where θ is the unit step function, i.e.,

$$\theta(t) = 0 \quad \text{when} \quad t < 0,$$
$$= 1 \quad \text{when} \quad t \geq 0.$$

Equations (11) and (13) hold in an arbitrary system of inertia. The step function in (13) implies that a source point (z_ρ) contributes to the field only at points inside a conical region given by $(x_3 - z_3) \leq -|a|^{\frac{1}{2}} \rho$.

III. TIME HARMONIC SOURCES, 3-DIMENSIONAL REPRESENTATION

Let the source be time harmonic in K: $S_i(x_r) = S_i(x_\rho)e^{-kx_4}$, where $k \equiv \omega/c$ and ω is the frequency. Omitting the time factor e^{-kx_4}, we derive from (9)

$$S_i(z_\rho, x_4 + \tau_-) = S_i(z_\rho) \exp[-ikb(z_3 - x_3)] \times \exp\{ik(n/a)[(z_3 - x_3)^2 + a\rho^2]^{\frac{1}{2}}\}, \quad (14)$$

where

$$b \equiv \frac{n\beta}{1 - (n\beta)^2}\left(n - \frac{1}{n}\right) = \frac{\beta\kappa c^2}{1 - (n\beta)^2}.$$

Furthermore, we derive

$$S_i(z_\rho, x_4 + \tau_-) + S_i(z_\rho, x_4 + \tau_+) = 2S_i(z_\rho)e^{-ikb(z_3 - x_3)} \cos\{(kn/a)[(z_3 - x_3)^2 + a_\rho^2]^{\frac{1}{2}}\}. \quad (15)$$

From the definition of S_i, and making use of the continuity equation $J_{\rho,\rho} = -J_{4,4} = i\omega\rho = kJ_4$, we get, for the spatial components of S_i,

$$S_\lambda(z_\rho) = \mu\left(\delta_{\lambda\nu} + \frac{\kappa}{n^2} U_\lambda U_\nu\right) J_\nu(z_\rho) + i\frac{\mu\kappa c^2}{n^2}\gamma U_\lambda J_{\nu,\nu(z_\rho)}. \quad (16)$$

Similarly,

$$\frac{c}{i} S_4(z_\rho) = \frac{\mu c^2}{i\omega}\left(1 - \frac{\kappa c^2}{n^2}\gamma^2\right) J_{\nu,\nu} + \frac{\mu\kappa c^2}{n^2}\gamma U_\nu J_\nu. \quad (17)$$

Substituting (14)–(17) into (11) and (13) leads to expressions which are in agreement with Ref. 1.

The field vector $\bar{E}$ may be obtained by using the equation $\bar{E} = -\nabla\Phi + i\omega\bar{A}$ (Ref. 1; it can be shown that $\bar{E} = -\nabla\Phi - \partial\bar{A}/\partial t$ holds in any system of inertia). By some calculation the results may be transformed to an expression as given in Ref. 8.

[1] K. S. H. Lee and C. H. Papas, J. Math. Phys. 5, 1668 (1964).
[2] R. T. Compton, Jr., J. Math. Phys. 7, 2145 (1966).
[3] C. T. Tai, J. Math. Phys. 8, 646 (1967).
[4] J. M. Jauch and K. M. Watson, Phys. Rev. 74, 951 (1948).
[5] C. Møller, *The Theory of Relativity* (Oxford U.P., London, 1952).
[6] J. L. Synge, *Relativity: The Special Theory* (North-Holland, Amsterdam, 1965).
[7] Equation (10) in Ref. 3 should contain a sum of two δ functions.
[8] C. T. Tai, IEEE Trans. Antennas Propagation 13, 322 (1965).

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Expansion of an n-Point Function as a Sum of Commutators*

DON B. ZAGIER†

Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts

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We write the n-point function of currents as a sum over nested commutators, a form more suitable for certain current algebra calculations.

