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INEQUALITIES FOR THE GINI COEFFICIENT OF COMPOSITE POPULATIONS

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Unlike other popular measures of income inequality, the Gini coefficient is not decomposable, i.e., the coefficient $G(\mathscr{X})$ of a composite population $\mathscr{X}=\mathscr{X}_1\cup\ldots\cup\mathscr{X}_r$ cannot be computed in terms of the sizes, mean incomes and Gini coefficients of the components $\mathscr{X}_i$. In this paper upper and lower bounds (best possible for $r=2$) for $G(\mathscr{X})$ in terms of these data are given. For example, $G(\mathscr{X}_1\cup\ldots\cup\mathscr{X}_r)\geqq\sum\alpha_i G(\mathscr{X}_i)$, where α_i is the proportion of the population in $\mathscr{X}_i$. One of the tools used, which may be of interest for other applications, is a lower bound for $\int_0^\infty f(x)g(x)\,\mathrm{d}x$ (converse to Cauchy's inequality) for monotone decreasing functions f and g.

1. Introduction

A very frequently used measure of the inequality of incomes in a population $\mathscr{X}$ is the *Gini coefficient*,[1]

$$G(\mathscr{X})=\frac{1}{2n^2\mu}\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i-x_j|. \tag{1}$$

(Here and in what follows we fix the following conventions: n denotes the size of $\mathscr{X}$, $x_1,\ldots,x_n$ are non-negative real numbers representing the individual incomes, and $X=\sum_{i=1}^n x_i$ and $\mu=X/n$ are the total and mean incomes of the population, respectively.) In many situations, the population $\mathscr{X}$ is the union of several population groups $\mathscr{X}_1,\ldots,\mathscr{X}_r$ (e.g. male and female wage earners, or the members of various racial or social groups) and one would like to know how much of the inequality of income distribution in $\mathscr{X}$ is due to the inequality within the separate groups $\mathscr{X}_i$ and how much to the disparateness of income between the groups. Such information might, for instance, be useful in gauging the effectiveness of a differentiated tax structure in reducing

[1]Note that we are averaging $|x_i-x_j|$ over all n^2 pairs i,j with $1\leqq i,j\leqq n$, not only over the $n(n-1)/2$ pairs with $i<j$ as is sometimes done.

inequality. To do this, one would like to know *a priori* bounds of the form

$$F_1(n_1, X_1, G_1; \dots; n_r, X_r, G_r; G_0) \leqq G(\mathscr{X})$$
$$\leqq F_2(n_1, X_1, G_1; \dots; n_r, X_r, G_r; G_0),$$

where F_1 and F_2 are universal functions of the sizes n_i, total incomes X_i and Gini coefficients G_i of the individual population groups $\mathscr{X}_i$ and of the 'between-group' Gini coefficient G_0 (=the Gini coefficient of the hypothetical population $\mathscr{X}'$ obtained from $\mathscr{X}$ by having the members of each subpopulation $\mathscr{X}_i$ share their incomes equally, so that they all have an income equal to $\mu_i = X_i/n_i$). This will describe the range of inequality possible for the composite population $\mathscr{X}$, and then seeing in a given case whether $G(\mathscr{X})$ is in fact closer to F_1 or F_2 will give us a feel for the nature of the sources of income inequality in $\mathscr{X}$.

The purpose of this paper is to give such bounds. In section 3 we prove five inequalities giving lower bounds for $G(\mathscr{X})$. In section 4 we show by examples that the result of section 3 (i.e., the maximum of the five bounds) is the best possible universal lower bound F_1 in the case $r=2$, and also establish the best possible universal upper bound F_2 in the same case. The simplest, though not the strongest, of the lower bounds in section 3 is the elegant convexity property

$$G(\mathscr{X}) \geqq \sum_{i=1}^{r} (n_i/n) G(\mathscr{X}_i), \tag{2}$$

which had been discovered experimentally by Sudhir Anand on the basis of data on the income distribution in Malaysia [Anand (1983)], the groups there being the Malay, Indian and Chinese populations. I would like to thank him for suggesting the problem of proving the inequality (2) and thereby stimulating my interest in the subject of this paper.

Note. This paper is part of the longer paper [Zagier (1982)], the rest of which is devoted to a complete classification of 'decomposable' indices of inequality, i.e., indices $I(\mathscr{X})$ which, as well as certain obviously desirable properties like the 'Pigou–Dalton condition' (see below), have the property that $I(\mathscr{X})$ for a composite population $\mathscr{X} = \mathscr{X}_1 \cup \dots \cup \mathscr{X}_r$ is given by an exact formula,

$$I(\mathscr{X}) = F(n_1, X_1, I_1; \dots; n_r, X_r, I_r; I_0),$$

in terms of the sizes, incomes and indices of the component populations and the between-group index I_0, where F is a universal function, assumed to be linear in the I_i $(0 \leqq i \leqq r)$. [An example of such an index is the Theil index $T(\mathscr{X}) = (1/n) \sum (x_i/\mu) \log(x_i/\mu)$.] The classification showed that all such I are in certain respects less attractive measures of inequality than G, so that we are justified in using the popular Gini coefficient even though it is not decomposable. The original version of the paper, with the inequalities for the Gini coefficient and the classification of decomposable indices, was submitted in 1976 to another journal and held up for several years (so-called JET lag). In the meantime, as Professor Atkinson has kindly informed me, the classification theorem, or essentially equivalent results, has been published by Cowell (1980), Bourguignon (1979) and Shorrocks (1980); I have therefore suppressed it in the present paper.

