\sigma,\tau) \;\in\; \bigl(\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0,1,\infty\}\bigr)^2 \mid \sigma\tau \neq 1\bigr\} &\xrightarrow{\ \sim\ } M_{0,5}\,, \\ (\sigma,\tau) &\mapsto (0,\,\infty,\,1/\tau,\,1,\,\sigma)\,, \end{aligned} \tag{9}$$

in which case the $\mathfrak{S}_5$-action becomes the one generated by the two automorphisms (6). (Explicitly, these generators correspond to the permutations (1 2) and (1 4 2 5 3), as one sees by applying to $\mathbb{P}^1(\mathbb{C})$ the fractional linear transformations $z \mapsto 1/z$ and $z \mapsto (1-\tau z)/(1-z)$, respectively. The ordering of the five variables here has been chosen to coincide with the one used in [**2**], to facilitate comparison of the results of the two papers. A full description of the action of $\mathfrak{S}_5$ can be found in Table 1 of [**2**].) Then the Wiman-Edge function

$$\Phi(\mathbf{z}) \;=\; \Phi(\sigma,\tau) \;=\; \frac{G(\sigma,\tau)}{H(\sigma,\tau)}\,, \tag{10}$$

with G and H defined as in (5), is invariant under even and anti-invariant under odd elements of $\mathfrak{S}_5$, as one checks by direct verification for these two generating automorphisms (and as must in any case be true to be compatible with the description of the lines on Q_ψ described in the introduction). Using the above identifications, we can view Φ either as a $(\mathrm{PGL}(2,\mathbb{C}) \times \mathfrak{A}_5)$-invariant function on the space of 5-tuples of distinct points $z_i \in \mathbb{P}^1(\mathbb{C})$, or as a $(\mathrm{GL}(2,\mathbb{C}) \times ((\mathbb{C}^*)^5 \rtimes \mathfrak{A}_5))$-invariant function on the space of 5-tuples of pairwise non-proportional vectors $(\xi_i,\eta_i) \in \mathbb{C}^2$.

Proposition 1. *Up to scalars, Φ is the unique rational function on $\overline{M_{0,5}}$ that is $\mathfrak{S}_5$-invariant up to sign and whose only singularities in $\mathbb{P}^1(\mathbb{C})^5$ are simple poles along the diagonals $z_i = z_j$.*

Proof. Any rational function on $\overline{M_{0,5}}$ can be written as a quotient of two polynomials in the Δ_{ij}'s that is homogeneous of degree 0 in each Z_i. If it has the other properties given, then its denominator must be proportional to the function $\Delta = \prod_{i<j}\Delta_{ij}$ and its numerator an element of the space $\mathbf{H}$ of $\mathfrak{S}_5$-invariant polynomials in the Δ_{ij} having

degree 4 in each Z_i. There are precisely 158 monomials in the Δ_{ij} of degree (4,4,4,4,4) in the Z_i's, forming seven orbits under the action of $\mathfrak{S}_5$, as follows:

Orbit	Size	Typical Monomial
1	1	$\prod_{i<j} \Delta_{ij}$
2	10	$(\Delta_{12}\Delta_{23}\Delta_{31})^2\Delta_{45}^4$
3	12	$(\Delta_{12}\Delta_{23}\Delta_{34}\Delta_{45}\Delta_{51})^2$
4	15	$\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}(\Delta_{23}\Delta_{45})^3$
5	30	$\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}\Delta_{24}\Delta_{35}(\Delta_{23}\Delta_{45})^2$
6	30	$(\Delta_{12}\Delta_{13})^2\Delta_{24}\Delta_{25}\Delta_{34}\Delta_{35}\Delta_{45}^2$
7	60	$(\Delta_{12}\Delta_{13})^2\Delta_{23}\Delta_{24}\Delta_{35}\Delta_{45}^3$

Hence if we let F_j $(1 \le j \le 7)$ be the sum of the monomials in the jth orbit, then $\mathbf{H}$ is spanned by $F_1, \ldots, F_7$, so $\dim \mathbf{H} \le 7$. But when we compute the polynomials F_j, we find that in fact they are all multiples of $F := F_2$ (explicitly, $(F_1, \ldots, F_7) = (0, 1, 1, 0, \frac{1}{2}, 1, 3)\, F$), so $\mathbf{H}$ is actually one-dimensional, and the computation also shows that the function $4F/\Delta$ equals Φ. q.e.d.

