A PASSION FOR MATHEMATICS

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A famous Hungarian number theorist once gave the following definition:

A mathematician is a machine for turning coffee into theorems.

Since there is no lack of mathematicians or theorems in my institute in Bonn, but good coffee is hard to come by, I have sometimes wondered whether our mathematicians couldn't be used for doing the opposite! That said, there are people who cannot stop turning coffee into theorems, while, for others, merely thinking about mathematics is sheer torture. I will come back to that point later on. But first, I would like to take a look at some other questions: What is mathematics? What is mathematics like today and what do we gain by doing it? How is mathematics beautiful? And how can we convey the joy of mathematics to others, including non-mathematicians?

WHAT KIND OF ACTIVITY IS MATHEMATICS?

It might seem naïve to ask what mathematics is, but it is a question that is in fact not at all easy to answer, and philosophers have been struggling over it for centuries. At the beginning of his *Critique of Pure Reason*, Kant even asks how pure mathematics is possible. Other sciences can be defined based on the objects they study: celestial bodies, living organisms, human relations, and so forth... In the case of mathematics, things are not that simple. To begin with, mathematics does not always study the same objects. Numbers, algebraic formulas, analytical functions, and geometric structures are certainly some of the things it studies, but many other objects are also examined and, strictly speaking, mathematical thought is actually the study of structures in general rather than the structure of predetermined objects. But the problem is even more complicated than that: unlike other disciplines, the *place* where our objects of study are to be found is not at all clear. Are these objects internal or external? Subjective or objective? Present only in our minds or somewhere out there in the real world? In other words, does the work of mathematicians consist of *inventing* things or *discovering* them?

On the side of "discovery," we have, first of all, the fact that mathematical results can be verified "objectively": the proof that a mathematician brings to a theorem will, as long as it does not contain any errors, convince any other mathematician of the truth it affirms. Another argument in favor of objectivity is the fact that when different mathematicians work on the same question, they always obtain the same answer, whatever their personality or individual taste. Finally, we can also say the same thing about whole cultures, since different cultures often developed the same mathematics independently of each other. The formulas for solving quadratic equations, "Pythagoras' theorem" (which was of course not called that everywhere!) or the algorithm for extracting cube roots were all discovered by many different ancient cultures.

But one can argue just as well for the point of view of "invention." First of all, there is a purely subjective argument: mathematicians often *feel* they have created something of their own. Secondly, different mathematicians are led by their personal tastes and experience to work on such different problems, and hence to such different results, that in many cases they can even be recognized by their mathematical theorems. In the same way, different cultures may sometimes take completely different mathematical routes and thus end up producing their own specific type of mathematics. For example, the Greeks invented and emphasized the notion of proof, while the Chinese, who often made the same discoveries as the Greeks, presented theirs in the form of algorithms or recipes for calculation. Or to take another example, we might mention the Egyptians who, like other ancient civilizations, developed calculation with rational numbers (fractions) which were needed in areas such as economics, surveying, and astronomy, but they did so in a very unusual way: instead of writing their fractions in the form of quotients of a numerator divided by a denominator, they permitted only pure reciprocals $1/n$ and represented each fraction as the sum of these kinds of numbers. Moreover, only different denominators were allowed: in other words, they did not write $2/5$ as $1/5 + 1/5$, but as $1/3 + 1/15$!

So where does the truth lie? For most mathematicians, it lies in a combination of both aspects. At each moment and for each problem, there is a huge number of possible deductions from the axioms and from what is already known, just as there exist many possible legal moves for each position in a chess game. All of these deductions are in a sense "already there," but one has to constantly decide which direction to take, and it is precisely these different choices that reveal the abilities, tastes and personality of the individual. The French mathematician Gustave Choquet formulated this beautifully by saying that the theorem one is seeking has existed from time immemorial, but that in order to *discover* it, one must *invent* a path.

MATHEMATICS: ART OR SCIENCE?

A related question, which also goes a long way back, is whether mathematics belongs to the arts or to the sciences. And once again, both points of view can be easily defended. On the side of "art" we might mention, first of all, the fact that mathematics often appears in art in the ordinary sense of the word. In the sphere of architecture, all we have to do is think of the pyramids, the Parthenon or the buildings designed by Christopher Wren, Le Corbusier, and many other architects. In the sphere of music, we think of works by Bach, Mozart, or Schoenberg; in painting, of those by Dürer or Leonardo da Vinci. But mathematics can also be intrinsically beautiful: we might think of the five regular polyhedra (fig. 1) which were already known to Plato, or, more recently, of the beautiful fractal images which many of you have surely seen (fig. 2).