2. Basic properties of the Gini coefficient

The Gini coefficient is the best-known and most frequently used measure of inequality; its properties have been studied by many authors [Anand (1983, annex A), Atkinson (1970, 1975), David (1968), Fei–Ranis (1974), Gastwirth (1972, 1975), Kendall–Stuart (1963), Pyatt (1976), Rao (1969); for surveys of the literature see Blümle (1975, II.1.4), Sen (1973, ch. 2), Theil (1967), von Weizsäcker (1967, part I, ch. II)]. In this section we review and re-prove some of its main properties and introduce a function $\xi(t)$ which will play a basic role in this paper.

There are several equivalent definitions of the Gini coefficient. For example, using the identity

$$|a-b| = a + b - 2\min(a,b),$$

we can rewrite (1) in the form

$$G = 1 - \frac{1}{nX} \sum_{i=1}^{n} \sum_{j=1}^{n} \min(x_i, x_j). \tag{3}$$

If we order the incomes x_i so that $x_i \geqq \ldots \geqq x_n$, then [since $\min(x_i, x_j) = x_{\max(i,j)}$ and there are $2k-1$ pairs (i,j) with $\max(i,j) = k$] (3) becomes

$$G = 1 - \frac{1}{nX} \sum_{k=1}^{n} (2k-1) x_k = \frac{n+1}{n} - \frac{2}{nX}(x_1 + 2x_2 + \ldots + nx_n).$$

This can be interpreted as saying that the Gini coefficient is based on a utility function which is a weighted sum of the individual income levels, the weight of the ith richest individual being proportional to i [cf. Sen (1973, p. 31)].

For very large populations $\mathscr{X}$, one often uses a continuous rather than a discrete formalism and describes $\mathscr{X}$ by a continuous frequency distribution function $f(x)$, where $f(x)\,\mathrm{d}x$ represents the proportion of the population with income between x and $x+\mathrm{d}x$. Clearly

$$\int_0^\infty f(x)\,\mathrm{d}x=1, \qquad \int_0^\infty x f(x)\,\mathrm{d}x=\mu. \tag{4}$$

In terms of $f(x)$, the eqs. (1) and (3) can be rewritten as

$$G=\frac{1}{2\mu}\int_0^\infty\int_0^\infty |x-y| f(x)f(y)\,\mathrm{d}x\mathrm{d}y, \tag{1'}$$

and

$$G=1-\frac{1}{\mu}\int_0^\infty\int_0^\infty \min(x,y) f(x)f(y)\,\mathrm{d}x\mathrm{d}y. \tag{3'}$$

Finally, the most common definition of the Gini coefficient is as twice the area between the 'Lorenz curve' and the diagonal,

$$G=2\int_0^1 (p-L(p))\,\mathrm{d}p, \tag{5}$$

where $L(p)$ is defined parametrically by

$$p=\int_0^x f(t)\,\mathrm{d}t, \qquad L(p)=\frac{1}{\mu}\int_0^x t f(t)\,\mathrm{d}t, \tag{6}$$

so that $L(p)$ represents the fraction of the total income which is earned by the poorest np people. The Lorenz curve, the graph of $L(p)$, is a convex curve going from $(0,0)$ to $(1,1)$, the slope at p being the relative income of the corresponding percentile of the population. Definition (5) shows that G satisfies the Pigou–Dalton condition [Sen (1973)] or principle of transfers, according to which a small transfer of income from a richer to a poorer individual should decrease the index of inequality. Indeed, by a well-known result of Atkinson (1970) [see also Rothschild–Stiglitz (1973)], based on a much older inequality of Hardy, Littlewood and Polya (1967), a distribution $\mathscr{Y}$ can be obtained from a distribution $\mathscr{X}$ by a sequence of transfers from rich to poor if and only if the Lorenz curve of $\mathscr{Y}$ is higher than that of $\mathscr{X}$.