I. INTRODUCTION AND RESULT

In this paper we consider the n-point function, defined as

$$T(q_1, \cdots, q_{n-1})_{\mu_1\cdots\mu_n} = \iint\cdots\int d^4x_1 \cdots d^4x_{n-1} e^{-iq_1x_1 - \cdots - iq_{n-1}x_{n-1}} \times \langle 0| T(j_{\mu_1}(x_1)\cdots j_{\mu_{n-1}}(x_{n-1})j_{\mu_n}(0))|0\rangle, \quad (1)$$

where $T(j_1 \cdots j_n)$ is the product of the n current operators $j_1, \cdots, j_n$, in the order of the time components of their points of evaluation:

$$T(j_1(x_1)\cdots j_n(x_n)) = \sum_{\pi\in S_n} \theta(x^0_{\pi(1)} - x^0_{\pi(2)}) \cdots \theta(x^0_{\pi(n-1)} - x^0_{\pi(n)}) \times j_{\pi(1)}(x_{\pi(1)}) \cdots j_{\pi(n)}(x_{\pi(n)}), \quad (2)$$

where the sum is over all permutations π in S_n, the symmetric group of order n. The function $\theta(t)$ is the usual step function. In Eq. (1) we only integrate over $n - 1$ space variables because we are using the translation invariance of the theory to work in a coordinate system where $x_n = 0$ and $q_1 + \cdots + q_n = 0$.

It is a straightforward calculation, which is given below, to rewrite Eq. (1) as a linear combination of products of n current operators (or, rather, of their Fourier transforms in momentum space) not involving the step functions θ. However, current algebra treats only commutators of operators rather than arbitrary products, so that it is desirable to express the n-point function as a linear combination of commutators of operators.

Bjorken[1] and Johnson and Low[2] pointed out that in the case $n = 2$ the 2-point function is asymptotically equal to a commutator term. They showed, specifically, that the leading term in the asymptotic expansion of

$$M_{\mu\nu}(q, \cdots) = -\int d^4x e^{-iq\cdot x}\langle A|\, T(j_\mu(x) j_\nu(0))\,|B\rangle, \quad (3)$$

as $q_0 \to \infty$ with $\mathbf{q}$ fixed, is

$$\frac{1}{q_0}\int d^3\mathbf{x} e^{-i\mathbf{q}\cdot\mathbf{x}}\langle A|\,[j_\mu(0,\mathbf{x}), j_\nu(0,\mathbf{0})]\,|B\rangle \quad (4)$$

and that the higher terms involve time derivatives of the currents; thus the next term is

$$\frac{1}{q_0^2}\, i\int d^3\mathbf{x} e^{-i\mathbf{q}\cdot\mathbf{x}}\left\langle A\left|\left[\frac{dj_\mu}{dt}(0,\mathbf{x}), j_\nu(0,\mathbf{0})\right]\right|B\right\rangle. \quad (5)$$

Not only the leading term, but all subsequent terms in the expansion of the n-point function, as the energies q_i^0 become infinite, can, in fact, be written as sums of equal-time commutators.[3] The expression we obtain is

$$\begin{aligned} T(q_1, \cdots, q_{n-1})_{\mu_1\cdots\mu_n} &= \frac{i^{n-1}}{n}\iint\cdots\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_{n-1} e^{-i\mathbf{q}_1\cdot\mathbf{x}_1-\cdots-i\mathbf{q}_{n-1}\cdot\mathbf{x}_{n-1}} \\ &\quad\times \sum_{\pi\in S_n}\left[\prod_{s=1}^{n-1}\left(E_{\pi(1)} + \cdots + E_{\pi(s)} + i\frac{\partial}{\partial t_{\pi(1)}} + \cdots + i\frac{\partial}{\partial t_{\pi(s)}}\right)^{-1}\right] \\ &\quad\times \langle 0|\,[[\cdots[j_{\mu_{\pi(1)}}, j_{\mu_{\pi(2)}}], \cdots], j_{\mu_{\pi(n)}}]\,|0\rangle\big|_{t_1=\cdots=t_n=0}, \end{aligned} \quad (6)$$

where the meaning of the right-hand side is that for each permutation π we expand each factor

$$\left(E_{\pi(1)} + \cdots + E_{\pi(s)} + i\frac{\partial}{\partial t_{\pi(1)}} + \cdots + i\frac{\partial}{\partial t_{\pi(s)}}\right)^{-1}$$

as a series

$$\sum_{r=0}^{\infty}\left(-i\frac{\partial}{\partial t_{\pi(1)}} - \cdots - i\frac{\partial}{\partial t_{\pi(s)}}\right)^r \Big/ (E_{\pi(1)} + \cdots + E_{\pi(s)})^{r+1},$$

