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An elementary proof of the Eichler–Selberg trace formula

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Abstract. We give a purely algebraic proof of the trace formula for Hecke operators on modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, using the action of Hecke operators on the space of period polynomials. This approach, which can also be applied for congruence subgroups, is more elementary than the classical ones using kernel functions, and avoids the analytic difficulties inherent in the latter (especially in weight two). Our main result is an algebraic property of a special Hecke element that involves neither period polynomials nor modular forms, yet immediately implies both the trace formula and the classical Kronecker–Hurwitz class number relation. This key property can be seen as providing a bridge between the conjugacy classes and the right cosets contained in a given double coset of the modular group.

1. Introduction

Our aim in this paper is to give a short, algebraic proof of the trace formula for Hecke operators on modular forms for the full modular group. We use the action of Hecke operators on the space of period polynomials associated to modular forms, bringing to completion an idea introduced by the second author 25 years ago [13]. A completely different (and considerably more complicated) proof based also on the action of Hecke operators on period polynomials was given in [11]. The proof given here depends on purely algebraic properties of a special Hecke element, independent of its action on period polynomials. The same Hecke element has been used by the first author in two sequels to this paper to obtain simple trace formulae on modular forms for congruence subgroups as well [8, 9]. Our approach is more elementary than the classical automorphic kernel method, and applies uniformly in all weights, whereas the classical approach requires additional technicalities in weight two.

Let Γ be the group $\mathrm{PSL}_2(\mathbb{Z})$, which is generated by the two matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $U = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, modulo the relations $S^2 = U^3 = 1$, and let $T = US = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. For $n \geqslant 1$, let $\mathcal{M}_n$ be the set of 2×2 matrices with integer entries of determinant n, modulo $\{\pm 1\}$, and let $\mathcal{R}_n = \mathbb{Q}[\mathcal{M}_n]$. The group Γ acts both on the left and on the right on the $\mathbb{Q}$-vector space $\mathcal{R}_n$.

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It was shown in [3] that an element $\widetilde{T}_n \in \mathcal{R}_n$ acts as the nth Hecke operator on period polynomials if

$$\text{(A)} \qquad (1-S)\widetilde{T}_n - T_n^\infty(1-S) \in (1-T)\mathcal{R}_n,$$

where $T_n^\infty = \sum_{M \in \mathcal{M}_n^\infty} M$, for $\mathcal{M}_n^\infty \subset \mathcal{M}_n$ any set of representatives for $\Gamma\backslash\mathcal{M}_n$ fixing ∞. It was stated in [13] that there exists such an element $\widetilde{T}_n$ which further satisfies the properties

$$\text{(B)} \qquad \begin{cases} \widetilde{T}_n(1+S) \in (1+U+U^2)\mathcal{R}_n, \\ \widetilde{T}_n(1+U+U^2) \in (1+S)\mathcal{R}_n, \end{cases}$$

and shown that any such element would lead to an explicit formula for the traces of Hecke operators on modular forms on Γ. The proof of existence, omitted in [13], will be given below in Lemma 3.

If $\mathcal{S}$ is any subset of $\mathcal{M}_n$ and $\xi = \sum c_M M \in \mathcal{R}_n$, we denote by $\langle \xi, \mathcal{S}\rangle$ the number $\sum_{M\in\mathcal{S}} c_M$. There are three natural actions of Γ on $\mathcal{R}_n$, by left multiplication, right multiplication, and conjugation, and we will be particularly interested in the cases when $\mathcal{S}$ is an orbit with respect to the second or third of these, i.e., a right coset $K = M_0\Gamma$ or a Γ-conjugacy class $X = \{\gamma^{-1}M_0\gamma : \gamma \in \Gamma\}$, respectively, where $M_0 \in \mathcal{M}$.

The Γ-conjugacy classes of elements $M \in \mathcal{M}_n$ are of five types: *scalar* ($M = \pm\sqrt{n}\cdot I_2$), *elliptic* ($\mathrm{tr}(M)^2 < 4n$), *split hyperbolic* ($\mathrm{tr}(M)^2 - 4n$ a positive square), *non-split hyperbolic* ($\mathrm{tr}(M)^2 - 4n$ a positive non-square), and *parabolic* ($\mathrm{tr}(M)^2 = 4n$, $M \neq$ scalar). We define the *weight* $w(M)$ in these five cases by the formulae

M	scalar	elliptic	split hyperbolic	non-split hyperbolic	parabolic
$w(M)$	1/6	$-1/\lvert\Gamma_M\rvert$	1	0	0

where Γ_M is the centralizer of M in Γ, which for elliptic M has order equal to 2 or 3 if M is conjugate to a matrix in $\mathbb{Z}I_2 + \mathbb{Z}S$ or $\mathbb{Z}I_2 + \mathbb{Z}U$, respectively, and to 1 in all other cases. Note that in the last three cases we can also write $w(M) = 1/|\Gamma_M|$, with the convention that $1/\infty = 0$ if $|\Gamma_M| = \infty$. Since $w(M)$ depends only on the conjugacy class X of M, we can also denote it by $w(X)$.

Our main result can then be stated as follows.

Theorem 1. *Let n be a positive integer, and let $\widetilde{T}_n \in \mathcal{R}_n$ satisfy both* (A) *and* (B).

(i) *For any right Γ-coset $K \subset \mathcal{M}_n$ we have $\langle \widetilde{T}_n, K\rangle = -1$.*

(ii) *For any Γ-conjugacy class X we have*

$$\text{(C)} \qquad \langle \widetilde{T}_n, X\rangle = w(X).$$

We will show in Section 2 that the theorem immediately implies the Eichler–Selberg trace formula for modular forms for Γ. As a warm-up, and to introduce the class numbers, we observe that computing $\langle \widetilde{T}_n, \mathcal{M}_n\rangle$ in two ways, using parts (i) and (ii) of the theorem, yields the Kronecker–Hurwitz class number relation in the form

$$\text{(1)} \qquad \sum_{X\subset\mathcal{M}_n} w(X) = -|\Gamma\backslash\mathcal{M}_n|,$$

where the sum is over all conjugacy classes X. To bring the formula to its classical form, we use the Γ-equivariant bijection $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \leftrightarrow cx^2 + (d-a)xy - by^2$ between integral matrices of determinant n and trace t and quadratic forms of discriminant $t^2 - 4n$ to write

$$\sum_{\substack{X \subset \mathcal{M}_n \\ \operatorname{tr}(X) = \pm t}} w(X) = \begin{cases} -2H(4n - t^2) & \text{if } t \neq 0, \\ -H(4n - t^2) & \text{if } t = 0, \end{cases} \tag{2}$$

where $\operatorname{tr}(X)$ is the trace of any element in the conjugacy class X (well-defined up to sign), and $H(D)$ is the Kronecker–Hurwitz class number, extended to all $D \in \mathbb{Z}$ as in [13]. That is, for $D > 0$, $H(D)$ equals the number of Γ-equivalence classes of positive definite integral binary quadratic forms of discriminant $-D$, with those classes that contain a multiple of $x^2 + y^2$ or of $x^2 - xy + y^2$ counted with multiplicity $1/2$ or $1/3$, respectively, $H(0) = -1/12$, $H(D) = 0$ if $D < 0$ is not the negative of a perfect square, and $H(-u^2) = -u/2$ if $u \in \mathbb{Z}_{>0}$.

Using (2), the relation above becomes $\sum_{t \in \mathbb{Z}} H(4n - t^2) = \sum_{d \mid n} d$, and the classical Kronecker–Hurwitz relation

$$\sum_{t^2 \leqslant 4n} H(4n - t^2) = \sum_{n = ad,\, a,d > 0} \max(a, d)$$

follows by observing that $\sum_{t^2 > 4n} H(4n - t^2) = \sum_{n = ad,\, d > a > 0} a - d$. We gave a yet different, simpler proof of (a refinement of) the Kronecker–Hurwitz formula in [10]. As a by-product of the proof of Theorem 1, we will obtain a different refinement here (Proposition 5).

We prove part (i) of the theorem in Section 3 (Corollary 2), after a preliminary study of the relation between operators $\widetilde{T}_n$ satisfying only one of properties (A), (B). The hardest part of this approach to the trace formula is the proof of part (ii), given in Section 4. We give an explicit element $\widetilde{T}_n$ satisfying part (ii) by construction, and then show that it satisfies part (i), as well as property (B). It then follows from the theory in Section 3 that $\widetilde{T}_n$ satisfies (A) as well, and Theorem 1 follows.

The proof of the trace formula in Section 2 does not require the full statement of Theorem 1, but only the existence of an element $\widetilde{T}_n$ satisfying properties (A), (B) and (C). For example, to finish the proof given in Section 2 for the trace of the first two Hecke operators, it is enough to check that the following elements satisfy (A)–(C):

$$\widetilde{T}_1 = I_2 - \frac{1}{2}(I_2 + S) - \frac{1}{3}(I_2 + U + U^2),$$

$$\widetilde{T}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

(Both of these expressions were given in [13]; the element $\widetilde{T}_2$ constructed in Section 4 is different from the above).