We give one other definition of the Gini coefficient, involving the function $\xi(t)$ mentioned at the beginning of this section. This function is defined as the number of people with income of at least t, i.e., in the discrete formalism

$$\xi(t)=\#\{i \mid x_i \geqq t\} \tag{7}$$

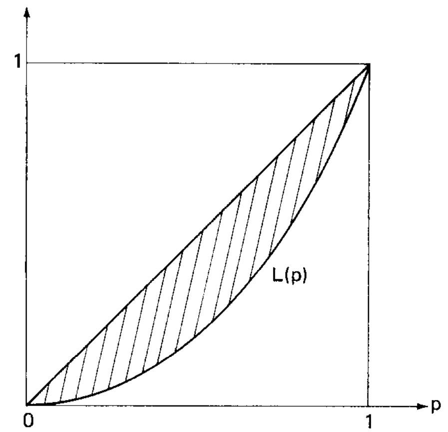

Fig. 1

(a step function), and in the continuous formalism

$$\xi(t)=n\int_t^\infty f(x)\,\mathrm{d}x \tag{7'}$$

(a differentiable function); it satisfies the properties

$$\begin{aligned}&\xi(t)\geqq 0, \qquad \xi \text{ monotone decreasing},\\ &\xi(0)=n, \qquad \int_0^\infty \xi(t)\,\mathrm{d}t=X=n\mu.\end{aligned} \tag{8}$$

Clearly the function ξ describes the income distribution $\mathscr{X}$ completely. W now find with definition (7)

$$\begin{aligned}\sum_{i=1}^n\sum_{j=1}^n \min(x_i,x_j) &= \sum_{i=1}^n\sum_{j=1}^n \int_0^{\min(x_i,x_j)} \mathrm{d}t\\ &=\int_0^\infty\Bigg(\sum_{\substack{i=1\ j=1\\ \min(x_i,x_j)\geqq t}}^{n\quad n} 1\Bigg)\mathrm{d}t\\ &=\int_0^\infty \xi(t)^2\,\mathrm{d}t,\end{aligned}$$

and similarly with definition (7′)

$$\int_0^\infty\int_0^\infty \min(x,y)f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^\infty\int_t^\infty f(x)\,\mathrm{d}x\int_t^\infty f(y)\,\mathrm{d}y\,\mathrm{d}t$$

$$=\frac{1}{n^2}\int_0^\infty \xi(t)^2\,\mathrm{d}t.$$

Combining this with (3) or (3′) gives

$$G=1-\frac{1}{n^2\mu}\int_0^\infty \xi(t)^2\,\mathrm{d}t \tag{9}$$

in both cases. This is the definition of G which will prove to be the most useful one for studying its decomposition properties. As a first application we give a short proof of the equivalence of formula (5) with the other definitions of the Gini coefficient: using (7′) and the parametrization (6) of the Lorenz curve we find

$$2\int_0^1 (p-L(p))\,\mathrm{d}p=1-2\int_0^\infty\left(\frac{1}{\mu}\int_0^x tf(t)\,\mathrm{d}t\right)f(x)\,\mathrm{d}x$$

$$=1+\frac{2}{n\mu}\int_0^\infty\left(\int_0^x tf(t)\,\mathrm{d}t\right)\mathrm{d}\xi(x)$$

$$=1-\frac{2}{n\mu}\int_0^\infty \xi(x)xf(x)\,\mathrm{d}x$$

(integration by parts)

$$=1+\frac{1}{n^2\mu}\int_0^\infty x\,\mathrm{d}(\xi(x)^2)$$

$$=1-\frac{1}{n^2\mu}\int_0^\infty \xi(x)^2\,\mathrm{d}x$$

(integration by parts again), and this equals G by eq. (9).

With a similar calculation, one sees that the inequality index $\frac{1}{6}\int_0^1(1-p)\times(p-L(p))\mathrm{d}p$ proposed by Mehran (1976) equals $1-1/(n^3\mu)\int_0^\infty \xi(t)^3\,\mathrm{d}t$ [or, equivalently, $1-1/(n^3\mu)\sum_{i,j,k}\min(x_i,x_j,x_k)$].

3. Lower bounds for the Gini coefficient of a many-component population

As in the introduction, consider a population $\mathscr{X}$ divided into r smaller groups $\mathscr{X}_i$ $(i=1,\ldots,r)$,

$$\mathscr{X}=\mathscr{X}_1\cup\ldots\cup\mathscr{X}_r, \tag{10}$$

and denote by n_i, X_i and μ_i the size, total income and mean income of $\mathscr{X}_i$, respectively. Thus

$$\sum_{i=1}^{r} n_i=n, \qquad \sum_{i=1}^{r} n_i\mu_i=\sum_{i=1}^{r} X_i=X=n\mu. \tag{11}$$

We are interested in relating the Gini coefficient of $\mathscr{X}$ to the Gini coefficients $G(\mathscr{X}_i)$ of the various population groups $\mathscr{X}_i$ and to the 'between-group index' G_0 (cf. section 1). By the Pigou–Dalton condition,

$$G(\mathscr{X})\geqq G_0; \tag{12}$$

we wish to strengthen this lower bound.