The above proof of the proposition does not really explain why the function Φ is unique, since there is no apparent reason, short of actually doing the computation, why the seven functions F_j should all be multiples of one of them. In fact the 1-dimensionality of the space in question is an old result, proved by Sylvester in 1846. To explain this, we recall the basics of the classical theory of invariants of binary forms, since we will need other parts of it afterward. A binary form of degree n (or binary n-ic) is a homogeneous polynomial

$$F(u,v) \;=\; A_0u^n \;+\; A_1u^{n-1}v \;+\; \cdots \;+\; A_nv^n \tag{11}$$

of degree n in two variables u and v with (for us) complex coefficients. The group $\mathrm{SL}(2,\mathbb{C})$ acts on the right on the vector space of such forms by defining $F\circ\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ as the form $F(au+bv, cu+dv)$. An invariant of binary n-ics is by definition a polynomial $I(A_0, \ldots, A_d)$ in the coefficients of the form (11) that is invariant under this action, an example being the discriminant, which is an invariant of degree $2n-2$ for all n. The space of all invariants, graded by degree, forms a graded $\mathbb{C}$-algebra, known to be finitely generated for all n. This was proved by Hilbert, long after Sylvester, but for small values of n it was already worked out in great detail in the 19th century. The generators for $n \le 6$ are as given in the following table:

$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
(none)	I_2	I_4	I_2, I_3	I_4, I_8, I_{12}, I_{18}	I_2, I_4, I_6, I_{10}, I_{15}

in which I_d denotes an invariant of degree d and the invariants listed generate the algebra of invariants freely in all cases except $n=5$ and $n=6$, for which the square of the last generator listed is a polynomial in the other ones. The result for $n=5$ is due to Sylvester and includes as its simplest corollary the statement that the invariant of binary quintics of lowest degree has degree 4 and is unique up to a scalar multiple, which is precisely the statement of Proposition 1 if we identify the space of binary quintic forms up to scalars with $M_{0,5}$ by setting $F(u,v) = A_0 \prod_{i=1}^5 (u - z_i v)$ or $F(u,v) = \text{const} \prod_{i=1}^5 (\eta_i u - \xi_i v)$ in the notation of (8). Sylvester's result also implies that the simplest $\mathrm{SL}(2,\mathbb{C})$-invariant rational function on the 5-dimensional projective space of binary quintics (in the sense of having numerator and denominator of lowest degree) is I_4^2/I_8, which is, up to a constant, the square of the function Φ. Thus the only $\mathfrak{S}_5$-invariant curves of degree ≤ 8 on this projective space are the ones given by $\Phi^2 = \varphi^2$ for some $\varphi \in \mathbb{C}$, and the only $\mathfrak{A}_5$-invariant curves in $M_{0,5}$ of degree ≤ 4 are the ones given by $\Phi = \varphi$ or $G = \varphi H$ for some $\varphi \in \mathbb{C}$, i.e., they are precisely the curves found by Wiman and Edge.

In the above table we observe that there is always an invariant of degree 2 when n is even. This is because the space of binary n-ic forms has an $\mathrm{SL}(2,\mathbb{C})$-invariant scalar product, given by

$$(F,\, G) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu}^{-1} A_\nu\, B_{n-\nu}$$

$$\text{for } F = \sum_{\nu=0}^{n} A_\nu\, u^{n-\nu}\, v^\nu,\ G = \sum_{\nu=0}^{n} B_\nu\, u^{n-\nu}\, v^\nu\,,$$

giving the invariant

$$I_2(F) \;=\; (F,F) \;=\; \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu}^{-1} A_\nu\, A_{n-\nu} \tag{12}$$

of degree 2. (Of course I_2 vanishes if n is odd.) This invariant will play a role in §2 and §4.