formally multiply these differential operators, apply the product operator to the commutator

$$[[\cdots[j_{\mu_{\pi(1)}}(t_{\pi(1)}, \mathbf{x}_{\pi(1)}), j_{\mu_{\pi(2)}}(t_{\pi(2)}, \mathbf{x}_{\pi(2)})], \cdots], j_{\mu_{\pi(n)}}(t_{\pi(n)}, \mathbf{x}_{\pi(n)})],$$

evaluate at $t_1 = \cdots = t_n = 0$, and finally Fourier-transform the space part of the result and divide by n. We have made the convention that

$$\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_n} = 0$$

(since x_n and hence t_n are identically zero, $\partial/\partial t_n$ is undefined), and t_i and E_i are the time components of x_i^μ and q_i^μ, respectively. Written out in full, our result is

$$\begin{aligned} T(E_1, \cdots, E_{n-1})_{1\cdots n} &= i^{n-1}\iint\cdots\int dt_1\cdots dt_{n-1} \\ &\quad\times e^{-iE_1t_1-\cdots-iE_{n-1}t_{n-1}}\langle 0|\, T(j_{\mu_1}(x_1)\cdots j_{\mu_n}(0))\,|0\rangle \\ &= \frac{1}{n}\sum_{\pi\in S_n}\sum_{r_1=0}^{\infty}\cdots\sum_{r_{n-1}=0}^{\infty} \\ &\quad\times\left(\prod_{s=1}^{\pi^{-1}(n)-1}\frac{(-i\partial/\partial t_{\pi(1)} - \cdots - i\partial/\partial t_{\pi(s)})^{r_s}}{(E_{\pi(1)} + \cdots + E_{\pi(s)})^{r_s+1}}\right) \\ &\quad\times\left(\prod_{s=\pi^{-1}(n)}^{n-1}\frac{(i\partial/\partial t_{\pi(s+1)} + \cdots + i\partial/\partial t_{\pi(n)})^{r_s}}{(E_{\pi(1)} + \cdots + E_{\pi(s)})^{r_s+1}}\right) \\ &\quad\times\langle 0|\,[[\cdots[j_{\mu_{\pi(1)}}(t_{\pi(1)}, \mathbf{x}_{\pi(1)}), j_{\mu_{\pi(2)}}(t_{\pi(2)}, \mathbf{x}_{\pi(2)})], \cdots], \\ &\qquad j_{\mu_{\pi(n)}}(t_{\pi(n)}, \mathbf{x}_{\pi(n)})]\,|0\rangle. \end{aligned}$$

Here we have omitted the integration over the space variables.

Before deriving this result, we will state it in a different form. Since the space variables and integrations do not affect the problem, we will cease to write them; similarly, we omit the brackets $\langle 0|\cdots|0\rangle$ denoting the vacuum state. Although the j_{μ_i} are, in fact, components of a single current, we do not use this, but treat them as separate functions; since the subscripts μ_i do not change, we omit them. Thus $j_{\mu_i}(t_i, \mathbf{x}_i)$ will be denoted $j_i(t_i)$ for $1 \le i \le n-1$. For convenience, we define j_n by $j_n(t) = j_{\mu_n}(0)\delta(t)$. We use the following Fourier transform:

$$\tilde{f}(E) = \frac{1}{2\pi}\int e^{-iEt}f(t)\,dt, \quad f(t) = \int e^{+iEt}\tilde{f}(E)\,dE. \quad (7)$$

With this definition of the Fourier transform, we have

$$\begin{aligned} (-i)^r\frac{d^r}{dt^r}f(t)\big|_{t=0} &= \int(-i)^r\frac{\partial^r}{\partial t^r}e^{iEt}\tilde{f}(E)\,dE\big|_{t=0} \\ &= \int E^r\tilde{f}(E)\,dE. \end{aligned} \quad (8)$$