While the methods of this paper are elementary, we point out that the formula we obtain has a cohomological interpretation with wide-ranging generalizations. Let V be a finite-dimensional complex $\mathrm{SL}_2(\mathbb{R})$-module. For a Γ-double coset Δ we denote by $[\Delta]$ the corresponding action on the group cohomology $H^i(\Gamma, V)$. Then we have the following formula for the Lefschetz number of the correspondence on the modular surface determined by the double coset Δ:

$$\sum_i (-1)^i \operatorname{tr}([\Delta], H^i(\Gamma, V)) = -\sum_{X \subset \Delta/\{\pm 1\}} w(X), \operatorname{tr}(M_X, V) \tag{3}$$

where the sum is over Γ-conjugacy classes X with representatives $M_X \in \Delta$. The results of this paper can be interpreted as proving this formula for irreducible representations V, and therefore for all finite-dimensional representations. Indeed, for the trivial module $V = \mathbb{C}$ only H^0 is nonzero and the formula reduces to the Kronecker–Hurwitz relation (1). If $V = V_{\mathsf{w}}$ is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ of odd dimension $\mathsf{w} + 1 \geqslant 3$, only H^1 is nonzero, and the formula above is proved in (4), taking into account the Eichler–Shimura isomorphism $H^1(\Gamma, V_{\mathsf{w}}) \simeq S_k \oplus M_k$, with M_k, S_k the spaces of modular forms, respectively cusp forms of weight $k = \mathsf{w} + 2$ for Γ. In a sequel to this paper [8], the first author has proved the cohomological trace formula (3) for arbitrary congruence subgroups Γ of the modular group, under a mild assumption on the double coset Δ. Surprisingly, Theorem 1 is central for the proof in the congruence subgroup case as well.

Another case where formula (3) is known for more general groups Γ is when $\Delta = \Gamma$, the trivial double coset, in which case the left hand side reduces to an Euler–Poincaré characteristic. For a large class of groups Γ including arithmetic subgroups of linear groups, it follows from work of Bass and Brown [1, 2] that (assuming for simplicity that Γ has trivial center)

$$\sum_i (-1)^i \dim H^i(\Gamma, V) = \sum_{M \in T(\Gamma)} \chi(\Gamma_M) \operatorname{tr}(M, V),$$

where $T(\Gamma)$ is a set of representatives for the conjugacy classes of Γ, Γ_M denotes the centralizer of M, and $\chi(G) \in \mathbb{Q}$ denotes the homological Euler–Poincaré characteristic of the group G. The sum on the right-hand side actually runs over finite order elements in $T(\Gamma)$, because of a theorem of Gottlieb–Stalling that states that $\chi(G) = 0$ if the group G contains an infinite order element in the center. Moreover, if G is finite, we have $\chi(G) = 1/|G|$, and $\chi(\mathrm{PSL}_2(\mathbb{Z})) = -1/6$, so for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ the formula specializes to the case $\Delta = \Gamma$ of (3). Therefore we have a geometric interpretation of the coefficients $w(X)$ in (3) – they are negatives of "local" Euler–Poincaré characteristics.

Perhaps the ultimate generalization of (3) is the topological trace formula of Goresky and MacPherson [4], where Γ can be any arithmetic subgroup of a reductive group, and the algebraic group cohomology can be replaced by other geometric cohomology theories. Our work can therefore also be seen as giving an elementary proof of an explicit version of the topological trace formula, in the special case of the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

2. Deduction of the trace formula

Let $\mathsf{w} \geqslant 0$ be an even integer, and let V_{w} be the space of complex homogeneous polynomials of degree w. The group $\mathrm{GL}_2(\mathbb{R})$ acts on V_{w} by $P|\gamma(X, Y) = P(aX + bY, cX + dY)$ for $\gamma = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, which extends by linearity to a right action of the algebra $\mathcal{R} = \bigoplus_{n \geqslant 1} \mathcal{R}_n$, where $\mathcal{R}_n = \mathbb{Q}[\mathcal{M}_n]$ as above. The space of period polynomials W_{w} is defined as

$$W_{\mathsf{w}} = \operatorname{Ker}(1 + S) \cap \operatorname{Ker}(1 + U + U^2).$$

For even $k > 2$, let M_k and S_k be the spaces of modular forms and cusp forms of weight k for Γ, respectively; so we have $M_k = \mathbb{C}G_k \oplus S_k$, with G_k a suitably normalized Eisenstein series of weight k. To each $f \in S_k$ we associate its period polynomial

$$P_f(X, Y) = \int_0^{i\infty} f(z)(X - zY)^{\mathsf{w}} \, dz,$$

where $\mathsf{w} = k - 2$, which is an element of W_{w}. Its even and odd parts, P_f^+ and P_f^-, also belong to W_{w}. One can also define P_f^+ for non-cuspidal f (see [12]), and we have

$$P^+_{G_k}(X, Y) = \lambda_k (X^{\mathsf{w}} - Y^{\mathsf{w}})$$

for a certain number $\lambda_k \neq 0$.

For $n \geqslant 1$, the Hecke operator T_n acts on M_k by $f|T_n = n^{k-1} \sum_{M \in \Gamma \backslash \mathcal{M}_n} f|_k M$, with $f|_k M(z) = f(Mz)(cz+d)^{-k}$ for $M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$. The Eichler–Shimura isomorphism can then be stated as follows. For a generalization to modular forms for congruence subgroups see [7].

Proposition 2. *Let $\mathsf{w} > 0$ be even. We have a Hecke-equivariant isomorphism*

$$M_{\mathsf{w}+2} \oplus S_{\mathsf{w}+2} \simeq W_{\mathsf{w}}, \quad (f, g) \mapsto P_f^+ + P_g^-,$$

where the action of the Hecke operator T_n on period polynomials $P \in W_{\mathsf{w}}$ is defined by $P|T_n = P|\widetilde{T}_n$ for any element $\widetilde{T}_n \in \mathcal{R}_n$ satisfying (A).

Proof. The fact that the map is an isomorphism and that $P_{f|T_n} = P_f|\widetilde{T}_n$ for $f \in S_{\mathsf{w}+2}$ and any $\widetilde{T}_n \in \mathcal{R}_n$ satisfying (A) is shown in [3] or [13]. For completeness we briefly sketch the proof of Hecke-equivariance: The Eichler integral

$$\widetilde{f}(z) = \int_z^{i\infty} f(t)(t-z)^{\mathsf{w}}\, dt \quad (\text{with } z \in \mathcal{H})$$

has the property that $\widetilde{f}|_{-\mathsf{w}}(1-S) = P_f(z,1)$ and $\widetilde{f}|_{-\mathsf{w}}(1-T) = 0$, so for any $\widetilde{T}_n$ satisfying (A) we have

$$P_f(z,1)|T_n = P_f(z,1)|\widetilde{T}_n = \widetilde{f}|_{-\mathsf{w}}(1-S)\widetilde{T}_n = \widetilde{f}|_{-\mathsf{w}} T_n^\infty (1-S) = P_{f|T_n}(z,1),$$

where the last equality follows from the easily verified equality $\widetilde{f}|_{-\mathsf{w}} T_n^\infty = \widetilde{f|T_n}$.

For the period polynomial $X^{\mathsf{w}} - Y^{\mathsf{w}} = Y^{\mathsf{w}}|(S-1)$ of the Eisenstein series, using (A) and the fact that $Y^{\mathsf{w}}|(1-T) = 0$ we obtain

$$Y^{\mathsf{w}}|(S-1)|T_n = Y^{\mathsf{w}}|(S-1)\widetilde{T}_n = Y^{\mathsf{w}}|T_n^\infty(S-1) = \sigma_{\mathsf{w}+1}(n) Y^{\mathsf{w}}|(S-1),$$

where $\sigma_{\mathsf{w}+1}(n) = \sum_{d|n} d^{\mathsf{w}+1}$ is the eigenvalue of $G_{\mathsf{w}+2}$ under T_n. $\square$

The next result, proved in [13], is the reason for introducing property (B). For completeness we give here a different, shorter proof.

