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A trace formula for Jacobi forms

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Dedicated to M. Eichler

Introduction

In the last few years Jacobi forms have begun to play a role in several contexts: the proof of the Saito-Kurokawa conjecture, the theory of modular forms of half-integral weight and, more recently, the theory of Heegner points. The theory of Jacobi forms was systematically developed in [E-Z].

The purpose of this paper is to state and prove a general trace formula for Jacobi forms. The exact statement of this trace formula can be found in § 1, especially Theorem 1. In this statement there occurs a certain quantity $G_m(\xi)$, the trace of a certain operator on a space of fundamental theta functions. This quantity is computed explicitly in § 4 (Theorems 2 and 3). In the course of these latter calculations two minor results, which may be of independent interest, incidentally drop out: a Gauss sum reciprocity law (Proposition 4. 2) and a nice formula for Gauss sums associated to binary quadratic forms (Theorem 3). Two proofs of the trace formula will be given, in § 2 and in § 3. The main ingredient of the first proof is the construction of a reproducing kernel function for Jacobi forms (Lemma 2. 1 and Proposition 2. 2), while in the second proof the trace formula for Jacobi forms is reduced to the trace formula for modular forms of half-integral weight as given in [Shi]. The first method is nicer because it is elementary and completely self-contained, but for convergence reasons we must assume that the weight k is greater than 3. The second proof works for all weights and is shorter modulo results in the literature, but it is considerably more abstract and is longer if one wants a complete proof "from scratch".

The specialisation of the general trace formula to Jacobi forms on the full modular group will be given in [S-Z]. It turns out that $J_{k,m}(SL_2(\mathbb{Z}))$ (= space of Jacobi forms on $SL_2(\mathbb{Z})$ of weight k and index m) is isomorphic as a Hecke module to a certain very natural subspace of $M_{2k-2}(\Gamma_0(m))$ (= space of modular forms on $\Gamma_0(m)$ of weight $2k-2$), the existence of which was apparently not noticed before. These liftings play a role in the relationship of Jacobi forms to Heegner points (cf. [G-K-Z]).

The original idea to study Jacobi forms and, especially, to develop a trace formula is due to M. Eichler. He proposed a different procedure to deduce such a trace formula: namely, to consider the kernel function

$$\mathscr{K}(\tau, z; \tau', z') = \sum_{\binom{ab}{cd} \in SL_2(\mathbb{Z})} \sum_{(\lambda,\mu)\in\mathbb{Z}^2} \frac{e^{2\pi i m\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z\right)}}{(c\tau+d)^k\left(\frac{a\tau+b}{c\tau+d}-\tau'\right)\left(\frac{z+\lambda\tau+\mu}{c\tau+d}-z'\right)}.$$

This function transforms like a Jacobi form (on the full modular group) of weight k and index m in the variables τ, z and, modulo a function with no poles in τ' and z', transforms with weight $2-k$ and index $-m$ in τ', z'; moreover the sum of the residues of

$$\mathscr{K}(\tau, z; \tau', z')\,\phi(\tau', z')\,d\tau'\,dz',$$

taken over a fundamental domain, equals $\phi(\tau, z)$ for any Jacobi form ϕ of weight k, index m. One then applies a Hecke operator (w.r.t. τ, z) to $\mathscr{K}$, sets $(\tau', z') = (\tau, z)$, and sums the residues over a fundamental domain. This approach, which is more geometrical than ours, works for the elliptic contributions of the trace formula but seems to be not so suitable for obtaining the parabolic contributions (one would have to study the behaviour of the line bundles whose sections are Jacobi forms when compactifying the cusps).

Originally the deduction of a trace formula for Jacobi forms and the study of its consequences were planned as chapter IV of the monograph [E-Z] (cf. [E-Z], p. 5, second paragraph). However, this project turned out to be much more time-consuming than expected and had to be dropped, to be taken up again by the present authors.

Finally, we mention the paper [E] of Eichler, which gives an interesting trace formula for Jacobi forms. However, his trace formula is of a completely different type and seems to be unsuitable for comparison with the usual trace formula for modular forms and therefore for proving the existence of the above mentioned lifting from $J_{k,m}(SL_2(\mathbb{Z}))$ to $M_{2k-2}(\Gamma_0(m))$.

It is a great pleasure to us to dedicate this paper to its initiator Martin Eichler.

§ 0. Preliminaries on Jacobi forms

As main reference for the basic facts and definitions from the theory of Jacobi forms we refer to [E-Z]. To fix the notation we briefly summarize those items that we shall need in the following.

By $\mathscr{J}(\mathbb{R})$ we denote the Jacobi group $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2 \cdot S^1$, i.e. the set of all triples (A, x, s) with $A \in SL_2(\mathbb{R})$, $x \in \mathbb{R}^2$ and $s \in S^1$ (the multiplicative group of complex numbers of modulus 1), supplied with the composition law

$$(A, x, s) \cdot (A', x', s') = \left(AA', xA' + x', ss'e\left(\begin{vmatrix} xA' \\ x' \end{vmatrix}\right)\right).$$

Here $xA'=(\lambda a'+\mu c', \lambda b'+\mu d')$ with $x=(\lambda, \mu)$, $A'=\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, and $\begin{vmatrix} xA' \\ x' \end{vmatrix}$ denotes the determinant of the matrix built from the row vectors xA' and x'. Furthermore, $e(\dots)$ always means $e^{2\pi i(\dots)}$ (and we shall use variations of this like $e^m(\dots)=e^{2\pi i m(\dots)}$ etc.).

We identify $SL_2(\mathbb{R})$ and S^1 with the subgroups $\{(A, 0, 1) | A \in SL_2(\mathbb{R})\}$ and $\{(1, 0, s) | s \in S^1\}$ respectively, and for $x \in \mathbb{R}^2$ we use $[x]$ for the element $(1, x, 1)$ of $\mathcal{J}(\mathbb{R})$. Then each element ξ of $\mathcal{J}(\mathbb{R})$ can be written uniquely as $\xi = A[x]s$ with suitable $A \in SL_2(\mathbb{R})$, $x \in \mathbb{R}^2$ and $s \in S^1$.

For subsets G, L, K of $SL_2(\mathbb{R})$, $\mathbb{R}^2$, S^1 respectively we set

$$G \ltimes L \cdot K := \{A[x]s | A \in G, x \in L, s \in S^1\}.$$

Obviously this defines a subgroup of $\mathcal{J}(\mathbb{R})$ if and only if G, L, K are subgroups of $SL_2(\mathbb{R})$, $\mathbb{R}^2$, S^1 respectively, L is invariant under G with respect to the usual action of G on $\mathbb{R}^2$ and K contains all numbers $e\left(\begin{vmatrix} x' \\ x \end{vmatrix}\right)$ with $x, x' \in L$. Special subgroups of this kind are the groups

$$\mathcal{J}(\mathbb{Q}) := SL_2(\mathbb{Q}) \ltimes \mathbb{Q}^2 \cdot S^1.$$

$$\Gamma^J := \Gamma \ltimes \mathbb{Z}^2 (= \Gamma \ltimes \mathbb{Z}^2 \cdot \{1\}), \quad \Gamma \text{ any subgroup of } SL_2(\mathbb{Z}).$$

We adopt the usual notations for special subgroups of $SL_2(\mathbb{Z})$:

$$\Gamma(n) = \{A \in SL_2(\mathbb{Z}) | A \equiv 1 \bmod n\}, \quad \Gamma_0(n) = \left\{A \in SL_2(\mathbb{Z}) | A = \begin{bmatrix} * & * \\ c & * \end{bmatrix} \text{ with } n | c\right\}.$$

$\mathfrak{H}$ denotes the upper half plane $\{\tau \in \mathbb{C} | \operatorname{Im}(\tau) > 0\}$. The Jacobi group $\mathcal{J}(\mathbb{R})$ acts on $\mathfrak{H} \times \mathbb{C}$ by

$$(A[x]s) \cdot (\tau, z) = \left(A\tau, \frac{z+\lambda\tau+\mu}{c\tau+d}\right), \quad \left(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, x = (\lambda, \mu), \tau \in \mathfrak{H}, z \in \mathbb{C}\right)$$

where $A\tau$ is given by the usual action of $SL_2(\mathbb{R})$ on $\mathfrak{H}$, i.e. $A\tau = \dfrac{a\tau+b}{c\tau+d}$.

Let k be a real number and m an integer. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}), \quad x = (\lambda, \mu) \in \mathbb{R}^2, \quad s \in S^1.$$

Then we have, generalizing the operator

$$(h|_k A)(\tau) = (c\tau+d)^{-k} h(A\tau)$$

on functions $h(\tau)$ on $\mathfrak{H}$, the operator

$$(1)\quad \begin{aligned}&(\phi|_{k,m}(A[x]\,s))(\tau,z)\\ &=(c\tau+d)^{-k}\,e^m\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\mu\right)s^m\,\phi\left(A\tau,\frac{z+\lambda\tau+\mu}{c\tau+d}\right)\end{aligned}$$

on functions $\phi(\tau,z)$ on $\mathfrak{H}\times\mathbb{C}$.

Here and in the following w^r ($w, r\in\mathbb{C}$, $w\neq 0$) is always defined by

$$w^r=e^{(i\operatorname{Arg}(w)+\log|w|)r},\quad (-\pi<\operatorname{Arg}(w)\leqq+\pi).$$

If k is an integer, then (1) defines an action of $\mathcal{J}(\mathbb{R})$ on the space of functions of $\mathfrak{H}\times\mathbb{C}$. In one or two instances we have to consider the case of half integral k, i.e. $k\in\frac{1}{2}+\mathbb{Z}$. In this case one has

$$(\phi|_{k,m}(A[x]\,s))|_{k,m}(B[y]\,r)=\sigma(A,B)\,\phi|_{k,m}((A[x]\,s)\,(B[y]\,t)),$$

where $\sigma(A,B)=\pm 1$; more precisely

$$(2)\quad \sigma(A,B)=\frac{(cB\tau+d)^{1/2}(c'\tau+d')^{1/2}}{(c''\tau+d'')^{1/2}},\quad \left(A=\begin{bmatrix}* & *\\ c & d\end{bmatrix},\ B=\begin{bmatrix}* & *\\ c' & d'\end{bmatrix},\ AB=\begin{bmatrix}* & *\\ c'' & d''\end{bmatrix}\right).$$

Let Γ be a subgroup of finite index in $SL_2(\mathbb{Z})$ and k, m be positive integers. Then $J_{k,m}(\Gamma)$ denotes the space of Jacobi forms of weight k and index m on Γ, i.e. the space of all holomorphic functions $\phi(\tau,z)$ on $\mathfrak{H}\times\mathbb{C}$ such that

(i) $\phi|_{k,m}\xi=\phi$ for all $\xi\in\Gamma^J$,

(ii) for each $A\in SL_2(\mathbb{Z})$ the function $\phi|_{k,m}A$ has a Fourier development of the form $\phi|_{k,m}A=\sum_{\substack{n,r\in\mathbb{Z}\\ 4m\frac{n}{t}-r^2\geqq 0}} c(n,r)\,q^{n/t}\zeta^r$ for some integer t.

Here and in the following $q^{n/t}$ and ζ^r denote the functions $e\left(\frac{n}{t}\tau\right)$ on $\mathfrak{H}$ and $e(rz)$ on $\mathbb{C}$ respectively.

If ϕ has for each $A\in SL_2(\mathbb{Z})$ a Fourier development (ii) satisfying the stronger condition $4m\frac{n}{t}-r^2>0$ then ϕ is called a cusp form. The subspace of cusp forms in $J_{k,m}(\Gamma)$ will be denoted by $S_{k,m}(\Gamma)$.

For integers ϱ, m with $m>0$ set

$$(3)\quad \theta_{m,\varrho}:=\sum_{\substack{r\in\mathbb{Z}\\ r\equiv\varrho \bmod 2m}} q^{r^2/4m}\zeta^r,$$

and denote by Th_m the span of the $\theta_{m,\varrho}(\varrho \in \mathbb{Z})$. Obviously Th_m is a $2m$-dimensional vector space.

As a consequence of the transformation law with respect to $\mathbb{Z}^2 (\subseteq \Gamma^J)$ every Jacobi form ϕ in $J_{k,m}(\Gamma)$ can be uniquely written as

$$\phi(\tau, z) = \sum_{\varrho=1}^{2m} h_\varrho(\tau)\, \theta_{m,\varrho}(\tau, z) \tag{4}$$

with suitable functions $h_\varrho(\tau)$ on $\mathfrak{H}$. It is a fact that the $h_\varrho(\tau)$ are elements of

$$M_{k-\frac{1}{2}}(\Gamma(4m) \cap \Gamma),$$

where $M_{k-\frac{1}{2}}(\Gamma)$ for any $\Gamma \subseteq \Gamma_0(4)$ denotes the space of modular forms of weight $k - \frac{1}{2}$ on Γ, i.e. the space of holomorphic functions $h(\tau)$ on $\mathfrak{H}$ satisfying

$$h(A\tau)\, j(A, \tau)^{1-2k} = h(\tau) \text{ for all } A \in \Gamma$$

and $h|_{k-\frac{1}{2}} A$ for any $A \in SL_2(\mathbb{Z})$ having a Fourier development of the form $\sum_{N \geqq 0} c(N) q^{N/t}$ (t a suitable integer). Here and in the following $j(A, \tau)$ denotes the usual theta multiplier which can be defined by

$$j(A, \tau) = \theta(A\tau)/\theta(\tau), \quad \left(A \in \Gamma_0(4),\ \theta = \sum_{r \in \mathbb{Z}} q^{r^2}\right).$$

One has $j(A, \tau) = \varepsilon(c\tau + d)^{1/2}$ $\left(A = \begin{bmatrix} * & * \\ c & d \end{bmatrix}\right)$ with a well-known fourth root of unity $\varepsilon = \varepsilon(A)$ (even $\varepsilon(A) \in \{\pm 1\}$ for $A \in \Gamma(4)$), but we shall not need such explicit formulas.

Conversely, let $h_\varrho(\tau)$ ($\varrho = 1, \ldots, 2m$) be given elements of $M_{k-\frac{1}{2}}(\Gamma(4m) \cap \Gamma)$ such that the function ϕ defined by (4) transforms with respect to Γ like an element of $J_{k,m}(\Gamma)$. Then it is easy to check that ϕ is indeed in $J_{k,m}(\Gamma)$.