However, when we speak of the "art" aspect in mathematics, we are thinking less about the relationships

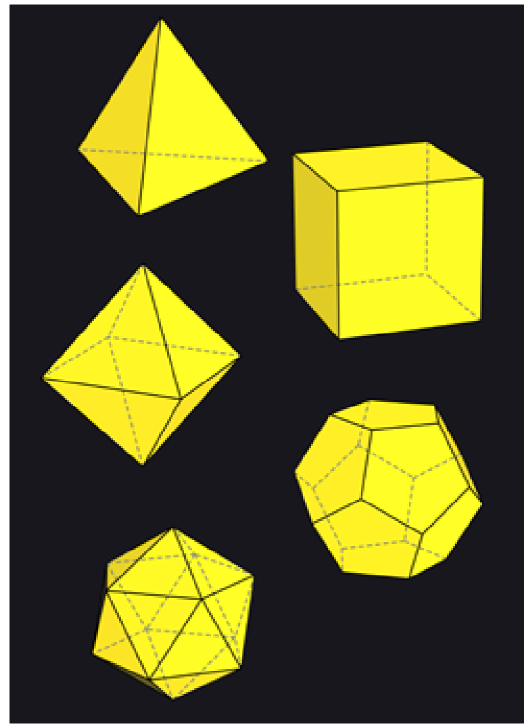

Fig. 1
The five regular polyhedra also known as the Platonic solids: the tetrahedron, the hexahedron (cube), the octahedron, the dodecahedron, and the icosahedron

between mathematics and the other arts, as varied and as interesting as they may be, but rather about the fact that mathematics is *itself* an art. The aesthetic criteria involved are not necessarily based on visual beauty—in spite of examples like the Platonic solids or fractals—but are much more abstract: concision, simplicity, clarity, and the absolute persuasiveness of the ideas and arguments that are employed. These criteria may seem to non-mathematicians to be more intellectual than artistic, but there is hardly anyone who has worked for a long time in mathematics without developing such feelings. Almost all mathematicians use words like "beauty" and "elegance," and they in fact use them more frequently than more scientific-sounding words like "convincing" or "correct." And, what is even more interesting, is that this feeling of mathematical beauty often seems to be the best guide to follow when trying to figure out which direction to take through the labyrinth of mathematics: a sort of Ariadne's thread. An artist can make his or her choices (what should I write? what should I paint? what should I compose?) based on aesthetic criteria. A scientist hardly ever has that kind of luxury since nature cannot always be expected to make choices that will be pleasing to humans, and scientists have to stick to reality. Mathematics is somewhere in between: it is not absolutely necessary to adhere to aesthetic criteria when doing mathematics, and the right solution to a problem may not always be the most beautiful, but in the vast majority of cases the right mathematical path actually turns out to be the best one from an aesthetic point of view. There is no better general strategy, when you want to do good mathematics, than to look for the *most beautiful* solution.

Mathematics can thus easily be seen as an art. But there are also convincing arguments in favor of the view of mathematics as a science. Indeed, mathematics has a level of objectivity that is scarcely attained in any of the other sciences: its results are absolutely guaranteed because they are proven, and its discoveries, once they have been made, never age—later developments may, of course, introduce new aspects, but they can never alter the truth once it has been discovered. We can even say that, in a way, mathematics is “more scientific than the other sciences” because it is less dependent on the accidental properties of the world. Sociology or Psychology depend on human society as it currently exists, Biology on organisms that have evolved on the earth, and even Chemistry and Physics on the laws of nature in our part of the universe, while Mathematics is, in a certain way, absolute.

PRESENT-DAY MATHEMATICS

I will confine myself here to three aspects: current research in mathematics, the applications of mathematics, and the influence of computers. Most people may not even know that research is still going on in mathematics and will be surprised to find out that everything has not been known for a long time. In fact, the opposite is true: not only do we produce thousands of new theorems every year, we also continue to solve old problems that have been around for decades or even centuries. The most famous example in recent times was the proof of what was called Fermat’s Last Theorem, which was formulated in 1637 but remained unsolved until 1995, when it was proved by Andrew Wiles. There are many other examples of this kind. Proof was found in 1976, after over a hundred years of research, for the so-called four-color theorem, which states that only four colors are required to color any geographical map, no matter how complicated, so that none of its regions will have the same color as any of the adjacent regions (fig. 3). Kepler’s conjecture, a hypothesis stating that the densest way of stacking balls is in pyramids such as those used to display oranges in markets, was confirmed just a few years ago. Recently, three Indian mathematicians came up with the first rapid method for determining the primality or compositeness of very large numbers. However, mathematicians do not only solve old problems; they are also constantly discovering new connections (for example, between algebraic geometry and number theory or between topology and mathematical physics) and even whole new areas in mathematics, recent examples being fractal theory, chaos theory, and complexity theory.