Theorem 1. With the above notations, we have the following lower bounds for the Gini coefficient of the composite population $\mathscr{X}$ in terms of the individual Gini coefficients:

(a) $$G(\mathscr{X})\geqq \sum_{i=1}^{r}\frac{n_i}{n}G(\mathscr{X}_i)$$

(b) $$G(\mathscr{X})\geqq \sum_{i=1}^{r}\frac{X_i}{X}G(\mathscr{X}_i)$$

(c) $$G(\mathscr{X})\geqq\left(\sum_{i=1}^{r}\sqrt{\frac{n_i}{n}\frac{X_i}{X}G(\mathscr{X}_i)}\right)^2$$

(d) $$G(\mathscr{X})\geqq 1-\left(\sum_{i=1}^{r}\sqrt{\frac{n_i}{n}\frac{X_i}{X}(1-G(\mathscr{X}_i))}\right)^2$$

(e) $$G(\mathscr{X})\geqq G_0+\sum_{i=1}^{r}\frac{n_i}{n}\frac{X_i}{X}G(\mathscr{X}_i).$$

Proof. It suffices to prove each of these formulas for $r=2$, since applying the two-group formulas to the decomposition $\mathscr{X}=(\mathscr{X}_1\cup\ldots\cup\mathscr{X}_{r-1})\cup\mathscr{X}_r$ one then obtains the general results by induction on r.

To prove (a), (b) and (d), we use the definition (9) of the Gini coefficient, where $\xi(t)$ is defined by (7) or (7'). Let $\xi_1(t)$, $\xi_2(t)$ be the corresponding functions for the income distributions $\mathscr{X}_1$ and $\mathscr{X}_2$. Then clearly

$$\xi(t)=\xi_1(t)+\xi_2(t),$$

so

$$\begin{aligned} nX(1-G(\mathscr{X})) &= \int_0^\infty \xi(t)^2\,\mathrm{d}t \\ &= \int_0^\infty \xi_1(t)^2\,\mathrm{d}t + \int_0^\infty \xi_2(t)^2\,\mathrm{d}t + 2\int_0^\infty \xi_1(t)\xi_2(t)\,\mathrm{d}t \\ &= n_1X_1(1-G_1)+n_2X_2(1-G_2)+2\int_0^\infty \xi_1(t)\xi_2(t)\,\mathrm{d}t, \end{aligned} \tag{13}$$

where we have written G_i for $G(\mathscr{X}_i)$. Using

$$\begin{aligned} 0 &\leqq \int_0^\infty (\xi_1(t)-\lambda\xi_2(t))^2\,\mathrm{d}t \\ &= \int_0^\infty \xi_1(t)^2\,\mathrm{d}t + \lambda^2\int_0^\infty \xi_2(t)^2\,\mathrm{d}t - 2\lambda\int_0^\infty \xi_1(t)\xi_2(t)\,\mathrm{d}t, \end{aligned}$$

we find that

$$nX(1-G(\mathscr{X}))\leqq(1+\lambda^{-1})n_1X_1(1-G_1)+(1+\lambda)n_2X_2(1-G_2),$$

for any $\lambda>0$. With $\lambda=X_1/X_2$ we obtain inequality (a), with $\lambda=n_1/n_2$ we obtain (b), and with $\lambda=\{n_1X_1(1-G_1)/n_2X_2(1-G_2)\}^{\frac{1}{2}}$ we obtain (d). Indeed, in view of (13) the inequality (d) is nothing else than the Cauchy–Schwartz inequality

$$\int_0^\infty \xi_1(t)\xi_2(t)\,\mathrm{d}t \leqq \left(\int_0^\infty \xi_1(t)^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^\infty \xi_2(t)^2\,\mathrm{d}t\right)^{\frac{1}{2}}. \tag{14}$$

Another, more illuminating, argument to prove (b) is the following, which was suggested to me by Professor von Weizsäcker. Assume for simplicity that $\mathscr{X}_1$ and $\mathscr{X}_2$ are of the same size m (so $n_1=n_2=m$, $n=2m$), and order the members of each by income: $x_1^{(1)}\leqq\ldots\leqq x_m^{(1)}$, $x_1^{(2)}\leqq\ldots\leqq x_m^{(2)}$. Now in the population $\mathscr{X}=\mathscr{X}_1\cup\mathscr{X}_2$ let the jth members of the two populations $\mathscr{X}_1$ and $\mathscr{X}_2$ share their income, so that we get a new income distribution $\mathscr{X}^*$ with $x_{2j-1}^*=x_{2j}^*=\frac{1}{2}(x_j^{(1)}+x_j^{(2)})$ for $j=1,\ldots,m$. By the Pigou–Dalton condition, we have $G(\mathscr{X})\geqq G(\mathscr{X}^*)$. On the other hand, one easily checks that the Lorenz curves of $\mathscr{X}_1$, $\mathscr{X}_2$ and $\mathscr{X}^*$ are related by

$$L^*(p) = \frac{X_1}{X} L_1(p) + \frac{X_2}{X} L_2(p),$$

so the definition of the Gini coefficient in terms of the Lorenz curve implies

$$G(\mathscr{X}^*) = \frac{X_1}{X} G(\mathscr{X}_1) + \frac{X_2}{X} G(\mathscr{X}_2).$$

If $\mathscr{X}_1$ and $\mathscr{X}_2$ have different sizes, say in the ratio $r_1 : r_2$, and each income is repeated $r_1 r_2$ times in each population (which can be assumed by replicating the populations sufficiently often), one can carry out the same argument, defining $\mathscr{X}^*$ by combining the incomes of r_1 members of $\mathscr{X}_1$ and r_2 members of $\mathscr{X}_2$ at a time.

