# Appendix by D. Zagier
# The Eichler–Selberg Trace Formula on $SL_2(\mathbf{Z})$

Throughout this appendix we let $\Gamma=\Gamma(1)=SL_2(\mathbf{Z})$. We let $F$ be a fundamental domain for $\Gamma$ in $\mathfrak{H}$. We fix a weight $k$ even $\geqslant 4$. We write $T(m)$ instead of $T_k(m)$ for the Hecke operator on the space of cusp forms $S_k=M_k^0$.

Let $h(z, z')$ be a function of two variables $z, z'$ in $\mathfrak{H}$, and assume that $h$ as a function of each variable is a cusp form of weight $k$. If $f\in S_k$ then we define $f*h$ as a function of $z'$ by

(1) $$f*h(z') = \int_F f(z)\overline{h(z, -\bar{z}')}(\mathrm{Im}\, z)^k \frac{dx\,dy}{y^2}.$$

Thus this operation is merely the Petersson scalar product of $f$ and $h$, viewed as a function of the first variable $z$. The purpose of this appendix is to show that the Hecke operator $T(m)$ can be represented by a kernel $h_m$, and to give a formula for its trace on $S_k$.

We let $f_1, \ldots, f_r$ be a basis of eigenfunctions for the Hecke operators, and assume that they are normalized, i.e.

(2) $$f_i = \sum_{n=1}^{\infty} a_n^i q^n, \qquad a_1^i=1, \qquad T(m) f_i=a_n^i f_i .$$

Note that this basis of eigenfunctions is orthogonal for the Petersson scalar product.

For each positive integer $m$ we define

(3) $$h_m(z, z') = \sum_{ad-bc=m} (czz'+dz'+az+b)^{-k},$$

where the sum is taken over all integer matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant $m$. We may also write $h_m$ in the form

(4) $$h_m(z, z') = \sum_{ad-bc=m} (cz+d)^{-k}\left(z'+\frac{az+b}{cz+d}\right)^{-k}.$$

The imaginary part of

$$z'+\frac{az+b}{cz+d}$$

Copyrighted material



is $>0$, so this expression never vanishes, and each term of (4) is holomorphic in $z, z'$. It is easily verified that the series is absolutely convergent because $k\geqslant 4$. The function $h_m(z, z')$ is therefore holomorphic in $z, z'$. It is also immediate from (4) that it is a cusp form in each variable separately.

**Theorem 1.** *Let*

(5) $$C_k = \frac{(-1)^{k/2}\pi}{2^{(k-3)}(k-1)}.$$

(i) *The function $C_k^{-1}m^{k-1}h_m(z, z')$ is a "kernel" for the operator*

$$T(m): S_k \to S_k .$$

*In other words, for every $f\in S_k$ we have*

(6) $$f*h_m(z') = C_k m^{-k+1}(T(m)f)(z').$$

(ii) *We have the identity*

(7) $$C_k^{-1}m^{k-1}h_m(z, z') = \sum_{i=1}^{r} a_m^i \frac{1}{\langle f_i, f_i\rangle} f_i(z) f_i(z').$$

(iii) *The trace* $\mathrm{Tr}\, T(m)$ *is given by*

(8) $$\mathrm{Tr}\, T(m) = C_k^{-1}m^{k-1}\int_F h_m(z, -\bar{z})\, \mathrm{Im}\,(z)^k \frac{dx\,dy}{y^2}.$$

*Proof.* Suppose first that $m=1$. If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$$

then from the definition of the operation $[\gamma]_k$ we get

$$(c\bar{z}+d)^{-k} f(z) y^k = f(\gamma z)\, \mathrm{Im}\,(\gamma z)^k .$$

From (4) we get

$$f(z)\, \overline{h_1(z, z')} y^k = \sum_{\gamma\in\Gamma} (\bar{z}'+\gamma\bar{z})^{-k} f(\gamma z)\, \mathrm{Im}\,(\gamma z)^k$$

and therefore

$$f*h_1(z') = \int_F \sum_{\gamma\in\Gamma} (-z'+\gamma\bar{z})^{-k} f(\gamma z)\, \mathrm{Im}\,(\gamma z)^k \frac{dx\,dy}{y^2}.$$