Proposition 3. *Let $\mathsf{w} > 0$ and $n \geqslant 1$ be integers with w even. If $\widetilde{T}_n \in \mathcal{R}_n$ satisfies* (B)*, then $\widetilde{T}_n$ preserves the subspace W_{w} of V_{w} and satisfies*

$$\mathrm{tr}(\widetilde{T}_n, W_{\mathsf{w}}) = \mathrm{tr}(\widetilde{T}_n, V_{\mathsf{w}}).$$

Proof. Property (B) implies that the operator $\widetilde{T}_n$ maps the subspaces $A = \mathrm{Ker}(1+S)$ and $B = \mathrm{Ker}(1+U+U^2)$ of V_{w} into each other, so it maps $W_{\mathsf{w}} = A \cap B$ into itself. On the other hand $A + B = V_{\mathsf{w}}$, because V_{w} is endowed with a natural, nondegenerate Γ-invariant inner product, and the orthogonal complement $(A+B)^\perp$ is the Γ-invariant subspace

$$V_{\mathsf{w}}^\Gamma = \mathrm{Ker}(1-S) \cap \mathrm{Ker}(1-U),$$

which is trivial for $\mathsf{w} > 0$. The claim now follows immediately from a simple linear algebra fact: if T is a linear transformation of a vector space to itself, and A, B are subspaces mapped into each other by T, then $\operatorname{tr}(T, A \cap B) = \operatorname{tr}(T, A + B)$. □

To state the trace formula, let $p_{\mathsf{w}}(t,n)$ be the Gegenbauer polynomial, defined by the power series expansion

$$(1 - tX + nX^2)^{-1} = \sum_{\mathsf{w}=0}^{\infty} p_{\mathsf{w}}(t,n) X^{\mathsf{w}}.$$

It satisfies $\operatorname{tr}(M, V_{\mathsf{w}}) = p_{\mathsf{w}}(\operatorname{tr} M, \det M)$ for any $M \in \mathrm{GL}_2(\mathbb{R})$ (in particular $p_{\mathsf{w}}(t,n)$ is an even function of t for w even).

Corollary 1 (Eichler–Selberg trace formula). *For all* $\mathsf{w} > 0$ *and* $n \geqslant 1$ *we have*

$$\operatorname{tr}(T_n, M_{\mathsf{w}+2}) + \operatorname{tr}(T_n, S_{\mathsf{w}+2}) = -\sum_{t \in \mathbb{Z}} p_{\mathsf{w}}(t,n) H(4n - t^2).$$

Proof. For odd w both sides vanish trivially, so we assume w even. Let $\widetilde{T}_n \in \mathcal{R}_n$ satisfy (A) and (B). Combining the two propositions and part (ii) of Theorem 1, we obtain for even $\mathsf{w} \geqslant 2$:

$$\begin{aligned} \operatorname{tr}(T_n, M_{\mathsf{w}+2}) + \operatorname{tr}(T_n, S_{\mathsf{w}+2}) &= \operatorname{tr}(\widetilde{T}_n, W_{\mathsf{w}}) = \operatorname{tr}(\widetilde{T}_n, V_{\mathsf{w}}) \\ &= \sum_{X \subset \mathcal{M}_n} \operatorname{tr}(M_X, V_{\mathsf{w}}) w(X), \end{aligned} \tag{4}$$

where the last sum is over all conjugacy classes X, and M_X is any element in X. The conclusion follows by rewriting the last sum using the property of $p_{\mathsf{w}}(t,n)$ above, together with formula (2). □

Note that $\operatorname{tr}(T_n, M_k) - \operatorname{tr}(T_n, S_k) = \sigma_{k-1}(n)$, so that Corollary 1 can be rewritten as a formula for either $\operatorname{tr}(T_n, M_k)$ or $\operatorname{tr}(T_n, S_k)$ ($k > 2$ even), which is the form usually given in the literature.

Remark. The formula in Corollary 1 is generalized to modular forms on congruence subgroup Γ with Nebentypus in [9], using the same operator $\widetilde{T}_n$ as in Theorem 1 acting on the space of period polynomials for Γ. The trace on the Eisenstein subspace is also explicitly computed there, yielding a simple formula for the trace of a composition of arbitrary Hecke and Atkin-Lehner operators on cusp forms for $\Gamma_0(N)$.

3. General properties of Hecke operators

In this section we analyze and generalize properties (A) and (B) of the introduction for arbitrary double cosets of Γ in the set $\mathcal{M}$ of 2×2 matrices with integer entries and positive determinant, modulo $\{\pm 1\}$. We let $\mathcal{R} = \mathbb{Q}[\mathcal{M}]$, which is a left and right module for the action of the group ring $\mathcal{R}_\Gamma = \mathcal{R}_1 = \mathbb{Q}[\Gamma]$.

Generalizing the definition of the subspace $W_{\mathsf{w}} \subset V_{\mathsf{w}}$ of period polynomials, we can consider for any (right) $\mathcal{R}_\Gamma$-module its *period subspace* $\operatorname{Ker} \pi_S \cap \operatorname{Ker} \pi_U$, where

$$\pi_S = \frac{1+S}{2}, \quad \pi_U = \frac{1+U+U^2}{3}$$

are the idempotents corresponding to the generators S and U of Γ of order 2 and 3. Equivalently, the period subspace of a (right) $\mathcal{R}$-module is the space annihilated by the right $\mathcal{R}_\Gamma$-ideal

$$\mathcal{I} = \pi_S \mathcal{R} + \pi_U \mathcal{R}, \tag{5}$$

and the most general subset of $\mathcal{R}$ which preserves the period subspace is

$$\mathcal{A} = \{\xi \in \mathcal{R} : \xi \mathcal{I} \subset \mathcal{I}\} = \{\xi \in \mathcal{R} : \xi \pi_S \in \mathcal{I},\ \xi \pi_U \in \mathcal{I}\}.$$

Note that elements $\widetilde{T}_n$ satisfying property (A) (and of course their multiples) belong to $\mathcal{A}$. Indeed multiplying (A) by π_S, π_U, and using the relation $(1-S)\pi_U = (1-T^{-1})\pi_U$ and the fact that $T_n^\infty(1-T) \in (1-T)\mathcal{R}$, we obtain $(1-S)\widetilde{T}_n \pi_S$ and $(1-S)\widetilde{T}_n \pi_U \in (1-T)\mathcal{R}$, so $\widetilde{T}_n \pi_S, \widetilde{T}_n \pi_U \in \mathcal{I}$ by the characterization of $\mathcal{I}$ in the next lemma.

Lemma 1 ([3]). *We have $\xi \in \mathcal{I}$ if and only if $(1-S)\xi \in (1-T)\mathcal{R}$.*

For the purpose of proving the trace formula, we introduce the subset $\mathcal{B}$ of $\mathcal{A}$ given by

$$\mathcal{B} = \{\xi \in \mathcal{R} : \xi \pi_S \in \pi_U \mathcal{R},\ \xi \pi_U \in \pi_S \mathcal{R}\}, \tag{6}$$

which contains those elements satisfying relation (B). To show that the set $\mathcal{B}$ contains elements satisfying (A) as well, we need a preliminary lemma.

Lemma 2. *The action of π_S and π_U on $\mathcal{R}$ satisfies*

$$\operatorname{Im}(\pi_S) \cap \operatorname{Im}(\pi_U) = \operatorname{Ker}(\pi_S) \cap \operatorname{Ker}(\pi_U) = \{0\}.$$

Proof. The first statement is clear since any element in $\pi_S \mathcal{R} \cap \pi_U \mathcal{R}$ is left invariant under both S and U, hence is Γ-invariant, and hence equal to 0. The second property, called "acyclicity" in [3], was proved there using the action of $\mathcal{R}$ on rational period functions. For completeness we give a shorter, more direct proof here.

Assume that $\xi \in \operatorname{Ker}(\pi_S) \cap \operatorname{Ker}(\pi_U)$, so in particular $(1+S)\xi = (1+U+U^2)\xi$. Setting $T' = U^2 S = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and recalling that $T = US$, we obtain $\xi = (T^{-1} + T'^{-1})\xi$. Letting $\xi = \sum c(M)M$, it follows that $c(M) = c(TM) + c(T'M)$ for all $M \in \mathcal{M}$, and this immediately leads to a contradiction if $c(M) \neq 0$: we would get $c(\gamma_i M) \neq 0$ for an infinite sequence of elements $\gamma_i \in \Gamma$ with $\gamma_0 = 1$ and $\gamma_{i+1} = T\gamma_i$ or $T'\gamma_i$, and this is impossible since the γ_i are distinct (they have non-negative coefficients whose sum increases strictly), and hence the $\gamma_i M$ are also all distinct. Therefore $\xi = 0$. □

Lemma 3. *For $\xi \in \mathcal{A}$, there are unique elements $\xi_S \in \pi_S \mathcal{R}$, $\xi_U \in \pi_U \mathcal{R}$ such that*

$$\xi \pi_S - \xi_S \in \pi_U \mathcal{R}, \quad \xi \pi_U - \xi_U \in \pi_S \mathcal{R}.$$

The map $P : \mathcal{A} \to \mathcal{A}$ given by $P(\xi) := \xi - \xi_S - \xi_U$ is a projection onto $\mathcal{B}$, and the image of the ideal $\mathcal{I}$ under this map is the set $\mathcal{J} \subset \mathcal{I}$ given by

$$\mathcal{J} = \pi_S \mathcal{R}(1-\pi_S) + \pi_U \mathcal{R}(1-\pi_U). \tag{7}$$

Note that the projection $P : \mathcal{A} \to \mathcal{B}$ defined in the lemma satisfies $P(\xi) - \xi \in \mathcal{I}$. This proves the existence of elements satisfying both (A) and (B), since any element $\widetilde{T}_n \in \mathcal{R}_n$ satisfying (A) belongs to $\mathcal{A}$ as we saw above, and $P(\widetilde{T}_n) \in \widetilde{T}_n + \mathcal{I}$ satisfies both (A) and (B).