Making all of these changes and substituting the definition (2) for the time-ordered product, we obtain as the theorem to be demonstrated,[4]

$$\begin{aligned} &i^{n-1}\iint\cdots\int dt_1\cdots dt_n e^{-iE_1t_1-\cdots-iE_nt_n} \\ &\quad\times\sum_{\pi\in S_n}\theta(t_{\pi(1)} - t_{\pi(2)})\cdots\theta(t_{\pi(n-1)} - t_{\pi(n)}) \\ &\quad\times j_{\pi(1)}(t_{\pi(1)})\cdots j_{\pi(n)}(t_{\pi(n)}) \\ &= \frac{2\pi}{n}\iint\cdots\int dE_1'\cdots dE_n'\delta(E_1' + \cdots + E_n') \\ &\quad\times\sum_{\pi\in S_n}\frac{[[\cdots[\tilde{j}_{\pi(1)}(E_1'), \tilde{j}_{\pi(2)}(E_2')], \cdots], \tilde{j}_{\pi(n)}(E_n')]}{\prod_{s=1}^{n-1}(E_{\pi(1)} + \cdots + E_{\pi(s)} - E_1' - \cdots - E_s')}. \end{aligned} \quad (9)$$

This is the real result, and the previous form, in which integrations had been done by using Eq. (8), is simply its expansion for large energies [which may not be valid; it is only the identity (9) that will be proved rigorously].

The proof of the theorem, Eq. (9), is purely algebraic in nature and so makes no reference to the actual existence of the commutators and T products which we consider. It has been shown[5] that, in fact, they do not exist in certain perturbation-theoretical models. We regard the question of their existence in general as being open at the present time. In the following, we assume that there is a theory for which the problems found in Ref. 5 do not exist.

II. PROOF

The proof of Eq. (9) will proceed in two stages: first, transforming the left-hand side to an expression involving a linear combination of products of n current operators, and then rewriting this as a sum of commutators. With the normalization of Eqs. (7), the Fourier transform of $\theta(t-t_0)$ is $e^{-iEt_0}/2\pi iE$, and the rule for transforming a product is

$$\widetilde{fg}(E)=\frac{1}{2\pi}\int f(t)g(t)e^{-iEt}\,dt=\int\tilde{f}(E')\tilde{g}(E-E')\,dE',\tag{10}$$

so that

$$\int\theta(t-t_0)f(t)e^{-iEt}\,dt=\int\frac{\tilde{f}(E')e^{-i(E-E')t_0}}{i(E-E')}\,dE'.\tag{11}$$

Applying this repeatedly, we obtain [abbreviating $\theta(t_i-t_j)$ to θ_{ij}]

$$\begin{aligned}
&\iint\cdots\int dt_1\cdots dt_n\\
&\quad\times e^{-iE_{\pi(1)}t_1-\cdots-iE_{\pi(n)}t_n}\theta_{12}\cdots\theta_{n-1,n}j_{\pi(1)}(t_1)\cdots j_{\pi(n)}(t_n)\\
&=\iint\cdots\int dt_2\cdots dt_n\\
&\quad\times e^{-iE_{\pi(2)}t_2-\cdots-iE_{\pi(n)}t_n}\theta_{23}\theta_{34}\cdots\theta_{n-1,n}\\
&\quad\times\int\frac{\tilde{j}_{\pi(1)}(E_1')e^{-i(E_{\pi(1)}-E_1')t_2}}{i(E_{\pi(1)}-E_1')}dE_1'j_{\pi(2)}(t_2)\cdots j_{\pi(n)}(t_n)\\
&=\iint\cdots\int dE_1'\,dE_2'\,dt_3\cdots dt_n\\
&\quad\times e^{-iE_{\pi(3)}t_3-\cdots-iE_{\pi(n)}t_n}e^{-i(E_{\pi(1)}+E_{\pi(2)}-E_1'-E_2')t_3}\\
&\quad\times\theta_{34}\cdots\theta_{n-1,n}\frac{\tilde{j}_{\pi(1)}(E_1')}{i(E_{\pi(1)}-E_1')}\\
&\quad\times\frac{\tilde{j}_{\pi(2)}(E_2')}{i(E_{\pi(1)}+E_{\pi(2)}-E_1'-E_2')}j_{\pi(3)}(t_3)\cdots j_{\pi(n)}(t_n)\\
&=\cdots\\
&=\frac{1}{i^{n-1}}\iint\cdots\int dE_1'\cdots dE_{n-1}'\,dt_n\\
&\quad\times e^{-i(E_{\pi(1)}+\cdots+E_{\pi(n)}-E_1'-\cdots-E_{n-1}')t_n}\\
&\quad\times\frac{\tilde{j}_{\pi(1)}(E_1')}{E_{\pi(1)}-E_1'}\frac{\tilde{j}_{\pi(2)}(E_2')}{E_{\pi(1)}+E_{\pi(2)}-E_1'-E_2'}\cdots\\
&\quad\times\frac{\tilde{j}_{\pi(n-1)}(E_{n-1}')}{E_{\pi(1)}+\cdots+E_{\pi(n-1)}-E_1'-\cdots-E_{n-1}'}j_{\pi(n)}(t_n).
\end{aligned}$$