The argument to prove (c) was pointed out to me by Professor Vind: Consider the set of all subpopulations $\mathscr{X}' \subset \mathscr{X}$ and for each such $\mathscr{X}'$ the point $(n', X') \in \boldsymbol{R}^2$, where n' is the size and X' the total income of $\mathscr{X}'$. The convex hull of this set of points in $\boldsymbol{R}^2$ is the region of $V \subset R$ (where R is the rectangle $[0, n] \times [0, X]$), which is bounded below by the rescaled Lorenz curve

$$\{(np, XL(p)) \,|\, 0 \leqq p \leqq 1\}$$

(corresponding to choosing for $\mathscr{X}'$ the poorest n' members of $\mathscr{X}$), and above by the reflection of this curve in the center of R (corresponding to choosing for $\mathscr{X}'$ the richest n' members of $\mathscr{X}$). By eq. (5), the Gini coefficient is the ratio of the area of V to that of R. But it is clear from the definition that the region V for $\mathscr{X} = \mathscr{X}_1 \cup \mathscr{X}_2$ is the (vector) sum of the corresponding regions,

$$V_1 \subset [0, n_1] \times [0, X_1], \qquad V_2 \subset [0, n_2] \times [0, X_2],$$

for $\mathscr{X}_1$ and $\mathscr{X}_2$, so that (c) is a consequence of the Brunn–Minkowski theorem [see Grünbaum (1967, p. 338)], which states that for any two convex sets $V_1, V_2 \subset \boldsymbol{R}^n$ the (n-dimensional) volume of the sum $V = V_1 + V_2$ is related to the volumes of V_1 and V_2 by

$$\operatorname{vol}(V)^{1/n} \geqq \operatorname{vol}(V_1)^{1/n} + \operatorname{vol}(V_2)^{1/n}.$$

[Conversely, one could interpret (d), which, as shown below, is stronger than (c), as an improvement of the Brunn–Minkowski theorem for the special case that $n = 2$ and V_1, V_2 are centrally symmetrical convex sets which are

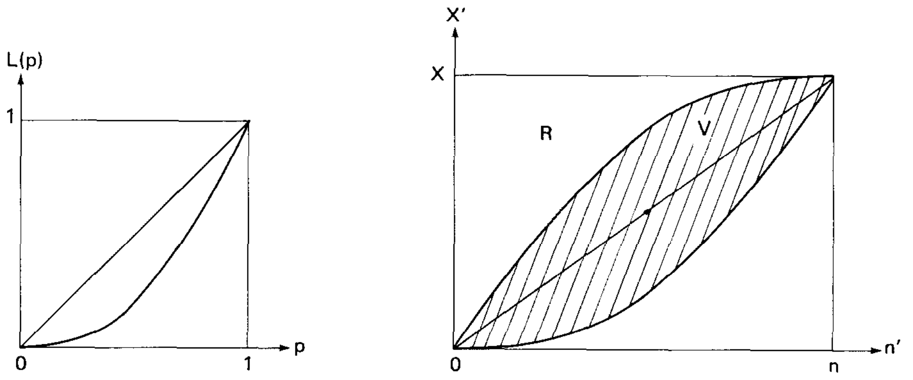

Fig. 2. The Lorenz curve and region V.

contained in, and contain the upwards diagonals of, two rectangles with parallel sides.]

Finally, we can deduce the last inequality (e)[2] from definition (10) of the Gini coefficient if we note that [by (8)]

$$\int_0^\infty \xi_1(t)\xi_2(t)\,\mathrm{d}t \leqq \xi_1(0)\int_0^\infty \xi_2(t)\,\mathrm{d}t = n_1 X_2,$$

and similarly

$$\int_0^\infty \xi_1(t)\xi_2(t)\,\mathrm{d}t \leqq \xi_2(0)\int_0^\infty \xi_1(t)\,\mathrm{d}t = n_2 X_1,$$

and hence [by (13)]

$$nX(1-G(\mathscr{X})) \leqq n_1X_1(1-G_1) + n_2X_2(1-G_2) + 2\min(n_1X_2, n_2X_1),$$

which is equivalent to (e). This completes the proof of Theorem 1.