Example. We have $1 \in \mathcal{A}$, and $1_S = \pi_S$, $1_U = \pi_U$, so $P(1) = 1 - \pi_S - \pi_U \in \mathcal{B}$.

Proof. The existence of ξ_S, ξ_U follows from the definition of $\mathcal{A}$, and their uniqueness from the first part of Lemma 2. The element $P(\xi)$ belongs to $\mathcal{B}$ because

$$P(\xi)\pi_S = (\xi\pi_S - \xi_S - \xi_U)\pi_S \in \pi_U\mathcal{R}$$

(using $\pi_S = \pi_S^2$), and similarly for $P(\xi)\pi_U$. If $\xi \in \mathcal{B}$, the uniqueness of ξ_S, ξ_U shows that $\xi_S = \xi_U = 0$, so $P(\xi) = \xi$.

We have $P(\mathcal{I}) = \mathcal{I} \cap \mathcal{B}$, which clearly contains $\mathcal{J}$. To show the other implication, let $\xi \in \mathcal{I} \cap \mathcal{B}$, that is, $\xi = \pi_S X + \pi_U Y$, with $\pi_S X \pi_S \in \pi_U\mathcal{R}$ and $\pi_U Y \pi_U \in \pi_S\mathcal{R}$. By the first part of Lemma 2 we get

$$\pi_S X \pi_S = 0,$$

and since $\mathrm{Ker}(\pi_S) = \mathrm{Im}(1 - \pi_S)$, we have $\pi_S X \in \mathcal{R}(1 - \pi_S)$. Multiplying on the left by $\pi_S = \pi_S^2$, we obtain $\pi_S X \in \pi_S\mathcal{R}(1 - \pi_S)$. Similarly one shows $\pi_U Y \in \pi_U\mathcal{R}(1 - \pi_U)$, and so $\xi \in \mathcal{J}$. □

We now come to the main results of this section, for which we need to consider elements $\widetilde{T}_\Delta$ satisfying analogues of (A) and (B), defined for any Δ which is a double coset $\Gamma\delta\Gamma$ ($\delta \in \mathcal{M}$), or more generally a left- and right-Γ-invariant subset of $\mathcal{M}_n$ (which is a finite union of double cosets).

We set $\mathcal{R}_\Delta = \mathbb{Q}[\Delta]$, $\mathcal{A}_\Delta = \mathcal{A} \cap \mathcal{R}_\Delta$, $\mathcal{B}_\Delta = \mathcal{B} \cap \mathcal{R}_\Delta$. As in the case $\Delta = \mathcal{M}_n$, we define

$$T_\Delta^\infty = \sum_{M \in \mathcal{M}_\Delta^\infty} M \in \mathcal{R}_\Delta,$$

where $\mathcal{M}_\Delta^\infty$ is a set of representatives for $\Gamma\backslash\Delta$ fixing ∞. The set Δ gives rise to a Hecke operator T_Δ acting on modular forms $f \in M_k$ by

$$f|T_\Delta = n^{k-1} \sum_{M \in \Gamma\backslash\Delta} f|_k M$$

(for $\Delta \subset \mathcal{M}_n$). One can show as in the case $\Delta = \mathcal{M}_n$ that the corresponding action on period polynomials is by any element $\widetilde{T}_\Delta$ satisfying

$$\text{(A}'\text{)} \qquad (1 - S)\widetilde{T}_\Delta \equiv T_\Delta^\infty(1 - S) \pmod{(1 - T)\mathcal{R}_\Delta},$$

and the existence of such elements follows as in [3]. The same proof given above for $\widetilde{T}_n$ shows that any such $\widetilde{T}_\Delta$ belongs to $\mathcal{A}_\Delta$. The next theorem shows that for a single double coset the converse is also true up to scaling.

Theorem A. *Let Δ be a double coset. Then any element $\mathcal{T} \in \mathcal{A}_\Delta$ satisfies*

$$(8) \qquad (1 - S)\mathcal{T} \equiv \alpha \cdot T_\Delta^\infty(1 - S) \pmod{(1 - T)\mathcal{R}_\Delta}$$

for a unique number $\alpha = \alpha(\mathcal{T}) \in \mathbb{Q}$.

Equation (8) says that $\mathcal{T} \equiv \alpha \widetilde{T}_\Delta \pmod{\mathcal{I}}$ for any $\widetilde{T}_\Delta$ satisfying (A$'$). The theorem can be reformulated as the existence of a linear map $\alpha : \mathcal{A}_\Delta \to \mathbb{Q}$ and of an exact sequence

$$0 \to \mathcal{I}_\Delta \to \mathcal{A}_\Delta \xrightarrow{\alpha} \mathbb{Q} \to 0,$$

where α is uniquely determined by the normalization $\alpha(\widetilde{T}_\Delta) = 1$ for any element $\widetilde{T}_\Delta$ satisfying (A$'$).

Proof. Suppose $\mathcal{T} \in \mathcal{A}$ (we will assume $\mathcal{T} \in \mathcal{A}_\Delta$ only at the end of the proof). By Lemma 1, we get

$$(9) \qquad (1-S)\mathcal{T}\pi_S \in (1-T)\mathcal{R}, \quad (1-S)\mathcal{T}\pi_U \in (1-T)\mathcal{R}.$$

Let Γ_∞, $\mathcal{M}_\infty$, and Δ_∞ be the subsets fixing ∞ of Γ, $\mathcal{M}$, and Δ, respectively, and for Γ_∞-orbits $K \in \Gamma_\infty \backslash \mathcal{M}$, we denote by $M_K \in K$ a fixed representative.

For any $\zeta \in \mathcal{R}$ we have $\zeta - \sum_{K \in \Gamma_\infty \backslash \mathcal{M}} \langle \zeta, K \rangle M_K \in (1-T)\mathcal{R}$, so

$$(10) \qquad \zeta \in (1-T)\mathcal{R} \iff \langle \zeta, K \rangle = 0 \text{ for all } K \in \Gamma_\infty \backslash \mathcal{M},$$

where the notation $\langle \cdot, \cdot \rangle$ was defined in the introduction.

Set $\xi = (1-S)\mathcal{T}$, and define the function $a = a_\xi : \Gamma_\infty \backslash \mathcal{M} \to \mathbb{Q}$ by $a(K) = \langle \xi, K \rangle$, so $a(K)$ is nonzero for finitely many K. From (9) and (10) we obtain

$$a(K) + a(KS) = 0, \quad a(K) + a(KU) + a(KU^2) = 0.$$

It follows that $a(K) = a(KT) + a(KT')$. Since elements in K share the same second row, we conclude as in the proof of Lemma 2 that $a(K) = 0$ unless one of the elements on the second row of matrices in K equals 0, that is, unless $K \subset \mathcal{M}_\infty$ or $K \subset \mathcal{M}_\infty S$. For $K \subset \mathcal{M}_\infty$ it follows that $a(KT') = 0$, so $a(K) = a(KT)$ for $K \in \Gamma_\infty \backslash \mathcal{M}_\infty$.

The set of orbits $\Gamma \backslash \mathcal{M}$ can be identified with $\Gamma_\infty \backslash \mathcal{M}_\infty$ by choosing for each $K \in \Gamma \backslash \mathcal{M}$ a representative $M_K \in \mathcal{M}_\infty$. With this identification, the function a gives rise to a function $\widetilde{a}$ on $\Gamma \backslash \mathcal{M}$, by $\widetilde{a}(K) = a(\Gamma_\infty M_K)$, and we have shown that $\widetilde{a}(K) = \widetilde{a}(KT)$ and

$$(11) \qquad \xi \equiv \sum_{K \in \Gamma \backslash \mathcal{M}} \widetilde{a}(K) M_K (1-S) \pmod{(1-T)\mathcal{R}}.$$

Recalling that $\xi = (1-S)\mathcal{T}$, we have $\langle \xi, K \rangle = 0$ for all $K \in \Gamma \backslash \mathcal{M}$. Therefore

$$\widetilde{a}(K) = \widetilde{a}(KS).$$

We now assume that $\mathcal{T} \in \mathcal{A}_\Delta$, so the sum in (11) is over $K \in \Gamma \backslash \Delta$. Since Δ is a double coset, the group Γ acts transitively on $\Gamma \backslash \Delta$ on the right. Since S and T generate Γ, from $\widetilde{a}(K) = \widetilde{a}(KS) = \widetilde{a}(KT)$ it follows that $\widetilde{a}(K) = \alpha$ is constant for $K \in \Gamma \backslash \Delta$. We conclude from (11) that $\mathcal{T}$ satisfies (8), and then Lemma 1 shows that $\mathcal{T} \equiv \alpha \widetilde{T}_\Delta \pmod{\mathcal{I}}$ for any $\widetilde{T}_\Delta$ satisfying (A$'$).

