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ANALYTIC TORSION AND THE ARITHMETIC TODD GENUS

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with an Appendix by D. ZAGIER

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INTRODUCTION

THE AIM of this article is to state a *conjectural* Grothendieck–Riemann–Roch theorem for metrized bundles on arithmetic varieties, which would extend the known results of Arakelov [3], Faltings [19] and Deligne [17] in the case of arithmetic surfaces. The project of looking for such a theorem was first advocated by Manin in [29].

Let X be an arithmetic variety (i.e. a regular scheme, quasi-projective and flat over $\mathbb{Z}$). In a previous paper [20] we defined *arithmetic Chow groups* $\widehat{CH}^p(X)$ for every integer $p \geq 0$, generated by pairs of cycles and "Green currents" (*loc. cit.*). We showed that these groups have basically the same formal properties as the classical Chow groups. They are covariant for proper maps (with a degree shift). In [21] we attached to any algebraic vector bundle E on X, endowed with an hermitian metric h on the associated holomorphic vector bundle, characteristic classes

$$\hat{\phi}(E, h) \in \bigoplus_{p \geq 0} \widehat{CH}^p(X) \otimes \mathbb{Q} = \widehat{CH}(X)_{\mathbb{Q}},$$

for every symmetric power series $\phi(T_1, \ldots, T_{rk(E)})$ with coefficients in $\mathbb{Q}$. For instance we have Chern character $\widehat{ch}(E, h) \in \widehat{CH}(X)_{\mathbb{Q}}$. We also introduced in [21] a group $\hat{K}_0(X)$ of *virtual hermitian vector bundles* on X and extended $\widehat{ch}$ to $\hat{K}_0(X)$.

To state a Grothendieck–Riemann–Roch theorem one still needs two notions. First, given a smooth projective morphism $f: X \to Y$ between arithmetic varieties, one needs a *direct image morphism*

$$f_!: \hat{K}_0(X) \to \hat{K}_0(Y).$$

Given (E, h) on X, to get the determinant of $f_!(E, h)$ amounts to defining a metric on the determinant of the cohomology of E (on the fibers of f). This question was solved by Quillen [35] using the Ray–Singer analytic torsion [36]. In §3 below we shall define higher analogs of Ray–Singer analytic torsion and get a reasonable definition of $f_!$ (this is a variant of ideas from our work with Bismut [8, 9, 10]).

The second question we have to ask is what will play the role of the Todd genus. For this we proceed in a way familiar to algebraic geometry (see for instance [25]), namely we compute both sides of the putative Riemann–Roch formula for the trivial line bundle on the projective spaces $\mathbb{P}^n$ over $\mathbb{Z}$, $n \geq 1$. This normalizes the *arithmetic Todd genus* uniquely. To

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the obvious candidate $\widehat{Td}(E, h)$, with

$$Td(x) = \frac{x}{1-e^{-x}} = 1 - \sum_{n \geq 1} \zeta(1-n) \frac{x^n}{(n-1)!},$$

where $\zeta(s)$ is the Riemann zeta function, it turns out that a secondary characteristic class has to be added. It is constructed using the following characteristic power series (see 1.2.3)

$$R(x) = \sum_{\substack{m \text{ odd} \\ m \geq 1}} (2\zeta'(-m) + \zeta(-m)(1 + \tfrac{1}{2} + \ldots + \tfrac{1}{m})) \frac{x^m}{m!}.$$

The computations which yield this power series are pretty involved. We got the first coefficients of $R(x)$ using a computer. To check the general expression we reduced the problem to a difficult combinatorial identity, that D. Zagier was able to prove in general (Appendix). The conclusion is that the Grothendieck–Riemann–Roch theorem we conjecture is true for the trivial line bundle on $\mathbb{P}^n$ (Theorem 2.1.1.).

The paper is organized as follows. In §1 we define Quillen's metric on the determinant of cohomology, recall the definitions from [20] and [21], and define the arithmetic Todd genus. We then give a conjecture computing the Quillen metric (1.3). The holomorphic variation of this equality is known to be true [35] [8, 9, 10] (sec. 1.4). When specialized to the moduli space of curves of a given genus, the conjecture 1.3 gives the value of some unknown constants in string theory (1.5). In fact, our computation on $\mathbb{P}^n$ extends the work of the string theorist Weisberger [38] on this question when $n = 1$.

In §2 we prove conjecture 1.3 for the trivial line bundle on $\mathbb{P}^n$ (Theorem 2.1.1) by reduction to an identity of Zagier. In §3 we define higher analytic torsion using results of [9], compute its holomorphic variation (3.1) and define the map $f_!$ (3.2). We then conjecture a general arithmetic Grothendieck–Riemann–Roch identity (3.3) the holomorphic variation of which holds.

§1. ON THE DETERMINANT OF COHOMOLOGY

1.1. Quillen's metric

Let X be a compact complex manifold of complex dimension n, g a Kähler metric on X, E an holomorphic vector bundle on X, and h a smooth hermitian metric on E. We orient X using the convention that $\mathbb{C}^n$ is oriented by $dx_1 dy_1 dx_2 dy_2 \ldots dx_n dy_n$, with $z_\alpha = x_\alpha + iy_\alpha$, $\alpha = 1, \ldots, n$, the complex coordinates. Define the normalized Kähler form ω on X to be

$$\omega = \frac{i}{2\pi} \sum_{\alpha, \beta} g\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}\right) dz_\alpha d\bar{z}_\beta,$$

for any choice of local coordinates z_α, $\alpha = 1, \ldots, n$. Let $\mu = \omega^n/n!$.

Consider the Dolbeault complex

$$\ldots \to A^{0q}(X, E) \xrightarrow{\bar\partial} A^{0,q+1}(X, E) \to \ldots,$$

where $A^{pq}(X, E)$ is the vector space of smooth forms of type (p, q) with coefficients in E, and $\bar\partial$ is the Cauchy–Riemann operator. For each $q \geq 0$ we define the hermitian scalar product on $A^{0q}(X, E)$ by the formula

$$\langle \eta, \eta' \rangle_{L^2} = \int_X \langle \eta(x), \eta'(x) \rangle \mu, \tag{1}$$

where $\langle \eta(x), \eta'(x) \rangle$ is the pointwise scalar product coming from the metric on E and the metric on differential forms induced by the metric on X.

The operator $\bar{\partial}$ admits an adjoint $\bar{\partial}^*$ for this scalar product:

$$\langle \bar{\partial}\eta, \eta' \rangle_{L^2} = \langle \eta, \bar{\partial}^*\eta' \rangle_{L^2}, \quad \eta \in A^{0q}(X, E), \ \eta' \in A^{0,q+1}(X, E).$$

Let $\Delta_q = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ be the Laplace operator on $A^{0q}(X, E)$ and $\mathscr{H}^{0q}(X, E) = \operatorname{Ker} \Delta_q$ the set of harmonic forms. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ be the eigenvalues of Δ_q on the orthogonal complement to $\mathscr{H}^{0q}(X, E)$, indexed in increasing order and taking into account multiplicities (these are finite and $\lambda_n > 0$ for all $n \geq 1$). For every complex number s such that $Re(s) > \dim_{\mathbf{C}} X$, the series

$$\zeta_q(s) = \sum_{n \geq 1} \lambda_n^{-s} \quad (= \zeta_q(X, E, s))$$

converges absolutely. This function of s admits a meromorphic continuation to the whole complex plane, the zeta function of the operator Δ_q. This function is holomorphic at $s = 0$, so it makes sense to consider its derivative $\zeta_q'(0)$. Following Ray and Singer [36] one considers the analytic torsion

$$\tau(E)^0 = \sum_{q \geq 0} (-1)^q q \zeta_q'(0).$$

Remark. Notice that $\tau(E)^0$ depends on the metrics chosen on E and X. The number $\exp(-\zeta_q'(0))$ may be taken as a definition for $\det' \Delta_q$, the determinant of Δ_q restricted to the orthogonal complement of $\mathscr{H}^{0q}(X, E)$, since, for every finite sequence $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_M$ of positive real numbers, the following holds:

$$\mu_1 \mu_2 \dots \mu_M = \exp\left(-\frac{d}{ds} \left(\sum_{n=1}^{M} \mu_n^{-s} \right) \bigg|_{s=0} \right).$$

1.1.2. Consider the cohomology groups $H^q(X, E)$ of X with coefficients in E, and the one-dimensional complex vector space

$$\lambda(E) = \bigoplus_{q \geq 0} \Lambda^{\max} H^q(X, E)^{(-1)^q}$$

(when L is a line bundle we denote by L^{-1} its dual). Since $H^q(X, E)$ is canonically isomorphic to the cohomology of the Dolbeault complex, hence to $\mathscr{H}^{0q}(X, E)$, the scalar product $\langle \ , \ \rangle_{L^2}$ gives rise to a metric h_{L^2} on $\lambda(E)$. Quillen [35] defined a new metric h_Q on $\lambda(E)$ by the formula

$$h_Q = h_{L^2} \exp\left(\sum_{q \geq 0} (-1)^{q+1} q \zeta_q'(0) \right) = h_{L^2} \exp(-\tau(E)^0).$$

1.1.3. Now let $f\colon X \to Y$ be a smooth proper map of complex analytic manifolds. Assume that every point $y \in Y$ has an open neighborhood U such that $f^{-1}(U)$ can be endowed with a Kähler structure.

On the relative tangent space $T_{X/Y}$ (a bundle on X) choose an hermitian metric $h_{X/Y}$ whose restriction to each fiber $X_y = f^{-1}(y)$, $y \in Y$, gives a Kähler metric. Let E be a holomorphic vector bundle on X and h an hermitian metric on E.

Let $\lambda(E) = \det Rf_*(E)$ be the determinant of the direct image of E, as defined in [28] and [10]. This is a holomorphic line bundle $\lambda(E)$ on Y such that, for every $y \in Y$,

$$\lambda(E)_y = \bigotimes_{q \geq 0} \Lambda^{\max} H^q(X_y, E)^{(-1)^q}$$

It is shown in [8, 9, 10], Theorem 0.1, that the Quillen metric h_Q on $\lambda(E)$ (defined fiberwise as in 1.1.2) is smooth. Furthermore its curvature was computed in *loc. cit.*

1.2. Characteristic classes

1.2.1. Let (A, Σ, F_∞) be an arithmetic ring in the sense of [20], i.e. A is an excellent noetherian integral domain, Σ is a non-empty finite set of imbeddings σ: $A \to \mathbb{C}$, and F_∞: $\mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$ is a conjugate linear involution fixing A (imbedded diagonally into $\mathbb{C}^\Sigma$). Let F be the fraction field of A.

Let X be an arithmetic variety (*loc. cit*) i.e. a regular quasi-projective flat scheme over A. Assume the generic fiber X_F is projective. Let X_σ be the set of complex points of X defined using the imbedding $\sigma \in \Sigma$ and $X_\infty = \coprod_{\sigma\in\Sigma} X_\sigma$. In [20] we defined arithmetic Chow groups $\widehat{CH}^p(X)$ for every integer $p \geq 0$, which generalize those introduced by Arakelov [2] for arithmetic surfaces. The group $\widehat{CH}^p(X)$ is generated by pairs (Z, g), where Z is a cycle of codimension p on X and g is a "Green current" for the corresponding cycle on X_∞ (i.e. $dd^c g$ plus the current given by integration on Z_∞ is a smooth form; see *loc. cit.* for the relations). There is a canonical morphism z: $\widehat{CH}^p(X) \to CH^p(X)$ to the usual Chow group of codimension p sending (Z, g) to Z. On the other hand, let $A^{pp}(X_\mathbb{R})$ be the set of real forms ω of type (p, p) on X_∞ such that $F_\infty^*(\omega) = (-1)^p\omega$. Denote by $\tilde{A}^{pp}(X_\mathbb{R})$ the quotient of $A^{pp}(X_\mathbb{R})$ by $\operatorname{Im}\partial + \operatorname{Im}\bar\partial$, $Z^{pp}(X_\mathbb{R})$ the kernel of $\partial\bar\partial$ in $A^{pp}(X_\mathbb{R})$ and $H^{pp}(X_\mathbb{R})$ the quotient of $Z^{pp}(X_\mathbb{R})$ by $\operatorname{Im}\partial + \operatorname{Im}\bar\partial$. According to [20] there is a morphism ω: $\widehat{CH}^p(X) \to Z^{pp}(X_\mathbb{R})$ and canonical exact sequences:

$$\tilde{A}^{p-1,p-1}(X_\mathbb{R}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{z} CH^p(X) \to 0 \qquad (1)$$

$$H^{p-1,p-1}(X_\mathbb{R}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{z\oplus\omega} CH^p(X) \oplus Z^{pp}(X_\mathbb{R}) \qquad (3)$$

Any projective map f: $X \to Y$ of arithmetic varieties which is smooth on X_F induces a direct image morphism f_*: $\widehat{CH}^p(X) \to \widehat{CH}^{p+\delta}(Y)$, where $-\delta$ is the relative dimension. Furthermore

$$\widehat{CH}(X)_\mathbb{Q} = \bigoplus_{p\geq 0} \widehat{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

has a graded ring structure, contravariant for all morphisms of arithmetic varieties. The product on $\widehat{CH}(X)_\mathbb{Q}$ satisfies the formula

$$a(x)y = a(x\omega(y)),\ x \in \tilde{A}(X) = \bigoplus_{p\geq 1} A^{p-1,p-1}(X_\mathbb{R}),\ y \in \widehat{CH}(X)_\mathbb{Q}. \qquad (4)$$

In particular $\operatorname{Im} a$ is a square zero ideal in $\widehat{CH}(X)_\mathbb{Q}$.