Copyrighted material

Interchanging the integral and summation, and using the invariance of $dx\,dy/y^2$ with respect to $\Gamma$ yields

$$(9)\qquad f * h_1(z') = \sum_{\gamma \in \Gamma} \int_{\gamma F} (-z' + \bar{z})^{-k} f(z) \operatorname{Im}(z)^k \frac{dx\,dy}{y^2}$$

$$= 2 \int_0^\infty \int_{-\infty}^\infty (x - iy - z')^{-k} f(x + iy) y^{k-2}\, dx\, dy\,.$$

This last equality comes from the fact that the upper half plane is equal to the union of transforms of the fundamental domain under $\Gamma$, disjoint except for boundary points of measure zero, and except for the fact that $\pm\gamma$ give the same transform, whence the factor of 2. Cauchy's formula and the fact that $f$ is holomorphic and sufficiently small at infinity imply that

$$\int_{-\infty}^\infty (x - iy - z')^{-k} f(x + iy)\, dx = \frac{2\pi i}{(k-1)!} f^{(k-1)}(2iy + z')\,.$$

Therefore the right-hand side of (9) is

$$= \frac{4\pi i}{(k-1)!} \int_0^\infty y^{k-2} f^{(k-1)}(2iy + z')\, dy$$

$$= \frac{4\pi i}{(k-1)!} \int_0^\infty \frac{1}{(2i)^{k-2}} (d/dt)^{k-2} f'(2ity + z')\Big|_{t=1} dy$$

$$= \frac{4\pi i}{(k-1)!} \frac{1}{(2i)^{k-2}} (d/dt)^{k-2} \int_0^\infty f'(2ity + z')\, dy\Big|_{t=1}$$

$$= \frac{4\pi i}{(k-1)!} \frac{1}{(2i)^{k-2}} (d/dt)^{k-2} \left(\frac{-f(z')}{2it}\right)\Big|_{t=1}$$

$$= C_k f(z')\,.$$

This proves the desired formula (6) in case $m = 1$. The general case is a consequence of the case $m = 1$, because one easily sees that

$$m^{k-1} h_m = T(m) h_1\,,$$

where $T(m)$ operates with respect to the first variable $z$ on the right-hand side.



Part (ii) now follows essentially from elementary linear algebra. The function $h_m$ being a cusp form with respect to each variable $z$, $z'$ can be written in the form

$$h_m(z, z') = \sum_{i,j=1}^{r} c_{ij} f_i(z) f_j(z')\,.$$

We apply Part (i) to a function $f_\mu$ (one of the normalized eigenfunctions), and Part (ii) follows at once using the orthogonality. Part (iii) follows trivially from Part (ii). This proves the theorem.

The second theorem will give an explicit expression for the trace. We need some definitions.

We define a function $H(n)$ for integers $n$ first by putting

$$H(n) = 0 \text{ if } n < 0 \quad \text{and} \quad H(0) = -1/12\,.$$

If $n > 0$, we let $H(n)$ be the number of equivalence classes with respect to $SL_2(\mathbf{Z})$ of positive definite binary quadratic forms

$$ax^2 + bxy + cy^2$$

with discriminant

$$b^2 - 4ac = -n\,,$$

counting forms equivalent to a multiple of $x^2 + y^2$ (resp. $x^2 + xy + y^2$) with multiplicity $\frac{1}{2}$ (resp. $\frac{1}{3}$).

If $n \equiv 1$ or 2 (mod 4) then $H(n) = 0$. We have the following table.

| $n$ | 0 | 3 | 4 | 7 | 8 | 11 | 12 | 15 | 16 | 19 | 20 | 23 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $H(n)$ | $-\frac{1}{12}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | 1 | 1 | 1 | $\frac{4}{3}$ | 2 | $\frac{3}{2}$ | 1 | 2 | 3 | 2 |

We also define a polynomial $P_k(t, N)$ ($k > 0$ even) as the coefficient of $x^{k-2}$ in the power series development of

$$(1 - tx + Nx^2)^{-1}\,.$$

We also have

$$P_k(t, N) = \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}}$$

where

$$\rho + \bar{\rho} = t \quad \text{and} \quad \rho\bar{\rho} = N\,.$$

For instance $P_2(t, N) = 1$ and $P_4(t, N) = t^2 - N$.