For the uniqueness of α, note from Lemma 1 that any two elements $\widetilde{T}_\Delta$ satisfying (A$'$) differ by an element in $\mathcal{I}_\Delta$, so it is enough to show that $\widetilde{T}_\Delta \notin \mathcal{I}_\Delta$. But $\widetilde{T}_\Delta \in \mathcal{I}_\Delta$ would imply that $T_\Delta^\infty(1-S) \in (1-T)\mathcal{R}_\Delta$, which is easily seen to contradict (10). □

Remark. An equivalent formulation of condition (A$'$) is due to L. Merel (for the case $\Delta = \mathcal{M}_n$), and it was used as the definition of Hecke operators acting on modular symbols in [6]. For a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we denote its adjoint by $M^\vee = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, and we extend the notation to elements of $\mathcal{R}$ by linearity. Then the operator $\mathcal{T} \in \mathcal{R}_\Delta$ satisfies (A′) if and only if its adjoint $\mathcal{T}^\vee$ satisfies Merel's condition:

$$\mathcal{T}_K^\vee([0]-[\infty]) = [0]-[\infty] \quad \text{for all } K \in \Delta/\Gamma,$$

where $\mathcal{T}_K^\vee = \sum_{M\in K} c_M M$ if $\mathcal{T}^\vee = \sum c_M M$. Here $\mathcal{M}$, and by linearity $\mathcal{R}$, act on the set of divisors supported on the cusps by fractional linear transformations.

Indeed, after taking adjoints condition (A′) is equivalent with

$$\mathcal{T}_K^\vee(1-S) - (M_K - SM_{SK}) \in M_K \cdot \mathcal{R}_\Gamma(1-T) \quad \text{for all } K \in \Delta/\Gamma\,,$$

where we denote by M_K the unique representative of the coset K which is in $(\mathcal{M}_\Delta^\infty)^\vee$. The previous relation is equivalent to Merel's condition by the following immediate consequence of the adjoint of equivalence (10): for $\xi \in \mathcal{R}_\Gamma$ we have $\xi[\infty] = 0$ if and only if $\xi \in \mathcal{R}_\Gamma(1-T)$.

We now study more closely elements in the subspace $\mathcal{B}_\Delta = \mathcal{B} \cap \mathbb{Q}[\Delta]$ of the vector space $\mathcal{B}$.

Theorem B. *Let $\Delta \subset \mathcal{M}$ be a double coset and let $\mathcal{T} \in \mathcal{B}_\Delta \subset \mathcal{A}_\Delta$.*

(a) *There exists $\beta(\mathcal{T}) \in \mathbb{Q}$ such that $\langle \mathcal{T}, K\rangle = \beta(\mathcal{T})$ for all $K \in \Delta/\Gamma$.*

(b) *We have $\beta(\mathcal{T}) = -\alpha(\mathcal{T})$, with $\alpha(\mathcal{T})$ defined in Theorem* A.

Moreover, the map $\beta : \mathcal{B}_\Delta \to \mathbb{Q}$ is surjective.

The theorem implies that the exact sequence stated after Theorem A can be a completed to a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_\Delta \arrow[r] & \mathcal{A}_\Delta \arrow[r, "\alpha"] & \mathbb{Q} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{J}_\Delta \arrow[r] \arrow[u, hook] & \mathcal{B}_\Delta \arrow[r, "-\beta"] \arrow[u, hook] & \mathbb{Q} \arrow[r] \arrow[u, equal] & 0, \end{tikzcd}$$

where $\mathcal{J}$, which was defined in (7), was already shown to equal $\mathcal{I} \cap \mathcal{B}$ in Lemma 3.

Proof. (a) For $\mathcal{T} \in \mathcal{R}_\Delta$, denote by $\mathcal{T}_K$ the part of $\mathcal{T}$ supported on a coset $K \in \Delta/\Gamma$. For $\mathcal{T} \in \mathcal{B}_\Delta$, we have $(1-U)\mathcal{T}\pi_S = (1-S)\mathcal{T}\pi_U = 0$, hence

$$\mathcal{T}_K\pi_S = U\mathcal{T}_{U^2K}\pi_S, \quad \mathcal{T}_K\pi_U = S\mathcal{T}_{SK}\pi_U$$

for all cosets $K \in \Delta/\Gamma$. It follows that $\langle \mathcal{T}, K\rangle = \langle \mathcal{T}, UK\rangle = \langle \mathcal{T}, SK\rangle$ for all cosets K. Since U and S generate Γ, which acts transitively on the cosets Δ/Γ, we obtain that $\langle \mathcal{T}, K\rangle$ is the same for all cosets K.

(b) To show $\beta = -\alpha$, and to prove surjectivity of β, we give a direct construction of elements $\mathcal{T} \in \mathcal{B}_\Delta$ having prescribed $\beta(\mathcal{T}) = \beta$. Without loss of generality assume $\beta = 1$, as we can always scale elements in $\mathcal{B}_\Delta$. Choose $A, A' \in \mathcal{R}_\Delta$ such that

$$(12) \qquad \langle A, K\rangle = \langle A', K\rangle = 1 \quad \text{for all } K \in \Delta/\Gamma, \qquad \text{and} \qquad \begin{cases} A\pi_S \in \pi_U \mathcal{R}_\Delta, \\ A'\pi_U \in \pi_S \mathcal{R}_\Delta. \end{cases}$$

(For example, letting $T_\Delta = \sum_{K\in\Delta/\Gamma} M_K$, $T'_\Delta = \sum_{K\in\Delta/\Gamma} M'_K$ with arbitrary representatives $M_K, M'_K \in K$, we can choose $A = \pi_U T_\Delta$, $A' = \pi_S T'_\Delta$.) Since S and U generate Γ, there are $B, B' \in \mathcal{R}_\Delta$ such that $A' - A = B(1-S) - B'(1-U)$, and it follows immediately that

$$\mathcal{T} = A + B(1-S) = A' + B'(1-U) \tag{13}$$

belongs to $\mathcal{B}_\Delta$, and $\beta(\mathcal{T}) = \beta(A) = 1$.

We now prove exactness in the middle of the second row in the diagram above. Let $\xi \in \mathcal{B}_\Delta$, and we show that $\xi \in \mathcal{J}_\Delta$. Indeed, we have $\langle \xi, K\rangle = 0$ for all cosets $K \in \Delta/\Gamma$, and since S and U generate Γ, we have

$$\xi = A(1-\pi_S) + B(1-\pi_U)$$

for some $A, B \in \mathcal{R}$. Since ξ satisfies the second relation in (6), setting $X = (1-\pi_S)A(1-\pi_S)$, we have $X\pi_U = 0$. But $X\pi_S = 0$ as well, and we conclude from Lemma 2 that $X = 0$. Since $\operatorname{Ker}(1-\pi_S) = \operatorname{Im}\pi_S$, it follows that $A(1-\pi_S) \in \pi_S\mathcal{R}$, and by multiplying on the right by $(1-\pi_S) = (1-\pi_S)^2$, we obtain $A(1-\pi_S) \in \pi_S\mathcal{R}(1-\pi_S)$. Similarly one shows $B(1-\pi_U) \in \pi_U\mathcal{R}(1-\pi_U)$, and we conclude that $\xi \in \mathcal{J}$.

Coming back to the proof of $\alpha = -\beta$, we assume without loss of generality that

$$\Delta = \Gamma\left(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix}\right)\Gamma.$$

All operators $\mathcal{T} \in \mathcal{B}_\Delta$ with $\beta(\mathcal{T}) = 1$ differ by an element in $\mathcal{J}_\Delta$, by the exactness proved in the previous paragraph, so they have the same value of $\alpha(\mathcal{T})$ as well. Therefore it is enough to prove that $\alpha(\mathcal{T}) = -1$ for a particular such element.

For $n = 1$, the claim can be verified using the example following Lemma 3. Hereafter we assume $n > 1$, and we construct an element $\mathcal{T} \in \mathcal{B}_\Delta$ as in (13), by making a particular choice of A, A'.