1.2.2. Let E be a vector bundle of rank n on the arithmetic variety X and h an hermitian metric on the associated holomorphic vector bundle E_∞ on X_∞. We assume h is invariant under F_∞. (We say that (E, h) is an hermitian vector bundle on X.) When $E = L$ is a line bundle one can define its first Chern class $\hat{c}_1(L, h) \in \widehat{CH}^1(X)$ ([17], [21] 2.5). More generally, let $\phi \in \mathbb{Q}[[T_1, \ldots, T_n]]$ be a symmetric power series in n variables. In [21] §4 we defined a class $\hat\phi(E, h) \in \widehat{CH}(X)_\mathbb{Q}$, characterized by the following properties:

(i) $\hat\phi(f^*E, f^*h) = f^*\hat\phi(E, h)$

(ii) Let ϕ_i, $i \geq 0$, be defined by the identity

$$\phi(T_1 + T, \ldots, T_n + T) = \sum_{i\geq 0} \phi_i(T_1, \ldots, T_n)T^i.$$

Then

$$\hat{\phi}(E \otimes L, h \otimes h') = \sum_{i \geq 0} \hat{\phi}_i(E, h)\hat{c}_1(L, h')^i,$$

for every line bundle L.

(iii) Given two metrics h and h' on E,

$$\hat{\phi}(E, h) - \hat{\phi}(E, h') = a(\tilde{\phi}(E, h, h')),$$

where $\tilde{\Phi}(E, h, h') \in \tilde{A}(X_{\mathbb{R}})$ is a secondary characteristic class introduced by Bott and Chern ([15, 18, 8, 21]).

(iv) When $(E, h) = (L_1 \oplus \ldots \oplus L_n, h_1 \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} h_n)$ is an orthogonal direct sum of hermitian line bundles,

$$\hat{\phi}(E, h) = \phi(\hat{c}_1(L_1, h_1), \ldots, \hat{c}_1(L_n, h_n)).$$

In particular the Chern character $\widehat{ch}(E, h) \in \widehat{CH}(X)_{\mathbb{Q}}$ is defined using

$$ch(T_1, \ldots, T_n) = \sum_{i=1}^{n} \exp(T_i)$$

and the Todd class $\widehat{Td}(E, h) \in \widehat{CH}(X)_{\mathbb{Q}}$ by means of

$$Td(T_1, \ldots, T_n) = \prod_{i=1}^{n} (T_i/(1 - \exp(-T_i))).$$

1.2.3. Let E be a holomorphic bundle on a complex manifold X. Let us define a characteristic class $R(E) \in H^{ev}(X)$ in the even complex cohomology of X by the following properties:

(i) $R(f^* E) = f^* R(E)$

(ii) Given any exact sequence $0 \to S \to E \to Q \to 0$ of vector bundles over X, we have

$$R(E) = R(S) + R(Q)$$

(iii) When L is a line bundle on X with $x = c_1(L) \in H^2(X)$ its first Chern class,

$$R(L) = \sum_{\substack{m \text{ odd} \\ m \geq 1}} (2\zeta'(-m) + \zeta(-m)(1 + \tfrac{1}{2} + \ldots + \tfrac{1}{m}))\frac{x^m}{m!} \tag{5}$$

Here $\zeta(s)$ is the Riemann zeta function and $\zeta'(s)$ its derivative.

Assume now (E, h) is a hermitian vector bundle on an arithmetic variety X as in 1.2.2. Then $R(E_{\mathbb{C}})$ lies in $H(X) = \bigoplus_{p \geq 1} H^{p-1, p-1}(X_{\mathbb{R}})$. We define the *arithmetic Todd genus* of (E, h) to be

$$Td^A(E, h) = \widehat{Td}(E, h)(1 - a(R(E_{\mathbb{C}}))) \text{ in } \widehat{CH}(X)_{\mathbb{Q}}. \tag{6}$$

1.3. A conjecture.

Let $f\colon X \to Y$ be a smooth projective morphism of arithmetic varieties. Choose a hermitian metric $h_{X/Y}$ on the relative tangent space $T_{X/Y}$ which induces a Kähler metric on each fiber $f^{-1}(y)$, $y \in Y_{\mathbb{C}}$. Let (E, h) be an hermitian vector bundle on X. The determinant line bundle

$$\lambda(E) = \det Rf_* E$$

on Y [28] is endowed with the Quillen metric h_Q as in 1.1.3. Given $x \in \widehat{CH}(Y)_{\mathbb{Q}}$, denote $x^{(p)}$ its component in $\widehat{CH}^p(Y) \otimes_Z \mathbb{Q}$.

Conjecture 1.3. $\hat{c}_1(\lambda(E), h_Q) = f_*(\widehat{ch}(E, h)Td^A(T_{X/Y}, h_{X/Y}))^{(1)}$

1.4. Some evidence for the conjecture

Let

$$\delta(E) = \hat{c}_1(\lambda(E), h_Q) - f_*(\widehat{ch}(E, h)Td^A(T_{X/Y}, h_{X/Y}))^{(1)}.$$

THEOREM 1.4

(i) [8, 9, 10] *The element $\delta(E)$ lies in $a(H(Y)) \subset \widehat{CH}(Y)_{\mathbb{Q}}$. It is independent of h and $h_{X/Y}$. Given any short exact sequence $0 \to S \to E \to Q \to 0$ on X, then $\delta(E) = \delta(S) + \delta(Q)$.*

(ii) *Let E^* be the dual of E, d the rank of $T_{X/Y}$, and $K = \Lambda^d T^*_{X/Y}$ the relative dualizing bundle. Then*

$$\delta(E) = (-1)^{d+1}\delta(K \otimes E^*).$$

(iii) *Let E' be any bundle on Y. Then*

$$\delta(E \otimes f^* E') = rk(E')\delta(E).$$

(iv) [17] *When f has relative dimension one and Σ contains a real imbedding, one has*

$$\delta(E) = c \,.\, rk(E)$$

where $c \in \mathbb{R}$ depends only on the genus of the fibers of f.

Proof

(i) By the Grothendieck–Riemann–Roch theorem for higher Chow groups [24] we get $z(\delta(E)) = 0$. On the other hand, we know from [21] 4.1. that

$$\omega(\hat{\phi}(E, h)) = \phi(E, h) \tag{7}$$

is the closed form in $A(X) = \bigoplus_{p \geq 0} A^{pp}(X_{\mathbb{R}})$ representing the ϕ-characteristic class of E_{∞} which is attached to the hermitian holomorphic connection on E_{∞}. Since ω is multiplicative and commutes with f_* [20] we get

$$\omega(\delta(E)) = c_1(\lambda(E), h_Q) - f_*(ch(E, h)Td(T_{X/Y}, h_{X/Y}))^{(2)}.$$

This is zero by [8, 9, 10], Theorem 0.1. We conclude from (7) that $\delta(E)$ lies in the image of a.

When $h_{X/Y}$ is replaced by $h'_{X/Y}$ we have

$$\hat{c}_1(\lambda(E), h_Q) - \hat{c}_1(\lambda(E), h'_Q) = a(\tilde{c}_1(\lambda(E), h_Q, h'_Q))$$

by 1.2.2 (iii). Similarly

$$Td^A(T_{X/Y}, h_{X/Y}) - Td^A(T_{X/Y}, h'_{X/Y}) = a(\widetilde{Td}(T_{X/Y}, h_{X/Y}, h'_{X/Y})).$$

Using (6) we get

$$\delta(E) - \delta'(E) = a(\tilde{c}_1(\lambda(E), h_Q, h'_Q) - f_*(ch(E, h)\widetilde{Td}(T_{X/Y}, h_{X/Y}, h'_{X/Y}))).$$

Theorem 0.3 in [8, 9, 10] gives $\delta(E) - \delta'(E) = 0$. By a similar argument, Theorem 0.2 in [8, 9, 10] implies that $\delta(E)$ does not depend on the metric h on E and $\delta(E) = \delta(S) + \delta(Q)$ for every exact sequence

$$0 \to S \to E \to Q \to 0.$$

(ii) For every point $y \in Y$, Serre's duality identifies $H^q(X_y, E)$ with the dual of $H^{d-q}(X_y, K \otimes E^*)$, $q \geq 0$. Hence one gets an isomorphism of line bundles on Y

$$\lambda(E) \simeq \lambda(K \otimes E^*)^{(-1)^{d+1}} \tag{8}$$

Up to sign, on X_∞, Serre's duality is induced by the pairing of Dolbeault complexes

$$A^{oq}(X_y, E) \otimes A^{o,d-q}(X_y, K \otimes E^*) \to \mathbb{C} \tag{9}$$

sending $\eta \otimes \eta'$ to

$$\left(\frac{i}{2\pi}\right)^d \int_{X_y} \eta \wedge \eta'.$$

(we forget the subscript ∞). Let us endow E with a hermitian metric h. From the definition of the (normalized) Kähler form ω and the L^2-metric (1.1), we see that the pairing (9) gives an isometry

$$A^{oq}(X_y, E) \xrightarrow{\sim} A^{o,d-q}(X_y, K \otimes E^*)$$

for the L^2 metrics. Furthermore

$$\zeta_q(X_y, E, s) = \zeta_{d-q}(X_y, K \otimes E^*, s).$$

Therefore (8) is an isometry for Quillen's metrics. Let $x \to x^\vee$ be the involution on $\widehat{CH}$ equal to $(-1)^p$ on $\widehat{CH}^p$. Then $\widehat{ch}((E,h)^*) = \widehat{ch}(E,h)^\vee$ ([21] 4.9) and, by a standard computation

$$\begin{aligned} f_*(\widehat{ch}(E,h)\widehat{Td}(T_{X/Y}, h_{X/Y}))^\vee &= (-1)^d f_*(\widehat{ch}(E,h)^\vee\, \widehat{Td}(T_{X/Y}, h_{X/Y})^\vee) \\ &= (-1)^d f_*(\widehat{ch}((E,h)^*)\,\widehat{ch}(K, \Lambda^d h_{X/Y})\widehat{Td}(T_{X/Y}, h_{X/Y})^\vee) \end{aligned}$$

Therefore

$$f_*(\widehat{ch}(E,h)\widehat{Td}(T_{X/Y}, h_{X/Y}))^\vee = (-1)^d f_*(\widehat{ch}((K \otimes E^*, \Lambda^d h_{X/Y} \otimes h^*)\widehat{Td}(T_{X/Y}, h_{X/Y})). \tag{10}$$

Furthermore

$$a(f_*(ch(E)Td(T_{X/Y})R(T_{X/Y})))^\vee = (-1)^d a f_*(ch(E)^\vee\, Td(T_{X/Y})^\vee\, R(T_{X/Y})^\vee).$$

Since $R(x) = -R(-x)$ we get $R(T_{X/Y})^\vee = R(T_{X/Y})$, hence

$$f_*(ch(E)Td(T_{X/Y})R(T_{X/Y}))^\vee = (-1)^d f_*(ch(K \otimes E^*)Td(T_{X/Y})R(T_{X/Y})). \tag{11}$$

Applying (10) and (11) in degree one we get

$$\begin{aligned} &f_*(\widehat{ch}(E,h)Td^A(T_{X/Y}, h_{X/Y}))^{(1)} \\ &\quad = (-1)^{d+1} f_*(\widehat{ch}(K \otimes E^*, \Lambda^d h_{X/Y} \otimes h^*)\widehat{Td}(T_{X/Y}, h_{X/Y}))^{(1)} \end{aligned}$$

and (i) follows.

(iii) From the algebraic isomorphisms

$$H^q(X_y, E \otimes f^*E') \simeq H^q(X_y, E) \otimes E'_y, \quad y \in Y, \ q \geq 0$$

we get

$$\lambda(E \otimes f^*E') \simeq \lambda(E)^{rk(E')}(\det E')^{\chi(E)}. \tag{12}$$

Let us endow E and E' with hermitian metrics h and h'. On X_∞ the isomorphism (12) is induced by L^2 isometries

$$A^{oq}(X_y, E \otimes f^*E') \simeq A^{oq}(X_y, E) \otimes E'_y$$

from which we conclude that (12) is an isometry for Quillen's metric.

On the other hand

$$f_*(\widehat{ch}((E,h)\otimes f^*(E',h'))Td^A(T_{X/Y},h_{X/Y}))^{(1)}$$
$$=[f_*(\widehat{ch}(E,h)Td^A(T_{X/Y},h_{X/Y}))\widehat{ch}(E',h')]^{(1)}$$
$$=\chi(E)\hat{c}_1(E',h')+rk(E')f_*(\widehat{ch}(E,h)Td^A(T_{X/Y},h_{X/Y}))^{(1)}.$$

This proves (iii).