Via (4) we occasionally identify $J_{k,m}(\Gamma)$ with the subspace of elements in

$$M_{k-\frac{1}{2}}(\Gamma(4m) \cap \Gamma) \otimes Th_m$$

which are fixed by Γ with respect to the obvious action "$|_{k-\frac{1}{2}} \otimes |_{\frac{1}{2},m}$" of Γ:

$$\begin{aligned} J_{k,m}(\Gamma) &\xrightarrow{\ \approx\ } (M_{k-\frac{1}{2}}(\Gamma(4m) \cap \Gamma) \otimes Th_m)^\Gamma, \\ \phi &\mapsto \sum_{\varrho=1}^{2m} h_\varrho \otimes \theta_{m,\varrho}. \end{aligned} \tag{5}$$

It is easily seen that by (5) the subspace $S_{k,m}(\Gamma)$ corresponds to

$$(S_{k-\frac{1}{2}}(\Gamma(4m) \cap \Gamma) \otimes Th_m)^\Gamma,$$

where $S_{k-\frac{1}{2}}(\Gamma)$ for any $\Gamma \subseteq \Gamma_0(4)$ denotes the subspace of cusp forms in $M_{k-\frac{1}{2}}(\Gamma)$. Also note that we may replace in (5) the group $\Gamma(4m) \cap \Gamma$ occurring on the right side by any subgroup Σ of finite index in $\Gamma(4m) \cap \Gamma$.

§ 1. The statement of the trace formula

Let k, m denote positive integers and let $\Delta \subseteq \mathscr{J}(\mathbb{Q}) = SL_2(\mathbb{Q}) \ltimes \mathbb{Q}^2 \cdot S^1$ be a finite union of double cosets with respect to a subgroup Γ^J of finite index in $SL_2(\mathbb{Z})^J$.

We define an operator $H_{k,m,\Gamma}(\Delta)$ on $J_{k,m}(\Gamma)$ by

$$\phi | H_{k,m,\Gamma}(\Delta) := \sum_{\xi \in \Gamma^J \backslash \Delta} \phi|_{k,m} \xi. \tag{1}$$

Here the sum is over a complete set of representatives ξ for the Γ^J-left cosets of Δ.

Proposition 1. 1. *The operator $H_{k,m,\Gamma}(\Delta)$ is well-defined (i.e. the sum in (1) is finite and does not depend on the choice of representatives ξ) and maps $J_{k,m}(\Gamma)$ to $J_{k,m}(\Gamma)$ and $S_{k,m}(\Gamma)$ to $S_{k,m}(\Gamma)$.*

Proof. We first show that $\Gamma^J \backslash \Gamma^J \xi \Gamma^J$ is finite for any $\xi \in \mathscr{J}(\mathbb{Q})$.

Let $\xi = A[x]s$, let $\gamma: \Gamma^J \xi \Gamma^J \to \Gamma A \Gamma$ be the canonical map $\eta = B[y]t \to B$ and let $\gamma_*: \Gamma^J \backslash \Gamma^J \xi \Gamma^J \to \Gamma \backslash \Gamma A \Gamma$ be the induced map. For $B \in \Gamma A \Gamma$ the fibre $\gamma^{-1}(B)$ is invariant by left and right multiplication with $\mathbb{Z}^2 (\subseteq \Gamma^J)$, and it is easily seen that a set of representatives for $\mathbb{Z}^2 \backslash \gamma^{-1}(B)$ defines also a set of representatives for the Γ^J-left cosets in $\gamma_*^{-1}(\Gamma B)$. Hence, in view of the well-known fact that $\Gamma \backslash \Gamma A \Gamma$ is finite, it suffices to show that $\mathbb{Z}^2 \backslash \gamma^{-1}(B)$ is finite for any $B \in \Gamma A \Gamma$.

A simple calculation shows that $\gamma^{-1}(B)$ is the union of all double cosets $\mathbb{Z}^2 B[xG] s \mathbb{Z}^2$ with $G \in A^{-1} \Gamma B \cap \Gamma$. The number of such double cosets is actually finite: if $l > 0$ is an integer such that $lx \in \mathbb{Z}$, and if $G, G' \in A^{-1} \Gamma B \cap \Gamma$ with $G \equiv G' \bmod l^2$, then $[xG]\mathbb{Z}^2 = [xG']\mathbb{Z}^2$. But $A^{-1} \Gamma B \cap \Gamma \bmod l^2 \subseteq SL_2(\mathbb{Z}/l^2\mathbb{Z})$ is finite.

Thus it is left to show that $\mathbb{Z}^2 \backslash \mathbb{Z}^2 \eta \mathbb{Z}^2$ is finite for any $\eta \in \mathscr{J}(\mathbb{Q})$. But this is immediate, since a set of representations for $\mathbb{Z}^2 \backslash \mathbb{Z}^2 \eta \mathbb{Z}^2$ is given by $\eta \eta'$ with η' running through a set of representatives for $(\eta^{-1} \mathbb{Z}^2 \eta \cap \mathbb{Z}^2) \backslash \mathbb{Z}^2$, and $\eta^{-1} \mathbb{Z}^2 \eta \supseteq N \mathbb{Z}^2$ for a suitably chosen positive integer N (depending on η).

Since Δ is a finite union of double cosets of the kind considered above, we deduce that $\Gamma^J \backslash \Delta$ is finite.

That $\phi | H_{k,m,\Gamma}(\Delta)$ in (1) is independent of the choice of representatives ξ is clear from the transformation law of $\phi \in J_{k,m}(\Gamma)$, and also it is obvious that $\phi | H_{k,m,\Gamma}(\Delta)$ transforms like a Jacobi form in $J_{k,m}(\Gamma)$ with respect to Γ^J.

In order to check that $(\phi | H_{k,m,\Gamma}(\Delta))|_{k,m} A$, $(A \in SL_2(\mathbb{Z}))$ has the correct Fourier development one writes for each ξ in (1) $\xi A = A' B[y] t$ with a suitable $A' \in SL_2(\mathbb{Z})$ and an upper triangular matrix $B \in SL_2(\mathbb{Q})$. Let

$$B = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}, \quad y = (\lambda, \mu).$$

By assumption $\phi|_{k,m}A' = \sum c(n,r)\, q^n \zeta^r$ with $c(n,r)=0$ unless $4mn - r^2 \geqq 0$. (Here r is integral and n is rational with bounded denominator.) Hence

$$\phi|_{k,m}\xi A = a^k t^m \sum c(n,r)\, e(abn + a\mu r + m\lambda\mu)\, q^{a^2 n + a\lambda r + m\lambda^2} \zeta^{ar + 2m\lambda},$$

and since $4m(a^2 n + a\lambda r + m\lambda^2) - (ar + 2m\lambda)^2 = a^2(4mn - r^2)$, we see that in the Fourier development of $(\phi|H_{k,m,\Gamma}(\Delta))|_{k,m}A$ only powers $q^{n'}\zeta^{r'}$ (n', r' rational numbers) occur where $4mn' - r'^2 \geqq 0$. Finally, using the fact that $(\phi|H_{k,m}(\Delta))|_{k,m}A$ is invariant with respect to $\mathbb{Z}^2$, so that only integer powers of ζ occur in the Fourier development of $(\phi|H_{k,m}(\Delta))|_{k,m}A$, one ends with the required Fourier development. Also, these arguments show that $H_{k,m,\Gamma}(\Delta)$ maps cusp forms to cusp forms.

Remark. Let Γ' be a subgroup of finite index in Γ. Then Δ is also a finite union of double cosets with respect to Γ'^J and we can consider the operator $H_{k,m,\Gamma'}(\Delta)$. It is obvious that the image of $J_{k,m}(\Gamma')$ by $H_{k,m,\Gamma'}(\Delta)$ is contained in $J_{k,m}(\Gamma)$ and that the restriction of $|\Gamma'\backslash\Gamma|^{-1} \cdot H_{k,m,\Gamma'}(\Delta)$ to $J_{k,m}(\Gamma)$ coincides with $H_{k,m,\Gamma}(\Delta)$. Thus we obtain

$$\text{(2)} \qquad \operatorname{tr}(H_{k,m,\Gamma}(\Delta), J_{k,m}(\Gamma)) = |\Gamma'\backslash\Gamma|^{-1} \operatorname{tr}(H_{k,m,\Gamma'}(\Delta), J_{k,m}(\Gamma')),$$

and the same applies to cusp forms.

The simple formula (2) will become important in the derivation of the trace formula for $H_{k,m,\Gamma}(\Delta)$ given in § 3.

The aim of this paragraph is to state a formula for the traces of the operators $H_{k,m,\Gamma}(\Delta)$. In order to do this we have to introduce another operator.

Recall that Th_m denotes the span of the theta series

$$\theta_{m,\varrho} = \sum_{r \equiv \varrho(2m)} q^{r^2/4m} \zeta^r, \quad (\varrho = 1, \ldots, 2m).$$

For each $\xi \in \mathcal{J}(\mathbb{Q})$ we define an operator $U_m(\xi)$ on Th_m by

$$\text{(3)} \qquad \theta|U_m(\xi) := |L\backslash\mathbb{Z}^2|^{-1} \sum_{x \in L\backslash\mathbb{Z}^2} \theta|_{1/2,m}\xi[x].$$

Here L is any subgroup of finite index in $\mathbb{Z}^2$ such that $\xi L \xi^{-1} \subseteqq \mathbb{Z}^2$. From the transformation law of θ with respect to $\mathbb{Z}^2$ it is clear that $\theta|U_m(\xi)$ depends neither on the choice of L nor on the choice of representatives x for $L\backslash\mathbb{Z}^2$. In § 4 we shall show that $U_m(\xi)$ actually maps Th_m to Th_m (cf. Prop. 4. 1).

Note that $U_m(A)$ for $A \in SL_2(\mathbb{Z})$ coincides with the usual projective action of $SL_2(\mathbb{Z})$ on Th_m. Using this it is easy to check that

$$U_m(A\xi A^{-1}) = \pm U_m(A)\, U_m(\xi)\, U_m(A)^{-1}$$

for any ξ and any A in $SL_2(\mathbb{Z})$. Thus the essential part of the trace of $U_m(\xi)$ should only depend on the $SL_2(\mathbb{Z})$-conjugacy class of ξ.

To be more precise set

$$\text{(4)} \qquad G_m(\xi) := \varepsilon(A) \operatorname{tr} U_m(\xi), \quad (\xi = A[x]\, s \in \mathcal{J}(\mathbb{Q}))$$

where

(5) $$\varepsilon(A)=\begin{cases}-1 & \text{if } c<0 \text{ and } \operatorname{tr}(A)<2,\\ +1 & \text{otherwise}\end{cases}\qquad \left(A=\begin{bmatrix}* & *\\ c & *\end{bmatrix}\right).$$

Lemma 1.2. *The expressions $G_m(\xi)$ defined by (4) satisfy*

$$G_m(M\xi M^{-1})=G_m(\xi)\quad \textit{and}\quad G_m(B\xi)=G_m(\xi)$$

for all $\xi\in\mathcal{J}(\mathbb{Q})$, $M\in SL_2(\mathbb{Z})$ and all parabolic $B\in\Gamma(4m)$.

Proof. Let $\xi=A[x]s$. Then

$$U_m(M)\,U_m(\xi)=\sigma(M,A)\,U_m(M\xi)=\sigma(M,A)\,\sigma(MAM^{-1},M)\,U_m(M\xi M^{-1})\,U_m(M)$$

with $\sigma(\cdot,\cdot)$ as in (0.2). Hence by (3) of the Appendix we find

$$\varepsilon(A)\,U_m(M)\,U_m(\xi)\,U_m(M)^{-1}=\varepsilon(MAM^{-1})\,U_m(M\xi M^{-1}),$$

which immediately implies the first assertion of the lemma. The second one is a simple consequence of the first and the fact that every parabolic $B\in\Gamma(4m)$ is $SL_2(\mathbb{Z})$-conjugate to a matrix of the form $\begin{bmatrix}1 & 4mt\\ 0 & 1\end{bmatrix}$, $(t\in\mathbb{Z})$, and that such a matrix obviously acts trivially on Th_m.

We can now state the trace formula.

Theorem 1. *Let $k>2$, $m>0$ be integers, $\Gamma\subseteqq SL_2(\mathbb{Z})$ a subgroup of finite index, and Δ a finite union of double Γ^J-cosets in $\mathcal{J}(\mathbb{Q})$. Then*

(6) $$\operatorname{tr}(H_{k,m,\Gamma}(\Delta),S_{k,m}(\Gamma))=\sum_{A\in P(\Delta)/\sim_{m,\Gamma}} I_{k,m,\Gamma}(A)\,g_m(\Delta,A),$$

where $P:\mathcal{J}(\mathbb{Q})\to SL_2(\mathbb{Q})$, $P(A[x]s)=A$, denotes the canonical projection, $\sim_{m,\Gamma}$ is the equivalence relation defined by

$$A\sim_{m,\Gamma}B\quad \textit{if and only if}\quad \begin{cases}A \textit{ and } B \textit{ are } \Gamma\textit{-conjugate}\\ \qquad\textit{or}\\ A \textit{ and } B \textit{ are parabolic and } GA \textit{ is}\\ \Gamma\textit{-conjugate to } B \textit{ for some } G\in\Gamma_A\cap\Gamma(4m),\end{cases}$$

and $g_m(\Delta,A)$, $I_{k,m,\Gamma}(A)$ are invariants of the $\sim_{m,\Gamma}$-equivalence class of A defined as follows:

— $$g_m(\Delta,A)=\sum_{\xi\in\mathbb{Z}^2\backslash P^{-1}(A)\cap\Delta/\mathbb{Z}^2}|\mathbb{Z}^2\backslash\mathbb{Z}^2\xi\mathbb{Z}^2|\,G_m(\xi),$$

with $G_m(\xi)$ as in (4) and the sum being over a set of representatives of the double cosets in $P^{-1}(A)\cap\Delta$ with respect to $\mathbb{Z}^2(\subseteqq\Gamma^J)$.