Let $K_0 \in \Delta/\Gamma$ be the coset $\left(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix}\right)\Gamma$. Clearly $UK_0 \neq K_0$ and $SK_0 \neq K_0$, so we can take A, A' in (12) to be given by

$$A = (1+U+U^2)\left(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix}\right) + \cdots, \quad A' = (1+S)\left(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix}\right) + \cdots,$$

where the part of A, A' supported on other cosets than $K_0, SK_0 = U^2K_0, UK_0$ can be chosen such that (12) is satisfied. We have

$$A_{K_0} = A'_{K_0} = \left(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix}\right), \quad A_{SK_0} = U^2 A_{K_0} = SA_{K_0}T^{-n}, \quad A'_{SK_0} = SA_{K_0}.$$

Using that $1 - T^{-n} = (1 + T^{-1} + \cdots + T^{-n+1})\big[(1-S) + S(1-U^2)\big]$, we get

$$A'_{SK_0} - A_{SK_0} = SA_{K_0}(1 + T^{-1} + \cdots + T^{-n+1})[(1-S) + S(1-U^2)],$$

which implies that $B_{SK_0}(1-S) = SA_{K_0}(1 + T^{-1} + \cdots + T^{-n+1})(1-S)$ in (13), and we conclude

$$\mathcal{T}_{K_0} - S\mathcal{T}_{SK_0} = \left(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix}\right)\big[1 - T^{-n} - (1 + T^{-1} + \cdots + T^{-n+1})(1-S)\big].$$

On the other hand, by Theorem A we know that $\mathcal{T} - \alpha\widetilde{T}_\Delta \in \mathcal{J}$ for some $\widetilde{T}_\Delta$ satisfying (A′) and a number $\alpha = \alpha(\mathcal{T})$. Taking only that part of (A′) supported on matrices in K_0 we have (using that $TK_0 = K_0$)

$$\mathcal{T}_{K_0} - S\mathcal{T}_{SK_0} - \alpha(\mathcal{T}) \sum_{b \ (\mathrm{mod}\ n)} \left(\begin{smallmatrix}1 & b\\ 0 & n\end{smallmatrix}\right)(1-S) \in (1-T)\mathcal{R}.$$

From the last two equations it follows that $(1+\alpha(\mathcal{T}))\cdot\sum_{b \pmod n}\left(\begin{smallmatrix}1 & b\\ 0 & n\end{smallmatrix}\right)(1-S)\in(1-T)\mathcal{R}$. Since the matrices in the last sum are in distinct Γ_∞-orbits in $\Gamma_\infty\backslash\Delta$, from (10) we obtain that $\alpha(\mathcal{T})=-1$. □

For the proof of Theorem 1, we only need the following immediate consequences of Theorem B.

Corollary 2. *Let* $\Delta=\mathcal{M}_n$ *and* $\mathcal{T}\in\mathcal{B}_\Delta$*, that is,* $\mathcal{T}$ *satisfies property* (B) *in the introduction. We have* $\langle\mathcal{T},K\rangle=-1$ *for all cosets* $K\in\Delta\backslash\Gamma$ *if and only if* $\mathcal{T}$ *satisfies* (A).

Corollary 3. *Let* $\Delta=\mathcal{M}_n$ *and* $\mathcal{T}=\widetilde{T}_n\in\mathcal{R}_n$ *an element satisfying both* (A) *and* (B). *Then for each conjugacy class* $X\subset\Delta$ *the quantity* $\langle\mathcal{T},X\rangle$ *depends only on* X *and not on the choice of* $\mathcal{T}$.

Proof of corollaries. We decompose $\mathcal{M}_n=\bigsqcup\Delta'$ into a finite disjoint union of double cosets Δ'. For $\mathcal{T}\in\mathcal{B}_\Delta$, we have a corresponding decomposition $\mathcal{T}=\sum\mathcal{T}_{\Delta'}$. Clearly $\mathcal{T}$ satisfies property (A) if and only if all such $\mathcal{T}_{\Delta'}$ satisfy (A), and the same holds for the condition $\langle\mathcal{T},K\rangle=-1$ for all cosets $K\in\Delta\backslash\Gamma$. Therefore Corollary 2 follows from part (a) of Theorem B.

If two elements $\mathcal{T},\mathcal{T}'\in\mathcal{B}_\Delta$ satisfy (A), by the exact sequence following Theorem B we have $\mathcal{T}_{\Delta'}-\mathcal{T}'_{\Delta'}\in\mathcal{J}_{\Delta'}$ for each double coset $\Delta'\subset\Delta$, so $\mathcal{T}-\mathcal{T}'\in\mathcal{J}_\Delta$. But $\mathcal{J}_\Delta$ is spanned by $M-\gamma M\gamma^{-1}$ for $\gamma\in\Gamma$, $M\in\Delta$, so $\langle\mathcal{T},X\rangle=\langle\mathcal{T}',X\rangle$ for each conjugacy class X, proving Corollary 3. □

4. An explicit Hecke operator

Explicit elements $\widetilde{T}_n\in\mathbb{Q}[\mathcal{M}_n]$ satisfying condition (A) were first given by Manin [5] using continued fractions (as re-interpreted in [11], where this condition was introduced), and other constructions were given in [13] and [6]. In this section we prove Theorem 1 by giving an explicit element satisfying all *three* properties (A), (B) and (C). Since property (C) is the hardest to prove, we start with an element that satisfies it by construction, and then show that it verifies (B) and property (i) of Theorem 1 as well. The corollaries at the end of the previous section then show that it satisfies (A) as well.

Since we want to give a uniform formula for all n, it is convenient to introduce

$$\widehat{\mathcal{R}}=\bigotimes_n\mathcal{R}_n,$$

the vector space of infinite formal linear combinations of elements of $\mathcal{R}$ with only finitely many elements of any fixed determinant. We look for $\widetilde{T}$ of the form

$$\widetilde{T}=-E+H+\sum c_i(M_i-\gamma_iM_i\gamma_i^{-1}),$$

where $E\in\widehat{\mathcal{R}}$ contains representatives of all elliptic, and $H\in\widehat{\mathcal{R}}$ of all hyperbolic split together with scalar conjugacy classes in $\mathcal{M}$, and such that $\langle -E+H,X\rangle=w(X)$ for all conjugacy classes $X\subset\mathcal{M}$, with $w(X)$ defined in the introduction. Any such $\widetilde{T}$ satisfies (C).

Choosing representatives for elliptic conjugacy classes amounts to choosing a fundamental domain of Γ acting on the upper half plane $\mathcal{H}$. Let χ be the characteristic function of the

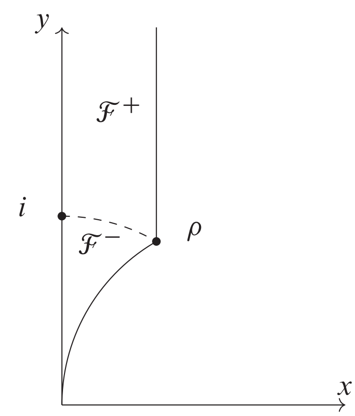

Figure 1. The fundamental domain $\mathscr{F} = \{z \in \mathscr{H} : 0 \leqslant \operatorname{Re} z \leqslant \frac{1}{2},\ |z-1| \geqslant 1\}$.

fundamental domain $\mathscr{F}$ shown in Figure 1, modified on the boundary by setting $\chi(z)$ equal to $1/2\pi$ times the angle subtended by $\mathscr{F}$ at z (i.e., χ is 1 in the interior of $\mathscr{F}$, 0 outside of $\mathscr{F}$, 1/2 on the boundary points different from the corner $\rho = e^{\pi i/3}$, and 1/3 at ρ). Our choice of $\widetilde{T}$ is then

$$\widetilde{T} = -E + H + X - SXS + Y - U^2YU + Z - U^2ZU, \tag{14}$$

with $E, H, X, Y, Z \in \widehat{\mathscr{R}}$ defined by

$$E = \sum_{M \text{ elliptic}} \chi(z_M)M = \left\langle \substack{0\leqslant a-d\leqslant -b;\ a-d\leqslant c \\ b<0<c} \right\rangle, \quad H = \left\langle \substack{a-d\leqslant -b\leqslant c \\ c=0<a} \right\rangle,$$

$$X = \left\langle \substack{0\leqslant a-d;\ -b\leqslant c \\ b<0<d} \right\rangle, \quad Y = \left\langle \substack{a-d\leqslant -b\leqslant c \\ 0<c<a} \right\rangle, \quad Z = \left\langle \substack{a-d\leqslant c\leqslant -b \\ 0<a;\ 0<c} \right\rangle.$$

Here z_M is the unique fixed point in $\mathscr{H}$ of an elliptic matrix M, and the notation $\langle\#\rangle$, where # is a collection of inequalities written on two lines, means the sum $\sum c(M)M \in \widehat{\mathscr{R}}$ over matrices $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries satisfying these inequalities, weighted with a coefficient $c(M) \in \{1, 1/2, 1/4, 1/3, 1/6\}$ according to the number of inequalities in the *first line* of # that become equalities for M: it is 1 if there are no equalities, $1/2$ if there is exactly one, and $1/4$, $1/3$ or $1/6$ if there are two and they are independent ($A \leqslant B$, $C \leqslant D$), overlapping ($A \leqslant B$, $A \leqslant C$), or nested ($A \leqslant B \leqslant C$), respectively. Note that in these definitions the first line of # involves only the coefficients of the quadratic form $Q_M = [c, d-a, -b]$ associated to M. Note also that the definition of the coefficient $c(\cdot)$ behaves correctly (additively) if # splits up as a union of two sets of inequalities, as we will use several times below. Recall that elements in $\mathscr{M}$ are defined up to ± 1; we always choose representatives with non-negative lower left entry.

As already explained, the above choice of $\widetilde{T}$ automatically satisfies property (C). The difficult part was to find a choice of elements X, Y, Z for which it satisfies (B) as well, and this particular choice was found numerically with the help of a computer. Before checking property (B), we first show that formula (14) hides considerable cancellation.

Lemma 4. *The following statements hold:*

(a) *The element* $\widetilde{T}$ *belongs to* $\widehat{\mathscr{R}}$, *namely* $\widetilde{T} = \sum_{n=1}^{\infty} \widetilde{T}_n$, *where* $\widetilde{T}_n \in \mathbb{Q}[\mathscr{M}_n]$.