The statement (iv) is Deligne's result [17], since, by [21] Theorem 4.10.1., the right hand side of Conjecture 1.3 is the class of the corresponding metrized line bundle introduced in [17]. q.e.d.

1.5. Consequences of the Conjecture

1.5.1. Under the hypotheses of 1.3 assume that the rank of $T_{X/Y}$ is one, i.e. f is a family of curves. Then Conjecture 1.3 is equivalent to the following:

Conjecture 1.5

$$\hat{c}_1(\lambda(E),h_Q)=f_*(ch(E,h)\widehat{Td}(T_{X/Y},h_{X/Y}))^{(1)}-a(rk(E)(1-g)(4\zeta'(-1)-\tfrac{1}{6})),$$

where $g(y)$ is the genus of $f^{-1}(y)$ for every $y\in Y_\infty$.

To see that the conjectures are equivalent notice that

$$f_*(\widehat{ch}(E,h)\widehat{Td}(T_{X/Y},h_{X/Y})a(R(T_{X/Y})))^{(1)}=a(f_*(ch(E)Td(T_{X/Y})R(T_{X/Y})))^{(1)}$$

by (4). Since $R(T_{X/Y})$ has degree at least 2 we get

$$a(f_*(rk(E)r_1c_1(T_{X/Y})))^{(1)}$$

where

$$r_1=2\zeta'(-1)+\zeta(-1)=2\zeta'(-1)-\tfrac{1}{12}$$

By the classical Riemann-Roch theorem in cohomology:

$$1-g=f_*(Td(T_{X/Y}))^{(0)}=\tfrac{1}{2}f_*(c_1(T_{X/Y})).$$

Hence Conjecture 1.5 is equivalent to Conjecture 1.3 when $rk(T_{X/Y})=1$.

1.5.2. We keep the hypotheses of 1.5.1 and let ω be the dual of $T_{X/Y}$, with the dual metric.

Proposition 1.5.2. *Assume Conjecture 1.5 holds. Then, for every $j\geq 1$, there is an isomorphism,*

$$M:\lambda(\omega^j)\to\lambda(\omega)^{6j^2-6j+1}$$

such that

$$h_Q(M(s),M(s'))=h_Q(s,s')\exp((1-g)(j^2-j)(24\zeta'(-1)-1)). \tag{13}$$

Proof. The algebraic isomorphism $\lambda(\omega^j)\simeq\lambda(\omega)^{6j^2-6j+1}$ is due to Mumford [32]. By a standard computation ([32, 12]) we get

$$f_*(\widehat{ch}(\omega^j)\widehat{Td}(T_{X/Y},h_{X/Y}))^{(1)}=(6j^2-6j+1)f_*(\widehat{ch}(\omega)\widehat{Td}(T_{X/Y},h_{X/Y}))^{(1)}.$$

Therefore, by applying the Conjecture 1.5 to ω and ω^j,

$$\hat{c}_1(\lambda(\omega^j),h_Q)=(6j^2-6j+1)\hat{c}_1(\lambda(\omega),h_Q)+(6j-6j^2)(1-g)(4\zeta'(-1)-\tfrac{1}{6}).$$

Let $\widehat{\mathrm{Pic}}(Y)$ be the group of hermitian line bundles on Y, modulo the algebraic isomorphisms which preserve the metrics. From [17] and [21] 2.5, we know that

$$\hat{c}_1 : \widehat{\mathrm{Pic}}(Y) \to \widehat{CH}^1(Y)$$

is an isomorphism. Hence the Proposition follows.

1.5.3. The Mumford isomorphism $M: \lambda(\omega^j) \simeq \lambda(\omega)^{6j^2-6j+1}$ is fixed up to sign when the base is $A = \mathbb{Z}$. In particular there is a unique metric on $\lambda(\omega^j)$ such that M is an isometry when $\lambda(\omega)$ has its L^2 metric. As shown in [5], when $j = 2$, this metric on $\lambda(\omega^2)$ gives rise to the Polyakov measure on the moduli space of curves of genus g (cf. also [12]). If Conjecture 1.5 would hold, it would then normalize the constant which appears in several expressions for the Polyakov measure ([5, 30, 14, 4, 31, 11, 1]). The meaning of such a normalization over $\mathbb{Z}$ for string theory is *a priori* unclear. However Weisberger in [38] argues that "unitarity" can be used to normalize the expression of the Polyakov measure. His method is based on the computation of the determinant of the Laplace operator on $\mathbb{P}^1$ (as in paragraph 2 below) and the constants he gets are similar to those in (13) (Proposition 1.5.2).

§2. PROJECTIVE SPACES

2.1. Statement of the results

2.1.1. THEOREM 2.1.1. *For every $n \geq 0$ let $f: \mathbb{P}^n \to \mathrm{Spec}(\mathbb{Z})$ be the projective space of dimension n over $\mathbb{Z}$. Then the conjecture 1.3 holds when E is the trivial line bundle $\mathcal{O}_{\mathbb{P}^n}$ on $\mathbb{P}^n$.*

2.1.2. *Remarks.* As shown in Theorem 1.4, it is enough to prove 2.1.1 with one choice of metrics. On $\mathbb{P}^n(\mathbb{C})$ we shall take the Fubini Study metric $h_{\mathbb{P}^n}$; and on $\mathcal{O}_{\mathbb{P}^n}$ we take the trivial metric.

Let $R'(x) = r_0 + r_1 x + r_2 x^2 + \ldots \in \mathbb{R}[[x]]$ be an arbitrary power series with real coefficients. Define a characteristic class $R'(E) \in H(X)$ as in 1.2.3, with $R(L) = R'(c_1(L))$ instead of (5). Let

$$Td^{A'}(E, h) = \widehat{Td}(E, h)(1 - a(R'(E_{\mathbb{C}}))).$$

In $\widehat{CH}^1(\mathrm{Spec}\ \mathbb{Z}) = \mathbb{R}$ consider the equation

$$\hat{c}_1(\lambda(\mathcal{O}_{\mathbb{P}^n}), h_Q) = f_*(Td^{A'}(T_{\mathbb{P}^n}, h_{\mathbb{P}^n}))^{(1)}. \tag{14}$$

For every $n \geq 0$ this is a linear equation in the variables $r_0, r_1, \ldots, r_n$ and the coefficient of r_n is not zero. Therefore there is a unique sequence $r_0, r_1, r_2, \ldots$ such that (14) holds for all $n \geq 0$. Theorem 2.1.1 computes these numbers, proving that $R' = R$ must be given by formula (14). This is quite similar to the way the Todd genus is defined in [25] for instance: the Todd genus is the unique multiplicative characteristic class such that the Riemann–Roch theorem holds for the trivial line bundle on $\mathbb{P}^n$ (cf. Lemma 1.7.1 in *loc. cit.*).

2.1.3. COROLLARY. *The conjecture 1.3 holds when f is the projection $\mathbb{P}^1_{\mathbb{Z}} \to \mathrm{Spec}\ \mathbb{Z}$.*

Proof. Let $\mathcal{O}(1)$ be the standard line bundle on $\mathbb{P}^1$. From Theorem 1.4(i) we only need to check $\delta(E) = 0$ for any E in $K_0(\mathbb{P}^1) = \mathbb{Z}^2$. Since $\delta(\mathcal{O}_{\mathbb{P}^1}) = 0$ (Theorem 2.1.1) we are left with showing $\delta(\mathcal{O}(1)) = 0$. From Theorem 1.4(ii) and Theorem 2.1.1 we get $\delta(K) = 0$.

From the relation

$$[\mathcal{O}_{\mathbb{P}^1}]-[K]+2[\mathcal{O}(1)]=0$$

in $K_0(\mathbb{P}^1)$ (see (15) below) the Corollary follows. q.e.d.

2.2. The right hand side

In this paragraph we shall compute

$$f_*(Td^A(T_{\mathbb{P}^n}, h_{\mathbb{P}^n}))^{(1)} \quad \text{in } \widehat{CH}^1(\operatorname{Spec} \mathbb{Z})=\mathbb{R}$$

(the identification being given by the map a of 1.2.1.).

2.2.1. First we compute

$$R_n=f_*(\widehat{Td}(T_{\mathbb{P}^n}, h_{\mathbb{P}^n})a(R(T_{\mathbb{P}^n})))=\int_{\mathbb{P}^n(\mathbb{C})} Td(T_{\mathbb{P}^n})R(T_{\mathbb{P}^n})$$

by (4) and (7). Consider the canonical exact sequence on $\mathbb{P}^n$:

$$\mathscr{E}_n: 0\to\mathbb{C}\to\mathcal{O}(1)^{n+1}\to T_{\mathbb{P}^n}\to 0. \tag{15}$$

Let $x=c_1(\mathcal{O}(1))\in H^2(\mathbb{P}^n)$. We get, from (15), $R(T_{\mathbb{P}^n})=(n+1)R(x)$. On the other hand

$$Td(T_{\mathbb{P}^n})=Td(x)^{n+1}, \quad \text{where } Td(x)=x/(1-e^{-x}),$$

and

$$\int_{\mathbb{P}^n(\mathbb{C})} x^k=\begin{cases} 1 & \text{if } k=n \\ 0 & \text{otherwise}\end{cases}.$$

So we have

Lemma 2.2.1. *$R_n=$ coefficient of x^n in $(n+1)\left(\dfrac{x}{1-e^{-x}}\right)^{n+1}R(x)$.*

2.2.2. Let us equip all bundles in $\mathscr{E}_n$ with the standard metric invariant under $SU(n+1)$. From [21] Theorem 4.8. and (15) we get

$$\widehat{Td}(T_{\mathbb{P}^n}, h_{\mathbb{P}^n})\widehat{Td}(\mathbb{C},|\cdot|)=\widehat{Td}(\mathcal{O}(1)^{n+1})+a(\widetilde{Td}(\mathscr{E}_n)). \tag{16}$$

Here $\widetilde{Td}(\mathscr{E}_n)$ is the secondary characteristic class considered in 1.2.2 (iii). Let

$$\hat{x}=\hat{c}_1(\mathcal{O}(1)).$$

We get from (16), since $\widehat{Td}(\mathbb{C},|.|)=1$,

$$\widehat{Td}(T_{\mathbb{P}^n}, h_{\mathbb{P}^n})=Td(\hat{x})^{n+1}+a(\widetilde{Td}(\mathscr{E}_n)).$$

In [21] 5.4.6. we computed

$$f_*(\hat{x}^k)^{(1)}=\begin{cases} \displaystyle\sum_{p=1}^{n}\sum_{j=1}^{p}\frac{1}{j} & \text{if } k=n+1 \\ 0 & \text{otherwise.}\end{cases}$$

So we have proved

LEMMA 2.2.2. *Let*

$$t_n = f_*(\widehat{Td}(\mathcal{O}(1))^{n+1})^{(1)}.$$

Then

$$t_n = \left(\sum_{p=1}^{n} \sum_{j=1}^{p} \frac{1}{j}\right) \cdot \left(\text{coefficient of } x^{n+1} \text{ in } \left(\frac{x}{1-e^{-x}}\right)^{n+1}\right).$$

2.2.3. We still need to compute

$$\tilde{Td}_n = f_* a(\tilde{Td}(\mathcal{E}_n))^{(1)}.$$

PROPOSITION 2.2.3. $\tilde{Td}_n =$ *coefficient of* x^n *in* $\displaystyle\int_0^1 \frac{\phi(t)-\phi(0)}{t}\,dt,$

where

$$\phi(t) = \left[\frac{1}{tx} - \frac{e^{-tx}}{1-e^{-tx}}\right]\left(\frac{x}{1-e^{-x}}\right)^{n+1}$$

Proof. To compute $\tilde{Td}(\mathcal{E}_n)$ we apply the method of [15], §4. Let

$$\mathcal{E}_n\colon\ 0 \to S \to E \to Q \to 0$$

be the exact sequence (15). The metrics on $S = \mathbb{C}$ and $Q = T_{\mathbb{P}^n}$ are induced by the metric on $E = \mathcal{O}(1)^{n+1}$, as in *loc. cit.* Let us write E as the orthogonal direct sum of S and $S^\perp \simeq Q$. The curvature of E $\left(\text{multiplied by } \dfrac{i}{2\pi}\right)$ decomposes as a 2 by 2 matrix $K = (K_{ij})$. Let K_S (resp. K_Q) the curvature of S (resp. Q) multiplied by $\dfrac{i}{2\pi}$. Let $Td(A) = \det(A/(1-e^{-A}))$ for any square matrix A. For every $t \in [0, 1]$ consider

$$\phi(t) = \text{coefficient of } \lambda \text{ in } Td\left[\begin{array}{c|c} tK_{11} + (1-t)K_S + \lambda & tK_{12} \\ \hline K_{21} & tK_{22} + (1-t)K_Q \end{array}\right].$$

and

$$I = \int_0^1 \frac{\phi(t)-\phi(0)}{t}\,dt.$$

As in [15] *loc. cit.* one checks that

$$\frac{i}{2\pi}\,\bar{\partial}\partial(I) = Td(K) - Td(K_S \oplus K_Q).$$

Moreover the characteristic properties of $\tilde{Td}$ given in [8] are easily seen to be satisfied by the class of I in $\tilde{A}(\mathbb{P}^n)$. Therefore $\tilde{Td}(\mathcal{E}_n) \equiv I$, modulo $Im\,\partial + Im\,\bar{\partial}$.