— *If $A=\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}$ then*

$$I_{k,m,\Gamma}(A)=[SL_2(\mathbb{Z}):\Gamma]\cdot a^{\frac{1}{2}-k}\cdot\frac{2k-3}{48}.$$

— *If A is elliptic,* $A=\begin{bmatrix} * & * \\ c & * \end{bmatrix}$, *let ϱ, ϱ' denote the eigenvalues of A such that* $\mathrm{Im}(\varrho)$ *and c have the same sign. Then*

$$I_{k,m,\Gamma}(A)=|\Gamma_A|^{-1}\,\mathrm{sign}(c)\,\frac{\varrho^{\frac{3}{2}-k}}{\varrho-\varrho'}.$$

— *If A is hyperbolic with $\mathrm{tr}(A)^2-4$ a square in $\mathbb{Q}$, let ϱ, ϱ' denote the eigenvalues of A such that $|\varrho|>|\varrho'|$. Then*

$$I_{k,m,\Gamma}(A)=-|\Gamma_A|^{-1}\,\frac{\varrho^{\frac{3}{2}-k}}{\varrho-\varrho'}.$$

— *If A is hyperbolic with $\mathrm{tr}(A)^2-4$ not a square in $\mathbb{Q}$, then $I_{k,m,\Gamma}(A)=0$.*

— *If A is parabolic, then there exists a $D\in SL_2(\mathbb{Z})$ and numbers $s>0$, r such that $D^{-1}\Gamma_A(4m)D$ is generated by $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ and $D^{-1}AD=\frac{1}{2}\,\mathrm{tr}(A)\cdot\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$. Then*

$$I_{k,m,\Gamma}(A)=-\frac{1}{2}\,[\Gamma_A:\Gamma_A(4m)]^{-1}\cdot(\mathrm{tr}(A)/2)^{\frac{1}{2}-k}\begin{cases} 1 & \text{if } \frac{r}{s}\in\mathbb{Z}, \\ 1-i\cot\pi\frac{r}{s} & \text{if } \frac{r}{s}\notin\mathbb{Z}. \end{cases}$$

Here Γ_A denotes the subgroup of all $G\in\Gamma$ such that $GAG^{-1}=A$, and $\Gamma_A(4m)$ the subgroup $\Gamma_A\cap\Gamma(4m)$.

Remarks. (i) The formulas for $I_{k,m,\Gamma}(A)$ are completely explicit. In § 4, we will calculate $G_m(\xi)$, thus obtaining a "ready to compute" formula for the trace of $H_{k,m,\Gamma}(\Delta)$.

(ii) To check that the statement of the theorem makes sense we need the following observations:

— the sum in (6) is finite, since the matrices in $P(\Delta)$ have bounded denominators and since there is no contribution from the conjugacy classes of the non-split hyperbolic elements in $P(\Delta)$.

— The expressions defining $I_{k,m,\Gamma}(A)$ make sense (because Γ_A is finite for elliptic or split hyperbolic A) and depend only on the $\sim_{m,\Gamma}$-conjugacy class of A (recall that the sign of the left lower entry of an elliptic A is invariant with respect to $SL_2(\mathbb{Z})$-conjugacy).

— The sum defining $g_m(\Delta, A)$ is finite (in the proof of Prop. 1. 1 we showed that the set $\mathbb{Z}^2\backslash P^{-1}(A)\cap\Delta$ is finite); also $g_m(\Delta, A)$ does not depend on the choice of representatives ξ for $\mathbb{Z}^2\backslash P^{-1}(A)\cap\Delta/\mathbb{Z}^2$ (it is obvious from the definition of $G_m(\xi)$ that $G_m([x]\,\xi)=G_m(\xi\,[x])=G_m(\xi)$ for all $x\in\mathbb{Z}^2$).

— One can easily deduce from Lemma 1. 2 that $g_m(\Delta, A)$ depends only on the $\sim_{m,\Gamma}$-conjugacy class of A.

(iii) There are also formulas for $\mathrm{tr}(H_{k,m,\Gamma}(\Delta),\ S_{k,m}(\Gamma))$ in the case $k=1,2$. For details of this the reader is referred to the end of § 3.

§ 2. First method: Kernel function for Jacobi forms

Throughout this paragraph Γ denotes a subgroup of finite index in $SL_2(\mathbb{Z})$, and k, m denote positive integers. We shall mostly assume that $k \geq 4$. This assumption is needed to ensure the convergence of several expressions occurring below. A proof of Theorem 1 which is valid also for smaller values of k will be given in § 3.

As indicated in the title the derivation of the formula for the trace of $H_{k,m,\Gamma}(\Delta)$ on $S_{k,m}(\Gamma)$ will depend on an explicit description of the reproducing kernel function of $S_{k,m}(\Gamma)$ with respect to the Petersson scalar product (which was introduced in [E-Z]). The procedure will be very similar to the derivation of the trace formula for Hecke operators on the space of cusp forms on $SL_2(\mathbb{Z})$ as given in [Z].

To fix the notation let us recall the definition of the Petersson scalar product for Jacobi forms.

For $\tau \in \mathfrak{H}$, $z \in \mathbb{C}$ let

$$\mu_{k,m}(\tau, z) := v^{k/2} e^{-2\pi m y^2/v}.$$

Here and henceforth we use

$$\tau = u + iv, \ z = x + iy \quad (u, v, x, y \in \mathbb{R}).$$

It is easily checked that

$$\begin{aligned}\mu_{k,m}(\xi \cdot (\tau, z)) &= \mu_{k,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right)\\ &= |c\tau+d|^{-k}\left|e^m\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z\right)\right| \mu_{k,m}(\tau, z)\end{aligned}$$

for all $\xi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} [\lambda, \mu] \in \mathscr{J}(\mathbb{R})$.

Thus, given any two functions ϕ, ψ on $\mathfrak{H} \times \mathbb{C}$ invariant with respect to the action of Γ^J given by "$|_{k,m}$", we deduce that $\phi\bar{\psi}\mu_{k,m}^2$ is a Γ^J-invariant function on $\mathfrak{H} \times \mathbb{C}$.

The $\mathscr{J}(\mathbb{R})$-invariant volume element in $\mathfrak{H} \times \mathbb{C}$ is given by

$$dV = v^{-3}\, du\, dv\, dx\, dy.$$

Note that the volume of $\Gamma^J \backslash \mathfrak{H} \times \mathbb{C}$ (with respect to dV) equals $|\Gamma \cap \{\pm 1\}|^{-1}$ times the volume of $\Gamma \backslash \mathfrak{H}$ (with respect to $v^{-2}\, du\, dv$); in particular, it is finite.

The Petersson scalar product of two cusp forms ϕ, $\psi \in S_{k,m}(\Gamma)$ is now defined by

$$\langle \phi, \psi \rangle = \int_{\Gamma^J \backslash \mathfrak{H} \times \mathbb{C}} \phi \bar{\psi} \mu_{k,m}^2\, dV.$$

The convergence of this integral follows from the easily proved fact that for any cusp form $\phi \in S_{k,m}(\Gamma)$ the expression $|\phi(\tau, z)\, \mu_{k,m}(\tau, z)|$ is bounded on $\mathfrak{H} \times \mathbb{C}$.

Finally, we define a function $h_{k,m}$ on $(\mathfrak{H} \times \mathbb{C})^2$ by

$$(1) \qquad h_{k,m}(\tau, z; \tau_0, z_0) := (\tau - \bar{\tau}_0)^{-k} e^m\left(\frac{-(z-\bar{z}_0)^2}{\tau - \bar{\tau}_0}\right).$$

It is invariant under $\mathscr{J}(\mathbb{R})$ acting by $|_{k,m}$ on (τ, z) and in the complex conjugate way on (τ_0, z_0), i.e.,

$$h_{k,m}(\xi \cdot (\tau, z); \xi \cdot (\tau_0, z_0)) \cdot (c\tau + d)^{-k} e^m\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right) s^m$$
$$\cdot \overline{(c\tau_0+d)^{-k} e^m\left(\frac{-c(z_0+\lambda\tau_0+\mu)^2}{c\tau_0+d} + \lambda^2\tau_0 + 2\lambda z_0 + \lambda\mu\right) s^m} = h_{k,m}(\tau, z; \tau_0, z_0)$$

for $\xi = \begin{bmatrix} * & * \\ c & d \end{bmatrix} [\lambda, \mu]\, s \in \mathscr{J}(\mathbb{R})$. This can be verified by a messy direct computation which becomes somewhat simpler if $c=0$, the only case we shall need.

The basic lemma is:

Lemma 2.1. *Let $k > 3$. Then one has for any $(\tau_0, z_0) \in \mathfrak{H} \times \mathbb{C}$:*

(i) *The integral*

$$\int_{\mathfrak{H} \times \mathbb{C}} |h_{k,m}(\tau, z; \tau_0, z_0)\, \mu_{k,m}(\tau, z)|\, dV$$

is finite.

(ii) *Let $\phi(\tau, z)$ be any holomorphic function on $\mathfrak{H} \times \mathbb{C}$ such that $|\phi(\tau, z)\, \mu_{k,m}(\tau, z)|$ is bounded in $\mathfrak{H} \times \mathbb{C}$. Then*

$$(2) \qquad \int_{\mathfrak{H} \times \mathbb{C}} \phi(\tau, z)\, \overline{h_{k,m}(\tau, z; \tau_0, z_0)}\, \mu_{k,m}(\tau, z)^2\, dV = \frac{2^{2-k}\pi i^k}{(2k-3)m}\, \phi(\tau_0, z_0).$$

(Note that the integral in (2) is absolutely convergent by (i) and the boundedness condition for ϕ.)

Proof. First of all we observe that it suffices to prove the lemma for $(\tau_0, z_0) = (i, 0)$, as we see by choosing $\xi \in \mathscr{J}(\mathbb{R})$ (with upper triangular $SL_2(\mathbb{R})$-component if so desired) sending $(i, 0)$ to (τ_0, z_0), replacing ϕ in (ii) by $\phi|_{k,m}\xi$, and using the transformation laws of $\mu_{k,m}$ and $h_{k,m}$. Write simply $h(\tau, z)$ for $h_{k,m}(\tau, z; i, 0)$. Now to prove (i) we observe

$$(3) \qquad |h(\tau, z)\, \mu_{k,m}(\tau, z)| = \frac{v^{k/2}}{|\tau + i|^k}\, e^{2\pi m\left(\operatorname{Im}\left\{\frac{1}{\tau+i}\right\} x^2 + 2\operatorname{Re}\left\{\frac{1}{\tau+i}\right\} xy - \left[\operatorname{Im}\left\{\frac{1}{\tau+i}\right\} + \frac{1}{v}\right] y^2\right)}.$$

Integrated over $\mathbb{C}$ with respect to $dx\,dy$ this yields $\frac{v^{k/2}}{|\tau+i|^k} \cdot \frac{\pi}{\sqrt{D}}$, where D denotes the absolute value of the discriminant of the negative definite quadratic form in the exponential of the right hand side of (3). A simple calculation shows $D = \frac{(4\pi m)^2}{v|\tau+i|^2}$, so

$$\int_{\mathfrak{H} \times \mathbb{C}} |h(\tau, z)\, \mu_{k,m}(\tau, z)|\, dV = \frac{1}{4m} \int_{\mathfrak{H}} \left(\frac{v^{1/2}}{|\tau + i|}\right)^{k-1} \frac{du\,dv}{v^2}.$$

Identifying the upper half-plane with the unit disk and introducting polar coordinates, i.e., setting $\frac{\tau-i}{\tau+i}=re^{i\theta}$, $(0\leqq r<1,\ 0\leqq\theta<2\pi)$, we find

$$\frac{v}{|\tau+i|^2}=\frac{1-r^2}{4},\quad \frac{du\,dv}{v^2}=\frac{4r}{(1-r^2)^2}\,d\theta\,dr \tag{4}$$

and hence finally

$$\int\limits_{\mathfrak{H}\times\mathbb{C}} |h(\tau,z)\,\mu_{k,m}(\tau,z)|\,dV=\frac{2^{2-k}\pi}{m}\int\limits_0^1 (1-r^2)^{\frac{k-5}{2}}\,r\,dr<\infty.$$

To prove (ii), denote the integral on the left-hand side of (2) by I; then

$$I=\int\limits_{\mathfrak{H}} \Phi(\tau)\,\frac{v^{k-\frac12}}{(\bar\tau-i)^{k-\frac12}}\,\frac{du\,dv}{v^2}.$$

where

$$\Phi(\tau)=\int\limits_{\mathbb{C}} \phi(\tau,z)\,g(\tau,z)\,dx\,dy$$

with

$$g(\tau,z)=(\bar\tau-i)^{-\frac12}\,e^m\left(\frac{\bar z^2}{\bar\tau-i}\right)v^{-\frac12}\,e^{-4\pi m y^2/v}.$$

To evaluate the integral I we again set $\frac{\tau-i}{\tau+i}=re^{i\theta}$ and use (4), obtaining

$$I=\int\limits_0^1\left(\int\limits_0^{2\pi}\Phi(\tau)\,(\tau+i)^{k-\frac12}\,d\theta\right)\left(\frac{1-r^2}{4}\right)^{k-\frac12}\frac{4r}{(1-r^2)^2}\,dr.$$

Let us assume for a moment that $\Phi(\tau)$ is holomorphic. Then the inner integral equals $2\pi(2i)^{k-\frac12}\,\Phi(i)$ by Cauchy's theorem, so

$$I=\frac{\pi}{2}\,(2i)^{k-\frac12}\,\Phi(i)\int\limits_0^1\left(\frac{1-r^2}{4}\right)^{k-\frac52} r\,dr=\frac{2^{\frac72-k}\,\pi\, i^{k-\frac12}}{2k-3}\,\Phi(i).$$

But

$$\begin{aligned}\Phi(i)&=(-2i)^{-\frac12}\int\limits_{\mathbb{C}}\phi(i,z)\,e^m\left(\frac{\bar z^2}{-2i}\right)e^{-4\pi m y^2}\,dx\,dy\\ &=(-2i)^{-\frac12}\int\limits_0^\infty\int\limits_0^{2\pi}\phi(i,z)\,e^{-2\pi m r^2+\pi m z^2}\,r\,d\theta\,dr,\quad (z=re^{i\theta}),\end{aligned}$$

and since $\phi(i,z)$ is holomorphic this gives finally

$$\Phi(i)=(-2i)^{-\frac12}\,2\pi\phi(i,0)\int\limits_0^\infty e^{-2\pi m r^2}\,r\,dr=\frac{(-2i)^{-\frac12}}{2m}\,\phi(i,0).$$

It remains to show that $\Phi(\tau)$ is holomorphic, or—equivalently—that

$$\frac{\partial}{\partial\bar\tau}\,\Phi(\tau)=\int_{\mathbb{C}}\phi(\tau,z)\,\frac{\partial}{\partial\bar\tau}\,g(\tau,z)\,dx\,dy \tag{5}$$

vanishes. A simple calculation gives

$$\frac{\partial}{\partial\bar\tau}\,g(\tau,z)=-\frac{1}{2}\,\frac{\partial}{\partial\bar z}\left\{\left(\frac{\bar z-z_0}{\bar\tau-\tau_0}+\frac{\bar z-z}{\bar\tau-\tau}\right)g(\tau,z)\right\}.$$

Introducing this in (5), taking into account that $\phi(\tau,z)$ is holomorphic and applying Stokes's Theorem, one obtains

$$\frac{\partial}{\partial\bar\tau}\,\Phi(\tau)=\frac{i}{4}\lim_{r\to\infty}\int_{|z|=r}\phi(\tau,z)\left(\frac{\bar z-z_0}{\bar\tau-\tau_0}+\frac{\bar z-z}{\bar\tau-\tau}\right)g(\tau,z)\,dz.$$

But the right hand side of this is obviously equal to zero by the boundedness condition for ϕ and since $g(\tau,z)=O(e^{-\varepsilon|z|^2}\,e^{-2\pi m y^2/v})$ (as function of z for fixed τ and $|z|\to\infty$) for a suitable $\varepsilon>0$. This completes the proof of the lemma.