**Theorem 2. (Trace Formula)** *let $k \geqslant 4$ be an even integer and let $m$ be an integer $>0$. Then the trace of the Hecke operator $T(m)$ on the space of cusp forms $S_k$ is given by*

$$\operatorname{Tr} T(m) = -\tfrac{1}{2} \sum_{t=-\infty}^{\infty} P_k(t, m) H(4m - t^2) - \tfrac{1}{2} \sum_{dd'=m} \min(d, d')^{k-1}.$$

**Note.** The first sum is in fact finite, because $H(4m - t^2) = 0$ for $t > 2\sqrt{m}$. The second sum is taken over all factorizations of $m$ as a product of two positive integers.

**Example.** For $k = 4$ the only cusp forms are 0, so the right-hand side of the formula is 0. This implies relations among the class numbers $H(m)$. For instance for $m = 5$, we find:

$$\sum (t^2 - m) H(4m - t^2) = -5H(20 - 8H(19) - 2H(16) + 8H(11) + 22H(4)$$
$$= -10 - 8 - 3 + 8 + 11 = -2,$$
$$\sum \min(d, d')^3 = 1^3 + 1^3 = 2.$$

The rest of this appendix is devoted to the proof of Theorem 2.

In Theorem 1 we have proved the identity

$$\operatorname{Tr} T(m) = C_k^{-1} m^{k-1} \int_F \sum_{ad-bc=m} \frac{y^k}{(c|z|^2 + d\bar{z} - az - b)^k} \frac{dx\,dy}{y^2}.$$

The sum on the right-hand side is invariant under $\Gamma$ (otherwise the integral would not be independent of the choice of fundamental domain $F$). Looking at the terms of this sum, we observe that replacing $z$ by $\gamma z$ amounts to replacing the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ by } \gamma^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\gamma.$$

These two matrices have the same determinant and the same trace. Therefore we may decompose the sum into pieces which are $\Gamma$-invariant, characterized by the condition $a + d =$ constant:

$$\operatorname{Tr} \mathrm{T}(m) = \sum_{t=-\infty}^{\infty} I(m, t),$$

where

(10) $$I(m, t) = C_k^{-1} m^{k-1} \int_F \sum_{\substack{ad-bc=m \\ a+d=t}} \frac{y^k}{(c|z|^2 + d\bar{z} - az - b)^k} \frac{dx\,dy}{y^2}.$$

Copyrighted material



We shall prove:

(11) $$\tfrac{1}{2}(I(m, t) + I(m, -t)) = \begin{cases} -\tfrac{1}{2} P_k(t, m) H(4m - t^2) & \text{for } t^2 - 4m < 0 \\ \dfrac{k-1}{24} m^{(k-2)/2} - \tfrac{1}{4} m^{(k-1)/2} & \text{for } t^2 - 4m = 0 \\ -\dfrac{1}{2}\left(\dfrac{|t| - u}{2}\right)^{k-1} & \text{for } t^2 - 4m = u^2,\ u > 0 \\ 0 & \text{for } t^2 - 4m > 0 \text{ non-square} \end{cases}$$

It is clear that these formulas imply the trace formula in Theorem 2. The numbers

$$\left|\frac{t+u}{2}\right| \quad \text{and} \quad \left|\frac{t-u}{2}\right|$$

play the role of $d, d'$ in the trace formula.

To study the integral (10), we first remark that there is a bijection between the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant $m$ and trace $t$, and the set of binary quadratic forms $g$ with discriminant

$$|g| = t^2 - 4m.$$

The bijection is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g(u, v) = cu^2 + (d - a)uv - bv^2$$

$$g(u, v) = \alpha u^2 + \beta uv + \gamma v^2 \mapsto \begin{pmatrix} \tfrac{1}{2}(t - \beta) & -\gamma \\ \alpha & \tfrac{1}{2}(t + \beta) \end{pmatrix}.$$

For every form $g(u, v) = \alpha u^2 + \beta uv + \gamma v^2$ and real $t$, $z = x + iy \in \mathfrak{H}$, we put

(12) $$R_g(z, t) = \frac{y^k}{(\alpha(x^2 + y^2) + \beta x + \gamma - ity)^k}.$$

Then

(13) $$I(m, t) = C_k^{-1} m^{k-1} \int_F \sum_{|g| = t^2 - 4m} R_g(z, t) \frac{dx\,dy}{y^2}.$$