In our case K is equal (in any frame) to the product of the first Chern form ω of $\mathcal{O}(1)$ by the identity matrix. Furthermore S has rank one and $K_S = 0$. Therefore we get

$$\phi(t) = \text{coefficient of } \lambda \text{ in } Td\left[\begin{array}{c|c} t\omega + \lambda & 0 \\ \hline 0 & t\omega + (1-t)K_Q \end{array}\right].$$

Since $Td(A \oplus B) = Td(A)Td(B)$ we get

$$\phi(t) = \frac{d}{d\lambda}\left[\frac{t\omega+\lambda}{1-e^{-t\omega-\lambda}}\right]_{|\lambda=0} Td(t\omega+(1-t)K_Q). \tag{17}$$

We define a characteristic class $Td_{u,v}(E)$ with coefficients in the ring $\mathbb{Q}[[u, v]]$ of power series in two variables by the formula

$$Td_{u,v}(E) = \det \frac{uK+v}{1-e^{-uK-v}}.$$

Then $Td_{u,v}$ is multiplicative on exact sequences. From $\mathscr{E}_n$ we get

$$Td_{u,v}(K_Q)\frac{v}{1-e^{-v}} = \left[\frac{u\omega+v}{1-e^{-u\omega-v}}\right]^{n+1}.$$

Specializing to $u = 1-t$ and $v = t\omega$ we get

$$Td(t\omega+(1-t)K_Q) = \frac{1-e^{-t\omega}}{t\omega}\left[\frac{\omega}{1-e^{-\omega}}\right]^{n+1}. \tag{18}$$

From (17) and (18) we get

$$\phi(t) = \left[\frac{1}{t\omega} - \frac{e^{-t\omega}}{1-e^{-t\omega}}\right]\left[\frac{\omega}{1-e^{-\omega}}\right]^{n+1}$$

Since

$$\int_{\mathbb{P}^n(\mathbb{C})} \omega^k = \begin{cases} 1 & \text{when } k=n \\ 0 & \text{otherwise,} \end{cases}$$

the Proposition 2.2.3 follows.

2.3. The left hand side.

2.3.1. Let ω be the (1, 1) form of the Fubini Study metric on $\mathbb{P}^n(\mathbb{C})$. By definition, ω is the first Chern form of $\mathcal{O}(1)$ (with its standard metric), with cohomology class $x = c_1(\mathcal{O}(1))$. The associated density is $\mu = \omega^n/n!$, hence

$$\int_{\mathbb{P}^n(\mathbb{C})} \mu = 1/n!. \tag{19}$$

Since

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \begin{cases} \mathbb{Z} & \text{if } q=0 \\ 0 & \text{otherwise,} \end{cases}$$

the line bundle $\lambda(\mathcal{O}_{\mathbb{P}^n})$ is trivial, with section $1 \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ of L^2-norm

$$h_{L^2}(1, 1) = \int_{\mathbb{P}^n(\mathbb{C})} \mu = 1/n!.$$

Therefore

$$-\hat{c}_1(\lambda(\mathcal{O}_{\mathbb{P}^n}), h_Q) = \log h_Q(1, 1) = -\log(n!) + \sum_{q\geq 0} (-1)^{q+1} q\zeta_q'(0), \tag{20}$$

where $\zeta_q(s)$ is the zeta function of the Laplace operator $\Delta_q = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ acting upon $A^{0q}(\mathbb{P}^n)$, i.e. forms of type $(0, q)$ on $\mathbb{P}^n(\mathbb{C})$.

2.3.2. The spectrum of Δ_q was computed by Ikeda and Taniguchi in [27]. Let $\Lambda_i = x_1 + \ldots + x_i$, $1 \leq i \leq n$, be the standard fundamental weights of the group $SU(n+1)$

and $\Lambda_0=0$ (hence $x_1,\ldots,x_{n+1}$ are the usual characters of the diagonal subgroup of $SU(n+1)$, and $x_1+x_2+\ldots+x_{n+1}=0$). When $k\geq q\geq 0$ denote by $\Lambda(k,0,q)$ the irreducible representation of $SU(n+1)$ of highest weight

$$(k-q)\Lambda_1+\Lambda_q+k\Lambda_n.$$

According to [27], Theorem 5.2, $A^{0q}(\mathbb{P}^n)$ contains as a dense subspace (stable under $SU(n+1)$) the following infinite direct sum:

$$\bigoplus_{k\geq 0}\Lambda(k,0,0)\qquad\text{when } q=0$$

$$\left(\bigoplus_{k\geq q}\Lambda(k,0,q)\right)\oplus\left(\bigoplus_{k\geq q+1}\Lambda(k,0,q+1)\right)\quad\text{when } 1\leq q<n,$$

and

$$\bigoplus_{k\geq n}\Lambda(k,0,n)\quad\text{when } q=n.$$

Furthermore the Laplace operator Δ_q acting on $\Lambda(k,0,q)$, $q>0$, is the multiplication by $k(k+n+1-q)$ (the subspace $\Lambda(k,0,q+1)$ of $A^{0q}(\mathbb{P}^n)$ is mapped isomorphically by $\bar\partial$ to $\Lambda(k,0,q+1)\subset A^{0,q+1}(\mathbb{P}^n)$, $q<n$). We define

$$d_{n,q}(k)=dim_{\mathbb{C}}\,\Lambda(k,0,q).$$

Therefore

$$\sum_{q\geq 0}(-1)^{q+1}q\zeta_q(s)=\sum_{\substack{q\geq 1\\ k\geq q}}(-1)^{q+1}\frac{d_{n,q}(k)}{(k(k+n+1-q))^s}. \tag{21}$$

2.3.3. Lemma 2.3.3. *When $k\geq q$ and $n\geq q$,*

$$d_{n,q}(k)=\left(\frac{1}{k}+\frac{1}{k+n+1-q}\right)\frac{(k+n)!\,(k+n-q)!}{k!\,(k-q)!\,n!(n-q)!\,(q-1)!}. \tag{22}$$

Proof. We apply the Hermann-Weyl formula. Let $\lambda=(k-q)\Lambda_1+\Lambda_q+k\Lambda_n$, δ the half sum of positive roots, and $(\,,)$ the invariant scalar product on the root system of $SU(n+1)$. Then

$$d_{n,q}(k)=\frac{\prod_{\alpha>0}(\lambda+\delta,\alpha)}{\prod_{\alpha>0}(\delta,\alpha)}.$$

The standard positive roots of $SU(n+1)$ are x_i-x_j, $1\leq i<j\leq n+1$. The basis Λ_i is dual for $(\,,)$ to the basis x_i-x_{i+1}, $i=1,\ldots,n$. We have [26],

$$\prod_{\alpha>0}(\delta,\alpha)=\prod_{1\leq i<j\leq n+1}(j-i)$$

and, if

$$\lambda=\sum_{i=1}^{n}m_i\Lambda_i,$$

$$\prod_{\alpha>0}(\lambda+\delta,\alpha)=\prod_{1\leq i<j\leq n+1}\left(\sum_{l=i}^{j-1}(m_l+1)\right).$$

The factor $\sum_{l=i}^{j-1}(m_l+1)$ is equal to $j-i$ unless $i=1, j=n+1$ or $1<i\le q<j\le n$. Hence we get

$$\begin{aligned} d_{n,q}(k) &= \left(\prod_{1<j\le q}\frac{k-q+j-1}{j-1}\right)\cdot\left(\prod_{q<j<n+1}\frac{k-q+j}{j-1}\right) \\ &\quad\cdot\left(\frac{2k-q+n+1}{n}\right)\cdot\left(\prod_{1<i\le q}\frac{k+n+2-i}{n+1-i}\right) \\ &\quad\cdot\left(\sum_{q<i\le n}\frac{k+n+1-i}{n+1-i}\right)\cdot\left(\prod_{1<i\le q<j\le n}\frac{1+j-i}{j-i}\right) \\ &= \frac{2k-q+n+1}{k(k+n-q+1)}\cdot\frac{(k+n)!\,(k+n-q)!}{k!\,(k-q)!\,n!(n-q)!(q-1)!} \end{aligned}$$

q.e.d.

2.3.4. To compute $\hat{c}_1(\lambda(\mathcal{O}_{\mathbb{P}^n}), h_Q)$ we need to know $\zeta_q'(0)$. For this we use a result of Vardi [37] (see also [38] when $n=1$, and an unpublished work of Bost [13]. Let $P(X)\in\mathbb{C}[X]$ be a polynomial and $a\ge 0$ an integer. Let $P(X)=\sum_{n\ge 0}c_nX^n$. Consider the real numbers

$$\zeta P=\sum_{n\ge 0}c_n\zeta'(-n),$$

where $\zeta(s)$ is the Riemann zeta function and $\zeta'(s)$ its derivative, and

$$P^*(a)=\sum_{n\ge 1}c_n\frac{a^{n+1}}{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\ \dots\ +\frac{1}{n}\right).$$

The series

$$Z(s)=\sum_{k\ge 1}P(k)\,(k(k+a))^{-s}$$

converges absolutely when $Re(s)$ is big enough, and extends meromorphically to the whole complex plane.

PROPOSITION 2.3.4 ([37], Prop. 3.1.)†

$$Z'(0)=\sum_{m=1}^{a}P(m-a)\log m+\zeta P+\zeta P(.-a)-\tfrac{1}{2}P^*(-a). \tag{23}$$

Remark. Consider the formal sum

$$\begin{aligned} &-\sum_{k\ge 1}P(k)\log k-\sum_{k\ge 1}P(k)\log(k+a) \\ &\qquad=-\sum_{k\ge 1}P(k)\log k-\sum_{m\ge 1}P(m-a)\log m+\sum_{m=1}^{a}P(m-a)\log m. \end{aligned}$$

If we replace in this formula $-\sum_{k\ge 1}k^n\log k$, $n\ge 0$, by $\zeta'(-n)$ we get $\zeta P+\zeta P(.-a)+\sum_{m=1}^{a}P(m-a)\log m$. The extra term $-\frac{1}{2}P^*(-a)$ in (23) is the effect of regularizing this sum by means of a zeta function.

† M. Wodzicki tells us he had proved this result in 1982.

2.4. Proof of Theorem 2.1.1.

When $1 \leq q \leq n$ we denote by $d_{n,q}(X)$ the polynomial such that $d_{n,q}(k)$ is given by (22) when k is an integer and $k \geq q$.

2.4.1. LEMMA 2.4.1.

(i) $d_{n,q}(k)=0$ *when* $k=q-n,\ q-n+1, \ldots,$ *or* $q-1$, *and* $k \neq 0$; $d_{n,q}(0)=(-1)^{q+1}$.

(ii) $d_{n,q}(X-n-1+q)=-d_{n,q}(-X)$

(iii) $1+\sum_{q \geq 1}(-1)^{q+1} d_{n,q}(k)=(n+1)\frac{(k+n)!}{n!\,k!}$

(k integer $\geq n$).

Proof.

(i) This follows from

$$d_{n,q}(k)=\left(\frac{1}{k}+\frac{1}{k+n-q+1}\right)\binom{n+k}{n}\frac{1}{(n-q)!\,(q-1)!}$$
$$\cdot(k+n-q)(k+n-q-1)\ldots(k-q+1).$$

(ii) One checks that

$$d_{n,q}(X)=\left(\frac{1}{X}+\frac{1}{X+n-q+1}\right)\phi(X)$$

with

$$\phi(X-n-1+q)=\phi(-X).$$

(iii) (The following proof, simpler than our original one, is due to D. Zagier). First notice that

$$d_{n,q}(k)+d_{n-1,q-1}(k)$$
$$=\frac{(n-1)!}{(n-q)!\,(q-1)!}\left(\frac{1}{k}+\frac{1}{k+n-q}\right)\frac{(k+n-1)!\,(k+n-q)!}{k!\,(k-1)!\,(n-1)!\,(n-q)!}\left[\frac{k+n}{n}+\frac{q-1}{k-q+1}\right]$$
$$=\binom{n-1}{q-1}\left[\binom{k+n-1}{n}\binom{k+n-q}{n-1}+\binom{k+n-1}{n-1}\binom{k+n-q+1}{n}\right].$$

Call L_n the left hand side of (iii). We get

$$L_n-L_{n-1}=\binom{k+n-1}{n}\left[\sum_{q=1}^{n}(-1)^{q+1}\binom{n-1}{q-1}\binom{k+n-q}{n-1}\right]$$
$$+\binom{k+n-1}{n-1}\left[\sum_{q=1}^{n}(-1)^{q+1}\binom{n-1}{q-1}\binom{k+n-q+1}{n}\right]$$
$$=\binom{k+n-1}{n}+\binom{k+n-1}{n-1}(k+1)$$
$$=(n+1)\binom{k+n}{n}-n\binom{k+n-1}{n-1}.$$

When $n=1$, (iii) is easily checked, therefore it follows by induction on n.