We observe that the inequality (d) is always superior to (a), (b) and (c). This follows by induction on r, since for $r=2$ one has

$$G_d - G_a = \left(\sqrt{\frac{n_1}{n}\frac{X_2}{X}(1-G_1)} - \sqrt{\frac{n_2}{n}\frac{X_1}{X}(1-G_2)}\right)^2 \geqq 0,$$

$$G_d - G_b = \left(\sqrt{\frac{n_2}{n}\frac{X_1}{X}(1-G_1)} - \sqrt{\frac{n_1}{n}\frac{X_2}{X}(1-G_2)}\right)^2 \geqq 0,$$

[2]Equivalent to formula 21 of Pyatt (1976).

$$G_d - G_c = \left(\sqrt{\frac{n_1}{n}\frac{X_2}{X}} - \sqrt{\frac{n_2}{n}\frac{X_1}{X}}\right)^2 + 2\sqrt{\frac{n_1}{n}\frac{n_2}{n}\frac{X_1}{X}\frac{X_2}{X}}$$

$$\times \frac{(\sqrt{G_1(1-G_2)} - \sqrt{G_2(1-G_1)})^2}{1+\sqrt{G_1 G_2}+\sqrt{(1-G_1)(1-G_2)}} \geqq 0,$$

where $G_a, \ldots, G_d$ denote the right-hand sides of (a),…,(d). We nevertheless have included all four bounds in our theorem because (a) and (b) are particularly simple and the proof of (c) involved an interesting idea. The lower bounds (d) and (e) are not comparable: if $\mu_1 = \mu_2$ (so that all inequality in $\mathscr{X}$ comes from inequality *within* the groups $\mathscr{X}_i$), then $G_0 = 0$ and (e) is clearly worse than any of the other inequalities, while in the other extreme case $G_1 = G_2 = 0$ (when all the inequality in $\mathscr{X}$ is due to the disparity of income *between* the $\mathscr{X}_i$), (e) gives the correct value $G = G_0$ and the other bounds all give something worse:

$$G_a = G_b = G_c = 0 \leqq G_d = \left(\sqrt{\frac{n_1}{n}\frac{X_2}{X}} - \sqrt{\frac{n_2}{n}\frac{X_1}{X}}\right)^2$$

$$\leqq G_0 = \left|\frac{n_1}{n}\frac{X_2}{X} - \frac{n_2}{n}\frac{X_1}{X}\right|.$$

4. Best possible upper and lower bounds for the Gini coefficient of a two-component population

In this section we show that (d) and (e) of Theorem 1 together give the best possible lower bound for the Gini coefficient of $\mathscr{X}$ in the case $r=2$ and give the corresponding result for the upper bound.

Theorem 2. The Gini coefficient of a population $\mathscr{X}$, consisting of two components $\mathscr{X}_1$ and $\mathscr{X}_2$, satisfies the inequalities

$$\max\left(\frac{n_1}{n}\frac{X_2}{X} + \frac{n_2}{n}\frac{X_1}{X} + \frac{n_1}{n}\frac{X_1}{X}G_1 + \frac{n_2}{n}\frac{X_2}{X}G_2\right.$$

$$\left. -2\sqrt{\frac{n_1}{n}\frac{n_2}{n}\frac{X_1}{X}\frac{X_2}{X}}(1-G_1)(1-G_2), \quad G_0 + \frac{n_1}{n}\frac{X_1}{X}G_1 + \frac{n_2}{n}\frac{X_2}{X}G_2\right)$$

$$\leqq G(\mathscr{X})$$

$$\leqq G_0 + \frac{n_1}{n}\frac{X_1}{X}G_1 + \frac{n_2}{n}\frac{X_2}{X}G_2 + 2\min\left(\frac{n_1}{n}\frac{X_2}{X}, \frac{n_2}{n}\frac{X_1}{X}\right)(G_1 + G_2 - G_1 G_2),$$

where n_i, X_i and G_i denote the population sizes, total incomes and Gini coefficients of $\mathscr{X}_i$, respectively, $n=n_1+n_2$ and $X=X_1+X_2$ denote the size and total income of $\mathscr{X}$, and

$$G_0=\left|\frac{n_1}{n}\frac{X_2}{X}-\frac{n_2}{n}\frac{X_1}{X}\right|=\left|\frac{n_1}{n}-\frac{X_1}{X}\right|=\left|\frac{n_2}{n}-\frac{X_2}{X}\right|$$

denotes the 'between-group' Gini coefficient. Both bounds are sharp for fixed values of n_1/n, n_2/n, X_1/X, X_2/X, G_1 and G_2.

Proof. The lower bound, which in the notation of the proof of Theorem 1 says that $G(X)\geqq\max(G_d,G_e)$, has already been proved. To show that it is sharp, we must find populations $\mathscr{X}_1$ and $\mathscr{X}_2$ with given n_i, X_i and G_i and with $G(\mathscr{X}_1\cup\mathscr{X}_2)$ arbitrarily close to $\max(G_d,G_e)$. We must distinguish two cases, according to the relative sizes of the mean incomes $\mu_i=X_i/n_i$ and of the Gini coefficients G_i.