Copyrighted material

where the sum is taken over all forms of discriminant $t^2 - 4m$. An element $\gamma \in \Gamma$ transforms a quadratic form $g$ into a form $\gamma g$ having the same discriminant, and one verifies that

$$R_{\gamma g}(z, t) = R_g(\gamma z, t)\,. \tag{14}$$

Therefore, for each discriminant $D$ (i.e. for each integer $D \equiv 0$ or $1 \bmod 4$) we have the equality

$$\sum_{|g|=D} R_g(z, t) = \sum_{\substack{|g|=D \\ \bmod \Gamma}} \sum_{\gamma \in \Gamma/\Gamma_g} R_{\gamma g}(z, t)$$
$$= \sum_{\substack{|g|=D \\ \bmod \Gamma}} \sum_{\gamma \in \Gamma/\Gamma_g} R_g(\gamma z, t)\,.$$

The first sum is taken over a set of representatives for classes of quadratic forms with discriminant $D$, and the second sum is taken over right cosets of $\Gamma$ with respect to the isotropy group $\Gamma_g$ of elements leaving $g$ fixed. For $D \neq 0$, the class number $h(D)$ is finite, and therefore the first sum is finite, giving

$$\int_F \sum_{|g|=D} R_g(z, t) \frac{dx\,dy}{y^2} = \sum_{\substack{|g|=D \\ \bmod \Gamma}} \int_{F_g} R_g(z, t) \frac{dx\,dy}{y^2}, \tag{15}$$

where

$$F_g = \bigcup_{\gamma \in \Gamma/\Gamma_g} \gamma F$$

is a fundamental domain for the operation of $\Gamma_g$ on $\mathfrak{H}$. The argument is the same as that used in the proof of Theorem 1.

For $D = 0$ we can take as a system of representatives for the forms of discriminant the forms $g_r$ $(r \in \mathbf{Z})$, where $g_r(u, v) = rv^2$. The isotropy group of $g_r$ is equal to $\Gamma$ for $r = 0$, and is equal to

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}$$

for $r \neq 0$. In this case, we find

$$\int_F \sum_{|g|=0} R_g(z, t) \frac{dx\,dy}{y^2} = \int_F R_{g_0}(z, t) \frac{dx\,dy}{y^2} + \int_{F_\infty} \sum_{r \neq 0} R_{g_r}(z \cdot t) \frac{dx\,dy}{y^2}, \tag{16}$$

where $F_\infty$ is a fundamental domain for the operation of $\Gamma_\infty$ on $\mathfrak{H}$, say the strip between 0 and 1. Here we cannot interchange the order of integration and summation, since for instance

$$\int_{F_\infty} R_{g_r}(z, t) \frac{dx\,dy}{y^2} = 0 \quad \text{for all } r\,,$$

but the integral of the sum is $\neq 0$, as we shall see below.

Copyrighted material



There remains to compute the right-hand side of (15) and (16) for $D = t^2 - 4m$. We distinguish four cases.

**Case 1.** $D < 0$.

In this case $\Gamma_g$ is finite for each form $g$ in (15), (and one even can prove that its order is 1, 2, or 3). For a quadratic form

$$g(u, v) = \alpha u^2 + \beta uv + \gamma v^2$$

with discriminant $D$ we therefore have

$$\int_{F_g} R_g(z, t) \frac{dx\,dy}{y^2} = \frac{1}{|\Gamma_g|} \int_{\mathfrak{H}} R_g(z, t) \frac{dx\,dy}{y^2}$$
$$= \frac{1}{|\Gamma_g|} \int_{\mathfrak{H}} \frac{y^k}{(|z|^2 - ity - \frac{1}{4}D)^k} \frac{dx\,dy}{y^2}.$$