In terms of the applications of mathematics, their most surprising aspect is that they are rarely the outcome of planning, but often emerge unexpectedly in areas that may have nothing obvious to do with the application. Again and again it turns out to be the very purest mathematics—mathematics that was done because it was so beautiful, showed such perfect internal cohesion, and therefore so delighted its discoverers—that provides the key to an important problem in science or technology. Thus, the two greatest discoveries of the 20th century in physics, the theory of relativity and quantum theory, would not have been possible if non-Euclidean geometry and abstract matrix calculus had not been developed a little earlier by mathematicians who had no idea at all of their potential applications. The technology we use in our daily lives is also full of examples: computers would be inconceivable without the very abstract developments that preceded them in mathematical logic and Boolean algebra; the theory of prime numbers supplied new methods for cryptography that are now crucial for electronic banking; the extremely sophisticated geometric theory of the Radon transform provides a basis for tomography, which has become indispensable in making medical diagnoses; and so-called “fuzzy mathematics” is what makes it possible for washing machines to be silent and passengers in high-speed trains to drink their cups of coffee without spilling them when the train goes around a bend.

Finally, I would like to say a few words about the influence of computers on present-day mathematics. This influence is in fact much less than is usually thought, and computers have no more replaced mathematicians than typewriters have replaced writers. They are only tools. Nevertheless, they are without question extremely useful. First of all, there is the obvious aspect: computers can perform lengthy numerical or algebraic calculations that no human being could or would ever want to do, and they are indispensable for simulating or modeling complex systems. But there is

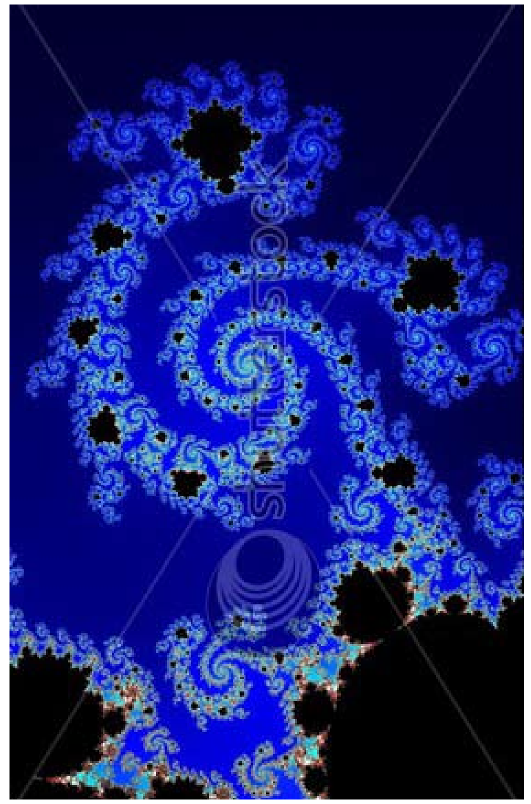

Fig. 2
An example of a fractal object, a mathematical concept introduced and developed by Benoît Mandelbrot during the 60’s and the 70’s

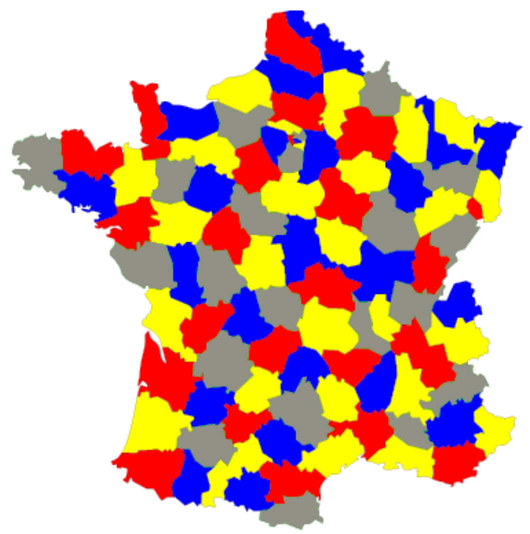

Fig. 3
The four-color theorem applied to the map of the departments of France

much more than that. First of all, computers make it possible to carry out experiments in order to discover or to verify mathematical statements. Of course, great mathematicians of the past such as Euler, Gauss, or Riemann[1] did many numerical experiments in order to find new results, but the incomparable speed of computers has greatly extended the possibilities offered by this method of investigation. Many deep conjectures that exist today were found in this way. And there is another thing: computers not only make it possible to carry out lengthy *calculations*, they are also able to produce complex *proofs*. A well-known example is the proof of the four-color theorem mentioned above: it is based on a complex strategy that involves distinguishing over 2,000 cases, which can then be examined and solved in a purely mechanical way through the use of a computer program. Finally, the existence of computers has, to a certain extent, changed our mathematical way of thinking in that they have given the notions of *algorithm* and *efficiency* much more weight than they used to have.