Case 1: $\min(\mu_1/\mu_2,\mu_2/\mu_1)\geqq(1-G_1)(1-G_2)$

Here $G_d\geqq G_e$, so we must construct populations $\mathscr{X}_i$ $(i=1,2)$ with the given values of n_i, X_i and G_i such that $G(\mathscr{X}_1\cup\mathscr{X}_2)<G_d+\varepsilon$. Choose an income x_0 such that

$$\mu_1(1-G_1)\leqq x_0\leqq\frac{\mu_1}{1-G_1},\qquad \mu_2(1-G_2)\leqq x_0\leqq\frac{\mu_2}{1-G_2},$$

(this is possible by virtue of the assumption on μ_1,μ_2,G_1,G_2), and define $\mathscr{X}_i$ $(i=1,2)$ by giving an income x_0 to a fraction $\{\mu_i(1-G_i)/x_0\}^{\frac{1}{2}}$ of the population, an income 0 to a fraction $1-\{\mu_i(1-G_i)/x_0\}^{\frac{1}{2}}$ of the population, and the remaining income $X_i(1-\{x_0(1-G_i)/\mu_i\}^{\frac{1}{2}})$ to a fraction 0 of the population. [We are formulating everything for the limiting case of very large populations, i.e., ignoring indivisibilities. A more precise formulation for finite populations n_i is that we give the income x_0 to n_i' people, where $n_i'\leqq n_i\{\mu_i(1-G_i)/x_0\}^{\frac{1}{2}}<n_i'+1$, an income 0 to $n_i-n_i'-1$ people, and the remaining income $X_i-n_i'x_0$ to the last person; this gives a Gini coefficient $G_i+\mathrm{O}(1/n_i)$, i.e., equal to G_i in the limit $n_i\to\infty$.] One easily checks that with this distribution one has $G(\mathscr{X}_1\cup\mathscr{X}_2)=G_d$ [respectively, $G_d+\mathrm{O}(1/n)$ in the case of a finite population].

Case 2: $\min(\mu_1/\mu_2,\mu_2/\mu_1)<(1-G_1)(1-G_2)$

Here $G_e>G_d$. Suppose for definiteness that $\mu_1\leqq\mu_2$, so that $\mu_1\mu_2^{-1}<(1-G_1)(1-G_2)$. We define the distribution $\mathscr{X}_1$ by assigning income 0 to a fraction G_1 of the population, and income $\mu_1/(1-G_1)$ to the remaining people. We define $\mathscr{X}_2$ by giving the income $\mu_1/(1-G_1)$ to a fraction

$$G_2(1-G_1)/\{1-G_1-\mu_1\mu_2^{-1}\}$$

of the population, and the (larger) income

$$\{(1-G_1)\mu_2-(1+G_2)\mu_1\}/\{(1-G_1)(1-G_2)-\mu_1\mu_2^{-1}\}$$

to the remaining fraction

$$\{(1-G_1)(1-G_2)-\mu_1\mu_2^{-1}\}/\{1-G_1-\mu_1\mu_2^{-1}\}$$

of the population. Then everyone in the second population is richer than any one in the first, and one easily deduces that $G(\mathscr{X}_1\cup\mathscr{X}_2)=G_e$.

We now turn to the upper bound. Using definition (9) of the Gini coefficient as in the proof of Theorem 1, we see that the upper bound is equivalent to the estimate

$$\int_0^\infty \xi_1(t)\xi_2(t)\,\mathrm{d}t \geqq \min(A_1A_2/n_1X_2, A_1A_2/n_2X_1) \tag{15}$$

for monotone decreasing functions $\xi_1(t)$, $\xi_2(t)$ on $[0,\infty)$, with

$$\xi_i(0)=n_i, \quad \int_0^\infty \xi_i(t)\,\mathrm{d}t=X_i, \quad \int_0^\infty \xi_i(t)^2\,\mathrm{d}t=A_i=n_iX_i(1-G_i).$$

Inequality (15), which can be thought of as a converse to the Cauchy–Schwarz inequality (14) for the class of monotone functions, is proved in Zagier (1977). However, since that paper is in Dutch, and since the proof of (15) is not very long and is actually simpler when formulated in terms of the standard definition (3) of the Gini coefficient than when stated in terms of the function ξ, we repeat the argument here. We have to show that

$$\begin{aligned} nX(1-G) \geqq\ & n_1X_1(1-G_1)+n_2X_2(1-G_2) \\ & + 2\min(n_1X_2, n_2X_1)(1-G_1)(1-G_2). \end{aligned}$$