The final expression contains no θ functions and, therefore, can be evaluated by the direct substitution of the first of Eqs. (7). We notice that

$$E_{\pi(1)}+\cdots+E_{\pi(n)}=E_1+\cdots+E_n=0$$

(since $q_1+\cdots+q_n$ was zero), so that the result is

$$\frac{2\pi}{i^{n-1}}\iint\cdots\int dE_1'\cdots dE_{n-1}'\times\frac{\tilde{j}_{\pi(1)}(E_1')\cdots\tilde{j}_{\pi(n-1)}(E_{n-1}')\tilde{j}_{\pi(n)}(-E_1'-\cdots-E_{n-1}')}{\prod_{s=1}^{n-1}(E_{\pi(1)}+\cdots+E_{\pi(s)}-E_1'-\cdots-E_s')},$$

which may be written more symmetrically as

$$\frac{2\pi}{i^{n-1}}\iint\cdots\int dE_1'\cdots dE_n'\delta(E_1'+\cdots+E_n')\times\frac{\tilde{j}_{\pi(1)}(E_1')\cdots\tilde{j}_{\pi(n)}(E_n')}{\prod_{s=1}^{n-1}(E_{\pi(1)}+\cdots+E_{\pi(s)}-E_1'-\cdots-E_s')}.$$

This completes the first stage. Multiplying our last equation by i^{n-1} and summing over all permutations π of $1, 2, \cdots, n$, for the left-hand side of Eq. (9), we obtain the expression

$$2\pi\iint\cdots\int dE_1'\cdots dE_n'\delta(E_1'+\cdots+E_n')\times\sum_{\pi\in S_n}\frac{\tilde{j}_{\pi(1)}(E_1')\cdots\tilde{j}_{\pi(n)}(E_n')}{\prod_{s=1}^{n-1}(E_{\pi(1)}+\cdots+E_{\pi(s)}-E_1'-\cdots-E_s')},$$

the desired expression as a sum of products of the $\tilde{j}_i$. Both in this expression and in Eq. (9) we could just as well have written $E_{\pi(i)}'$ for E_i' in the denominators, since for each permutation π one could relabel the symmetric expression

$$\iint\cdots\int dE_1'\cdots dE_n'\delta(E_1'+\cdots+E_n').$$

Therefore, it suffices to prove the purely algebraic identity

$$n\sum_{\pi\in S_n}\frac{\tilde{j}_{\pi(1)}(E_{\pi(1)}')\cdots\tilde{j}_{\pi(n)}(E_{\pi(n)}')}{\prod_{s=1}^{n-1}(E_{\pi(1)}+\cdots+E_{\pi(s)}-E_{\pi(1)}'-\cdots-E_{\pi(s)}')}=\sum_{\pi\in S_n}\frac{[[\cdots[\tilde{j}_{\pi(1)}(E_{\pi(1)}'),\tilde{j}_{\pi(2)}(E_{\pi(2)}')],\cdots],\tilde{j}_{\pi(n)}(E_{\pi(n)}')]}{\prod_{s=1}^{n-1}(E_{\pi(1)}+\cdots+E_{\pi(s)}-E_{\pi(1)}'-\cdots-E_{\pi(s)}')}.\tag{12}$$