2.4.2. To compute $\sum_{q \geq 0}(-1)^{q+1} q\zeta_q'(0)$ using (21), (22) and Proposition 2.3.4, we have to study two different terms.

The first term involves logarithms of integers. This is

$$\sum_{q=1}^{n}(-1)^{q+1}\sum_{m=1}^{n-q+1} d_{n,q}(m-n-1+q)\log m$$
$$=\sum_{q=1}^{n}(-1)^q(-1)^q\log(n+1-q)\quad\text{(by Lemma 2.4.1 (i))}$$
$$=\log(n!).$$

This term cancels with $\log h_{L^2}(1,1)=-\log(n!)$ in (20).

The second term involves values of $\zeta'(s)$. First, since $d_{n,q}(X-n-1+q)=-d_{n,q}(-X)$ (Lemma 2.4.1 (ii)), the Proposition 2.3.4, when applied to $P=d_{n,q}$ and $a=n+1-q$, gives

$$\hat{c}_1(\lambda(\bar{\mathcal{O}}_{\mathbb{P}^n}),h_Q)=2\sum_{q=1}^{n}(-1)^{q+1}\zeta(d_{n,q}^{\text{odd}})-\frac{1}{2}\sum_{q=1}^{n}(-1)^{q+1}d_{n,q}^{*}(q-n-1),$$

where

$$2d_{n,q}^{\text{odd}}(X)=d_{n,q}(X)-d_{n,q}(-X).$$

From Lemma 2.4.1 (iii) we get

$$2\sum_{q=1}^{n}(-1)^{q+1}\zeta(d_{n,q}^{\text{odd}})=2(n+1)\zeta\left(\left(\frac{(k+n)!}{k!\,n!}\right)^{\text{odd}}\right). \tag{24}$$

On the right hand side, from Lemma 2.2.1 and the definition (5) we get

$$R_n=\zeta P-s_n \tag{25}$$

with $P(k)=$coefficient of x^n in

$$2(n+1)\left(\frac{x}{1-e^{-x}}\right)^{n+1}\sum_{\substack{m\text{ odd}\\ m\geq 1}}k^m\frac{x^m}{m!}$$

and

$$s_m=\text{coefficient of } x^n \text{ in} \tag{26}$$

$$(n+1)\left(\frac{x}{1-e^{-x}}\right)^{n+1}\left[\sum_{\substack{m\text{ odd}\\ m\geq 1}}-\zeta(-m)\left(1+\frac{1}{2}+\ \dots\ +\frac{1}{m}\right)\frac{x^m}{m!}\right]$$

Clearly

$$P(k)=\left[\text{coefficient of } x^n \text{ in } 2(n+1)\left(\frac{x}{1-e^{-x}}\right)^{n+1}e^{kx}\right]^{\text{odd}}$$
$$=2(n+1)\left[\int\frac{e^{kx}}{(1-e^{-x})^{n+1}}\,dx\right]^{\text{odd}}$$

where the integral is taken on a small circle around the origin in the complex plane. We perform the change of variable $u=1-e^{-x}$ and we get

$$\int\frac{e^{kx}}{(1-e^{-x})^{n+1}}\,dx=\int\frac{(1-u)^{-k-1}}{u^{n+1}}\,du=\frac{(n+k)!}{k!\,n!}.$$

Hence

$$P(k)=2(n+1)\left[\frac{(k+n)!}{k!\,n!}\right]^{\text{odd}} \tag{27}$$

From (21), (23), 2.4.1(iii), (25) and (27) we conclude that the terms involving $\zeta'(s)$ are the same on the left and right hand sides, and Theorem 2.1.1 is equivalent to the identity

$$\frac{1}{2}\sum_{q=1}^{n}(-1)^{q+1}d_{n,q}^{*}(q-n-1)=s_n+t_n+\tilde{Td}_n, \tag{28}$$

where s_n, t_n and $\tilde{Td}_n$ are defined in (26), 2.2.2, and 2.2.3 respectively.

2.4.3. LEMMA 2.4.3. *Let T, y be two variables related by $T=1-e^{-y}$. Define coefficients β_l, σ_n and λ_n by the generating functions*

$$\sum_{l\geq 0}\beta_l y^l=y(1-T)/T$$

$$\sum_{n\geq 0}\sigma_n T^n=y/(1-T),$$

and

$$\sum_{n\geq 0}\lambda_n T^n=y^{-1}T/(1-T).$$

Then the following holds:

(i) $\displaystyle\sum_{n\geq 1}\frac{s_n}{n+1}T^n=\frac{1}{1-T}\sum_{k\geq 2}\sigma_{k-1}\beta_k y_{k-1}$

(ii) $\displaystyle\sum_{n\geq 1}\frac{\tilde{Td}_n}{n+1}T^{n+1}=-\sum_{k\geq 2}\frac{\beta_k}{k(k-1)}y^k$

(iii) $t_n=(n+1)\lambda_{n+1}(\sigma_{n+1}-1)$

Proof.

(i) Let

$$\psi(x)=\sum_{\substack{m\text{ odd}\\ m\geq 1}}-\zeta(-m)\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right)\frac{x^m}{m!}.$$

From (26) we get

$$s_n=(n+1)\int_C\psi(x)\frac{dx}{(1-e^{-x})^{n+1}},$$

the integral being taken on a small oriented loop C around $0\in\mathbb{C}$. Define a new variable $u=1-e^{-x}$. Then

$$\sum_{n\geq 1}\frac{s_n}{n+1}T^n=\sum_{n\geq 1}\left[\int_C\frac{\psi(x)}{1-u}\frac{du}{u^{n+1}}\right]T^n$$
$$=\frac{1}{1-T}\psi(y).$$

Since

$$\sigma_k=1+\frac{1}{2}+\ldots+\frac{1}{k}$$

and

$$\beta_k=\begin{cases}-\zeta(-m)/(m!) & \text{when } m=k-1 \text{ is odd}\\ 0 & \text{otherwise,}\end{cases}$$

we get (i).

(ii) We have

$$\frac{e^{-tx}}{1-e^{-tx}}-\frac{1}{tx}=\sum_{l\geq 1}\beta_l t^{l-1}x^{l-1}.$$

Define

$$\psi(x)=-\int_0^1\sum_{l\geq 2}\beta_l t^{l-2}x^{l-1}\,dt.$$

From Proposition 2.2.3 we get

$$\tilde{Td}_n=\int_C\psi(x)\frac{dx}{(1-e^{-x})^{n+1}},$$

hence, as in (i) above,

$$\sum_{n\geq 1}\tilde{Td}_n T^n=\frac{1}{1-T}\psi(y).$$

Therefore

$$\sum_{n\geq 1}\frac{\tilde{Td}_n}{n+1}T^{n+1}=\int_0^y\psi(z)dz$$
$$=-\sum_{k\geq 2}\frac{\beta_k}{k(k-1)}y^k.$$

(iii) We have

$$\sum_{p=1}^{n}\sum_{j=1}^{p}\frac{1}{j}=(n+1)\left(1+\frac{1}{2}+\ \dots\ +\frac{1}{n}\right)-n$$
$$=(n+1)(\sigma_{n+1}-1).$$

Furthermore, the coefficient of x^{n+1} in $(x/(1-e^{-x}))^{n+1}$ is

$$\int\frac{dx}{x(1-e^{-x})^{n+1}}=\int\frac{du}{x(u-1)u^{n+1}}=\lambda_{n+1}.$$

Using Lemma 2.2.2, we get (iii). q.e.d.

2.4.4. The equality (28), $n\geq 1$, is proved by D. Zagier in the Appendix (using the notation $\delta_{n,r}=d_{n,n+1-r}$ and the definition of s_n, $\tilde{Td}_n$ and t_n coming from Lemma 2.4.3). This concludes the proof of Theorem 2.1.1.

3. HIGHER DIRECT IMAGES OF HERMITIAN HOLOMORPHIC VECTOR BUNDLES

3.1. Higher analytic torsion.

Let $f\colon X\to Y$ be a smooth proper map of complex analytic manifolds, TX the tangent bundle to X, and $T_{X/Y}$ the relative tangent bundle. Let $h_{X/Y}$ be a metric on $T_{X/Y}$ whose restriction to each fiber $X_y=f^{-1}(y)$, $y\in Y$, is Kähler. Call $\omega_{X/Y}$ the associated (1, 1) form.

Let T^HX be a smooth sub-bundle of TX such that $TX=T_{X/Y}\oplus T^HX$. We shall assume that $(f, h_{X/Y}, T^HX)$ is a *Kähler fibration* in the sense of [9], Def. 2.4. p. 50, i.e. there exists a closed form ω on X such that $T_{X/Y}$ and T^HX are orthogonal with respect to ω, and ω restricts to $\omega_{X/Y}$ on $T_{X/Y}$.

Let now E be a holomorphic vector bundle on X which is *f-acyclic* i.e. the coherent sheaf R^qf_*E vanishes for every $q\geq 1$, and h a metric on E. By the semi-continuity of the Euler characteristic the sheaf $f_* E = R^0 f_* E$ is locally free on Y. We shall define a form $\tau(E)$ in

$$A(Y,\mathbb{C}) = \bigoplus_{p \geq 0} A^{pp}(Y,\mathbb{C})$$

whose component of degree zero is the Ray–Singer analytic torsion $\tau(E)^0$ considered in §1. Let $T^*_{X/Y}$ be the complexification of the dual of $T_{X/Y}$, $T^{*(0,1)}_{X/Y}$ its antiholomorphic component, $\Lambda^q T^{*(0,1)}_{X/Y}$ its q-th exterior power, $q \geq 0$, and $\mathscr{D}^q$ the infinite dimensional C^∞ bundle on Y whose sections on any open $U \subset Y$ are

$$\mathscr{D}^q(U) = C^\infty(f^{-1}(U), \Lambda^q T^{*(0,1)}_{X/Y} \otimes E). \tag{29}$$

Let $\bar{\partial}: \mathscr{D}^q \to \mathscr{D}^{q+1}$ be the Cauchy–Riemann operator of E along the fibers of f. We shall be interested in the *relative Dolbeault complex*:

$$\mathscr{D}^0 \xrightarrow{\bar{\partial}} \mathscr{D}^1 \xrightarrow{\bar{\partial}} \mathscr{D}^2 \to \dots\ .$$

Let $\mathscr{D}$ be the graded bundle $\bigoplus_{q \geq 0} \mathscr{D}^q$. Each fiber X_y having a Kähler metric, hence a density μ_y as in 1.1.1, we may define an L^2-metric on $\mathscr{D}^q_y = A^{oq}(X_y, E)$ by the formula (1) in 1.1.1. We let $f_*(h)$ denote the metric on the smooth bundle $f_*(E)_\infty \subset \mathscr{D}^0$ attached to $f_*(E)$ which is induced by the L^2-metric on $\mathscr{D}$, and we denote by $\bar{\partial}^*$ the adjoint of $\bar{\partial}$.

We now turn $\Lambda T^{*(0,1)}_{X/Y} \otimes E$ into a Clifford module under the action of the smooth sections of $T_{X/Y}$ as follows. ([9] (2.42) and (2.43)). If v is a relative tangent vector of type $(1,0)$, let $v^* \in T^{*(0,1)}_{X/Y}$ be the one form sending $w \in T_{X/Y}$ to its scalar product with v, and $c(v)$ the endomorphism of $\Lambda T^{*(0,1)}_{X/Y} \otimes E$ sending η to $2v^* \Lambda \eta$. On the other hand, when v is a relative tangent vector of type $(0,1)$, we let $c(v)$ be the interior product by $-2v$. The map c extends by linearity to the whole tangent space.

Now let v and w be two vector fields on y. Call v^H and w^H the vector fields on X obtained by lifting v and w to $T^H X$. Let $[v,w]$ be their commutator and $T(v,w) \in T_{X/Y}$ be the projection along $T^H X$ of $-[v,w]$. The map T defines a tensor in $C^\infty(X, T_{X/Y} \otimes \Lambda^2(T^H X)^*)$. The action of T by Clifford multiplication on $\mathscr{D}$ and the exterior product of forms on Y define an operator $c(T)$ in the algebra

$$\operatorname{End}_{\mathbb{C}}(\mathscr{D}) \underset{\mathbb{C}}{\otimes} A^*(Y,\mathbb{C}),$$

where

$$A^*(Y,\mathbb{C}) = \bigoplus_{n \geq 0} A^n(Y,\mathbb{C})$$

(see [7], 3.Def.1.8., and [9] (3.7.) p. 69).

Let $T = T^{(1,0)} + T^{(0,1)}$ be the decomposition of T according to its type in $T_{X/Y}$ and

$$c(T) = c(T^{(1,0)}) + c(T^{(0,1)})$$

the corresponding decomposition of $c(T)$.