By (3), this is equivalent to

$$\sum_{\alpha_1=1}^{n_1}\sum_{\alpha_2=1}^{n_2}\min(x_{\alpha_1}^{(1)},x_{\alpha_2}^{(2)})\geqq A_1A_2/\max(n_1X_2,n_2X_1), \tag{16}$$

where $x_\alpha^{(i)}$ $(i=1,2,\ 1\leqq\alpha\leqq n_i)$ are the elements of $\mathscr{X}_i$ and

$$A_i=n_iX_i(1-G_i)=\sum_{\alpha=1}^{n_i}\sum_{\beta=1}^{n_i}\min(x_\alpha^{(i)},x_\beta^{(i)}),\quad i=1,2.$$

For fixed α_1 and α_2 $(1\leqq\alpha_i\leqq n_i)$ we have

$$\sum_{\beta_1=1}^{n_1}\sum_{\beta_2=1}^{n_2}\min(x_{\alpha_1}^{(1)},x_{\beta_1}^{(1)})\min(x_{\alpha_2}^{(2)},x_{\beta_2}^{(2)})$$

$$\leqq\sum_{\beta_1=1}^{n_1}\sum_{\beta_2=1}^{n_2}x_{\alpha_1}^{(1)}x_{\beta_2}^{(2)}=n_1X_2x_{\alpha_1}^{(1)}\leqq\max(n_1X_2,n_2X_1)x_{\alpha_1}^{(1)}. \tag{17}$$

By symmetry, the left-hand side of (17) is also bounded by $\max(n_1X_2,n_2X_1)x_{\alpha_2}^{(2)}$, and hence in fact smaller than $\max(n_1X_2,n_2X_1)\cdot\min(x_{\alpha_1}^{(1)},x_{\alpha_2}^{(2)})$. Summing this inequality for $\alpha_1=1,\dots,n_1$ and $\alpha_2=1,\dots,n_2$, we obtain (16). This proves the desired upper bound.

To show that it is the best possible, we take for $\mathscr{X}_1$ and $\mathscr{X}_2$ the following distributions (assuming without loss of generality that $\mu_1=X_1/n_1\leqq\mu_2=X_2/n_2$): in the first population we give a fraction G_1 of the total income to a fraction 0 of the population (say to one person) and distribute the remaining income $X_1(1-G_1)$ equally among the remainder of the population [so that each person receives the income $\mu_1(1-G_1)$]; in the second population we give an income 0 to a fraction G_2 of the population and income $\mu_2/(1-G_2)$ to each of the other $n_2(1-G_2)$ people. Then one easily checks [using the fact that $\mu_2/(1-G_2)\geqq\mu_1(1-G_1)$] that the Gini coefficient of $\mathscr{X}_1\cup\mathscr{X}_2$ is given exactly by the upper bound in Theorem 2. In other words, we obtain the largest degree of inequality for $\mathscr{X}_1\cup\mathscr{X}_2$ by achieving the individual Gini coefficients G_1 and G_2 in as different ways as possible (in $\mathscr{X}_1$ by having one millionaire and a large middle class, in $\mathscr{X}_2$ by having no very rich but a large percentage of paupers), whereas we obtained the lower bound by choosing the distributions $\mathscr{X}_1$ and $\mathscr{X}_2$ to be as similar as possible within the limits permitted by their relative mean incomes and Gini coefficients.

Remark 1. By (a) or (b) of Theorem 1, the Gini coefficient of $\mathscr{X}_1\cup\mathscr{X}_2$ can never be less than the individual Gini coefficients. It can, however, be quite a bit larger. For instance, for populations $\mathscr{X}_1$ and $\mathscr{X}_2$, with $n_1=n_2$, $X_1=X_2$, $G_1=G_2=\gamma$, we obtain from Theorem 2 the sharp estimates

$$\gamma\leqq G(\mathscr{X}_1\cup\mathscr{X}_2)\leqq\tfrac{3}{2}\gamma-\tfrac{1}{2}\gamma^2,$$

so that — even in the case of two population components of the same size and with the same mean incomes and Gini coefficients — the Gini coefficient of the combined population can be almost 50% greater than those of the individual components.

Remark 2. We observe that the Gini coefficient of a population made up of r components is given by

$$G(\mathscr{X}_1 \cup \ldots \cup \mathscr{X}_r) = \sum_{i=1}^{r} \frac{n_i}{n} \frac{X_i}{X} G(\mathscr{X}_i) + \sum_{1 \leq i < j \leq r} \left(\frac{n_i + n_j}{n} \frac{X_i + X_j}{X} G(\mathscr{X}_i \cup \mathscr{X}_j) - \frac{n_i}{n} \frac{X_i}{X} G(\mathscr{X}_i) - \frac{n_j}{n} \frac{X_j}{X} G(\mathscr{X}_j) \right),$$

[this follows easily from either (2) or (3) or (9)], so that one can always reduce to the case $r=2$. Thus Theorem 2 also implies upper and lower bounds for the Gini coefficient of a population made up of more than two components. In general, however, these bounds will not be the best possible.

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