2. Lines on the Dwork quintic

We now come to the parametrization of lines on the threefold Q_ψ by certain points in $M_{0,5}$. In the paper by Candelas *et al*, the line parametrized by $(\sigma,\tau) \in C_\varphi$ is obtained by setting

$$(x_1 : x_2 : x_3 : x_4 : x_5) = (c_1 u : c_2 v : c_3(\tau u - v) : c_4(u - v) : c_5(u - \sigma v))$$

with $(u : v)$ running over $\mathbb{P}^1(\mathbb{C})$, where $c_1, \ldots, c_5$ are numbers whose fifth powers are given (up to an irrelevant common factor) as explicit rational functions of σ and τ. (Cf. equation (2.1) of [**2**] and the following calculations.) In view of (9), that statement corresponds to:

Proposition 2. *Any five distinct lines in $\mathbb{C}^2$ can be given by the vanishing of five linear forms $L_1, \ldots, L_5$ such that $\sum_{i=1}^5 L_i^5$ is a multiple of $\prod_{i=1}^5 L_i$.*

Proof. Let u, v be the coordinates on $\mathbb{C}^2$ and write the five given lines as $u = z_i v$ or $\eta_i u = \xi_i v$ $(i = 1, \ldots, 5)$ with $z_i \in \mathbb{P}^1(\mathbb{C})$ and $(\xi_i, \eta_i) \in \mathbb{C}^2$, with the obvious interpretation of the equation $u = z_i v$ as the equation $v = 0$ if $z_i = \infty$. For notational convenience we will ignore the last possibility (which is non-generic) and use the coordinates z_i rather than the homogeneous coordinates (ξ_i, η_i) to parametrize the lines in question. Then, similarly writing z for u/v, we can rewrite the statement of the proposition as the statement that the polynomials $(z - z_1)^5, \ldots, (z - z_5)^5$ and $(z - z_1) \cdots (z - z_5)$ are linearly dependent. Choosing the monomials $(-1)^\nu \binom{5}{\nu} z^{5-\nu}$ $(0 \le \nu \le 5)$ as a basis for the space of all polynomials of degree ≤ 5 in z, and writing $\prod_{i=1}^5 (z - z_i) = \sum_{\nu=0}^5 (-1)^\nu \sigma_\nu z^{5-\nu} =: \sum_{\nu=0}^5 (-1)^\nu \binom{5}{\nu} \widetilde{\sigma}_\nu z^{5-\nu}$, we can rewrite this as the statement that the determinant of the matrix

$$B_5(\mathbf{z}) \;=\; B_5(z_1, \ldots, z_5) \;=\; \begin{pmatrix} 1 & \cdots & 1 & 1 \\ z_1 & \cdots & z_5 & \widetilde{\sigma}_1 \\ \vdots & \ddots & \vdots & \vdots \\ z_1^5 & \cdots & z_5^5 & \widetilde{\sigma}_5 \end{pmatrix} \tag{13}$$

is zero. But this is obvious, because the vector $(\sigma_5, -\sigma_4, \sigma_3, -\sigma_2, \sigma_1, -1)$ $B_5(\mathbf{z})$ vanishes. q.e.d.

Remark 1. The kernel of the matrix $B_5(\mathbf{z})$ is always 1-dimensional, because this matrix has an upper left principle minor that is a Vandermonde matrix with determinant $\Delta = \prod_{i<j}(z_i - z_j) \neq 0$, and hence cannot have rank less than 5. This means that the forms L_j^5 in Proposition 2 are always unique up to a common scalar factor, but of course the forms themselves can be multiplied by any 5th root of unity without affecting their defining property, so altogether there are $5^4 = 625$ solutions up to common scalar factors.