Proposition 2. 2. *Let $k\geqq 4$. For $(\tau,z),(\tau_0,z_0)\in\mathfrak{H}\times\mathbb{C}$ define*

$$K(\tau,z;\tau_0,z_0):=\lambda_{k,m}\sum_{\xi\in\Gamma^J}(h_{k,m}|^{(1)}_{k,m}\,\xi)\,(\tau,z;\tau_0,z_0), \tag{6}$$

where $h_{k,m}$ is a function defined by (1), $\lambda_{k,m}=\dfrac{(2k-3)mi^k}{2^{2-k}\pi}$, *and where* "$|^{(1)}_{k,m}$" *denotes the action of Γ^J with respect to the first pair of variables (τ,z). Then one has*:

(i) *For any $(\tau_0,z_0)\in\mathfrak{H}\times\mathbb{C}$ the series on the right hand side of* (6)*—considered as a series of functions in (τ,z)—is normally convergent on every compact subset of $\mathscr{H}\times\mathbb{C}$.*

(ii) *K is the reproducing kernel function for $S_{k,m}(\Gamma)$ with respect to the Petersson scalar product $\langle\cdot,\cdot\rangle$,* i.e.:

— *for any $(\tau_0,z_0)\in\mathfrak{H}\times\mathbb{C}$ the function $K(\cdot;\tau_0,z_0)$ is a cusp form in $S_{k,m}(\Gamma)$*

— *for any $(\tau_0,z_0)\in\mathfrak{H}\times\mathbb{C}$ and any cusp form $\phi\in S_{k,m}$ one has*

$$\langle\phi,K(\cdot,\tau_0,z_0)\rangle=\phi(\tau_0,z_0). \tag{7}$$

Proof. The proposition is an immediate consequence of the foregoing lemma. Indeed, to prove (i), let K be a compact subset of $\mathfrak{H}\times\mathbb{C}$. We have to show that on K the series is majorized by a series whose terms are independent of τ and z. Choose a compact subset $K'\subseteqq\mathfrak{H}\times\mathbb{C}$ such that each point of K is an interior point of K', and by standard function theory there exists a constant c_1, such that for any holomorphic function f on $\mathfrak{H}\times\mathbb{C}$ and all $(\tau,z)\in K$ one has

$$|f(\tau,z)|^2\leqq c_1\int_{K'}|f(\tau',z')|^2\,du'\,dv'\,dx'\,dy',\ (\tau'=u'+iv',\ z'=x'+y',\ u',v',x',y'\in\mathbb{R}),$$

and hence

$$|f(\tau, z)|^2 \leqq \frac{c_1}{c_2} \int_{K'} |f(\tau', z')|^2 \, \mu_{k,m}(\tau', z') \, dV(\tau', z'),$$

where c_2 denotes the minimum value of $\mu_{k,m}(\tau', z') v'^{-3}$ on K'. Applying the last inequality to $f(\tau, z) = (h_{k,m}|_{k,m}^{(1)} \xi)(\tau, z; \tau_0, z_0)^{1/2}$ yields

$$\begin{aligned} &\sum_{\xi \in \Gamma^J} |(h_{k,m}|_{k,m}^{(1)} \xi)(\tau, z; \tau_0, z_0)| \\ (8) \qquad &\leqq \frac{c_1}{c_2} \sum_{\xi \in \Gamma^J} \int_{K'} |(h_{k,m}|_{k,m}^{(1)} \xi)(\tau', z'; \tau_0, z_0)| \cdot \mu_{k,m}(\tau', z') \, dV(\tau', z') \\ &= \frac{c_1}{c_2} \sum_{\xi \in \Gamma^J} \int_{\xi \cdot K'} |h_{k,m}(\tau', z', \tau_0, z_0)| \, \mu_{k,m}(\tau', z') \, dV(\tau', z'). \end{aligned}$$

But this last series is convergent: since K' is compact it can be cut into a finite number—say n— of pieces, each of which is contained in a fundamental domain for $\Gamma^J \backslash \mathfrak{H} \times \mathbb{C}$; hence the series in the right hand side of (8) can be estimated by

$$n \cdot \int_{\mathfrak{H} \times \mathbb{C}} |h_{k,m}(\tau', z'; \tau_0, z_0)| \mu_{k,m}(\tau', z') \, dV(\tau', z'),$$

which is finite by Lemma 2. 1 (i). This proves part (i). It now follows that for fixed $(\tau_0, z_0) \in \mathfrak{H} \times \mathbb{C}$ the function $K(\cdot; \tau_0, z_0)$ is holomorphic on $\mathfrak{H} \times \mathbb{C}$, and from the definition it is clear that it transforms like a Jacobi form in $J_{k,m}(\Gamma)$. That it is even a cusp form is easily deduced from

$$\begin{aligned} &\lambda_{k,m}^{-1} \int_{\Gamma^J \backslash \mathfrak{H} \times \mathbb{C}} |K(\tau, z; \tau_0, z_0)| \mu_{k,m}(\tau, z) \, dV \\ &\qquad \leqq \int_{\Gamma^J \backslash \mathfrak{H} \times \mathbb{C}} \sum_{\xi \in \Gamma^J} |(h_{k,m}|_{k,m}^{(1)} \xi)(\tau, z; \tau_0, z_0)| \cdot \mu_{k,m}(\tau, z) \, dV \\ &\qquad = \int_{\mathfrak{H} \times \mathbb{C}} |h_{k,m}(\tau, z; \tau_0, z_0)| \mu_{k,m}(\tau, z) \, dV < \infty. \end{aligned}$$

Also, by the same procedure of "unfolding the integral" and by Lemma 2. 1 (ii) one deduces (7).

Since the reproducing kernel function K for $S_{k,m}$ can also be written in the form

$$K(\tau, z; \tau_0, z_0) = \sum_i \phi_i(\tau, z) \overline{\phi_i(\tau_0, z_0)},$$

where ϕ_i runs through any orthonormal basis of $S_{k,m}(\Gamma)$, it is clear that the trace of an operator $H_{k,m,\Gamma}(\Delta)$ on $S_{k,m}(\Gamma)$ is given by the integral of

$$\sum_{\eta \in \Gamma^J \backslash \Delta} (K|_{k,m}^{(1)} \eta)(\tau, z; \tau, z) \, \mu_{k,m}(\tau, z)^2$$

over $\Gamma^J \backslash \mathfrak{H} \times \mathbb{C}$ with respect to dV, i.e.:

Corollary 2.3. *Let $k \geqq 4$ and the notation as in Theorem 1 and the following proposition. Then one has:*

$$\operatorname{tr}(H_{k,m,\Gamma}(\Delta), S_{k,m}(\Gamma)) = \lambda_{k,m} \int_{\Gamma^J \backslash \mathfrak{H} \times \mathbb{C}} \sum_{\xi \in \Delta} (h_{k,m}|_{k,m}^{(1)} \xi)(\tau, z; \tau, z)\, \mu_{k,m}(\tau, z)^2\, dV. \tag{9}$$

We must now evaluate the integral in (9). First of all we rewrite (9) in the form

$$\begin{aligned}\operatorname{tr}(H_{k,m,\Gamma}(\Delta), S_{k,m}(\Gamma)) = \lambda_{k,m} \cdot n \int_{\Gamma \backslash \mathfrak{H}} \Bigg\{ \sum_{A \in P(\Delta)} \int_{\mathbb{Z}\tau + \mathbb{Z} \backslash \mathbb{C}} \sum_{\xi \in \mathbb{Z}^2 \backslash P^{-1}(A) \cap \Delta} \\ \cdot \sum_{\eta \in \mathbb{Z}^2} (h_{k,m}|^{(1)} \eta \xi)(\tau, z; \tau, z)\, \mu_{k,m}(\tau, z)^2 \frac{dx\, dy}{v} \Bigg\} \frac{du\, dv}{v^2},\end{aligned} \tag{10}$$

where, as in the notation of Theorem 1, P denotes the canonical projection from the Jacobi group onto $SL_2(\mathbb{R})$, and where $n = |\Gamma \cap \{\pm 1\}|^{-1}$ (the factor n comes in since Γ^J identifies (τ, z) and $(\tau, -z)$ if $-1 \in \Gamma$).

Using the formula

$$\sum_{\mu \in \mathbb{Z}} e\left(\frac{-(z+\mu)^2}{\tau}\right) = \left(\frac{\tau}{2i}\right)^{1/2} \sum_{r \in \mathbb{Z}} e\left(\frac{\tau}{4} r^2 + z \cdot r\right) \qquad (\tau \in \mathfrak{H},\ z \in \mathbb{C}),$$

one easily verifies that

$$\begin{aligned}\sum_{\eta \in \mathbb{Z}^2} (h_{k,m}|_{k,m}^{(1)} \eta)(\tau, z; \tau_0, z_0) &= (\tau - \bar\tau_0)^{-k} \sum_{\lambda, \mu \in \mathbb{Z}} e^m\left(\frac{-(z + \lambda\tau + \mu - \bar z_0)^2}{\tau - \bar\tau_0} + \lambda^2 \tau + 2\lambda z\right) \\ &= (\tau - \bar\tau_0)^{-k} \sum_{\lambda \in \mathbb{Z}} \left(\frac{\tau - \bar\tau_0}{2mi}\right)^{1/2} \sum_{r \in \mathbb{Z}} e\left(\frac{\tau - \bar\tau_0}{4m} r^2 + (z + \lambda\tau - \bar z_0) r + m(\lambda^2 \tau + 2\lambda z)\right) \\ &= (2mi)^{-1/2} (\tau - \bar\tau_0)^{1/2-k} \sum_{\lambda, r \in \mathbb{Z}} e\left(\frac{\tau}{4m} (r + 2m\lambda)^2 + z(r + 2m\lambda)\right) e\left(\frac{-\bar\tau_0}{4m} r^2 - \bar z_0 r\right) \\ &= (2mi)^{-1/2} (\tau - \bar\tau_0)^{1/2-k} \sum_{\varrho=1}^{2m} \theta_{m,\varrho}(\tau, z)\, \overline{\theta_{m,\varrho}(\tau_0, z_0)}.\end{aligned} \tag{11}$$

Thus the inner integral in (10) may be written as

$$(2mi)^{-\frac{1}{2}} (A\tau - \bar\tau)^{\frac{1}{2}-k} (c\tau + d)^{\frac{1}{2}-k} \int_{\mathbb{Z}\tau + \mathbb{Z} \backslash \mathbb{C}} \sum_{\xi \in \mathbb{Z}^2 \backslash P^{-1}(A) \cap \Delta} \sum_{\varrho=1}^{2m} (\theta_{m,\varrho}|_{\frac{1}{2},m} \xi)(\tau, z) \cdot \overline{\theta_{m,\varrho}(\tau, z)}\, \mu_{k,m}(\tau, z)^2 \frac{dx\, dy}{v},$$

or — in terms of the operators $U_m(\xi)$ introduced in § 1, (3) — as

$$(2mi)^{-\frac{1}{2}} (A\tau - \bar\tau)^{\frac{1}{2}-k} (c\tau + d)^{\frac{1}{2}-k} \int_{\mathbb{Z}\tau + \mathbb{Z} \backslash \mathbb{C}} \sum_{\xi \in \mathbb{Z}^2 \backslash P^{-1}(A) \cap \Delta / \mathbb{Z}^2} |\mathbb{Z}^2 \backslash \mathbb{Z}^2 \xi \mathbb{Z}^2| \cdot \sum_{\varrho=1}^{2m} (\theta_{m,\varrho}|U_m(\xi))(\tau, z)\, \overline{\theta_{m,\varrho}(\tau, z)}\, \mu_{k,m}(\tau, z)^2 \frac{dx\, dy}{v}, \tag{12}$$

c, d denoting the lower entries of A.