(For this last equality, we used the substitution $z \mapsto (2z - \beta)/2\alpha$.) Let $I$ denote the value of the integral. It depends only on $D$ and $t$. The right-hand side of (15) is therefore equal to

$$\sum_{\substack{|g|=D \\ \bmod \Gamma}} \frac{1}{|\Gamma_g|} I = 2H(-D)I\,.$$

The factor 2 comes from the fact that in the definition of $H(n)$ we counted positive definite forms, whereas here we count all forms, positive or negative. Finally, using the formula

$$\int_{-\infty}^{\infty} (x^2 + A)^{-k}\,dx = \frac{\pi}{(k-1)!} \tfrac{1}{2} \tfrac{3}{2} \cdots (k - \tfrac{3}{2}) A^{-k+1/2}\,,$$

obtained by differentiating $k - 2$ times with respect to $A$ in the corresponding formula for $k = 2$, we obtain:

$$I = \int_0^\infty y^{k-2} \int_{-\infty}^{\infty} (x^2 + y^2 - ity - \tfrac{1}{4}D)^{-k}\,dx\,dy$$
$$= \frac{\pi}{(k-1)!} \tfrac{1}{2} \tfrac{3}{2} \cdots (k - \tfrac{3}{2}) \int_0^\infty (y^2 - ity - \tfrac{1}{4}D)^{-k+1/2} y^{k-2}\,dy$$
$$= \frac{\pi i^{k-2}}{2(k-1)!} (d/dt)^{k-2} \int_0^\infty (y^2 - ity - \tfrac{1}{4}D)^{-3/2}\,dy$$

Copyrighted material

$$= \frac{\pi i^{k-2}}{2(k-1)!}(d/dt)^{k-2}\left(\frac{4}{t^2-D}\frac{y-\frac{1}{2}it}{\sqrt{y^2-ity-\frac{1}{4}D}}\Bigg|_0^\infty\right)$$

$$= \frac{\pi i^{k-2}}{2(k-1)!}(d/dt)^{k-2}\left(\frac{4}{\sqrt{|D|}}\frac{1}{\sqrt{|D|}-it}\right)$$

$$= \frac{2\pi}{k-1}\frac{1}{\sqrt{|D|}}\frac{1}{(\sqrt{|D|}-it)^{k-1}}.$$

Formula (13) then gives

$$I(m,t) = C_k^{-1}m^{k-1}2H(4m-t^2)\frac{2\pi}{k-1}\frac{1}{\sqrt{4m-t^2}}\frac{1}{(\sqrt{4m-t^2}-it)^{k-1}}$$

$$= \frac{\bar{\rho}^{k-1}}{\rho-\bar{\rho}}H(4m-t^2), \quad \text{where } \rho = \tfrac{1}{2}(t+i\sqrt{4m-t^2}).$$

This proves the first formula in (11).

**Case 2.** $D = 0$

We now use formula (16). The first term is equal to $(-1)^{k/2}\pi/6t^k$, because

$$R_{g_0}(z,t) = (i/t)^k \quad \text{and} \quad \int_F \frac{dx\,dy}{y^2} = \frac{\pi}{3}.$$

The second term is equal to

$$\int_0^\infty\int_0^1 y^{k-2}\sum_{\substack{r\in\mathbf{Z}\\ r\neq 0}}(r-ity)^{-k}\,dx\,dy = \frac{i^{k-2}}{(k-2)!}(d/dt)^{k-2}\sum_{\substack{r\in\mathbf{Z}\\ r\neq 0}}(r-ity)^{-2}\,dy$$

$$= \frac{i^{k-2}}{(k-1)!}(d/dt)^{k-2}\int_0^\infty\left(\frac{1}{t^2y^2}-\frac{\pi^2}{\sinh^2\pi ty}\right)dy$$

$$= \frac{i^{k-2}}{(k-1)!}(d/dt)^{k-2}\left(\frac{\pi}{|t|}\right)$$

$$= (-1)^{(k-2)/2}\frac{\pi}{k-1}|t|^{-k+1}.$$

For $t = \pm 2\sqrt{m}$ we get the value

$$I(m,t) = C_k^{-1}m^{k-1}\int_F\sum_{|g|=0}R_g(z,t)\frac{dx\,dy}{y^2} = \frac{k-1}{24}m^{(k-2)/2} - \tfrac{1}{4}m^{(k-1)/2}.$$

This is precisely the second formula in (11).



**Case 3.** $D = u^2$ and $u > 0$.

As in the case $D < 0$ there is only a finite number of classes of forms with discriminant $D$, and $\Gamma_g$ is a finite group, so in the right-hand side of (15) we should again be able to replace the integral over $F_g$ by $|\Gamma_g|^{-1}$ times the integral of the same integrand over all of $\mathfrak{H}$. The problem is that the combination of integral and sum in (15) is not absolutely convergent, so that the interchange implicit in that equation is not necessarily justified and the left- and right-hand sides, though both absolutely convergent, do not have to be the same. In fact, we shall see that these two expressions, as they stand, differ by a factor $-1$ !