THE JOY OF MATHEMATICS

To conclude, I would like to talk briefly about the reasons why mathematics gives us so much joy. The answer that immediately comes to mind, and which undoubtedly has some truth to it, is simply that solving hard problems is a lot of fun. In addition to that, there is the aesthetic feeling mentioned earlier, the joy inspired by the elegance and beauty of the results and arguments that one reads in the works of others or discovers for oneself. However, it seems to me that what gives the adherents of mathematics the most satisfaction of all is the special feeling of being able to discover, without the use of any external expedients, a “piece of truth,” to see into one of nature’s mysteries. As a simple example, let us recall the famous proof of the existence of infinitely many prime numbers as it was formulated long ago by Euclid.

Suppose that there is only a finite number of prime numbers, for example, 2, 3, 5, 7, etc. up to 31. Multiply all of these primes (2, 3,…31) together and add 1 to the product. The result of this operation will not be divisible by

[1] Gauss calculated the first 100,000 prime numbers (or had them calculated) in order to formulate a conjecture about their distribution that would not be confirmed until forty years after his death, and Riemann also discovered his famous Hypothesis via numerical calculations.

any of the primes 2, 3,…31 because it will be greater by 1 than a multiple of each one. However, like any number, it has to either be a prime itself or have a prime factor smaller than itself, which would, contrary to the initial hypothesis, be a prime that is not on our original list.

Whether or not one is able to grasp all of the details of this argument after such a brief exposition, one can surely see that something extraordinary has been achieved: we start off with a question (is there a finite or an infinite quantity of prime numbers?) that we, as humans, should not actually be able to answer because we can never study more than a tiny, finite part of the prime numbers, and yet, in a few simple, albeit very subtle, sentences we are able to find the answer and prove it in an irrefutable way. Mathematics, which comes "from the inside" while at the same time describing something on the outside, is the only science in which one is able to find the truth (and even *prove* it!) through thought and thought alone, in other words, by, as it were, looking inside oneself.[2] And being able to do *that* is a wonderful feeling!

I have described the reasons why mathematics gives certain people such a deep feeling of joy. But it is indeed true only for certain people: mathematics is not for everyone. Unlike good food or music, for example, which almost everyone more or less appreciates—although some people may be more passionate and others less enthusiastic—mathematics inspires vastly different feelings in most human beings: those who have discovered how fascinating it can be are smitten forever, while the vast majority are unable to see any connection whatsoever between mathematics and joy. I do not wish to go into the causes of this phenomenon here, even though some very interesting studies have been done on this subject. But it is quite clear that it is largely cultural and that a love of mathematics exists in a potential state in many more people than is usually believed.
The main problem is that most people have never seen "genuine" mathematics. The mathematics that is taught in school is almost always presented as a series of recipes to be applied to everyday life or, at best, to science. It rarely has anything to do with the "beauty" of mathematics. But in order to understand this beauty, one must first encounter it: would you be able to imagine how beautiful music is if you had never heard a single melody? Fortunately, it is perfectly possible for non-mathematicians to encounter "real" mathematics. It could be, for instance, through the Platonic solids that we mentioned above, or through Euler's formula, which is discussed on pages 172–173; or through Lagrange's theorem, which states that any natural number is the sum of (at most) four square numbers; or through the mysterious Möbius strip that has only one side and one edge (fig. 4). Seeing objects such as these is bound, in my opinion, to kindle a certain amount of interest in mathematics in many people, both young and old, who have never felt it before. And that is of course the aim of the exhibition *Mathematics, A Beautiful Elsewhere*: to provide an encounter with "beautiful" mathematics.

Fig. 4
A Möbius strip

[2] Nowadays this point of view is often called "Platonism" from the famous passage in Plato's *Meno* in which Socrates uses a series of clever questions to lead an uneducated slave boy to understand and prove the theorem that a square drawn on the diagonal of a given square will have twice the area of the original square—though Plato drew from this the, to us, rather odd conclusion that the boy had an immortal soul and was simply remembering the proof from a previous life!