Having simplified the problem, we will again simplify our notation. Since the character of $\tilde{j}$ as a function of energy no longer interests us, we will omit the argument and tilde and write simply j_i for $\tilde{j}_i(E_i')$. Next, we have $E_1' + \cdots + E_n' = 0$. [Because of the presence of the Dirac δ, it does not matter whether or not the sum in Eq. (9) can be written as a sum of commutators off the subspace $E_1' + \cdots + E_n' = 0$; indeed, simple examples show that it cannot.] Of course, $E_1 + \cdots + E_n$ is always zero. Furthermore, because we were able to group together E's and E''s with the same subscript in Eq. (12), we can now define a new set of numbers $F_i = E_i - E_i'$, and the identity to be demonstrated becomes

$$n \sum_{\pi\in S_n} \frac{j_{\pi(1)}\cdots j_{\pi(n)}}{F_{\pi(1)}(F_{\pi(1)}+F_{\pi(2)})\cdots(F_{\pi(1)}+\cdots+F_{\pi(n-1)})} = \sum_{\pi\in S_n} \frac{[[\cdots[j_{\pi(1)},j_{\pi(2)}],\cdots],j_{\pi(n)}]}{F_{\pi(1)}(F_{\pi(1)}+F_{\pi(2)})\cdots(F_{\pi(1)}+\cdots+F_{\pi(n-1)})}, \tag{13}$$

where the F_i are n numbers such that their sum is zero, but no subsum is zero [the identity is meaningless if any subsum vanishes, but it suffices to prove it in the converse case since the region where some-subsum vanishes has zero measure in the $(n-1)$-dimensional space $E_1' + \cdots + E_n' = 0$], the j_i are noncommuting quantities, and the sums extend over all permutations of $1, 2, \cdots, n$.

To prove Eq. (13), we evidently have to expand the commutators on the right and then rearrange the sum so that we can pick out the coefficient of a given product $j_{\pi(1)}\cdots j_{\pi(n)}$ and check that it is indeed

$$n/F_{\pi(1)}(F_{\pi(1)}+F_{\pi(2)})\cdots(F_{\pi(1)}+\cdots+F_{\pi(n-1)}).$$

A commutator of n operators has 2^{n-1} terms, half of them positive and half negative, and we must start by finding the rule which determines which of the $n!$ possible permutations appear and with what sign. If we expand a commutator such as $[[[[h_1, h_2], h_3], h_4], h_5]$, then typical terms are $h_5h_4h_1h_2h_3$ and $-h_3h_1h_2h_4h_5$. Inspecting the terms, we see that each one has descending subscripts up to h_1 and then ascending, so that it is in the form $h_{\sigma(1)}h_{\sigma(2)}\cdots h_{\sigma(n)}$, where $\sigma(1) > \cdots > \sigma(k) = 1 < \sigma(k+1) < \cdots < \sigma(n)$ for some k, and that the sign of such a term is $(-1)^{k-1}$. Now, for a given value of k, we can choose any $k-1$ of the $n-1$ numbers $2, 3, \cdots, n$ to precede $\sigma(k) = 1$, but then their order is determined; thus the set $S_{n,k}$ of permutations σ with $\sigma(1) > \cdots > \sigma(k) = 1 < \sigma(k+1) < \cdots < \sigma(n)$ has $\binom{n-1}{k-1}$ members and, since

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1},$$

if all the terms of the commutator are of the special form considered, then all terms of this form appear in the commutator. That this is, in fact, the case can be seen easily by induction: We have to prove

$$[[\cdots[h_1, h_2], \cdots], h_n] = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\sigma\in S_{n,k}} h_{\sigma(1)}\cdots h_{\sigma(n)}, \tag{14}$$

an identity plainly valid for n equal to one or two. If it is valid for n, then $[[[\cdots[h_1, h_2], \cdots], h_n], h_{n+1}]$ is given by

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^{k-1}\sum_{\sigma\in S_{n,k}}(h_{\sigma(1)}h_{\sigma(2)}\cdots h_{\sigma(n)}h_{n+1} - h_{n+1}h_{\sigma(1)}h_{\sigma(2)}\cdots h_{\sigma(n)})\\
&= \sum_{k=1}^{n}(-1)^{k-1} \sum_{\substack{\pi\in S_{n+1}\\ \pi(1)>\cdots>\pi(k)\\ =1<\cdots<\pi(n+1)=n+1}} h_{\pi(1)}\cdots h_{\pi(n+1)}
+ \sum_{k=2}^{n+1}(-1)^{k-1} \sum_{\substack{\pi\in S_{n+1}\\ n+1=\pi(1)>\cdots>\pi(k)\\ =1<\cdots<\pi(n+1)}} h_{\pi(1)}\cdots h_{\pi(n+1)}\\
&= \sum_{k=1}^{n+1}(-1)^{k-1}\sum_{\pi\in S_{n+1,k}} h_{\pi(1)}\cdots h_{\pi(n+1)},
\end{aligned}$$

there being the last equality because any permutation π of $S_{n+1,k}$ must have either $\pi(1)$ or $\pi(n+1)$ equal to $n+1$ [since any other $\pi(i)$ is smaller than one of these]. This completes the proof of Eq. (14).