Remark 2. The above proof was based on the useful principle that a square matrix having a known element in its kernel must also have a vector in the kernel of its transpose. In general this principle is nonconstructive and does not tell one how to get the second vector from the first one. In our case, by direct calculation we can give the element in $\operatorname{Ker} B_5(\mathbf{z})$, and hence the linear relation among the six fifth-degree polynomials in question, explicitly. The result, which will be generalized to higher dimensions in §4, is as follows. For $(z_1,\dots,z_4)\in\mathbb{C}^4$, define

$$i_2(z_1,\dots,z_4) \;=\; \sigma_2^2 \,-\, \sigma_1\sigma_3 \,+\, 12\sigma_4\,,$$

where $\sigma_\nu=\sigma_\nu(z_1,\dots,z_4)$ is the νth elementary symmetric polynomial in the z_i. Up to a factor of 6, this is the value of the invariant I_2 defined at the end of §1 for the quartic form $\prod_{i=1}^4(u-z_iv)$. (Here we are using the inhomogeneous notation with $z_i\in\mathbb{P}^1(\mathbb{C})$ rather than the homogeneous notation with $(\xi_i,\eta_i)\in\mathbb{C}^2$ for convenience, but of course if some z_i equals infinity we must replace the corresponding factor $u-z_iv$ by v; we will not mention this minor point again.) Then the relation whose existence is asserted in Proposition 2 for the lines $u=z_iv$ $(i=1,\dots,5)$ is given by

$$\sum_{i=1}^{5}\frac{i_2(z_1,\dots,\widehat{z_i},\dots,z_5)}{\prod_{j\neq i}(z_i\,-\,z_j)}\,(u\,-\,z_iv)^5 \;=\; 10\,\prod_{i=1}^{5}(u\,-\,z_iv)^5\,. \tag{14}$$

By taking the 5th roots of the five terms on the left of (14) as the coordinates of a point $\mathbf{x}=(x_1:\cdots:x_5)$ in $\mathbb{P}^4(\mathbb{C})$, we obtain the desired description of the pencils of lines:

Theorem 1. *For every point $\mathbf{z}\in M_{0,5}$ there are $5^4=625$ lines in $\mathbb{P}^4(\mathbb{C})$ that have coordinates proportional to the coordinates of $\mathbf{z}$ and that lie on some hypersurface Q_ψ. The values of ψ that occur are the roots of the equation* (4), *where $\varphi=\Phi(\mathbf{z})$.*

Proof. Given the point $\mathbf{z}=(z_1,\dots,z_5)$, we see from (14) that for each of the $5^5=3125$ choices of solutions $\mathbf{a}=(a_1,\dots,a_5)$ of the equations

$$a_j^5 \;=\; \frac{i_2(z_1,\dots,\widehat{z_i},\dots,z_5)}{\prod_{j\neq i}(z_i\,-\,z_j)} \tag{15}$$

we obtain a line

$$\ell_{\mathbf{a}} \;=\; \bigl\{\,\bigl(\,a_1(u-z_1v)\;:\;\cdots\;:\;a_5(u-z_5v)\bigr)\;\big|\;(u:v)\in\mathbb{P}^1(\mathbb{C})\bigr\} \tag{16}$$

in $\mathbb{P}^4(\mathbb{C})$ that is contained in the hypersurface Q_ψ with $\psi=2/a_1\cdots a_5$. Then the product of $32/\psi^5$ with the discriminant $D_5=\prod_{i<j}(z_i-z_j)^2$ of the form $F_{\mathbf{z}}(u,v)=\prod_i(u-z_iv)$ is an expression $\prod_i I_2(z_1,\dots,\widehat{z_i},\dots,z_5)$ that is obviously $\mathfrak{S}_5$-invariant and by construction is an invariant of the form $F_{\mathbf{z}}$, easily seen to be of degree 8. By what we know about the invariants of binary quintic forms, this expression is a linear combination of I_4^2 and D_5, where I_4 is the 4th degree invariant discussed in §1, so $32/\psi^5$ is a linear combination of $\Phi(\mathbf{z})^2$ and 1, with coefficients that are easily verified to be those given in (4). Of course only 5^4 of the 5^5 lines $\ell_{\mathbf{a}}$ are distinct. q.e.d.