By the formula

$$\int_{\mathbb{Z}\tau+\mathbb{Z}\backslash\mathbb{C}} \theta_{m,\varrho}(\tau,z)\,\overline{\theta_{m,\sigma}(\tau,z)}\,\mu_{k,m}(\tau,z)^2\,\frac{dx\,dy}{v}=\delta_{\varrho,\sigma}(4m)^{-\frac{1}{2}}v^{k-\frac{1}{2}}$$

(here $\varrho,\sigma\in\mathbb{Z}/2m\mathbb{Z}$ and $\delta_{\varrho,\sigma}$ denotes the Kronecker δ; for the simple proof cf. [E-Z]) we now find that (12) is nothing else but

$$\frac{i^{-\frac{1}{2}}}{2^{\frac{3}{2}}m}\,\frac{v^{k-\frac{1}{2}}}{(A\tau-\bar{\tau})^{k-\frac{1}{2}}(c\tau+d)^{k-\frac{1}{2}}}\sum_{\xi\in\mathbb{Z}^2\backslash P^{-1}(A)\cap\Delta/\mathbb{Z}^2}|\mathbb{Z}^2\backslash\mathbb{Z}^2\xi\mathbb{Z}^2|\operatorname{tr}(U_m(\xi)).$$

Inserting the last formula in (10), we thus arrive at

$$\text{(13)}\qquad \operatorname{tr}(H_{k,m,\Gamma}(\Delta),S_{k,m}(\Gamma))=\lambda_k\cdot n\cdot\int_{\Gamma\backslash\mathfrak{H}}\sum_{A\in P(\Delta)}f_A(\tau)\,g_m(\Delta,A)\,\frac{du\,dv}{v^2},$$

where $g_m(\Delta,A)$ is the expression explained in the notation of Theorem 1, where

$$f_A(\tau)=\varepsilon(A)\,\frac{v^{k-\frac{1}{2}}}{(A\tau-\bar{\tau})^{k-\frac{1}{2}}(c\tau+d)^{k-\frac{1}{2}}}$$

with $\varepsilon(A)$ as defined in § 1, (5), and

$$\lambda_k=\frac{(2k-3)\,i^{k-\frac{1}{2}}}{2^{\frac{7}{2}-k}\pi}.$$

Using Proposition A. 1 it is easily verified that

$$f_A(M\cdot\tau)=f_{M^{-1}AM}(\tau),\qquad (A,M\in SL_2(\mathbb{R})).$$

Also, by Lemma 1. 2, one knows that $g_m(\Delta,A)$ only depends on the $\sim_{m,\Gamma}$-equivalence class of A, where "$\sim_{m,\Gamma}$" is the relation explained in Theorem 1. Therefore— disregarding questions of convergence — we may write the integral in (13) in the notations of Theorem 1 (and with $\mathscr{R}$ denoting a set of representatives for $P(\Delta)$ mod. $\sim_{m,\Gamma}$) as

$$\begin{aligned}&\sum_{A\in\mathscr{R}}g_m(\Delta,A)\int_{\Gamma\backslash\mathfrak{H}}\sum_{\substack{B\\ B\sim_{m,\Gamma}A}}f_B(\tau)\,\frac{du\,dv}{v^2}\\ &=\sum_{A\in\mathscr{R}_{np}}g_m(\Delta,A)\int_{\Gamma\backslash\mathfrak{H}}\sum_{M\in\Gamma_A\backslash\Gamma}f_{M^{-1}AM}(\tau)\,\frac{du\,dv}{v^2}\\ &+\sum_{A\in\mathscr{R}_p}g_m(\Delta,A)\int_{\Gamma\backslash\mathfrak{H}}\sum_{M\in\Gamma_A\backslash\Gamma}\sum_{B\in\Gamma_A(4m)}f_{M^{-1}BAM}(\tau)\,\frac{du\,dv}{v^2},\end{aligned}$$

where $\mathscr{R}_{np}$ and $\mathscr{R}_p$ denote the subsets of non-parabolic and parabolic elements of $\mathscr{R}$ respectively.

Now an integral of the form $\int_{\Gamma\backslash\mathfrak{H}}\sum_{M\in\Gamma_A\backslash\Gamma}f_{M^{-1}AM}(\tau)\,\frac{du\,dv}{v^2}$ can be written as

$$\int_{\Gamma\backslash\mathfrak{H}}\sum_{M\in\Gamma_A\backslash\Gamma}f_A(M\tau)\,\frac{du\,dv}{v^2}=\int_{\Gamma_A\backslash\mathfrak{H}}f_A(\tau)\,\frac{du\,dv}{v^2},$$

and similarly for the integral belonging to parabolic A. Hence we find

$$\begin{aligned}
&\operatorname{tr}(H_{k,m,\Gamma}(\Delta), S_{k,m}(\Gamma)) \\
(14) \qquad &= \sum_{A \in \mathfrak{R}_{np}} \left\{ \lambda_k \cdot n \cdot \int_{\Gamma_A \backslash \mathfrak{H}} f_A(\tau) \frac{du\,dv}{v^2} \right\} g_m(\Delta, A) \\
&+ \sum_{A \in \mathfrak{R}_p} \left\{ \lambda_k \cdot n \cdot \int_{\Gamma_A \backslash \mathfrak{H}} \sum_{B \in \Gamma_A(4m)} f_{BA}(\tau) \frac{du\,dv}{v^2} \right\} g_m(\Delta, A).
\end{aligned}$$

Unfortunately, the integrals appearing in (14) diverge for split-hyperbolic and parabolic A. To compensate for this, one has to interpret the corresponding integrals $\int_{\Gamma_A \backslash \mathfrak{H}}$ in the sense $\lim_{\varepsilon \to 0} \int_{\Gamma_A \backslash (\mathfrak{H} - C(\varepsilon))}$, where $C(\varepsilon)$ is the interior of the (one or two) horocycles of radius ε tangent to the (one or two) fixed points of A (if one fixed point of A equals ∞, then "interior of the horocycle of radius ε tangent to ∞" means the region $\left\{\tau \in \mathfrak{H} \,|\, \operatorname{Im}\tau > \frac{1}{\varepsilon}\right\}$). It can then be shown that the integrals in (14) make sense, and also the above deduction of (14) from (13) can be justified. For details of this, the reader is referred to [Z] or [O], where integrals exactly as in (13) were treated (the suspicious reader will also need some estimate for the number $g_m(\Delta, A)$: this is easily provided by the explicit formulae in § 4).

Finally, the reader will easily verify (or cf. [O]) that the expressions in brackets in (14) equal the expressions $I_{k,m,\Gamma}(A)$ as stated in Theorem 1.

Thus, in the case $k \geqq 4$, Theorem 1 is proved.

Remark. The reader who is acquainted with the Eichler-Selberg trace formula for modular forms of one variable will have noticed that in the last part of the foregoing proof of Theorem 1 there has been a strong trend towards modular forms of half integral weight. The starting point of this was (11), i.e. to write the kernel function K in the form

$$(15) \qquad K(\tau, z; \tau_0, z_0) = \frac{(2k-3)\, i^{k-\frac{1}{2}} \sqrt{m}}{2^{\frac{5}{2}-k}\, \pi} \sum_{A \in \Gamma} (\Theta|_{k,m}^{(1)} A)(\tau, z; \tau_0, z_0),$$

where

$$\Theta(\tau, z; \tau_0, z_0) = (\tau - \bar{\tau}_0)^{\frac{1}{2}-k} \sum_{\varrho=1}^{2m} \theta_{m,\varrho}(\tau, z)\, \overline{\theta_{m,\varrho}(\tau_0, z_0)}.$$

This is the pendant to the fact that each Jacobi form ϕ can be expanded as

$$\phi(\tau, z) = \sum_{\varrho=1}^{2m} h_\varrho(\tau)\, \theta_{m,\varrho}(\tau, z).$$

Indeed, starting with K as defined in (15), it is not hard to verify directly that K is a reproducing kernel function for $S_{k,m}(\Gamma)$. Also, (15) makes sense even for $k=3$.

In principle, this is the point of view that we shall adopt in the next paragraph; but we shall even go a step further and shall reduce the proof of Theorem 1 directly to the Eichler-Selberg trace formula for modular forms of half integral weight.

Nevertheless, we think that the presentation of this paragraph has its own rights, since it shows the possibility of a completely self-contained proof of Theorem 1 without passing to modular forms of half integral weight. Indeed, to proceed more systematically, it would have been possible to compute the inner integral in (10) without referring to the $\theta_{m,\varrho}$ and the operators $U_m(\xi)$. However, the result and the procedure would essentially have been the same as in § 4 below.

§ 3. Second method: Reduction to half-integral weight

In order to derive a formula for the trace of $H_{k,m,\Gamma}(\Delta)$ we can as well consider $H_{k,m,\Sigma}(\Delta)$ where Σ is any subgroup of finite index in Γ (cf. § 1, (2)). But if we choose $\Sigma \subseteqq \Gamma(4m)$, say $\Sigma = \Gamma \cap \Gamma(4m)$, then we have the canonical isomorphism

$$J_{k,m}(\Sigma) \approx M_{k-1/2}(\Sigma) \otimes Th_m$$

as explained in § 0. It is reasonable to expect that via this isomorphism $H_{k,m,\Sigma}(\Delta)$ can be written in terms of double coset operators on $M_{k-1/2}(\Sigma)$ and operators on Th_m. This turns out to be true, and hence we may apply the Eichler-Selberg trace formula for double coset operators on modular forms of half integral weight to derive the desired trace formula for $H_{k,m,\Gamma}(\Delta)$. This is the idea of the proof that we shall now sketch.

By $\widetilde{SL_2(\mathbb{R})}$ we denote the well-known double cover of $SL_2(\mathbb{R})$, i.e.

$$\widetilde{SL_2(\mathbb{R})} = \left\{ \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, w(\tau) \right) \middle| \begin{array}{l} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}),\ w(\tau) \text{ a holomorphic function} \\ \text{on } \mathfrak{H} \text{ satisfying } w(\tau)^2 = c\tau + d \end{array} \right\},$$

equipped with the composition law

$$(A, w(\tau)) \cdot (A', w'(\tau)) = (AA', w(A'\tau)\, w'(\tau)).$$

Let $\Sigma \subseteqq \Gamma(4)$ be a subgroup of finite index and let

$$\Sigma^* := \{(A, j(A, \tau)) \mid A \in \Sigma\}$$

($j(A, \tau)$ the theta automorphy factor as explained in § 0). Then Σ^* is a subgroup of $\widetilde{SL_2(\mathbb{R})}$.

For $A \in SL_2(\mathbb{R})$ denote by $\tilde{A}$ the element $(A, w(\tau)) \in \widetilde{SL_2(\mathbb{R})}$, where

$$w(\tau) = (c\tau + d)^{1/2} \quad \text{for} \quad A = \begin{bmatrix} * & * \\ c & d \end{bmatrix},$$

$(c\tau + d)^{1/2}$ being that square root of $c\tau + d$ with $-\frac{\pi}{2} < \operatorname{Arg}(c\tau + d)^{1/2} \leqq \frac{\pi}{2}$ (cf. § 0).

Let $A \in SL_2(\mathbb{Q})$. Then it is known ([Shi 1], Prop. 1. 1) that the canonical projection $\Sigma^* \tilde{A} \Sigma^* \to \Sigma A \Sigma$ $((A', w(\tau)) \mapsto A')$ is one to one. Thus there is a map

$$(1) \qquad t_A : \Sigma A \Sigma \to \{\pm 1\},$$

such that for all $A' \in \Sigma A \Sigma$ the element $\tilde{A}' \cdot (1, t_A(A'))$ is the inverse image of A' by the canonical projection $\Sigma^* \tilde{A} \Sigma^* \to \Sigma A \Sigma$.

With these notations one then has for any $A \in SL_2(\mathbb{Q})$ the operator $T_{k-\frac{1}{2},\Sigma}(A)$ on $M_{k-\frac{1}{2}}(\Sigma)$ given by

$$(2) \qquad h | T_{k-\frac{1}{2},\Sigma}(A) := \sum_{A' \in \Sigma \backslash \Sigma A \Sigma} t_A(A')\, h|_{k-\frac{1}{2}} A', \quad (h \in M_{k-\frac{1}{2}}(\Sigma)).$$

The operator $T_{k-\frac{1}{2},\Sigma}(A)$ is well-defined and maps $M_{k-\frac{1}{2}}(\Sigma)$ to $M_{k-\frac{1}{2}}(\Sigma)$ and cusp forms to cusp forms (see e.g. [Shi]).

Lemma 3. 1. *Let the notations be as in Theorem* 1. *Let Σ be a subgroup of finite index in $\Gamma \cap \Gamma(4m)$. Then, via the identification $J_{k,m}(\Sigma) \approx M_{k-\frac{1}{2}}(\Sigma) \otimes Th_m$, one has*

$$H_{k,m,\Gamma}(\Delta) = \sum_{A \in \Sigma \backslash P(\Delta)/\Sigma} T_{k-\frac{1}{2},\Sigma}(A) \otimes \left\{ \sum_{\xi \in \mathbb{Z}^2 \backslash P^{-1}(A) \cap \Delta / \mathbb{Z}^2} |\mathbb{Z}^2 \backslash \mathbb{Z}^2 \xi \mathbb{Z}^2| U_m(\xi) \right\}.$$

Proof. Let $h \in M_{k-\frac{1}{2}}(\Sigma)$, $\theta \in Th_m$. Writing out the definition of $H_{k,m,\Sigma}(\Delta)$ one finds

$$(3) \qquad \begin{aligned} (h \otimes \theta) | H_{k,m,\Sigma}(\Delta) &= \sum_{\xi \in \Sigma^J \backslash \Delta} (h \otimes \theta)|_{k,m} \xi \\ &= \sum_{A \in \Sigma \backslash P(\Delta)/\Sigma} \sum_{A' \in \Sigma \backslash \Sigma A \Sigma} \sum_{\xi \in \mathbb{Z}^2 \backslash P^{-1}(A') \cap \Delta} (h \otimes \theta)|_{k,m} \xi, \end{aligned}$$

where we have used once more the fact that a complete set of representatives for the Σ^J-left cosets in a fibre $\gamma_*^{-1}(\Sigma B)$ of the canonical map $\gamma_* : \Sigma^J \backslash \Delta \to \Sigma \backslash P(\Delta)$ is given by a complete set of representatives for $\mathbb{Z}^2 \backslash P^{-1}(B) \cap \Delta$ (cf. the proof of Proposition 1. 1).