Now we can expand the right-hand side of Eq. (13) to obtain

$$\sum_{\pi\in S_n} \frac{[[\cdots[j_{\pi(1)},j_{\pi(2)}],\cdots],j_{\pi(n)}]}{F_{\pi(1)}(F_{\pi(1)}+F_{\pi(2)})\cdots(F_{\pi(1)}+\cdots+F_{\pi(n-1)})} = \sum_{\pi\in S_n}\sum_{k=1}^{n}\sum_{\sigma\in S_{n,k}}(-1)^{k-1} \times \frac{j_{\pi\sigma(1)}\cdots j_{\pi\sigma(n)}}{F_{\pi(1)}(F_{\pi(1)}+F_{\pi(2)})\cdots(F_{\pi(1)}+\cdots+F_{\pi(n-1)})}.$$

The coefficient of $j_{\tau(1)}j_{\tau(2)}\cdots j_{\tau(n)}$ in this is obtained by noting that for each σ in $S_{n,k}$ there is a unique π in S_n (namely, $\pi = \tau\sigma^{-1}$) such that $\pi\sigma = \tau$; thus, the coefficient is

$$\sum_{k=1}^{n}\sum_{\sigma\in S_{n,k}} \frac{(-1)^{k-1}}{F_{\tau\sigma^{-1}(1)}(F_{\tau\sigma^{-1}(1)}+F_{\tau\sigma^{-1}(2)})\cdots(F_{\tau\sigma^{-1}(1)}+\cdots+F_{\tau\sigma^{-1}(n-1)})}.$$

To prove Eq. (13), we must show that this sum equals $n \prod_{s=1}^{n-1} (F_{\tau(1)} + \cdots + F_{\tau(s)})^{-1}$. By expanding the commutator, we have gotten rid of the operators and have reduced the problem to an algebraic identity among ordinary numbers. Having fastened our attention on a single permutation τ, we need no longer carry it as a subscript, but set $G_i = F_{\tau(i)}$. If we can prove

$$\sum_{\sigma\in S_{n,k}} \frac{1}{G_{\sigma^{-1}(1)}(G_{\sigma^{-1}(1)} + G_{\sigma^{-1}(2)}) \cdots (G_{\sigma^{-1}(1)} + \cdots + G_{\sigma^{-1}(n-1)})} = \frac{(-1)^{k-1}}{G_1(G_1 + G_2)\cdots(G_1 + \cdots + G_{n-1})} \quad (15)$$

for $1 \leq k \leq n$, then the desired equality will follow on summation from 1 to n.

We will prove Eq. (15) by another simple induction. For $n = 2$, it reduces to $+1/G_1 = +1/G_1$ or $+1/G_2 = -1/G_1$, depending on whether k is one or two, and both are true since $G_1 + G_2$ is zero. Our previous induction hinged on the fact that, for σ in $S_{n,k}$, either $\sigma(1)$ or $\sigma(n)$ must be n, since each $\sigma(i)$ is smaller than one of them; this one depends on the fact that $\sigma(k-1)$ or $\sigma(k+1)$ must be 2 since every $\sigma(i)$, except $\sigma(k) = 1$, is greater than one of them. Hence the left-hand side of Eq. (15) is

$$\frac{1}{G_k} \sum_{\substack{\sigma\in S_{n,k}\\ \sigma(k-1)=2}} \frac{1}{(G_k + G_{k-1})(G_k + G_{k-1} + G_{\sigma^{-1}(3)}) \cdots (G_k + G_{k-1} + \cdots + G_{\sigma^{-1}(n-1)})}$$
$$+ \frac{1}{G_k} \sum_{\substack{\sigma\in S_{n,k}\\ \sigma(k+1)=2}} \frac{1}{(G_k + G_{k+1})(G_k + G_{k+1} + G_{\sigma^{-1}(3)}) \cdots (G_k + G_{k+1} + \cdots + G_{\sigma^{-1}(n-1)})}. \quad (16)$$