3. The hypersurface swept out by the lines

Since $M_{0,5}$ is a surface, the union of the lines in Theorem 1 is 3-dimensional, so its closure $\mathcal{L}$ in $\mathbb{P}^4(\mathbb{C})$ is a hypersurface, cut out by the vanishing of some homogeneous polynomial $F(\mathbf{x})$ in the variables $\mathbf{x} = (x_1, \ldots, x_5)$. The surface in the quintic hypersurface Q_ψ swept out by the Mustaţă-Candelas-de la Ossa-van Geemen-van Straten lines is then given by the same equation $F(\mathbf{x}) = 0$, restricted to Q_ψ. In this section we describe the polynomial F. In particular, we will show:

Theorem 2. *The hypersurface $\mathcal{L} \subset \mathbb{P}^4(\mathbb{C})$ has degree* 250.

Proof. Since $\mathcal{L}$ is invariant under the action of both $\mathfrak{S}_5$ (permutations of the variables) and $\boldsymbol{\mu}_5^5$ (multiplication of the variables by 5th roots of unity), the polynomial $F(\mathbf{x})$ can be expressed as a weighted homogeneous polynomial $P(s_1, \ldots, s_5)$ in the elementary symmetric functions

$$s_j \;=\; \sigma_j(x_1^5, \ldots, x_5^5) \qquad (j \;=\; 1, \ldots, 5)$$

of the fifth powers of the components of $\mathbf{x}$. We can therefore, for each value $D = 1, 2, \ldots$, simply take the $p_5(D)$-dimensional dimensional vector space ($p_5(D)$ = number of partitions of D into parts ≤ 5) spanned by all monomials $s_1^a s_2^b s_3^c s_4^d s_5^e$ with $a+2b+3c+4d+5e = D$ and look for a linear relation among the values of these polynomials when $s_1, \ldots, s_5$ are replaced by the elementary symmetric functions in the five terms on the left-hand side of (14), for generic $z_1, \ldots, z_5, u, v \in \mathbb{C}$. We find that there is no such relation for $D < 50$ and exactly one (up to a scalar factor) if $D = 50$. The computation is quite complicated, even by computer, because the number of monomials that one has to consider, in the first case that succeeds, is rather large, namely $p_5(50) = 3765$. One can make things a little easier by working numerically rather than algebraically, i.e., instead of thinking of $s_1, \ldots, s_5$ as polynomials in variables u and v with coefficients in $\mathbb{Q}(z_1, \ldots, z_5)$, we simply evaluate them all at a large number N ($N > p_5(D)$) of randomly chosen points $(z_1, \ldots, z_5, u, v) \in \mathbb{Q}^7$ and then look for linear dependences of the resulting N vectors in $\mathbb{Q}^{p_5(D)}$. Even this becomes computationally difficult for D as large as 50, so in practice we reduce everything modulo a moderately large prime p (we used a "random" 19-digit prime) and look for linear relations among vectors with coefficients in $\mathbb{F}_p^{p_5(D)}$. The first value of D where even such a mod p relation is found is $D = 50$, corresponding to degree 250 for the original polynomial F. We then lift the numerically obtained polynomial $P \pmod{p}$, suitably normalized, back to $\mathbb{Q}$, choosing the coefficients to be of small height (this actually turned out to be very easy, since if we normalize P so that the coefficient of s_1^{50} is 1 then it turns out that all of the further coefficients are integers), and check that this polynomial $P \in \mathbb{Q}[s_1, \ldots, s_5]$ does indeed vanish when each s_j is replaced by the jth elementary symmetric function of the terms on the left of (14). q.e.d.