The inner sum on the right hand side of (3) may be written in terms of the operators $U_m(\xi)$ as

$$h|_{k-\frac{1}{2}} A' \otimes \left\{ \sum_{\xi \in \mathbb{Z}^2 \backslash P^{-1}(A') \cap \Delta / \mathbb{Z}^2} |\mathbb{Z}^2 \backslash \mathbb{Z}^2 \xi \mathbb{Z}^2|\ \theta |\ U_m(\xi) \right\}.$$

Let $A' = G_1 A G_2$ for suitable $G_1, G_2 \in \Sigma$. Then

(4)

$$\sum_{\xi \in \mathbb{Z}^2 \backslash P^{-1}(A') \cap \Delta / \mathbb{Z}^2} |\mathbb{Z}^2 \backslash \mathbb{Z}^2 \xi \mathbb{Z}^2| U_m(\xi) = \sum_{\xi \in \mathbb{Z}^2 \backslash P^{-1}(A) \cap \Delta / \mathbb{Z}^2} |\mathbb{Z}^2 \backslash \mathbb{Z}^2 \xi \mathbb{Z}^2| U_m(G_1 \xi G_2).$$

Now, using that $\theta|_{\frac{1}{2},m} G = \frac{j(G,\tau)}{(c\tau+d)^{\frac{1}{2}}}\,\theta$ for all $G = \begin{bmatrix} * & * \\ c & d \end{bmatrix} \in \Gamma(4m) \supseteqq \Sigma$, one easily verifies that for a ξ on the right hand side of (4) one has

$$U_m(G_1 \xi G_2) = t_A(A')\, U_m(\xi).$$

Thus the inner sum on the right hand side of (3) equals

$$t_A(A')\, h|_{k-\frac{1}{2}} A' \otimes \sum_{\xi \in \mathbb{Z}^2 \backslash P^{-1}(A) \cap \Delta / \mathbb{Z}^2} |\mathbb{Z}^2 \backslash \mathbb{Z}^2 \xi \mathbb{Z}^2|\; \theta | U_m(\xi),$$

and in view of the definition of $T_{k-\frac{1}{2},\Sigma}(A)$ (in (2)) this is exactly what we had to prove.

Equation (2) of § 1 combined with Lemma 3. 1 now immediately implies:

Corollary 3. 2. *Let the notations as in Theorem* 1 *and let Σ be any subgroup of finite index in $\Gamma \cap \Gamma(4m)$. Then, for any integer k,*

(5)

$$\operatorname{tr}(H_{k,m,\Gamma}(\Delta), S_{k,m}(\Gamma)) = |\Sigma \backslash \Gamma|^{-1} \sum_{A \in \Sigma \backslash P(\Delta)/\Sigma} \varepsilon(A)\, \operatorname{tr}(T_{k-\frac{1}{2},\Sigma}(A), S_{k-\frac{1}{2}}(\Sigma)) \cdot g_m(\Delta, A),$$

where $T_{k-\frac{1}{2},\Sigma}(A)$ is the operator defined by (2) *and $\varepsilon(A)$ as in* § 1, (5).

We shall now apply the Eichler-Selberg trace formula for the traces of the operators $T_{k-\frac{1}{2},\Sigma}(A)$ occurring in (5) (cf. [Shi], or for $k \geqq 3$, [O]). Using our notations it reads

$$\operatorname{tr}(T_{k-\frac{1}{2},\Sigma}(A), S_{k-\frac{1}{2}}(\Sigma)) - \overline{\operatorname{tr}(T_{\frac{5}{2}-k,\Sigma}(A), M_{\frac{5}{2}-k}(\Sigma))} \tag{6}$$
$$= \sum_{A' \in \Sigma A \Sigma / \sim_{m,\Sigma}} I_{k,m,\Sigma}(A')\, t_A(A')\, \varepsilon(A')$$

Here $t_A(A')$ is as in (1), $\varepsilon(A')$ as in § 1, (5), $I_{k,m,\Sigma}(A')$, "$\sim_{m,\Sigma}$" as explained in Theorem 1, and one has to sum over a set of representatives A' for the "$\sim_{m,\Sigma}$"-equivalence classes of $\Sigma A \Sigma$.

Note that we have not quoted the Eichler-Selberg trace formula in exactly the form given in [Shi] or [O]. Apart from some obvious simplifications coming from the assumption $\Sigma \subseteqq \Gamma(4m)$ (which also implies that our "$\sim_{m,\Sigma}$" coincides with the notation of "Σ-equivalence" in the sense of the Eichler-Selberg trace formula) we have made use of our Proposition A. 1. Using these remarks the reader will find it easy to identify (6) with the formulas given in the literature.

Substituting (6) into (5) and using.

$$\varepsilon(A)\, g_m(\Delta, A) = \varepsilon(A')\, t_A(A')\, g_m(\Delta, A'), \qquad (A' \in \Sigma A \Sigma)$$

(cf. the proof of Lemma 3. 1) we obtain

$$\operatorname{tr}(H_{k,m,\Gamma}(\Delta)\, S_{k,m}(\Gamma)) - |\Sigma\backslash\Gamma|^{-1} \sum_{A\in\Sigma\backslash P(\Delta)/\Sigma} \varepsilon(A)\, \overline{\operatorname{tr}(T_{\frac{5}{2}-k,\Sigma}(A), M_{\frac{5}{2}-k}(\Sigma))} \cdot g_m(\Delta, A)$$

$$(7)\qquad = \sum_{A'\in P(\Delta)/\sim_{m,\Sigma}} \{|\Sigma\backslash\Gamma|^{-1}\, I_{k,m,\Sigma}(A')\}\, g_m(\Delta, A')$$

Theorem 1 is now a consequence of (7) by noticing that $M_{\frac{5}{2}-k}(\Sigma)=0$ for $k\geqq 3$, that $g_m(\Delta, A)$ only depends on the $\sim_{m,\Gamma}$-equivalence class of A, and by the following

Lemma 3. 3. *Let Σ be a subgroup of finite index in Γ, and $A\in SL_2(\mathbb{Q})$. Then*

$$(8)\qquad |\Sigma\backslash\Gamma|^{-1} \sum_{\substack{A'\in SL_2(\mathbb{Q})/\sim_{m,\Sigma}\\ A'\sim_{m,\Gamma} A}} I_{k,m,\Sigma}(A') = I_{k,m,\Gamma}(A).$$

The easy proof of this is left to the reader (for non-parabolic A the equation (8) is almost obvious; for parabolic A one needs the identity $\sum_{t (\mathrm{mod}\, n)} c\left(\frac{z+t}{n}\right) = n c(z)$ for $z\in\mathbb{C}$, $n\in\mathbb{N}$, where $c(z)$ denotes $\cot\pi z$ if $z\notin\mathbb{Z}$, 0 if $z\in\mathbb{Z}$).

We observe that the preceding arguments also give formulas for $k=1$ and $k=2$, in particular (specializing (5) to $k=1$)

$$(9)\qquad \operatorname{tr}(H_{1,m,\Gamma}(\Delta), S_{1,m}(\Gamma)) = |\Sigma\backslash\Gamma|^{-1} \sum_{A\in\Sigma\backslash P(\Delta)/\Sigma} \varepsilon(A)\operatorname{tr}(T_{1/2,\Sigma}(A), S_{1/2}(\Sigma))\, g_m(\Delta, A)$$

and (inserting (7) for $k=2$ in (5) and applying Lemma 3. 3)

$$\operatorname{tr}(H_{2,m,\Gamma}(\Delta), S_{2,m}(\Gamma)) = (\text{Same formula as in Theorem 1 with } k=2)$$

$$(10)\qquad + |\Sigma\backslash\Gamma|^{-1} \sum_{A\in\Sigma\backslash P(\Delta)/\Sigma} \varepsilon(A)\, \overline{\operatorname{tr}(T_{1/2,\Sigma}(A), M_{1/2}(\Sigma))}\, g_m(\Delta, A),$$

where Σ again denotes any subgroup of finite index in $\Gamma\cap\Gamma(4m)$.

If Γ is a congruence subgroup of $SL_2(\mathbb{Z})$, then one can choose Σ to be a congruence subgroup also, and then one can apply the theorem of [S-S], which gives an explicit description of the spaces $M_{1/2}(\Sigma)$, to obtain from (9) and (10) "ready-to-compute" formulas for $\operatorname{tr}(H_{k,m,\Gamma}(\Delta), S_{k,m}(\Gamma))$ with $k=1$ or 2.

Finally, we mention an interpretation of the correction term in (10) which is helpful in explicit calculations (cf. [S-Z]). Denote by $J^*_{1,m}(\Gamma)$ the subspace of functions $\phi(\tau, z)$ in

$$\overline{M_{1/2}(\Gamma\cap\Gamma(4m))} \otimes Th_m$$

(here $\overline{M_{1/2}}$ is the space of complex conjugates of forms in $M_{1/2}$) satisfying $\phi|^*_{1,m}\xi = \phi$ for all $\xi\in\Gamma^J$, where the action $|^*_{1,m}$ is defined like $|_{k,m}$ but with $(c\tau+d)^{-k}$ replaced by $|c\tau+d|^{-1}$ (i.e., $|^*_{1,m}$ is $\overline{|_{1/2}} \otimes |_{1/2,m}$), and define an operator $H^*_{1,m}(\Delta)$ on $J^*_{1,m}(\Gamma)$ by

$$\phi \to \sum_{\Gamma^J\backslash\Delta} \phi|^*_{1,m}\xi.$$

Then the arguments of this paragraph can be repeated to show that the second term in (10) is the trace of $H^*_{1,m}(\Gamma)$ (we leave the details to the reader), so (10) can also be put in the form

$$\text{(11)}\qquad \begin{aligned}&\operatorname{tr}(H_{2,m,\Gamma}(\Delta),\, S_{2,m}(\Gamma))\\ &= \sum_{A\in P(\Delta)/\sim_{m,\Gamma}} I_{2,m,\Gamma}(A)\, g_m(\Delta, A) + \operatorname{tr}(H^*_{1,m}(\Delta),\, J^*_{1,m}(\Gamma)).\end{aligned}$$

§ 4. Evaluation of $G_m(\xi)$

Throughout this paragraph we shall use the following notation: if $f(x)$ is a periodic function on $\mathbb{Z}^r$, then $\mathcal{A}v_x f(x)$ denotes the average value of f, i.e.

$$\begin{aligned}\mathcal{A}v_x f(x) &= \lim_{N\to\infty}\Bigg(\sum_{\substack{x\in\mathbb{Z}^r\\|x|\leqq N}} f(x) \Big/ \sum_{\substack{x\in\mathbb{Z}^r\\|x|\leqq N}} 1\Bigg)\\ &= |L\backslash\mathbb{Z}^r|^{-1} \sum_{x\in L\backslash\mathbb{Z}^r} f(x),\end{aligned}$$

where in the second formula $L\subseteqq\mathbb{Z}^r$ is any lattice such that $f(x+y)=f(x)$ for all $x\in\mathbb{Z}^r$, $y\in L$.

The main result of this paragraph is summarized in the following

Theorem 2. *Let $\xi = A[x_0]\,s\in\mathcal{J}(\mathbb{Q})$, let $t=\operatorname{tr}(A)$ and Q_A the binary quadratic form*

$$Q_A(\lambda,\mu) = b\lambda^2 + (d-a)\lambda\mu - c\mu^2 \quad \left(A = \begin{bmatrix} a & b\\ c & d\end{bmatrix}\right).$$

Let $\varepsilon(Q_A) = -1$ if Q_A is negative definite and $\varepsilon(Q_A) = +1$ otherwise. Then one has the following formula for the trace $G_m(\xi)$ (cf. § 1, (4)):

$$G_m(\xi) = \begin{cases} s^m\cdot \operatorname{sign}(t-2)\,(t-2)^{1/2}\,\mathcal{A}v_x\, e\left(\dfrac{m}{t-2}\, Q_A(x+x_0) + m\begin{vmatrix} x_0\\ x\end{vmatrix}\right) & \text{if } t\neq 2,\\[2ex] s^m\cdot m\,\varepsilon(Q_A)\ \operatorname{sign}(t+2)\,(t+2)^{1/2}\,\mathcal{A}v_x\, e\left(\dfrac{1}{m(t+2)}\, Q_A(x+m x_0) + \begin{vmatrix} x_0\\ x\end{vmatrix}\right) & \text{if } t\neq -2,\end{cases}$$

the two expressions on the right hand side being equal for $t\neq\pm 2$.

(Recall that $(t-2)^{1/2}$ and $(t+2)^{1/2}$ denote those square roots which are positive or have positive imaginary part.)

To make Theorem 2 completely explicit, we need to compute $\mathcal{A}v_x\, e(p(x))$ where $p(x)$ is a polynomial of degree $\leqq 2$ with rational coefficients. As usual with Gauss sums, one can reduce to the case that $p(x)$ is a homogeneous quadratic form. Such a form can be written $p(x)=\frac{1}{M}Q(x)$ where M is a natural number and Q is a binary quadratic form with integer coefficients which is primitive modulo M, i.e.:

$$Q(\lambda,\mu)=a\lambda^2+b\lambda\mu+c\mu^2 \quad \text{with} \quad (a,b,c,M)=1.$$

The formula for $\mathcal{A}v_x\, e(p(x))$ in this case is an easy consequence of the classical calculations of Gauss sums in one variable. The result, which will be proved below, is

Theorem 3. *Let Q be an integral binary quadratic form with discriminant Δ and M be a positive integer such that Q is primitive modulo M. Then*

$$(1) \qquad \mathcal{A}v_x\, e\left(\frac{Q(x)}{M}\right)=\frac{1}{M}\sum_{\Delta_1}\left(\frac{\Delta_1}{A}\right)\left(\frac{\Delta/\Delta_1}{M/(M,\Delta)}\right)\Delta_1^{1/2},$$

where A is any integer represented by Q and prime to M and the sum extends over all integers Δ_1 such that $|\Delta_1|=(M,\Delta)$ and Δ_1, Δ/Δ_1 are both congruent to 0 or 1 mod 4. (Note that the sum has at most two terms.)

The first step towards Theorem 2 is to derive explicit formulas for the operators $U_m(\xi)$ (and to prove that $U_m(\xi)$ actually maps Th_m to Th_m).

Proposition 4.1. *Let $A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in SL_2(\mathbb{Q})$, $x_0=(\lambda_0,\mu_0)\in\mathbb{Q}^2$. Then one has for any $1\leqq\varrho\leqq 2m$:*

$$\theta_{m,\varrho}|U_m(A[x_0])=\sum_{\sigma=1}^{2m}K_{\varrho,\sigma}\theta_{m,\sigma},$$

where

$$(2) \quad K_{\varrho,\sigma}=(2mci)^{-1/2}l^{-2}\sum_{\substack{\sigma'\bmod 2ml\\ \sigma'\equiv\sigma(2m)}}\ \sum_{\substack{\varrho'\bmod 2mcl\\ \varrho'\equiv\varrho(2m)}} e\left(\frac{1}{4mc}\left[a\varrho'^2-2\varrho'(\sigma'-2m\lambda_0)+d(\sigma'-2m\lambda_0)^2\right]+(\sigma'-m\lambda_0)\mu_0\right)$$

if $c\neq 0$ and

$$(3) \quad K_{\varrho,\sigma}=d^{-1/2}l^{-1}\sum_{\substack{\sigma'\bmod 2ml\\ \sigma'\equiv\sigma(2m)\\ ld(\sigma'-2m\lambda_0)\equiv l\varrho(2ml)}} e\left(\frac{bd(\sigma'-2m\lambda_0)^2}{4m}+(\sigma'-m\lambda_0)\mu_0\right)$$

if $c=0$. Here l is any positive integer such that lA and lx_0 have integral entries.