If k is 1, the first sum is empty and, if k is n, the second is also empty; but if this is kept in mind, the following proof still is applicable. In any case Eq. (15) is almost trivial for $k = 1$ or $k = n$. To evaluate the two sums in Eq. (16), we use the fact that both can be transformed to special cases of Eq. (15) for $n - 1$. Thus, if we define numbers H_i for $1 \leq i \leq n - 1$ and [for each σ in the first sum in Eq. (16)] a permutation π of S_{n-1} by

$$\begin{aligned} H_i &= G_i, & 1 \leq i \leq k-2, \quad & \pi(i) = \sigma(i) - 1, & 1 \leq i \leq k-2,\\ &= G_{k-1} + G_k, & i = k-1, \quad & = 1, & i = k-1,\\ &= G_{i+1}, & k \leq i \leq n-1, \quad & = \sigma(i+1) - 1, & k \leq i \leq n-1, \end{aligned}$$

and notice that

$$H_1 + \cdots + H_{n-1} = G_1 + \cdots + (G_{k-1} + G_k) + \cdots + G_n = 0,$$

we can rewrite the first sum in Eq. (16) as

$$\sum_{\pi\in S_{n-1,k-1}} \frac{1}{(H_{\pi^{-1}(1)})(H_{\pi^{-1}(1)} + H_{\pi^{-1}(2)}) \cdots (H_{\pi^{-1}(1)} + \cdots + H_{\pi^{-1}(n-2)})} = \frac{(-1)^{k-2}}{H_1(H_1 + H_2)\cdots(H_1 + \cdots + H_{n-2})},$$

the equality following from the induction hypothesis. Hence the first term in Eq. (16) is

$$(-1)^{k-2}/G_k G_1(G_1 + G_2)\cdots(G_1 + \cdots + G_{k-2}) \times (G_1 + \cdots + G_{k-1} + G_k)\cdots(G_1 + \cdots + G_{n-1}).$$

Exactly similarly, the second term in Eq. (16) is

$$(-1)^{k-1}/G_k G_1(G_1 + G_2)\cdots(G_1 + \cdots + G_{k-1}) \times (G_1 + \cdots + G_k + G_{k+1})\cdots(G_1 + \cdots + G_{n-1}).$$

Adding these, we see that the expression (16), which represents the left-hand side of Eq. (15), equals

$$\frac{(-1)^{k-1}(G_1 + \cdots + G_k) + (-1)^{k-2}(G_1 + \cdots + G_{k-1})}{G_k G_1(G_1 + G_2)\cdots(G_1 + \cdots + G_{n-1})} = \frac{(-1)^{k-1}}{G_1(G_1 + G_2)\cdots(G_1 + \cdots + G_{n-1})},$$

which is the desired right-hand side of Eq. (15). It is interesting that the crucial hypothesis $G_1 + \cdots + G_n = 0$ did not enter the proof except to establish the case $n = 2$, and for larger n it was only needed to be able to apply the induction hypothesis.

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† Present address: New College, Oxford, England.

[1] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[2] K. Johnson and F. Low, Progr. Theoret. Phys. (Kyoto), Suppl. Nos. **37–38**, 74 (1966).

[3] Some partial results of this type have been used by P. Oleson [Phys. Rev. **175**, 2165 (1968)] and A. Sirlin [Phys. Rev. **176**, 1875 (1968)]. Our main result, Eq. (9), agrees with the specific examples discussed by these authors.

[4] The reason for the factor 2π in the right-hand side of Eq. (9) is that, with the Fourier transform of Eq. (7), the transform $\tilde{j}_n(E)$ is not $j_{\mu_n}(0)$ but $(2\pi)^{-1}j_{\mu_n}(0)$. [$j_n(t)$ was defined as $j_{\mu_n}(0)\delta(t)$.]

[5] R. Jackiw and G. Preparata, Phys. Rev. Letters **22**, 975 (1969); S. Adler and W.-K. Tung, Phys. Rev. Letters **22**, 978 (1969).