The polynomial P obtained by the above procedure is considerably simpler than it might have been, since it actually turns out to involve only 645 of the 3765 monomials of weighted degree 50, but it is still very complicated and we were not able to find any conceptual description of it. Because tabulating even 645 coefficients seems a bit excessive, we will not write out the full equation here, but since we failed to find an intrinsic description and hope that some reader may succeed (here and then hopefully also for the higher-degree analogues treated in §4), we will provide a partial table of coefficients and a list of various further properties that we observed.

In the table below we list the first ten and last two of the coefficients of P when considered as a polynomial in the variable s_1. Since these coefficients are typically divisible by large powers of 5, we have removed these powers from the coefficient and presented the data in the form of a table giving the coefficients $C(M)$ of suitable monomials of the form $M = 5^i s_1^a$.

a	i	Coefficient of $5^i s_1^a$ in $P(s_1, \dots, s_5)$
50	0	1
49	0	0
48	0	0
47	4	$-2\,s_3$
46	5	$-16\,s_4$
45	5	$2\,s_2s_3 - 210\,s_5$
44	8	$2\,s_2s_4 + s_3^2$
43	9	$22\,s_2s_5 + 26\,s_3s_4$
42	9	$-2\,s_2s_3^2 - 4\,s_2^2s_4 + 1028\,s_3s_5 + 173\,s_4^2$
41	10	$-73\,s_2^2s_5 - 75\,s_2s_3s_4 - 226\,s_4s_5$
$\vdots$	$\vdots$	$\vdots$
1	50	$-2^{21}s_5^6\,(s_3^2s_4 - s_2s_3s_5 + s_5^2)\,(s_2s_3s_4 - s_2^2s_5 + 2\,s_4s_5)$
0	50	$2^{20}s_5^6\,(s_3^2s_4 - s_2s_3s_5 + s_5^2)^2$

Here are some observations concerning these coefficients and the remaining, unlisted ones:

1. The coefficients of P (normalized to begin with s_1^{50}) are of course all rational numbers, because the space of polynomials of this degree vanishing on $\mathcal{L}$ is 1-dimensional and invariant under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, but in fact they are all integral, and the coefficient of s_1^a is divisible by 5^{50-a} for all $0 \le a \le 50$.

2. Although we were not able to find a rule describing completely which 645 of the 3765 possible monomials $s_1^a s_2^b s_3^c s_4^d s_5^e$ with $a+2b+3c+4d+5e =$

50 occur in P, we did find a number of inequalities satisfied by their exponents, some of which are listed in the following table

$b+2c$	$b+2d$	$b+e$	$c+d$	$c+e$	$d+e$	$2b+3c+4d$
≤ 10	≤ 10	≤ 12	≤ 5 $(*)$	≤ 10 $(*)$	≤ 10	≤ 20

(and hence in particular $b \le 10$, $c \le 5$, $d \le 5$), where in each of the two cases marked by an asterisk there is one exceptional pair of exponents, namely $(c,d) = (4,2)$ and $(c,e) = (5,6)$, respectively, the terms having these exponents being

$$5^{25}(9s_1^5 + 50000s_5)^5 s_3^4 s_4^2 s_5 \quad \text{(6 monomials)} \qquad \text{and}$$
$$-\,2^{21}3^3 5^{45}\, s_1^5 s_3^5 s_5^6 \quad \text{(1 monomial)},$$

respectively. These inequalities, taking into account the seven exceptional monomials, cut down the number of possible monomials from 3765 to 751. But this still falls somewhat short of explaining why only 645 collections of exponents actually occur, and also we have no explanation of the inequalities.

3. In several cases where the inequalities just given are sharp, the coefficient of the corresponding monomial in two of the five variables factors non-trivially, as we just saw above for the coefficients of $s_3^4 s_4^2$ and of $s_3^5 s_5^6$. Further examples are:

- For the extreme or near-extreme cases $b \in \{9,10\}$, $c = 5$, $d = 5$, and $e = 10$ we have
$$\begin{aligned}
[s_2^{10}](P) &= 5^{30} s_1^{20} s_5^2\,,\\
[s_2^{9}](P) &= -2\cdot 5^{29} s_1^{17} s_5^2\,(3s_1^5 + 100000s_5)\,,\\
[s_3^{5}](P) &= 3^3\,5^{20} s_1^5 s_5\,(s_1^5 - 3125s_5)^2\,(3s_1^5 - 400000s_5)^3\,,\\
[s_4^{5}](P) &= 2^6\,5^{20}\,s_1^5\,(s_1^5 - 3125s_5)^2\,(3s_1^5 - 400000s_5)^3\,,\\
[s_5^{10}](P) &= 2^{20}\,5^{30}\,.
\end{aligned}$$
- If $b{+}2c = 10$, then the coefficient of $s_2^b s_3^c$ in P equals $5^{28}\lambda_c s_1^{2b} s_5^2 (9s_1^5 + 50000s_5)^c$ if $c \le 4$, where $(\lambda_0, \dots, \lambda_4) = (25, -50, 35, -10, 1)$. (The value for $c = 5$ was already given above.)
- If $b + 2d = 10$, then the coefficient of $s_2^b s_4^d$ in P is divisible by $(3s_1^5 - 400000s_5)^{d-2}$. (This assertion is non-empty only for $3 \le d \le 5$, and was already given above for $d = 5$.)
- The coefficient of $s_3 s_4^4$ in P equals $-2^4 \cdot 3 \cdot 5^{20} s_1^6\,(s_1^5 - 3125s_5)^2 (3s_1^5 - 400000s_5)^3$.

The last inequality $2b + 3c + 4d \le 20$ in the table in **2** above is especially interesting, because it can be restated as $a + 5e \ge 30$ and this tells us that if we restrict $P(s_1, \dots, s_5)$ to $s_1^5 = ks_5$ for some constant k, then it is divisible by s_1^{30}. (More explicitly, it factors as

$s_1^{30}\widetilde{P}(s_1,\ldots,s_4;k)$ where the polynomial $\widetilde{P}(s_1,\ldots,s_4;k)$ has degree 10 in k, with leading coefficient $(2s_1)^{20}$, and is weighted homogeneous of degree 20 in $s_1,\ldots,s_4$, where s_i has weight i.) But setting $s_1^5 = ks_5$ corresponds precisely to restricting to the Dwork quintic Q_ψ, where $k=(5\psi)^{-5}$, so this tells us that if we restrict the polynomial F to any Dwork quintic, then its degree drops from 250 to 100. We state this as a theorem.

Theorem 3. *The subvariety swept out by the continuous families of lines on any Dwork quintic is a hypersurface of degree 100.*

4. Lines on the hypersurface $Q_\psi^{(n)}$ in $\mathbb{P}^{n-1}(\mathbb{C})$

We now consider what happens when we change the degree and number of variables in our variety. We first note that the analogue of Proposition 2 holds also with 5 replaced by 3, the identity corresponding to (14) being

$$\sum_{i \pmod 3} (z_{i+1}-z_{i+2})^3(z-z_i)^3 \;=\; 3\prod_{i \pmod 3}(z_{i+1}-z_{i+2})(z-z_i). \tag{17}$$

Surprisingly enough, it also holds with 5 replaced by any odd number n.

Proposition 3. *Any n distinct lines, where $n \geq 3$ is odd, can be given by the vanishing of n linear forms $L_1,\ldots,L_n$ such that $\sum_{i=1}^n L_i^n$ is a multiple of $\prod_{i=1}^n L_i$.*

Proof. Identical to the proof of Proposition 2, but with 5 replaced by n everywhere. q.e.d.

Remark 3. Just as in the case of Proposition 2, the kernel of the matrix $B_n(\mathbf{z})$ is one-dimensional, so that the linear forms L_j whose existence is asserted in Proposition 2 are unique up to nth roots of unity and a common factor.