We now define a connection $\tilde{\nabla}$ on the bundles $\mathscr{D}^q$, $q \geq 0$ ([7] Def. 1.10., [9] Def. 2.1.3.). The metric on $T_{X/Y}$ gives an isomorphism between $T^{*(0,1)}_{X/Y}$ and the holomorphic relative tangent bundle $T^{(1,0)}_{X/Y}$, hence a holomorphic structure on every bundle $\Lambda^q T^{*(0,1)}_{X/Y}$, $q \geq 0$. We let ∇ be the unique unitary connection on $\Lambda^q T^{*(0,1)}_{X/Y} \otimes E$ which is compatible to its holomorphic structure. Let σ be a smooth section of $\mathscr{D}^q$ on some open subset $U \subset Y$, i.e. a section of $\Lambda^q T^{*(0,1)}_{X/Y} \otimes E$ over $f^{-1}(U)$ (cf. (29)). For every $x \in X$, $y = f(x)$ and $v \in T_y Y$ denote by $v^H \in T_x^H X$ the horizontal lifting of v. We define

$$\tilde{\nabla}_v(\sigma) = \nabla_{v^H}(\sigma).$$

From [9], Theorem 1.14., we know that $\tilde{\nabla}$ is unitary.

Let now $p: X \times \mathbb{C}^* \to X$ be the first projection, δ the differential on $\mathbb{P}^1$, and $\tilde{\nabla} + \delta$ the connexion on $p^*\mathscr{D}$ induced by $\tilde{\nabla}$. By the Leibnitz' rule we extend $\tilde{\nabla} + \delta$ to get an operator in

$$\mathscr{A} = \operatorname{End}_{\mathbb{C}}(\mathscr{D} \underset{\mathbb{C}}{\otimes} A^*(Y \times \mathbb{C}^*, \mathbb{C})). \tag{30}$$

For every non zero complex number $z \in \mathbb{C}^*$ we consider the following element of $\mathscr{A}$ (a superconnection in the sense of Quillen [34]):

$$A_z = \tilde{\nabla} + \delta + z\bar{\partial} + \bar{z}\bar{\partial}^* - \frac{1}{4\bar{z}} c(T^{(1,0)}) - \frac{1}{4z} c(T^{(0,1)}).$$

The curvature $-A_z^2$ defines an element in $\operatorname{End}(\mathscr{D}) \otimes A(Y \times \mathbb{C}^*, \mathbb{C})$ whose exponential $\exp(-A_z^2)$ happens to be trace class (see below). Let

$$a(z) = Tr_s \exp(-A_z^2) \in A^*(Y \times \mathbb{C}^*, \mathbb{C}). \tag{31}$$

be its supertrace for the $\mathbb{Z}/2$ grading on $\mathscr{D} \otimes A(Y, \mathbb{C})$. For every positive real number $\varepsilon > 0$, we let

$$I(\varepsilon) = \int_{|z| > \varepsilon} a(z) \log |z|^2$$

in $A(Y, \mathbb{C})$. As we shall see below this integral happens to converge and to have a finite asymptotic development of the type

$$I(\varepsilon) = \sum_{j \le 0} a_j \varepsilon^j + \sum_{j \le 0} b_j \varepsilon^j \log \varepsilon + O(\varepsilon) \tag{32}$$

which is uniform on every compact subset of Y. We let $I(0) = a_0$ be the finite part of $I(\varepsilon)$.

We now define two new characteristic classes ch' and Td' as follows. The first is

$$ch' = \sum_{q \ge 0} (-1)^q q\, ch_q,$$

where ch_q is the component of degree q of the Chern character. The second one is the one coming from the invariant polynomial function on square matrices A:

$$Td'(A) = \frac{d}{dt} Td(A + t\, Id).$$

Given any form

$$\eta = \sum_{p \ge 0} \eta^p \quad \text{in} \bigotimes_{p \ge 0} A^{2p}(Y, \mathbb{C}) \qquad \text{and } \lambda \in \mathbb{C}^*$$

we let

$$\delta_\lambda(\eta) = \sum_{p \ge 0} \lambda^{-p} \eta^p.$$

If γ is the Euler constant, we define

$$\tau(E) = \delta_{2i\pi} I(0) + \gamma f_*(ch(E, h)\, Td'(T_{X/Y}, h_{X/Y})) - \gamma ch'(f_* E, f_* h).$$

The following is a variant of [10] Theorem 1.27.

THEOREM 3.1.

(i) *The form* $\tau(E)$ *lies in* $A(Y,\mathbb{C})=\bigoplus_{p\geq 0} A^{pp}(Y,\mathbb{C})$ *and satisfies the equation*

$$dd^c\tau(E)=f_*(ch(E,h)\,Td(T_{X/Y},h_{X/Y}))-ch(f_*E,f_*h). \tag{33}$$

(ii) *The degree zero component of* $\tau(E)$ *is the Ray–Singer analytic torsion* $\tau(E)^0$.

Proof

(i) Since f_* commutes with dd^c and since $(ch(E,h),Td'(T_{X/Y},h_{X/Y}))$ and $ch'(f_*E,f_*h)$ are killed by dd^c, we just need to prove (33) with $\tau(E)$ replaced by $\delta_{2i\pi}I(0)$.

The first thing we show is that $\exp(-A_z^2)$ is trace class for $z\in\mathbb{C}^*$. For this we notice that

$$A_z^2=|z|^2\Delta+\Phi,$$

where $\Delta=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial$ and Φ is nilpotent (since it has positive degree as form over Y). From Duhamel's formula we can write $\exp(-A_z^2)$ as a finite sum

$$\exp(-A_z^2)=\sum_{n\geq 0}\int_{0\leq t_1\leq\ldots\leq t_n\leq 1} e^{-t_1|z|^2\Delta}\Phi e^{(t_1-t_2)|z|^2\Delta}\Phi\ldots\Phi e^{(t_n-1)|z|^2\Delta}dt_1\ldots dt_n.$$

Using the fact that the heat kernel $e^{-|z|^2\Delta}$ is a smooth family of smoothing operators, we conclude that $\exp(-A_z^2)$ is trace class. Furthermore a similar argument for derivatives with respect to $Y\times\mathbb{C}^*$ shows that $a(z)$ is a smooth form on $Y\times\mathbb{C}^*$.

The form $a(z)$ is closed since

$$d\,tr_s\exp(-A_z^2)=tr_s[A_z,\exp(-A_z^2)]=0$$

where we use the fact that the supertrace vanishes on supercommutators (denoted $[\,,]_s$). For more details on this argument see [34] and [7] Prop. 2.9.

From the identities of [9] Theorem 2.6, we get

$$A_z^2=\left[\tilde\nabla^{(1,0)}+\frac{\partial}{\partial z}dz+z\bar\partial-\frac{1}{4\bar z}c(T^{(1,0)}),\tilde\nabla^{(0,1)}+\frac{\partial}{\partial\bar z}d\bar z+\bar z\bar\partial^*-\frac{1}{4z}c(T^{(0,1)})\right]. \tag{34}$$

For any $\theta\in\mathbb{R}$ let r_θ: $Y\times\mathbb{C}^*\to Y\times\mathbb{C}^*$ be the automorphism sending (y,z) to $(y,e^{i\theta}z)$.

The vector space $\mathcal{S}\otimes A^*(Y\times\mathbb{C}^*,\mathbb{C})$ is graded by $\mathbb{N}^3$ with

$$\mathcal{S}\otimes A^*(Y\times\mathbb{C}^*,\mathbb{C}))^{(n,p,q)}=\mathcal{S}^n\otimes_{\mathbb{C}} A^{pq}(Y\times\mathbb{C}^*,\mathbb{C}).$$

Therefore the algebra $\mathcal{A}$ is also graded by $\mathbb{N}^3$. We denote by $\mathcal{B}\subset\mathcal{A}$ the subalgebra generated (as complex vector space) by elements α of degree (n,p,q) such that $q=p+n$ and $r_\theta^*(\alpha)=e^{in\theta}\alpha$ (see [8]). From (34) we conclude that A_z^2 lies in $\mathcal{B}$. Since tr_s vanishes on the subspace of $\mathcal{B}$ spanned by elements of degree (n,p,q), $n>0$, we conclude that $a(Z)$ lies in

$$A(Y\times\mathbb{C}^*,\mathbb{C})=\bigotimes_{p\geq 0} A^{pp}(Y\times\mathbb{C}^*,\mathbb{C})$$

and

$$r_\theta^*(a(z))=a(z).$$

Let us study the behaviour of $a(z)$ as $r=|z|$ goes to infinity. For this we use a result of Berline and Vergne ([5], Theorem 2.4). For any real number $t>0$ let δ_t be the automorphism of $A^n(Y\times\mathbb{C}^*,\mathbb{C})$ acting by multiplication by $t^{-n/2}$. Extend δ_t to an automorphism of $\mathcal{A}$. Then (*loc. cit.*) as t goes to infinity the operator $\delta_t\exp(-t\,A_z^2)$ converges (for the operator norm in $\mathscr{A}$) to the orthogonal projection of $\exp(-(\tilde{\nabla}+\delta)^2)$ on the kernel of $\bar{\partial}$. In particular the component $[\delta_t \exp(-t\, A_z^2)]^{(1,1)}$ of degree $(1,1)$ with respect to $\mathbb{C}^*$ converges to zero. In fact, looking at the proof of *loc. cit.* (see Lemma 1.1.1) we get

$$[\delta_t \exp(-t\, A_z^2)]^{(1,1)} = O(t^{-1/2})$$

as $t \to \infty$.

Now, since $r_\theta^*(a(z)) = a(z)$ for every θ, we can write, with $z = re^{i\theta}$,

$$a(z) = tr_s \exp -(R(r) + S(r)dr),$$

where

$$R(r) = \left(\tilde{\nabla} + \frac{\partial}{\partial\theta} d\theta + r(\bar{\partial} + \bar{\partial}^*) - \frac{1}{4r} c(T)\right)^2$$

and

$$S(r) = \frac{\partial}{\partial r}(A_r) = \bar{\partial} + \bar{\partial}^* + \frac{1}{4r^2} c(T)$$

Therefore the component involving dr in

$$tr_s(\delta_t \exp(-t A_z^2))dr$$

is

$$\begin{aligned} & tr_s \delta_t(-tS(r)\exp(-tR(r))) \\ & = tr_s(((\sqrt{t}(\bar{\partial}+\bar{\partial}^*) + \frac{1}{4r^2\sqrt{t}} c(T))\exp(-R(r\sqrt{t})))dr \\ & = O(t^{-1/2})dr. \end{aligned}$$

Dividing by $\sqrt{t}$ and putting $r=1$ and $s=t^{-1/2}$ we get

$$tr_s(((\bar{\partial}+\bar{\partial}^*) + \frac{s^2}{4} c(T))\exp(-R(1/s))) = O(s^{-2}),$$

i.e. the form $tr_s(S(r)\exp(-R(r)))dr$ is bounded as r goes to infinity (take $s = 1/r$). Similarly $tr_s(\exp(-R(r)))$ is bounded as r goes to infinity, i.e. $a(z)$ remains bounded as a form on $X \times \mathbb{C}^*$ as $|z|$ goes to infinity. Similar arguments apply to any derivative of $a(z)$ with respect to the parameter space $Y \times S^1$ and the bounds we get are uniform on any compact subset of $Y \times S^1$.

We now consider the behaviour of $a(z)$ as $r = |z|$ goes to zero. We have

$$\delta_r(A_z^2) = r^2 A_1^2 = r^2(A - B\,dr),$$

where A and B are smooth families of differential operators on $Y \times S^1$, with A elliptic and positive definite. From [22] and [23] we conclude that $tr_s \exp(-r^2 A)$ and $tr_s(B\exp(-r^2 A))$ have finite asymptotic expansions in powers of r^2, which converge uniformly on any compact subset of $Y \times S^1$ as r goes to zero. Therefore the same holds for

$$a(z) = \delta_r^{-1}(tr_s \exp(-r^2 A) + tr_s(B\exp(-r^2 A))dr).$$

Now let $\varepsilon > 0$ be any positive real number. The integral

$$I(\varepsilon) = \int_{|z|>\varepsilon} a(z) \log|z|^2$$

converges and is C^∞ on Y since $a(z)$ and its derivatives with respect to Y are bounded as $|z|$ goes to infinity. From the asymptotic development of $a(z)$ we may write $I(\varepsilon)$ as in (32), the convergence being uniform on any compact subset of Y. Since $a(z)$ lies in $A(Y\times\mathbb{C}^*, \mathbb{C})$, we conclude that $I(\varepsilon)$ and $I(0)$ lie in $A(Y, \mathbb{C})$.