(b) *The element* $\widetilde{T}$ *simplifies to*

$$\widetilde{T} = \left\langle \substack{a-d\leqslant -b\leqslant c \\ 0\leqslant c<a} \right\rangle - \left\langle \substack{-b\leqslant a-d\leqslant c \\ b<d\leqslant 0} \right\rangle - \left\langle \substack{0\leqslant a-d\leqslant c\leqslant -b \\ a\leqslant 0<c} \right\rangle - \left\langle \substack{0\leqslant a-d\leqslant -b\leqslant c \\ d\leqslant 0<-b} \right\rangle. \tag{15}$$

Proof. (a) The terms E and H clearly contain finitely many elements of a given determinant. For X and Y, and for Z in the case $d > a$, we have that $a, d > 0$, and $c, -b > 0$, so $ad - bc = n$ has finitely many solutions for each n. For Z in the case $0 \leqslant a - d$ the corresponding matrix is elliptic with fixed point in $\mathcal{F}$, so the same conclusion holds.

(b) Splitting $\mathcal{F}$ into the parts $\mathcal{F}^+$, $\mathcal{F}^-$ for which $|z| \geqslant 1$, $|z| \leqslant 1$, as in Figure 1, we have

$$E = \left\langle {0\leqslant a-d\leqslant c\leqslant -b \atop 0<c} \right\rangle + \left\langle {0\leqslant a-d\leqslant -b\leqslant c \atop b<0} \right\rangle =: A + B.$$

We decompose: $A = A_1 + A_2$, the parts of A for which $a > 0$, respectively $a \leqslant 0$; $B = B_1 + B_2$, the parts of B for which $d > 0$, respectively $d \leqslant 0$; $Z = Z_1 + Z_2$, the parts for which $a \leqslant d$, respectively $a \geqslant d$ (with boundary coefficients as indicated below); and

$$U^2 Z U = \left\langle {-b\leqslant a-d\leqslant c \atop b<0;\, b<d} \right\rangle = Z_3 + Z_4,$$

the parts for which $d > 0$, respectively $d \leqslant 0$. We check easily that

$$Z_1 = SXS = \left\langle {a-d\leqslant 0;\, c\leqslant -b \atop 0<a;\, 0<c} \right\rangle, \quad Z_2 = A_1 = \left\langle {0\leqslant a-d\leqslant c\leqslant -b \atop 0<a;\, 0<c} \right\rangle,$$
$$X = B_1 + Z_3 + U^2 Y U = \left\langle {0\leqslant a-d\leqslant -b\leqslant c \atop b<0<d} \right\rangle + \left\langle {-b\leqslant a-d\leqslant c \atop b<0<d} \right\rangle + \left\langle {-b\leqslant c\leqslant a-d \atop b<0<d} \right\rangle.$$

The term H in (14) can be absorbed in the sum Y as the boundary term for which $c = 0$, yielding the first term in (15), and (14) simplifies to (15), where the second, third, and fourth terms are Z_4, A_2, and B_2, respectively. □

Theorem 4. *The following statements hold:*

(a) *For each* $n \geqslant 1$ *the element* $\widetilde{T}$ *satisfies*

$$\widetilde{T}\pi_S \in \pi_U \widehat{\mathcal{R}}, \quad \widetilde{T}\pi_U \in \pi_S \widehat{\mathcal{R}},$$

namely each component $\widetilde{T}_n$ *satisfies* (B).

(b) *We have* $\langle \widetilde{T}, K \rangle = -1$ *for all right cosets* $K = M\Gamma \in \mathcal{M}/\Gamma$.

Theorem 1 immediately follows: by construction, each component $\widetilde{T}_n$ of $\widetilde{T}$ satisfies (C), and the previous theorem and Corollary 2 show that it satisfies (A) as well. Corollary 3 then shows that *any* element satisfying (A), (B) also satisfies (C).

Proof. (a) Write $\widetilde{T} = T_1 - T_2 - T_3 - T_4$ in (15). We will show more precisely that

$$\widetilde{T} = -F - UFS - U^2 F + T_1(1 - S), \tag{16}$$
$$\widetilde{T} = -G - SGU + T_1(1 - U) + \frac{1}{12}\left[{a-d=c=-b \atop d\leqslant 0<a} \right](1 - U),$$

so that $\widetilde{T}\pi_S = -(1 + U + U^2)F\pi_S$, $\widetilde{T}\pi_U = -(1 + S)G\pi_U$, and part (a) follows. Here

$$F = \left\langle {0\leqslant a-d\leqslant c;\, a-d\leqslant -b \atop d\leqslant b<0<c} \right\rangle, \quad G = \left\langle {0\leqslant a-d\leqslant c;\, a-d\leqslant -b;\, a\leqslant -b-c \atop a\leqslant 0;\, b<0<c} \right\rangle^*,$$

with $\langle \# \rangle^* \in \widehat{\mathcal{R}}$ denoting the sum of matrices $\left({a \atop c}\, {b \atop d} \right)$, weighted by $1/2$ if at least one inequality on the first line of # becomes equality, except that matrices with $a - d = c = -b$ or $a = d = -b - c$ are weighted by $1/4$, and $[\#] \in \widehat{\mathcal{R}}$ denoting the corresponding unweighted sum of matrices.

For a sum $\xi \in \widehat{\mathcal{R}}$ given by inequalities # on two lines, as in $\langle\#\rangle$ or $\langle\#\rangle^*$ above, we denote by ξ° its "interior", that is, the subsum of the terms for which at most one inequality on the first line becomes equality (therefore the terms are weighted by either 1 or $1/2$ if there is no equality or exactly one equality, respectively).

To prove the first formula, we decompose $T_3 = T_{3,1} + T_{3,2}$, $T_4 = T_{4,1} + T_{4,2}$, where $T_{3,1}$, $T_{4,1}$ and $T_{3,2}$, $T_{4,2}$ have the added inequality $b < d$ and $d \leqslant b$, respectively. We have $F = T_{3,2} + T_{4,2}$ and

$$T_{3,1} = \left\langle {0 \leqslant a-d \leqslant c \leqslant -b \atop a \leqslant 0 < c;\, b<d} \right\rangle, \quad T_{4,1} + T_2 = \left\langle {0 \leqslant a-d \leqslant c;\, -b \leqslant c \atop b<d \leqslant 0} \right\rangle =: T_5,$$

so $\widetilde{T} = T_1 - T_{3,1} - T_5 - F$. We also have

$$T_1 S = \left\langle {d \leqslant a \leqslant b+c \atop b<d \leqslant 0} \right\rangle, \ \ UFS = \left\langle {d \leqslant a \leqslant c+d;\, b+c \leqslant a \atop a \leqslant 0 < c;\, b<d} \right\rangle, \ \ U^2 F = \left\langle {a-d \leqslant c;\, -b \leqslant c \leqslant a-b \atop b<d \leqslant 0 < a} \right\rangle,$$

and we now decompose the terms by adding inequalities on the first line as follows:

$$T_1 S = T_{1,1} + T_{1,2},$$

where $T_{1,1}$, $T_{1,2}$ are obtained by imposing the extra inequality $-b \geqslant c$, respectively $-b \leqslant c$; and

$$T_5 = T_{5,1} + T_{5,2}, \quad T_{3,1} = T_{3,1a} + T_{3,1b},$$

with the range of summation split according as $a \leqslant b + c$ (for $T_{5,1}$ and $T_{3,1a}$), or $a \geqslant b + c$ (for $T_{5,2}$ and $T_{3,1b}$). We check that $T_{1,1}^\circ = T_{3,1a}^\circ$, $T_{1,2}^\circ = T_{5,1}^\circ$, and

$$\begin{aligned}
T_5 &= T_{5,1}^\circ + T_{5,2}^\circ &&+ \frac{1}{3}\left[{a-d=c=-b \atop b<d \leqslant 0}\right] + \frac{1}{4}\left[{a=d;\, c=-b \atop b<d \leqslant 0}\right] + \frac{1}{2}\left[{a=0;\, c=-b \atop b<d<0}\right],\\
T_{3,1} &= T_{3,1a}^\circ + T_{3,1b}^\circ &&+ \frac{1}{4}\left[{a=d;\, c=-b \atop b<d \leqslant 0}\right] + \frac{1}{2}\left[{a=d=b+c \atop b<d \leqslant 0}\right],\\
T_1 S &= T_{1,1}^\circ + T_{1,2}^\circ &&+ \frac{1}{2}\left[{a=d;\, c=-b \atop b<d \leqslant 0}\right] + \frac{1}{6}\left[{a=d=b+c \atop b<d \leqslant 0}\right] + \frac{1}{2}\left[{a=0;\, c=-b \atop b<d<0}\right],\\
UFS &= T_{3,1b}^\circ &&+ \frac{1}{3}\left[{a=d=b+c \atop b<d \leqslant 0}\right],\\
U^2 F &= T_{5,2}^\circ &&+ \frac{1}{3}\left[{a-d=c=-b \atop b<d \leqslant 0}\right]
\end{aligned}$$

(in the sum $T_{3,1b}^\circ$ there is an ambiguity whether to include the term with $a = 0 = b + c$, and we choose to include it, by taking the first line inequalities to be $0 \leqslant a - d \leqslant c \leqslant a - b$; the inequality $c \leqslant -b$, which was part of the first line of $T_{3,1}$, is then implied by $a \leqslant 0$ on the second line). We obtain that

$$T_{3,1} + T_5 = T_1 S + UFS + U^2 F,$$

so the first equation in (16) follows.