Remark 4. For n even the proposition is false, the matrix $B_n(\mathbf{z})$ being invertible in this case. If we multiply $B_n(\mathbf{z})$ on the left by the same row vector $v=(\sigma_n,-\sigma_{n-1},\ldots,-\sigma_1,1)$ as used in the proof of Proposition 3, then for general n we get $(0,\ldots,0,I_2(F_{\mathbf{z}}))$, where $I_2(F_{\mathbf{z}}) = \sum_{\nu=0}^n(-1)^\nu\sigma_{n-\nu}\widetilde{\sigma}_\nu$ is the invariant of the binary n-ic form $F_{\mathbf{z}}(u,v) = \prod_{i=1}^n(u-z_iv)$ defined at the end of §1. This invariant vanishes for n odd, which is why B_n has a kernel in that case, but is non-zero for n even, and one can show that the determinant of $B_n(\mathbf{z})$ is the product of $\Delta=\prod_{i<j}(z_i-z_j)$ and $I_2(F_{\mathbf{z}})$.

Remark 5. Just as in the case $n=5$, one can write down explicitly the linear relation among the polynomials $(z-z_i)^n$ and $\prod_i(z-z_i)$ whose existence is asserted by Proposition 3, namely

$$\sum_{i=1}^{n} \frac{I_2(z_1, \ldots, \widehat{z_i}, \ldots, z_n)}{\prod_{j\neq i}(z_i - z_j)} (u - z_i v)^n = \frac{2n}{n+1} \prod_{i=1}^{n} (u - z_i v)^n .$$

Using Proposition 3, we can generalize the rest of the story given in §2 for the quintic to the case of the hypersurface (7) of degree n in $\mathbb{P}^{n-1}(\mathbb{C})$. Denote by $M_{0,n}$ the moduli space of curves of genus 0 with n marked points, identified as in §1 with the left quotient by $\mathrm{PGL}(2, \mathbb{C})$ of the space of n-tuples of distinct points $z_1, \ldots, z_n$ in $\mathbb{P}^1(\mathbb{C})$.

Theorem 4. *For every point $\mathbf{z} \in M_{0,n}$ there are n^{n-1} lines in $\mathbb{P}^{n-1}(\mathbb{C})$ that have coordinates proportional to the coordinates of $\mathbf{z}$ and that lie on some hypersurface $Q_\psi^{(n)}$.*

Proof. Identical to that of the first statement of Theorem 1, but with 5 replaced by n. q.e.d.

Just as in the case $n = 5$, the fact that $M_{0,n}$ has dimension $n - 3$ implies that the union of the lines occurring in Theorem 4 is a hypersurface in $\mathbb{P}^{n-1}(\mathbb{C})$, and hence is given by a single homogeneous equation, again necessarily of the form $P(s_1, \ldots, s_n) = 0$ for some weighted homogeneous polynomial $P = P_n$ in the elementary symmetric functions s_j of $x_1^n, \ldots, x_n^n$. In view of the complexity of this polynomial already for $n = 5$, we did not even attempt to find it for higher values of n. But if a conceptual description of the polynomial P_5 discussed in §3 were to be found, then one could hope to extend that description to arbitrary odd values of n. The two specific questions of most interest here, in analogy with Theorems 2 and 3 of §3, are: what is the degree of the hypersurface in $\mathbb{P}^{n-1}(\mathbb{C})$ swept out by all the lines on the subvarieties $Q_\psi^{(n)} \subset \mathbb{P}^{n-1}(\mathbb{C})$ as ψ varies? and what is the degree of the hypersurface swept out by the lines on each individual variety $Q_\psi^{(n)} \subset \mathbb{P}^{n-1}(\mathbb{C})$?

Finally, we remark that the pencil of Dwork quintics is, of course, famous above all in the context of mirror symmetry, which began in 1991 with the analysis of this pencil and of its associated Picard-Fuchs differential equation by Candelas *et al* [**1**]. The higher-dimensional pencils (7) were also studied in the context of mirror symmetry a few years later by Greene *et al* ([**4**], esp. §3.1). It seems natural to wonder whether there could be any connection between the mirror aspects of these varieties, and in particular their Picard-Fuchs differential equations, and the families of lines on them studied in [**5**], [**2**], and in this paper.

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