Now we compute $dd^cI(0)$. Since $a(z)$ is closed on $Y\times\mathbb{C}^*$ we have

$$(d+\delta)(a(z))=0 \tag{35}$$

If δ^c is the "complex conjugate" of δ, it follows that

$$(d^c+\delta^c)(a(z))=0, \tag{36}$$

since $a(z)$ lies in $A(Y\times\mathbb{C}^*, \mathbb{C})$. Let $R>\varepsilon$ be any real number and

$$I(\varepsilon, R)=\int_{\varepsilon<|z|<R} a(z)\log|z|^2.$$

We get from (35) and (36)

$$\begin{aligned} dd^cI(\varepsilon, R)=&\int_{|z|=\varepsilon}\delta^c(a(z))\log|z|^2-\int_{|z|=R}\delta^c(a(z))\log|z|^2\\ &-\frac{1}{2\pi i}\int_{|z|=\varepsilon}a(z)\delta\log|z|^2+\frac{1}{2\pi i}\int_{|z|=R}a(z)\delta\log|z|^2\\ &+\int_{\varepsilon<|z|<R}a(z)\delta^c\delta\log|z|^2. \end{aligned} \tag{37}$$

Now $\delta^c\delta\log|z|^2=0$, $\delta^c a(z)=-d^c a(z)$ is bounded as $|z|$ goes to infinity, and the component of $a(z)$ of degree zero with respect to $\mathbb{P}^1$ has a limit $a(\infty)$ when z goes to infinity ([6] Thm. 2.4.). Therefore, letting R go to infinity in (37), we get

$$dd^cI(\varepsilon)=-2\log(\varepsilon)\int_{|z|=\varepsilon}d^ca(z)-\frac{1}{2\pi i}\int_{|z|=\varepsilon}a(z)\delta\log|z|^2+a(\infty). \tag{38}$$

The component of $d^ca(z)$ which does not involve dr has a finite asymptotic development in powers of r, therefore the first summand in (38) is a sum of type

$$\sum_{k\gg-\infty}\beta_k\log(\varepsilon)\varepsilon^k+O(\varepsilon).$$

The component of $a(z)$ of degree zero with respect to $\mathbb{P}^1$ is

$$tr_s\exp-\left(\tilde{\nabla}+r(\bar{\partial}+\bar{\partial}^*)-\frac{1}{4r}c(T)\right)^2.$$

By the local index theorem for families [7] we know that this form has a limit $a(0)$ as r goes to zero. By the unicity of the asymptotic development of $I(\varepsilon)$ we get from (38);

$$dd^cI(0)=a(0)-a(\infty).$$

Now from [7] we have

$$\delta_{2\pi i}a(0)=f_*(ch(E,h)Td(T_{X/Y},h_{X/Y}))$$

and from [6]

$$\delta_{2\pi i}a(\infty)=ch(f_*E,f_*h). \qquad \text{q.e.d.}$$

(ii) Let $a(z)^0$ be the component of $a(z)$ of degree zero with respect to Y. We have

$$a(z)^0 = tr_s \exp -(\delta + z\bar{c} + \bar{z}\bar{c}^*)^2.$$

Since $a(z)^0$ is invariant under the rotations r_θ we get

$$a(z)^0 = tr_s \exp -(r^2\Delta + (\bar{c} + \bar{c}^*)dr + ir(\bar{c} - \bar{c}^*)d\theta).$$

Since $\bar{c} + \bar{c}^*$ commutes with Δ, the component of $a(z)^0$ of degree 2 with respect to $\mathbb{C}^*$ is

$$a(z)^{02} = tr_s \exp[(\bar{c} - \bar{c}^*)(\bar{c} + \bar{c}^*)\exp(-r^2\Delta)]r\,dr\,d\theta. \tag{39}$$

Let N be the operator acting on $\mathscr{D}^q$ by multiplication by q. We have (see [8]).

$$[N, \bar{c}] = \bar{c},$$

and

$$[N, \bar{c}^*] = -\bar{c}^*,$$

hence

$$\bar{c} - \bar{c}^* = [N, \bar{c} + \bar{c}^*]. \tag{40}$$

Since tr_s vanishes on supercommutators and $\bar{c} + \bar{c}^*$ commutes with Δ we get from (39) and (40):

$$a(z)^{02} = 2tr_s(N\Delta \exp(-r^2\Delta))r\,dr\,d\theta.$$

Let $u = r^2$. We get

$$\int_{\mathbb{C}^*} a(z)^0 \log|z|^2 = \int_0^\infty \int_0^{2\pi} tr_s(N\Delta \exp(-r^2\Delta)) \log(u)\,du\,dr$$
$$= 2\pi \int_0^\infty tr_s(N\Delta \exp(-u\Delta)) \log(u)\,du.$$

Now let Q be the orthogonal projection of $\mathscr{D}$ onto the orthogonal complement of $f_*(E)_{\infty}$. Define

$$\zeta(s) = \frac{-1}{\Gamma(s)} \int_0^\infty tr_s(QN \exp(-u\Delta))u^{s-1}\,du. \tag{42}$$

Clearly

$$\zeta(s) = \sum_{q \geq 0} (-1)^q q\zeta_q(s),$$

where $\zeta_q(s)$ is the zeta function of the Laplace operator Δ_q as in §1.1. From the fact that $Tr_S(QN \exp(-u\Delta))$ has a finite aymptotic development in powers of u as u goes to zero, it follows that

$$J(\varepsilon) = \int_\varepsilon^\infty u^{-1} tr_s(QN \exp(-u\Delta))\,du$$

has a finite asymptotic development in terms of $\varepsilon^k \log \varepsilon$ and ε^k, $k \in \mathbb{Z}$. Furthermore its finite part $J(0)$ satisfies

$$J(0) = \zeta'(0) + \gamma\alpha_0, \tag{43}$$

where γ is the Euler constant and α_0 is the finite part of $tr_s(QN \exp(-u\Delta))$ as u goes to zero (for a similar argument, see [16] 3.5).

Integrating $J(\varepsilon)$ by parts we get

$$J(\varepsilon) = [\log(u) tr_s(QM \exp(-u\Delta))]_\varepsilon^\infty - \int_\varepsilon^\infty \log(u) Tr_s(QN\Delta \exp(-u\Delta))\,du$$

The first term in this expression has a finite asymptotic development in terms of $\log(\varepsilon)\,\varepsilon^k$ and $Q\Delta = \Delta$, therefore

$$J(0) = -\int_0^\infty tr_s(N\Delta \exp(-u\Delta))\log(u)du$$
$$= -2\pi \int_{\mathbb{C}^*} a(z)^0 \log|z|^2.$$

Since $\Delta = 0$ on $f_*(E)_\infty = (1-Q)(\mathscr{S})$, we get

$$tr_s(QN \exp(-u\Delta)) = tr_s(N\exp(-u\Delta)) - tr_s(N \text{ on } f_*(E)_\infty).$$

It was shown in [9], Thm. 3.1.6, p. 87, that the finite part of $tr_s(N\exp(-u\Delta))$ as u goes to zero is the component of degree zero in

$$f_*(ch(E, h)Td'(T_{X/Y}, h_{X/Y})).$$

Since $Tr_s(N \mid f_*(E)_\infty)$ is the component of degree zero of $ch'(f_*E, f_*h)$ we conclude, using (43) and (44), that

$$\tau(E)^0 = \delta_{2i\pi}\zeta'(0)$$

is the analytic torsion considered in §1.1.

Remarks. Assume $R^0 f_* E = 0$. Then an argument similar to the proof of (ii) above shows that $\tau(E) = \delta_{2i\pi}\tilde{\zeta}'(0)$, where $\tilde{\zeta}(s)$ is the form-valued zeta function considered in [9] Thm. 3.20. Therefore (i) follows from *loc. cit.*

One may wonder whether the class of $\tau(E)$ in $\tilde{A}(Y, \mathbb{C})$ depends on the choice of the horizontal tangent space $T^H X$ (see Conjecture 3.3 below).

3.2. *Arithmetic K-theory*

Let (A, Σ, F_∞) be an arithmetic ring (1.2.1). Given any arithmetic variety X over A we defined in [21], §6, a group $\hat{K}_0(X)$ of *virtual hermitian vector bundles* over X as follows. A generator of $\hat{K}_0(X)$ is a triple (E, h, η), where (E, h) is a hermitian vector bundle on X and $\eta \in \tilde{A}(X)$. The relations are the following. Let

$$\mathscr{E}: 0 \to S \to E \to Q \to 0$$

be an exact sequence of vector bundles on X, h', h, h'' arbitrary metrics on S, E, Q respectively and $\bar{\mathscr{E}} = (\mathscr{E}, h', h, h'')$. Then, given any $\eta', \eta'' \in \tilde{A}(X)$ one has, in $\hat{K}_0(X)$,

$$(S, h', \eta') + (Q, h'', \eta'') = (E, h, \eta' + \eta'' + \tilde{ch}(\bar{\mathscr{E}})).$$

Here $\tilde{ch}(\bar{\mathscr{E}}) \in \tilde{A}(X)$ denotes (as in 1.2.2 above) the solution of the equation

$$-dd^c\tilde{ch}(\bar{\mathscr{E}}) = ch(E, h) - ch(S, h') - ch(Q, h'') \tag{45}$$

defined by Bott and Chern in [15], and studied in [18], [8] and [20] §1. One can then define a morphism

$$ch: \hat{K}_0(X) \to A(X)$$

by the formula

$$ch(E, h, \eta) = ch(E, h) + dd^c(\eta)$$

(by (45) this is compatible with the relations defining $\hat{K}_0(X)$).

Let now

$$f: X \to Y$$

be a smooth projective map between arithmetic varieties over A. Let $T_{X/Y}$ be the relative tangent bundle, and $h_{X/Y}$ a metric on the associated holomorphic bundle on X_∞. Let $f_\infty: X_\infty \to Y_\infty$ be the map of complex varieties induced by f and $T^H X_\infty$ a smooth sub-bundle of TX_∞ such that the triple $(f_\infty, h_{X/Y}, T^H X_\infty)$ is a Kähler fibration in the sense of 3.1.

We shall now define a direct image morphism from $\hat{K}_0(X)$ to $\hat{K}_0(Y)$. Given any triple (E, h, η) on X with $R^q f_* E = 0$ when $q > 0$, we define $f_!(E, h, \eta)$ in $\hat{K}_0(Y)$ to be the class of

$$(f_* E, f_* h, \tau(E) + f_!(\eta)),$$

where $f_* h$ is defined as in 3.1. (the L^2-metric on $f_* E$), $\tau(E)$ is the class in $\tilde{A}(Y)$ of the higher analytic torsion introduced in Theorem 3.1 and

$$f_!(\eta) = f_*(\eta\, Td(T_{X/Y}, h_{X/Y})) \in \tilde{A}(Y).$$

THEOREM 3.2. *The map $f_!$ induces a group morphism*

$$f_!: \hat{K}_0(X) \to \hat{K}_0(Y)$$

such that the following formula holds in $A(Y)$:

$$ch(f_!(\alpha)) = f_*(ch(\alpha)\, Td(T_{X/Y}, h_{X/Y})) \tag{46}$$

for any $\alpha \in \hat{K}_0(X)$.

Proof. We know already from Theorem 3.1 and the definition of ch that formula (46) holds when α is replaced by (E, h, η), with $R^q f_* E = 0$ when $q > 0$.

Consider an exact sequence

$$\mathscr{E}: 0 \to S \to E \to Q \to 0$$

of bundles on X, with $R^q f_* S = R^q f_* E = R^q f_* Q = 0$ for every $q > 0$. Choose arbitrary metrics h', h, h'' on S, E, Q respectively. Taking the direct images by f we get an exact sequence of vector bundles on Y:

$$f_* \mathscr{E}: 0 \to f_* S \to f_* E \to f_* Q \to 0$$

with metrics $f_* h', f_* h, f_* h''$. Let

$$\bar{\mathscr{E}} = (\mathscr{E}, h', h, h'') \quad \text{and} \quad f_* \bar{\mathscr{E}} = (f_* \mathscr{E}, f_* h', f_* h, f_* h'').$$

We shall prove below that the following equation holds in $\tilde{A}(Y)$:

$$\tau(E) - \tau(S) - \tau(Q) - c\tilde{h}(f_* \bar{\mathscr{E}}) = -f_!(c\tilde{h}(\bar{\mathscr{E}})). \tag{47}$$

Since f is projective, any vector bundle on X has a finite resolution by vector bundles E which are acyclic for f, i.e. $R^q f_* E = 0$ when $q > 0$ ([33], 7.27). Therefore $\hat{K}_0(X)$ is generated by triples (E, h, η) with E acyclic for f, and the relation (47) means that, in $\hat{K}_0(Y)$,

$$\begin{aligned} f_!(S, h', 0) + f_!(Q, h'', 0) &= (f_* S, f_* h', \tau(S)) + (f_* Q, f_* h'', \tau(Q)) \\ &= (f_* E, f_* h, \tau(S) + \tau(Q) + c\tilde{h}(f_* \bar{\mathscr{E}})) \\ &= (f_* E, f_* h, \tau(E) + f_!(c\tilde{h}(\bar{\mathscr{E}}))) \\ &= f_!(E, h, c\tilde{h}(\bar{\mathscr{E}})). \end{aligned}$$

In other words, $f_!$ preserves the defining relations in $\hat{K}_0(X)$, and by (33) (Theorem 3.1), Theorem 3.2 follows.