To prove the second equation, we first note that $T_2 = T_1 U$, so

$$\widetilde{T} = -T_3 - T_4 + T_1(1 - U).$$

We decompose $T_4 = T_{4,1} + T_{4,2}$, the subsums for which we add $a \leqslant -b - c$, respectively $-b - c \leqslant a$ on the first line of T_4. We have

$$SGU = \left\langle {0 \leqslant a-d \leqslant -b \leqslant c;\, -b-c \leqslant a \atop -c<d \leqslant 0<-b} \right\rangle^*,$$

and we check the following relations:

$$\begin{aligned}
T_3 &= T_3^\circ \qquad\qquad\quad + \frac{1}{6}\left[\substack{a-d=c=-b\\ a\leqslant 0<c}\right] + \frac{1}{4}\left[\substack{a=d;\, c=-b\\ a\leqslant 0<c}\right],\\
T_4 &= T_{4,1}^\circ + T_{4,2}^\circ + \frac{1}{6}\left[\substack{a-d=c=-b\\ d\leqslant 0<c}\right] + \frac{1}{4}\left[\substack{a=d;\, c=-b\\ d\leqslant 0<c}\right] + \frac{1}{2}\left[\substack{a=d=-b-c\\ d<0<-b}\right]\\
&\qquad + \frac{1}{2}\left[\substack{a=0;\, c=-b\\ 0<-d<c}\right],\\
G &= T_3^\circ + T_{4,1}^\circ \quad + \frac{1}{4}\left[\substack{a-d=c=-b\\ a\leqslant 0<c}\right] + \frac{1}{2}\left[\substack{a=d;\, c=-b\\ a<0<c}\right] + \frac{1}{4}\left[\substack{a=d=-b-c\\ d\leqslant 0<-b}\right],\\
SGU &= T_{4,2}^\circ \qquad\qquad + \frac{1}{4}\left[\substack{a-d=c=-b\\ d\leqslant 0<a}\right] + \frac{1}{4}\left[\substack{a=d=-b-c\\ d\leqslant 0<-b}\right] + \frac{1}{2}\left[\substack{a=0;\, c=-b\\ 0<-d<c}\right],
\end{aligned}$$

hence $T_3+T_4-G-SGU$ involves only the four terms with $a-d=c=-b$ above. Moreover,

$$\left[\substack{a-d=c=-b\\ d\leqslant 0<c}\right] = \left[\substack{a-d=c=-b\\ a\leqslant 0<c}\right] + \left[\substack{a-d=c=-b\\ d\leqslant 0<a}\right], \qquad \left[\substack{a-d=c=-b\\ d\leqslant 0<a}\right]\cdot U = \left[\substack{a-d=c=-b\\ a\leqslant 0<c}\right],$$

and we conclude that

$$T_3 + T_4 - G - SGU = \frac{1}{12}\left[\substack{a-d=c=-b\\ d\leqslant 0<a}\right](U-1),$$

yielding the second line in (16).

(b) We have a decomposition of $\mathcal{M}$ into a disjoint union of double cosets

$$\mathcal{M} = \bigcup_{a,m\geqslant 1} \Gamma\left(\begin{smallmatrix} a & 0\\ 0 & am\end{smallmatrix}\right)\Gamma,$$

and a corresponding decomposition $\widetilde{T} = \sum_{a,m} \widetilde{T}_{a,m}$, with $\widetilde{T}_{a,m}$ the part of $\widetilde{T}$ supported on the corresponding double coset. Each $\widetilde{T}_{a,m}$ satisfies (B) by part (a), and $\widetilde{T}_{a,m} = \left(\begin{smallmatrix} a & 0\\ 0 & a\end{smallmatrix}\right)\widetilde{T}_{1,m}$, so by Theorem B it is enough to show that $\langle \widetilde{T}_{1,m}, K_0\rangle = -1$ for all m, for the particular coset $K_0 = \left(\begin{smallmatrix} 1 & 0\\ 0 & m\end{smallmatrix}\right)\Gamma$.

For $m = 1$ we have

$$\widetilde{T}_{1,1} = \widetilde{T}_1 = 1 - \pi_S - \pi_U,$$

and the claim is clear. Assuming therefore $m > 1$, we have to find the matrices

$$M = \left(\begin{smallmatrix} a & b\\ mc & md\end{smallmatrix}\right) \in K_0$$

with $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$, in each of the four terms in (15). For the first two terms we have $\langle T_1 - T_2, K_0\rangle = 0$, as $T_2 = T_1 U$. For the sum T_3 we write $m = \det M$ as follows:

$$m = mc(-b) + (-a)(a - md) + a^2 \geqslant m^2c^2 \geqslant m^2,$$

so there are no matrices from K_0 in the sum T_3. For T_4 we write

$$m = mc(-b) + md(a - md) + m^2d^2 \geqslant b^2 + m^2d^2 - m|db| \geqslant \frac{3m^2d^2}{4},$$

so $d = 0$, $c = -b = 1$, and $0 \leqslant a \leqslant 1$. Therefore

$$\langle T_4, K_0\rangle = \frac{1}{2} + \frac{1}{2} = 1,$$

so $\langle \widetilde{T}_{1,m}, K_0\rangle = -1$. $\square$

We already pointed out in the introduction that Theorem 1 immediately implies the classical Kronecker–Hurwitz class number formula. In fact, the proof of Theorem 4 implies a refinement of this class number formula, similar in spirit to the refinement proved by different means in [10]. More precisely, in [10] the class number formula was proved by establishing a correspondence between an easily countable subset of the right cosets $\mathcal{M}_n/\Gamma$ and half the elliptic conjugacy classes of matrices in $\mathcal{M}_n$, whereas now we obtain a "weighted" bijection between *all* right cosets and an easily countable subset of the elliptic conjugacy classes, namely those conjugacy classes containing the matrices in the third and fourth sums in (15). We indicate the argument briefly.

Define a weighting function α on $\mathcal{M}$ by setting $\alpha(M) = 0$ for non-elliptic M, while $\alpha(M)$ for an elliptic matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}$ with $c > 0$ is given by

$$\alpha(M) = \chi^{+}(z_M) \cdot \delta(a) + \chi^{-}(z_M) \cdot \delta(d),$$

where $\chi^{\pm}$ are the characteristic functions of the half-fundamental domains $\mathcal{F}^{\pm}$ in Figure 1 (having boundary values given as for χ in terms of the angle subtended), and δ is the characteristic function of the set of non-positive integers. Conjugating by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ and simultaneously changing the sign of the matrix to preserve the condition $c > 0$ interchanges matrices with fixed point in $\mathcal{F}^{+}$ with those with fixed point in $\mathcal{F}^{-}$ and interchanges a and $-d$ in the formula for α, so we have

$$\begin{aligned} \sum_{M \in \mathcal{M}_n} \alpha(M) &= \sum_{M \in \mathcal{M}_n} \chi^{-}(z_M)\bigl(\delta(-d) + \delta(d)\bigr) \\ &= \sum_{M \in \mathcal{M}_n} \chi^{-}(z_M) + \sum_{\substack{0 \leqslant a \leqslant -b \leqslant c \\ n = -bc}} \alpha\bigl(\left(\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right)\bigr) \\ &= \frac{1}{2} \sum_{t^2 \leqslant 4n} H(4n - t^2) + \frac{1}{2} \sum_{\substack{n = bc \\ b > 0}} \min(b, c), \end{aligned} \tag{17}$$

where the final equality follows from the well-known Γ-equivariant bijection between matrices and binary quadratic forms. (The calculation of the final sum must be modified slightly for the terms with $-b = c$ when n is a square, but this is compensated by the term $H(0) = -1/12$.) The Kronecker–Hurwitz formula can be rephrased by saying that the expression in the last line of (17) equals $\sigma_1(n) = |\mathcal{M}_n/\Gamma|$. The above-mentioned refinement is then:

Proposition 5. *The sum $\sum_{M \in K} \alpha(M)$ equals 1 for each right coset $K \in \mathcal{M}/\Gamma$.*

Proof. In the proof of Theorem 4 we saw that $\widetilde{T} = T_1(1-U) - T_3 - T_4$, so part (b) of the theorem gives $\langle T_3 + T_4, K \rangle = 1$ for each right coset $K \in \mathcal{M} \backslash \Gamma$. But

$$T_3 + T_4 = \sum_{M \in \mathcal{M}} \alpha(M) M$$

by the very definition of α. □

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