Note that (2) and (3) immediately give formulas for the trace of $U_m(A[x_0])$. Namely in (2) one has to sum over all $\sigma'\bmod 2ml$ in the first sum and over all $\varrho'\bmod 2mcl$ with $\varrho'\equiv\sigma'(2m)$ in the second sum. Writing λ for σ' and summing over $\lambda+2m\mu$, $\mu\bmod cl$ in the second sum one finds

$$(4) \quad \operatorname{tr}U_m(A[x_0])=\left(\frac{2mc}{i}\right)^{1/2}e\left(\frac{md}{c}\lambda_0^2-m\lambda_0\mu_0\right)$$
$$\times\mathcal{A}v_{\lambda,\mu}\, e\left(\frac{t-2}{4mc}\lambda^2+\frac{a-1}{c}\lambda\mu+\frac{ma}{c}\mu^2+\left(\frac{1-d}{c}\lambda_0+\mu_0\right)\lambda+\lambda_0\frac{2m}{c}\mu\right)$$

if $c \neq 0$. A similar formula can be derived in the case $c = 0$; however, we shall need such formulas only in the case $A = -1$ or $\mathrm{tr}(A) = 2$ $\left(\text{i.e. } A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right)$. Here (3) simplifies and one easily sees

(5)
$$\operatorname{tr} U_m\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}[x_0]\right) = \begin{cases} 2m\,\mathscr{A}v_\lambda\, e\left(\dfrac{b(\lambda - 2m\lambda_0)^2}{4m} + (\lambda - m\lambda_0)\,\mu_0\right) & \text{if } \lambda_0 \in \mathbb{Z}, \\ 0 \quad \text{otherwise,} \end{cases}$$
$$\operatorname{tr} U_m((-1)[x_0]) = \begin{cases} -2i & \text{if } \; m x_0 \in \mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Proposition 4.1. By definition of $U_m(A[x_0])$ we have

$$\theta_{m,\varrho} | U_m(A[x_0]) = l^{-2} \sum_{\lambda, \mu \bmod l} \theta_{m,\varrho}|_{1/2, m}(A[x_0]\,[\lambda, \mu]).$$

We consider the case $c \neq 0$.

Observing that $\theta_{m,\varrho}|A$ is invariant with respect to $l\mathbb{Z}^2$, hence invariant by $z + l \longrightarrow z$, we obtain

(6)
$$\theta_{m,\varrho}|A = \sum_{s \in \mathbb{Z}} \zeta^{s/l}\, l^{-1} \int_0^l e\left(-\frac{sx}{l}\right) \cdot (\theta_{m,\varrho}|A)(\tau, x)\, dx.$$

Now

$$e\left(-\frac{sx}{l}\right) \cdot (\theta_{m,\varrho}|A)(\tau, x) = (c\tau + d)^{-1/2} \sum_{r \equiv \varrho(2m)} e(P_r(x)),$$

where

$$P_r(x) = A\tau\, \frac{r^2}{4m} + \frac{xr}{c\tau + d} - \frac{mcx^2}{c\tau + d} - \frac{sx}{l}.$$

Using $A\tau = \dfrac{a}{c} - \dfrac{1}{c(c\tau + d)}$ we find

$$P_r(x) = \tau\, \frac{(s/l)^2}{4m} + \frac{1}{4mc}\,(ar^2 - 2r(s/l) + d(s/l)^2)$$
$$- \frac{mc}{c\tau + d}\left(x - \frac{r}{2mc} + \frac{(s/l)\,(c\tau + d)}{2mc}\right)^2.$$

Inserting this in (6) gives

(7)
$$\theta_{m,\varrho}|A = \sum_{s \in \mathbb{Z}} q^{(s/l)^2/4m} \zeta^{s/l} \sum_{\substack{\varrho' \bmod 2mcl \\ \varrho' \equiv \varrho(2m)}} e\left(\frac{1}{4mc}\,[a\varrho'^2 - 2\varrho'(s/l) + d(s/l)^2]\right) \cdot C(s)$$

with

$$C(s) = (c\tau + d)^{-1/2}\, l^{-1} \int_0^l \sum_{r \equiv \varrho'(2mcl)} e\left(-\frac{mc}{c\tau + d}\left(x - \frac{r}{2mc} + \frac{(s/l)(c\tau + d)}{2mc}\right)^2\right) dx$$
$$= (c\tau + d)^{-1/2}\, l^{-1} \int_{-\infty}^{-\infty} e\left(-\frac{mc}{c\tau + d}\left(x - \frac{\varrho'}{2mc} + \frac{(s/l)\,(c\tau + d)}{2mc}\right)^2\right) dx$$
$$= (2mci)^{-1/2}\, l^{-1}.$$

From (7) we obtain

$$\theta_{m,\varrho}|A[x_0] = \sum_{s\in\mathbb{Z}} q^{(s/l)^2/4m+\lambda_0(s/l)+m\lambda_0^2}\zeta^{s/l+2m\lambda_0} e((s/l+m\lambda_0)\mu_0)$$
$$\times(2mci)^{-1/2}l^{-1}\sum_{\substack{\varrho' \bmod 2mcl\\ \varrho'\equiv\varrho(2m)}} e\left(\frac{1}{4mc}[a\varrho'^2-2\varrho'(s/l)+d(s/l)^2]\right),$$

and hence, replacing s by $s-2ml\lambda_0$,

$$\theta_{m,\varrho}|A[x_0] = \sum_{s\in\mathbb{Z}} q^{(s/l)^2/4m}\zeta^{s/l}\times(2mci)^{-1/2}l^{-1}\sum_{\substack{\varrho' \bmod 2mcl\\ \varrho'\equiv\varrho(2m)}} f(\varrho', s),$$

$$f(\varrho', s) = e\left(\frac{1}{4mc}[a\varrho'^2-2\varrho'(s/l-2m\lambda_0)+d(s/l-2m\lambda_0)^2]+(s/l-m\lambda_0)\mu_0\right).$$

Finally

$$\theta_{m,\varrho}|(A[x_0][\lambda,\mu])$$
$$=\sum_{s\in\mathbb{Z}} q^{(s/l+2m\lambda)^2/4m}\zeta^{s/l+2m\lambda} e\left(\frac{s\mu}{l}\right)\times(2mci)^{-1/2}l^{-1}\sum_{\varrho'}(\ldots),$$

and after summing over $\mu \bmod l$ only the terms with $l|s$ survive.

Thus, replacing s by $l(s-2m\lambda)$, we get finally

$$\theta_{m,\varrho}|U_m(A[x_0]) = \sum_{s\in\mathbb{Z}} q^{s^2/4m}\zeta^s\times(2mci)^{-1/2}l^{-2}\sum_{\lambda \bmod l}\sum_{\substack{\varrho' \bmod 2mcl\\ \varrho'\equiv\varrho(2m)}} f(\varrho', l(s-2m\lambda)).$$

Here the coefficient of $q^{s^2/4m}\zeta^s$ only depends on s modulo $2m$, which shows that $\theta_{m,\varrho}|U_m(A[x_0])$ is an element of Th_m and gives the formula stated in the proposition.

The case $c=0$ is left to the reader.

In order to bring the formula (4) into a more useful form we obviously need some lemmas on Gauss sums. We shall derive these lemmas from the transformation formula in the foregoing proposition.

Proposition 4.2. (i) *Let A, B, C, D, E be rational numbers, $A \neq 0$, and let $\Delta = B^2-4AC$. Then one has*

$$\mathcal{A}v_{\lambda,\mu}\, e(A\lambda^2+B\lambda\mu+C\mu^2+D\lambda+E\mu)$$
(8)
$$=(2Ai)^{1/2} e\left(\frac{-D^2}{4A}\right)\mathcal{A}v_{\lambda,\mu}\, e^{-1}\left(\frac{1}{4A}\lambda^2+\frac{B}{2A}\lambda\mu+\frac{\Delta}{4A}\mu^2+\frac{D}{2A}\lambda+\frac{BD-2AE}{2A}\mu\right).$$

(ii) *Let Q be a rational binary quadratic form with discriminant $\Delta \neq 0$, let $\gamma\in\mathbb{Q}$ and $x_0\in\mathbb{Q}^2$. Then*

$$(9)\qquad \mathcal{A}v_x e\left(Q(x+\gamma x_0)+\begin{vmatrix}x_0\\ x\end{vmatrix}\right)=\varepsilon(Q)\Delta^{1/2}\mathcal{A}v_x e\left(\frac{1}{\Delta}Q(x+x_0)+\gamma\begin{vmatrix}x_0\\ x\end{vmatrix}\right)$$

($\varepsilon(Q)$ *as in Theorem* 1, *i.e.* $\varepsilon(Q)=-1$ *if Q is negative definite and* $\varepsilon(Q)=+1$ *otherwise*).

Proof. (i) Let $x_0=(\lambda_0, \mu_0)\in \mathbb{Q}^2$, $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in SL_2(\mathbb{Q})$ with $a, c\neq 0$, let $S=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, hence $SM=\begin{bmatrix} -c & -d \\ a & b \end{bmatrix}$.

The formula (8) will result from the identity

$$(10)\quad U_m(S)\,U_m(M[x_0])=\varepsilon\, U_m(SM[x_0]),\ \text{where}\ \varepsilon=\left(\frac{a\tau+b}{c\tau+d}\right)^{1/2}\frac{(c\tau+d)^{1/2}}{(a\tau+b)^{1/2}},$$

by applying the transformation formula (2) in the case $m=1$.

Namely, let $\theta_{1,\varrho}|U_m(S)=\sum_{\sigma=1}^{2} K_{\varrho,\sigma}(S)\,\theta_{1,\varrho}$ and similarly $K_{\varrho,\sigma}(M[x_0])$, $K_{\varrho,\sigma}(SM[x_0])$. Then (10) gives

$$\sum_{\sigma=1}^{2} K_{0,\sigma}(S)\,K_{\sigma,0}(M[x_0])=\varepsilon K_{0,0}(SM[x_0]),$$

and writing this out we find (l denotes a positive integer such that lM, lx_0 are integral):

$$\sum_{\sigma=1}^{2}(2i)^{-1/2}(2ci)^{-1/2}l^{-2}\sum_{\substack{\sigma' \bmod 2l\\ \sigma'\equiv 0(2)}}\ \sum_{\substack{\varrho' \bmod 2cl\\ \varrho'\equiv\sigma(2)}} u(\sigma',\varrho')$$

$$=\varepsilon(2ai)^{-1/2}l^{-2}\sum_{\substack{\sigma' \bmod 2l\\ \sigma'\equiv 0(2)}}\ \sum_{\substack{\varrho' \bmod 2al\\ \varrho'\equiv 0(2)}} v(\sigma',\varrho'),$$

$$u(\sigma',\varrho')=e\left(\frac{1}{4c}\left[a\varrho'^2-2\varrho'(\sigma'-2\lambda_0)+d(\sigma'-2\lambda_0)^2\right]+(\sigma'-\lambda_0)\,\mu_0\right),$$

$$v(\sigma',\varrho')=e\left(\frac{1}{4a}\left[-c\varrho'^2-2\varrho'(\sigma'-2\lambda_0)+b(\sigma'-2\lambda_0)^2\right]+(\sigma'-\lambda_0)\,\mu_0\right),$$

i.e. (setting $\sigma'=2\mu$, $\varrho'=\lambda$ on the left and $\sigma'=2\mu$, $\varrho'=2\lambda$ on the right)

$$(11)\quad \begin{aligned}&(2i)^{-1/2}(2ci)^{-1/2}2|c|\; e\left(\frac{d}{c}\lambda_0^2\right)\mathscr{A}v_{\lambda,\mu}\, e\left(\frac{a}{4c}\lambda^2-\frac{1}{c}\lambda\mu+\frac{d}{c}\mu^2+\frac{\lambda_0}{c}\lambda+2\left(-\frac{d}{c}\lambda_0+\mu_0\right)\mu\right)\\ &=\varepsilon(2ai)^{-1/2}|a|\; e\left(\frac{b}{a}\lambda_0^2\right)\mathscr{A}v_{\lambda,\mu}\, e^{-1}\left(\frac{c}{a}\lambda^2+\frac{2}{a}\lambda\mu-\frac{b}{a}\mu^2-\frac{2\lambda_0}{a}\lambda+2\left(\frac{b}{a}\lambda_0-\mu_0\right)\mu\right).\end{aligned}$$

Now assume that $B\neq 0$. Then taking in (11)

$$\begin{matrix} a=-4AB^{-1} & b=\Delta B^{-1} \\ c=-B^{-1} & d=-CB^{-1} \end{matrix},\quad \lambda_0=-DB^{-1},\quad \mu_0=\frac{1}{2}E-CDB^{-1}$$

and checking the factors in front of the Gauss sums in (11) yields (8).

If $B=0$ then choose an integer $N \neq 0$ such that $\frac{N}{4A}$ and $\frac{ND}{2A}$ are integral, write

$$\mathscr{A}v_{\lambda,\mu}\, e(A\lambda^2 + C\mu^2 + D\lambda + E\mu) = \mathscr{A}v_{\lambda,\mu}\, e(A\lambda^2 + N\lambda\mu + C\mu^2 + D\lambda + E\mu)$$

and apply (8) to the right hand side of this.

(ii) Applying (8) two times, where in the second application the $\frac{\Delta}{4A}$ plays the role of the A in (8), yields after a simple calculation the identity (9) for $\gamma=0$ and for all Q with $Q(1,0) \neq 0$. But since the class modulo $GL_2(\mathbb{Z})$ of any quadratic form with $\Delta \neq 0$ contains such a Q, and since it obviously suffices to prove (9) for one element of a given class modulo $GL_2(\mathbb{Z})$ in order to prove it for the whole class, we have proved (9) in the case $\gamma=0$.