The correct way to interpret (15) is to consider the left-hand side as the limit as $\varepsilon \to 0$ of the same integral taken over the truncated fundamental domain

$$F^\varepsilon = \{x+iy \in F \mid y \geq \varepsilon^{-1}\}.$$

Then $F_g$ must be replaced by a fundamental domain for the action of the finite group $\Gamma_g$ on the subset $\mathfrak{H}_\varepsilon$ of $\mathfrak{H}$ obtained by removing all points with imaginary part $> 1/\varepsilon$ or lying in the interior of a circle of radius $\varepsilon/c$ tangent to the real axis at any rational point $d/c$. This gives

$$\int_F\sum_{|g|=D}R_g(z,t)\frac{dx\,dy}{y^2} = \lim_{\varepsilon\to 0}HI_\varepsilon,$$

where

$$H = \sum_{\substack{|g|=D\\ \text{mod }\Gamma}}\frac{1}{|\Gamma_g|} \quad \text{and} \quad I_\varepsilon = \int_{\mathfrak{H}_\varepsilon}\frac{y^k}{(|z|^2-ity-\frac{1}{4}D)^k}\frac{dx\,dy}{y^2}.$$

We have $H = u$, because the groups $\Gamma_g$ are trivial in this case, and because there are $u$ classes of quadratic forms with discriminant $u^2$. In the integral defining $I_\varepsilon$, the only poles of the integrand are at $z = \pm\frac{1}{2}u$, so we can shrink to 0 all discs in $\mathfrak{H}\backslash\mathfrak{H}_\varepsilon$ except those tangent to the real line at these two points. Hence

$$\lim_{\varepsilon\to 0}I_\varepsilon = I - \lim_{\varepsilon\to 0}I_{\varepsilon,\frac{1}{2}u} - \lim_{\varepsilon\to 0}I_{\varepsilon,-\frac{1}{2}u}$$

where

$$I = \int_0^\infty\left(\int_{-\infty}^\infty(x^2+y^2-ity-\tfrac{1}{4}u^2)^{-k}\,dx\right)y^{k-2}dy$$

and

$$I_{\varepsilon,\pm\frac{1}{2}u} = \int_0^{2\varepsilon}\left(\int_{\pm\frac{1}{2}u-\sqrt{2\varepsilon y-y^2}}^{\pm\frac{1}{2}u+\sqrt{2\varepsilon y-y^2}}(x^2+y^2-ity-\tfrac{1}{4}u^2)^{-k}\,dx\right)y^{k-2}dy$$

(integrals taken first over $x$ with $y$ fixed, then over $y$). For the first integral we obtain just as in the case $D < 0$ the value

$$I = \frac{\pi i^{k-2}}{2(k-1)!}(d/dt)^{k-2}\left(\frac{4}{t^2-D}\frac{y-\frac{1}{2}it}{\sqrt{y^2-ity-\frac{1}{4}D}}\Bigg|_0^\infty\right)$$

Copyrighted material

Copyrighted material

but for $D > 0$ the expression in parentheses is equal to

$$\frac{-4}{\sqrt{D}}\frac{1}{\sqrt{D}+|t|} \quad \text{and not} \quad \frac{4}{\sqrt{|D|}}\frac{1}{\sqrt{|D|}+it} \quad \text{as before}.$$

(The fact that the integral here depends only on $|t|$ is due to the fact that the value of $\sqrt{y^2 - ity - \frac{1}{4}D}$ for $y = 0$ depends on the sign of $t$, because we must choose the branch of the square root which has positive real part for $y \to \infty$.) We therefore have

$$(17) \qquad I = (-1)^{k/2}\frac{2\pi}{k-1}u^{-1}(u+|t|)^{-k+1}.$$

To evaluate $I_{\varepsilon,\pm\frac{1}{2}u}$, we first make the substitution $x = \pm\frac{1}{2}u + \varepsilon a$, $y = \varepsilon b$, finding

$$I_{\varepsilon,\pm\frac{1}{2}u} = \int_0^2 \int_{-\sqrt{2b-b^2}}^{\sqrt{2b-b^2}} \frac{b^{k-2}}{(\pm ua - itb + \varepsilon(a^2+b^2))^k}\,da\,db,$$