If $\gamma \neq 0$, then let $x_1 \in \mathbb{Q}^2$ such that $B(x_0, x) = \begin{vmatrix} x_1 \\ x \end{vmatrix}$ for all x, where B denotes the bilinear form associated to Q. With this x_1 we may write

$$\mathscr{A}v_x\, e\left(Q(x+\lambda x_0) + \begin{vmatrix} x_0 \\ x \end{vmatrix}\right) = e(\gamma^2 Q(x_0)) \times \mathscr{A}v_x\, e\left(Q(x) + \begin{vmatrix} \gamma x_1 + x_0 \\ x \end{vmatrix}\right),$$

and by the case we just have proved this equals

$$\varepsilon(Q)\, \Delta^{1/2}\, e(\gamma^2 Q(x_0)) \times \mathscr{A}v_x\, e\left(\frac{1}{\Delta}\, Q(x+\gamma x_1 + x_0)\right)$$

$$= \varepsilon(Q)\, \Delta^{1/2}\, e\left(\gamma^2 \left(Q(x_0) + \frac{1}{\Delta}\, Q(x_1)\right)\right) \times \mathscr{A}v_x\, e\left(\frac{1}{\Delta}\, Q(x+x_0) + \frac{\gamma}{\Delta}\, B(x_1, x+x_0)\right).$$

Now using the easily proved fact that $B(x_1, x) = \Delta \begin{vmatrix} x_0 \\ x \end{vmatrix}$ for all x one immediately obtains (9).

As an application of the formulae (8), (9) we give the

Proof of Theorem 3. The Gauss sum on the left hand side of (1) depends only on the $SL_2(\mathbb{Z})$-equivalence class of Q and furthermore, using standard Galois theory, it is easily seen that it suffices to consider the case of primitive Q and $Q \geqq 0$ if $\Delta \leqq 0$. Thus we can restrict to $Q(\lambda, \mu) = A\lambda^2 + B\lambda\mu + C\mu^2$ with A relative prime to Δ and $A=1$, $B=C=0$ if $\Delta=0$.

If $\Delta=0$, then by (8)

$$\mathscr{A}v_x\, e\left(\frac{Q(x)}{M}\right) = \mathscr{A}v_\lambda\, e\left(\frac{\lambda^2}{M}\right) = \left(\frac{2i}{M}\right)^{1/2} \mathscr{A}v_\lambda\, e\left(\frac{-M\lambda^2}{4}\right) = \frac{(1+i)(1+i^{-M})}{2\sqrt{M}}$$

which may be written as

$$\mathscr{A}v_x\, e\left(\frac{Q(x)}{M}\right) = \frac{1}{M} \sum_{\substack{\Delta_1 \\ |\Delta_1| = M,\, \Delta_1 \equiv 0,1 \bmod 4}} \Delta_1^{1/2},$$

i.e. in the form (1).

If $\Delta \neq 0$, then by (9)

$$\mathscr{A} v_x\, e\left(\frac{Q(x)}{M}\right) = \frac{\Delta^{1/2}}{M}\, \mathscr{A} v_x\, e\left(\frac{M}{\Delta}\, Q(x)\right).$$

Now $Q(\lambda, \mu)$ is $SL_2(\mathbb{Z})$-equivalent to a form which is congruent modulo Δ to $A\lambda^2$ if Δ is odd and to $A\lambda^2 + \frac{\alpha\Delta}{4}\mu^2$ (α an integer with $\alpha A \equiv -1 \bmod \Delta$) if Δ is even. Hence

$$\mathscr{A} v_x\, e\left(\frac{Q(x)}{M}\right) = \frac{\Delta^{1/2}}{M}\, \mathscr{A} v_\lambda\, e\left(\frac{MA}{\Delta}\lambda^2\right) \cdot \begin{cases} 1 & \text{if } \Delta \text{ is odd}, \\ \mathscr{A} v_\mu\, e\left(\frac{\alpha\Delta}{4}\mu^2\right) & \text{if } \Delta \text{ is even}, \end{cases}$$

i.e.—by the one variable case proved above—

$$\mathscr{A} v_x e\left(\frac{Q(x)}{M}\right) = \frac{\Delta^{1/2}}{M}\, \frac{(M, \Delta)}{|\Delta|} \sum_{\substack{\Delta_0 \\ |\Delta_0| = |\Delta|/(M,\Delta) \\ \Delta_0 \equiv 0,1 \bmod 4}} \left(\frac{\Delta_0}{\varepsilon A}\right)\left(\frac{\Delta_0}{M/(M,\Delta)}\right) \Delta_0^{1/2} \cdot \begin{cases} 1 & \text{if } \Delta \text{ is odd}, \\ \frac{1+i^{\alpha\Delta}}{2} & \text{if } \Delta \text{ is even}, \end{cases}$$

where $\varepsilon = \operatorname{sign}(\Delta)$.

Now it is an easy exercise to check that the last equation can be written in the form (1). (Consider the six cases according as Δ is odd or even and $\Delta/(M, \Delta)$ is odd, exactly divisible by 2 or divisible by 4, and make use of $\left(\frac{\Delta}{A}\right) = 1$.)

We can now complete the proof of Theorem 2.

Proof of Theorem 2. To begin with we consider the case $c \neq 0$, $t \neq 2$ (notations as in the statement of the theorem).

First of all replace λ by $\lambda - m\mu$ in the formula (4) for the trace of $U_m(A[x_0])$ to obtain

$$\begin{aligned} \operatorname{tr} U_m(A[x_0]) = \left(\frac{2mc}{i}\right)^{1/2} & e\left(\frac{md}{c}\lambda_0^2 - m\lambda_0\mu_0\right) \\ \times \mathscr{A} v_{\lambda,\mu}\, e\Bigg(&\frac{t-2}{4mc}\lambda^2 + \frac{a-d}{2c}\lambda\mu + m\frac{t+2}{4c}\mu^2 \\ &+ \left(\frac{1-d}{c}\lambda_0 + \mu_0\right)\lambda + m\left(\frac{d+1}{c}\lambda_0 - \mu_0\right)\mu\Bigg). \end{aligned}$$

Next apply to this the inversion formula (8) which yields

$$\begin{aligned} \operatorname{tr} U_m(A[x_0]) = \left(\frac{2mc}{i}\right)^{1/2} & e\left(\frac{md}{c}\lambda_0^2 - m\lambda_0\mu_0\right) \\ \times \left(\frac{t-2}{2mc}\, i\right)^{1/2} & e\left(\frac{-mc}{t-2}\left[\frac{1-d}{c}\lambda_0 + \mu_0\right]^2\right) \\ \times \mathscr{A} v_{\lambda,\mu}\, e\Bigg(&\frac{-m}{t-2}\Bigg\{c\lambda^2 + (a-d)\lambda\mu - b\mu^2 + 2c\left(\frac{1-d}{c}\lambda_0 + \mu_0\right)\lambda \\ &+ 2c\left[\frac{a-d}{2c}\left(\frac{1-d}{c}\lambda_0 + \mu_0\right) - \frac{t-2}{2mc}\left(\frac{d+1}{c}\lambda_0 - \mu_0\right)\right]\mu\Bigg\}\Bigg). \end{aligned}$$

Some simple calculations show

$$\left(\frac{2mc}{i}\right)^{1/2}\left(\frac{t-2}{2mc}\,i\right)^{1/2}=\varepsilon(A)\,\mathrm{sign}\,(t-2)\,(t-2)^{1/2}$$

($\varepsilon(A)$ as in (1. 5)),

$$\frac{md}{c}\,\lambda_0^2-m\lambda_0\mu_0-\frac{mc}{t-2}\left[\frac{1-d}{c}\,\lambda_0+\mu_0\right]^2=\frac{m}{t-2}\,(b\lambda_0^2+(d-a)\,\lambda_0\mu_0-c\mu_0^2),$$

$$2c\left(\frac{1-d}{c}\,\lambda_0+\mu_0\right)=(a-d)\,\lambda_0+2c\mu_0-(t-2)\,\lambda_0,$$

$$2c\left[\frac{a-d}{2c}\left(\frac{1-d}{c}\,\lambda_0+\mu_0\right)-\frac{t-2}{2mc}\,m\left(\frac{d+1}{c}\,\lambda_0-\mu_0\right)\right]=-2b\lambda_0+[(a-d)+t-2]\,\mu_0.$$

Thus the last expression for $\mathrm{tr}\,U_m(A[x_0])$ equals

$$\mathrm{tr}\;U_m(A[x_0])=\varepsilon(A)\,\mathrm{sign}\,(t-2)\,(t-2)^{1/2}$$
$$\times\,\mathscr{A}v_{\lambda,\mu}\,e\left(\frac{-m}{t-2}\,\{c(\lambda+\mu_0)+(a-d)\,(\lambda+\mu_0)\,(\mu+\lambda_0)-b(\mu+\lambda_0)^2-(t-2)\,(\lambda_0\lambda-\mu_0\mu)\}\right),$$

and using $G_m(\xi)=\varepsilon(A)\,\mathrm{tr}\,U_m(A[x_0]\,s)=s^m\varepsilon(A)\,\mathrm{tr}\,U_m(A[\dot{x}_0])$ we see that we have proved the first equation of Theorem 1—of course, for the present only in the case $c\neq 0$. But $G_m(\xi)$ depends only on the $SL_2(\mathbb{Z})$-conjugacy class of A, the same is obviously true for the expressions given for $G_m(\xi)$ in Theorem 2, and every $A\in SL_2(\mathbb{Q})$, $A\neq\pm 1$ is $SL_2(\mathbb{Z})$-conjugate to a matrix with non-vanishing left lower entry. Thus we deduce that the first equation in Theorem 2 is true for all $A\neq -1$ (and with $\mathrm{tr}\,A\neq 2$).

If $A=-1$ or $\mathrm{tr}\,A=2$, $c=0$, then the corresponding formulas of Theorem 2 are easily verified by using (5), and again we can drop the assumption $c=0$ (because any $SL_2(\mathbb{Z})$-conjugacy class with trace ± 2 contains an upper triangular matrix). Finally, the equality of the two expressions for $G_m(\xi)$ in Theorem 2 follows from (9) of Proposition 4. 2. This completes the proof in all cases.

Appendix: Conjugacy classes in the double cover of $SL_2(\mathbb{R})$

For $A=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in SL_2(\mathbb{R})$ define $J(A,\tau)=(c\tau+d)^{1/2}$. Recall that $SL_2(\mathbb{R})$ consists of all pairs $(A, w(\tau))$, where $A\in SL_2(\mathbb{R})$ and $w(\tau)$ is a holomorphic function on $\mathfrak{H}$ satisfying $w(\tau)^2=J(A,\tau)^2$, i.e. $w(\tau)=\pm J(A,\tau)$, together with the composition law

$$(A,\,w(\tau))\cdot(A',\,w'(\tau))=(A\,A',\,w(A'\tau)\,w'(\tau)).$$

Finally recall the definition (cf. § 1, (5))

$$\varepsilon(A)=\begin{cases}-1 & \text{if } c<0 \text{ and } \mathrm{tr}(A)<2,\\ +1 & \text{otherwise,}\end{cases}\qquad \left(A=\begin{bmatrix} * & *\\ c & *\end{bmatrix}\in SL_2(\mathbb{R})\right).$$

Proposition A.1. *For $\alpha \in \widetilde{SL_2(\mathbb{R})}$ define*

$$\varrho(\alpha) := \varepsilon(A) \cdot s \quad (\alpha = (A, s \cdot J(A, \tau)),\ s = \pm 1).$$

Then $\varrho(\alpha)$ depends only on the $\widetilde{SL_2(\mathbb{R})}$-conjugacy class of α, i.e.

$$(1) \qquad \varrho(\alpha) = \varrho(\mu \alpha \mu^{-1})$$

for all α and μ in $\widetilde{SL_2(\mathbb{R})}$.

Remark. Equation (1) is obviously equivalent to

$$(2) \qquad \varepsilon(A) = \varepsilon(MAM^{-1}) \frac{J(M, AM^{-1}\tau)\, J(A, M^{-1}\tau)\, J(M, M^{-1}\tau)^{-1}}{J(MAM^{-1}, \tau)}$$

$(\alpha = (A, *),\ \mu = (M, *))$. In terms of the cocycle $\sigma(\cdot, \cdot)$, introduced in § 0, (2), this can also be written as

$$(3) \qquad \varepsilon(A)\, \sigma(MAM^{-1}, M) = \varepsilon(MAM^{-1})\, \sigma(M, A).$$

Proof of Proposition A. 1. Let $\mathscr{C}$ be any conjugacy class in $\widetilde{SL_2(\mathbb{R})}$. It suffices to show that $\mathscr{C}$ contains one element $\alpha = (A, *)$ such that (1) is true for all $\mu \in \widetilde{SL_2(\mathbb{R})}$, i.e. that (2) is true for all M.

If $\mathscr{C}$ contains only elliptic elements, then let α be one of these. It is clear (e.g. by continuity) that for any $M \in SL_2(\mathbb{R})$ the signs of the left lower entries of A and MAM^{-1} are equal, i.e. $\varepsilon(A) = \varepsilon(MAM^{-1})$. Thus, for $\tau = M\tau_0$, where τ_0 denotes the fixed point of A in the upper half plane, equation (2) becomes $J(A, \tau_0) = J(MAM^{-1}, M\tau_0)$, and this can again be verified by a simple calculation or by continuity considerations.

If the elements in $\mathscr{C}$ are not elliptic then $\mathscr{C}$ contains an element α where the left lower entry of A vanishes, say $A = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$. For this A equation (2) is easily checked if the left lower entry of M is 0. Otherwise let $M = \begin{bmatrix} * & * \\ u & v \end{bmatrix}$, $MAM^{-1} = \begin{bmatrix} * & * \\ c' & * \end{bmatrix}$ and $\tau = Mit$, $(t \in \mathbb{R})$. Then (2) becomes

$$1 = \varepsilon(MAM^{-1}) \frac{[ua^2 i + (uab + v)/t]^{1/2} [a^{-1}]^{1/2} [ui + v/t]^{-1/2}}{\left[a - \dfrac{c'}{u(uit + v)}\right]^{1/2}}.$$

Considering this for $t \to \infty$ one obtains

$$\varepsilon(MAM^{-1}) = \begin{cases} -1 & \text{if } c' < 0,\ a < 0, \\ +1 & \text{otherwise.} \end{cases}$$

But this is correct since $\operatorname{tr}(MAM^{-1}) - 2 = \operatorname{tr}(A) - 2 = a^{-1}(a-1)^2$.

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