so

$$\lim_{\varepsilon\to 0} I_{\varepsilon\pm\frac{1}{2}u} = \int_0^2 \int_{-\sqrt{2b-b^2}}^{\sqrt{2b-b^2}} \frac{b^{k-2}}{(\pm ua - itb)^k}\,da\,db$$

$$= -\frac{1}{k-1}u^{-1}\int_0^2 \left\{\left(u\sqrt{2b-b^2} - itb\right)^{-k+1} + \left(u\sqrt{2b-b^2}+itb\right)^{-k+1}\right\}b^{k-2}\,db$$

$$= -\frac{2}{k-1}u^{-1}\int_{-\infty}^{\infty}(uv+it)^{-k+1}\frac{v\,dv}{v^2+1},$$

where in the last line we have the substitution $b = \frac{2}{v^2+1}$. The latter integral can be evaluated easily by contour integration (for example, if $t > 0$ then the only pole of the integrand in the upper half plane is at $v = i$) and equals the negative of expression (17), giving finally

$$I(m,t) = -C_k^{-1}m^{k-1}HI = -\tfrac{1}{2}\left(\frac{|t|-u}{2}\right)^{k-1}.$$

This proves the third formula in (11).

**Case 4.** $D > 0$ and non-square

Here we again have only a finite number of classes of quadratic forms, but the isotropy groups are infinite cyclic. Intuitively we have

$$H = \sum\frac{1}{|\Gamma_g|} = \frac{1}{\infty} = 0,$$

Copyrighted material



and thus $HI = 0$. We now assert that for each $g$ of discriminant $D$, we in fact have

$$(17) \qquad \int_{F_g} R_g(z,t)\frac{dx\,dy}{y^2} + \int_{F_g} R_g(z,-t)\frac{dx\,dy}{y^2} = 0.$$

Let $g(u,v) = \alpha u^2 + \beta uv + \gamma v^2$ be such a quadratic form, and let $w > w'$ be the roots of the equation $\alpha u^2 + \beta u + \gamma = 0$. Then the matrix

$$\gamma = (w-w')^{-1/2}\begin{pmatrix} w' & -w \\ 1 & 1\end{pmatrix} \in SL_2(\mathbf{R})$$

transforms $g$ into $\gamma g$, with

$$\gamma g(u,v) = \sqrt{D}\,uv.$$

The conjugate of $\Gamma_g$ by $\gamma$ operates on the upper half plane as the infinite cyclic group generated by $z \mapsto \varepsilon^2 z$, where $\varepsilon > 1$ is the fundamental unit of the order in $\mathbf{Q}(\sqrt{D})$ associated with $g$. We can therefore choose the fundamental domain $F_g$ so that $\gamma^{-1}F_g$ is an annulus defined by

$$y > 0 \quad \text{and} \quad r_0 \leqslant |z| \leqslant \varepsilon^2 r_0.$$

Then

$$\int_{F_g} R_g(z,t)\frac{dx\,dy}{y^2} = \int_{F_g} R_{\gamma g}(\gamma^{-1}z,t)\frac{dx\,dy}{y^2} \quad \text{(by (14))}$$

$$= \iint_{\text{annulus}} (\sqrt{D}\,x - ity)^{-k}y^{k-2}\,dx\,dy.$$

We write $z = x + iy$ in polar coordinates, $z = r\,e^{i\theta}$ to obtain

$$= \int_0^{\pi}\int_{r_0}^{\varepsilon^2 r_0} (\sqrt{D}\cos\theta - it\sin\theta)^{-k}(\sin\theta)^{k-2}\frac{dr}{r}\,d\theta$$

$$= (\log\varepsilon^2)\int_0^{\pi}(\sqrt{D}\cos\theta - it\sin\theta)^{-k}(\sin\theta)^{k-2}\,d\theta.$$

To prove (17) it suffices therefore to verify that

$$\int_{-\pi}^{\pi}(\sqrt{D}\cos\theta - it\sin\theta)^{-k}(\sin\theta)^{k-2}\,d\theta = 0,$$

which is easily done by putting $\zeta = e^{i\theta}$ and using the residue theorem.

The last formula in (11) follows easily from (13), (15) and (17). This concludes the proof of the trace formula.

Copyrighted material