So let us prove (47). For this we may assume that the ground ring is $\mathbb{C}$. We use a definition of $c\tilde{h}(\bar{\mathscr{E}})$ introduced in [8] and [21]. Let $\mathbb{P}^1$ be the complex projective line, $\mathcal{O}(1)$

the standard line bundle of degree one on $\mathbb{P}^1$ and σ a section of $\mathcal{O}(1)$ vanishing only at infinity. Let z be the standard complex parameter on $\mathbb{P}^1$, and $i_z\colon X \to X \times \mathbb{P}^1$ the map sending x to (x, z). On $X \times \mathbb{P}^1$ consider the bundle

$$\tilde{E} = (E \oplus S(1))/S,$$

where S is embedded in E as in $\mathscr{E}$ and in $S(1) = S \otimes \mathcal{O}(1)$ by $\mathrm{id} \otimes \sigma$. Choose on $\tilde{E}$ a metric $\tilde{h}$ for which the isomorphisms $i_0^* \tilde{E} \simeq E$ and $i_\infty^* \tilde{E} \simeq S \oplus Q$ are isometries ($S \oplus Q$ being equipped with the orthogonal direct sum $h' \oplus h''$). Then $\tilde{ch}(\bar{\mathscr{E}})$ is the class in $\tilde{A}(X)$ of

$$-\int_{\mathbb{P}^1} ch(\tilde{E}, \tilde{h}) \log |z|^2$$

(cf. *loc. cit.*)

Now consider the following commutative diagram of proper smooth analytic maps

$$\begin{tikzcd} X \times \mathbb{P}^1 \arrow[r] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\ Y \times \mathbb{P}^1 \arrow[r] & Y \end{tikzcd}$$

where $\tilde{f} = f \times id_{\mathbb{P}^1}$ and the horizontal maps are the first projections. We get

$$f_*(\tilde{ch}(\mathscr{E}) Td(T_{X/Y}, h_{X/Y})) = -\int_{\mathbb{P}^1} \tilde{f}_*(ch(\tilde{E}, \tilde{h}) \log |z|^2 \, Td(T_{X/Y}, h_{X/Y})). \tag{48}$$

The pull back of $T''X$ and $h_{X/Y}$ by the projection $X \times \mathbb{P}^1 \to X$ define with $\tilde{f}$ a Kähler fibration in the sense of 3.1, and we have, for every $\eta \in \tilde{A}(X \times \mathbb{P}^1)$,

$$\tilde{f}_!(\eta) = \tilde{f}_*(\eta \, Td(T_{X/Y}, h_{X/Y})).$$

Furthermore $R^q \tilde{f}_* \tilde{E} = 0$ when $q > 0$ since $i_z^* \tilde{E}$ is either E or $S \oplus Q$. Therefore we may apply Theorem 3.1 to $\tilde{f}$ and $(\tilde{E}, \tilde{h})$. We get

$$\tilde{f}_!(ch(\tilde{E}, \tilde{h})) = dd^c \tau(\tilde{E}) + ch(f_* \tilde{E}, f_* \tilde{h}). \tag{49}$$

From (48) and (49) we deduce

$$\begin{aligned} f_! \tilde{ch}(\bar{\mathscr{E}}) &= -\int_{\mathbb{P}^1} dd^c \tau(\tilde{E}) \log |z|^2 - \int_{\mathbb{P}^1} ch(f_* \tilde{E}, f_* \tilde{h}) \log |z|^2 \\ &= -\int_{\mathbb{P}^1} \tau(\tilde{E}) dd^c \log |z|^2 - \int_{\mathbb{P}^1} ch(f_* \tilde{E}, f_* \tilde{h}) \log |z|^2. \end{aligned}$$

We now use the equation of currents

$$dd^c \log |z|^2 = \delta_0 - \delta_\infty,$$

where δ_z is the Dirac mass at $z \in \mathbb{P}^1$, and we obtain

$$f_! \tilde{ch}(\bar{\mathscr{E}}) = -i_0^* \tau(\tilde{E}) + i_\infty^* \tau(\tilde{E}) - \int_{\mathbb{P}^1} ch(f_* \tilde{E}, f_* \tilde{h}) \log |z|^2.$$

By definition of τ, $\tilde{E}$ and $\tilde{h}$ we get

$$i_0^* \tau(\tilde{E}) = \tau(i_0^* \tilde{E}) = \tau(E)$$

and

$$i_\infty^* \tau(\tilde{E}) = \tau(i_\infty^* \tilde{E}) = \tau(S \oplus Q) = \tau(S) + \tau(Q).$$

Finally

$$f_* \tilde{E} = (f_* E \oplus f_*(S)(1))/f_*(S),$$

$$i_0^*(f_* \tilde{h}) = f_* h \quad \text{and} \quad i_\infty^*(f_* \tilde{h}) = f_* h' \oplus f_* h'',$$

therefore

$$\int_{\mathbf{P}^1} ch(f_* \tilde{E}, f_* \tilde{h}) \log|z|^2 = c\tilde{h}(f_* \bar{\mathscr{E}}).$$

We conclude that

$$f_!(c\tilde{h}(\bar{\mathscr{E}})) = -\tau(E) + \tau(S) + \tau(Q) - c\tilde{h}(f_* \bar{\mathscr{E}})$$

as stated in (47). q.e.d.

3.3. A Conjecture.

We keep the notations of 3.2. The Conjecture 1.3 may be extended to higher degrees as follows.

Conjecture 3.3. *For any* $\alpha \in \hat{K}_0(X)$, *the following holds in* $\widehat{CH}(Y)_{\mathbf{Q}}$:

$$\widehat{ch}(f_!(\alpha)) = f_*(\widehat{ch}(\alpha) Td^A(T_{X/Y}, h_{X/Y})). \tag{50}$$

From Theorem 3.2, the Grothendieck–Riemann–Roch Theorem in Chow groups, and the exact sequence (3), we know that the difference between both sides of (50) lies in the image of a.

The Conjecture 1.3 is a special case of Conjecture 3.3 since

$$\hat{c}_1(f_!(\bar{E})) = \hat{c}_1(\lambda(E), h_Q)$$

(using Theorem 3.1(ii))

APPENDIX BY D. ZAGIER: PROOF OF THE IDENTITY (28)

§ 1. PRELIMINARIES

We will consistently use the notation, T, y for two variables related by

$$T = 1 - e^{-y} = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l!} y^l, \quad y = \log \frac{1}{1-T} = \sum_{n=1}^{\infty} \frac{1}{n} T^n.$$

Define coefficients $S_1(n, l)$, $S_2(l, n)$ $(n, l \geqq 0)$ by the generating functions

$$y^l = \sum_{n=0}^{\infty} S_1(n, l) T^n, \quad T^n = \sum_{l=0}^{\infty} S_2(l, n) y^l,$$

so that $\{S_1(n, l)\}_{n, l \geq 0}$ and $\{S_2(l, n)\}_{l, n \geq 0}$ are mutually inverse infinite triangular matrices. Define coefficients β_l, s_n, λ_n by the generating functions

$$\sum_{l=0}^{\infty} \beta_l y^l = \frac{1-T}{T} y, \quad \sum_{n=0}^{\infty} \sigma_n T^n = \frac{1}{1-T} y, \quad \sum_{n=0}^{\infty} \lambda_n T^n = \frac{T}{1-T} y^{-1}.$$

Alternatively, we can define these numbers by the recursions

$$nS_1(n, l) = lS_1(n-1, l-1) + (n-1)S_1(n-1, l), \quad lS_2(l, n) = nS_2(l-1, n-1) - nS_2(l-1, n)$$

$$\beta_l = -\sum_{k=0}^{l-1} \frac{\beta_k}{(l+1-k)!}, \quad \sigma_n = \sigma_{n-1} + \frac{1}{n}, \quad \lambda_n = 1 - \sum_{m=0}^{n-1} \frac{\lambda_m}{n+1-m} \quad (n, l \geqq 1)$$

with the initial conditions

$$S_1(r, 0) = S_1(0, r) = S_2(r, 0) = S_2(0, r) = \delta_{r,0}, \ \beta_0 = 1, \sigma_0 = 0, \lambda_0 = 1.$$

Thus $l!\beta_l$ is the lth Bernoulli number, $\sigma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, and $\frac{n!}{l!}S_1(n, l)$ and $\frac{l!}{n!}S_2(l, n)$ are (up to sign) the integers known as Stirling numbers of the first and second kind (= number of permutations of $\{1, 2, \ldots, n\}$ having exactly l cycles and number of partitions of $\{1, 2, \ldots, l\}$ into exactly n non-empty subsets, respectively). The numbers $S_1(n, l)$ are also the Taylor coefficients of binomial coefficients:

$$\binom{x}{n} = \sum_{l=0}^{n} \frac{(-1)^{n-l}}{l!} S_1(n, l)x^l, \quad \binom{x+n-1}{n} = \sum_{l=0}^{n} \frac{1}{l!} S_1(n, l)x^l.$$

The first few values are as follows (note $S_1(n, l) = 0$ if $n < l$, $S_2(l, n) = 0$ if $l < n$):

r	β_r	s_r	λ_r
0	1	0	1
1	$-\frac{1}{2}$	1	$\frac{1}{2}$
2	$\frac{1}{12}$	$\frac{3}{2}$	$\frac{5}{12}$
3	0	$\frac{11}{6}$	$\frac{3}{8}$
4	$-\frac{1}{720}$	$\frac{25}{12}$	$\frac{251}{720}$
5	0	$\frac{137}{60}$	$\frac{95}{288}$
6	$\frac{1}{30240}$	$\frac{49}{20}$	$\frac{19087}{60480}$

l \ n	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
2	0	$-\frac{1}{2}$	1	1	$\frac{11}{12}$	$\frac{5}{6}$	$\frac{137}{180}$
3	0	$\frac{1}{6}$	-1	1	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{15}{8}$
4	0	$-\frac{1}{24}$	$\frac{7}{12}$	$-\frac{3}{2}$	1	2	$\frac{17}{6}$
5	0	$\frac{1}{120}$	$-\frac{1}{4}$	$\frac{5}{4}$	-2	1	$\frac{5}{2}$
6	0	$-\frac{1}{720}$	$\frac{31}{360}$	$-\frac{3}{4}$	$\frac{13}{6}$	$-\frac{5}{2}$	1

$S_1(n, l)$

$S_2(l, n)$

For $n \geq 1$, $l \geq 1$, and $v \in \{-1, 0, 1\}$ define

$$\alpha_v(n, l) = \sum_{m=1}^{n} m^v S_1(n, m) S_2(l+m, n).$$

We will need the following proposition, which says that for fixed l the function $\alpha_v(n, l)$ becomes constant (respectively zero, respectively linear) for $n > l$ and $v = 0$ (respectively $v = -1$, respectively $v = 1$), and also gives the values for $n = l, l-1$.

PROPOSITION. *For $1 \leq l \leq n+1$ and $-1 \leq v \leq 1$, $\alpha_v(n, l)$ is given by*

$$\alpha_0(n, l) = \beta_l + \begin{cases} 0 & (l \leq n), \\ \lambda_n - \lambda_{n+1} & (l = n+1), \end{cases}$$

$$\alpha_{-1}(n, l) = \begin{cases} 0 & (l < n), \\ -\frac{1}{n}\lambda_n & (l = n), \\ \lambda_{n+1} - \frac{1}{2}\lambda_n & (l = n+1), \end{cases}$$

$$\alpha_1(n, l) = -n[(l-1)\beta_l + \beta_{l-1}] - l\beta_l + \begin{cases} 0 & (l \leq n), \\ \lambda_{n+1} - \lambda_n & (l = n+1). \end{cases}$$

Proof. Form the generating function $A_v(x, y) = \sum_{n \geq 1, l \geq 0} \alpha_v(n, l)x^n y^l$. Then the definitions of $S_1(n, l)$ and $S_2(l, n)$ give

$$\begin{aligned} A_v(x, y) &= \sum_{m=1}^{\infty} m^v \sum_{n=m}^{\infty} S_1(n, m) \left(\sum_{l=0}^{\infty} S_2(l+m, n) y^l \right) x^n \\ &= \sum_{m=1}^{\infty} m^v \sum_{n=m}^{\infty} S_1(n, m) y^{-m} T^n x^n = \sum_{m=1}^{\infty} m^v \left(\frac{1}{y} \log \frac{1}{1-xT} \right)^m \end{aligned}$$

or

$$A_0(x,y)=H(x,y)-1,\ A_{-1}(x,y)=\log H(x,y),\ A_1(x,y)=H(x,y)^2-H(x,y)$$

with

$$H(x,y)=\left(1+\frac{1}{y}\log(1-xT)\right)^{-1}=y\left(\log\frac{1-xT}{1-T}\right)^{-1}.$$

We now develop everything in powers of $1-x$, obtaining

$$\log\frac{1-xT}{1-T}=\log\left(1+\frac{T}{1-T}(1-x)\right)=\frac{T}{1-T}(1-x)\cdot\sum_{r=0}^{\infty}\frac{1}{r+1}\left(\frac{T}{1-T}\right)^r(x-1)^r$$

and hence

$$A_0(x,y)=y\frac{1-T}{T}\cdot\frac{1}{1-x}-1-\sum_{r=0}^{\infty}\mu_{r+1}y\left(\frac{T}{1-T}\right)^r(x-1)^r,$$

$$A_{-1}(x,y)=\log\left(y\frac{1-T}{T}\right)+\log\left(\frac{1}{1-x}\right)-\sum_{r=1}^{\infty}\kappa_r\left(\frac{T}{1-T}\right)^r(x-1)^r,$$

$$A_1(x,y)=\left(y\frac{1-T}{T}\right)^2\frac{1}{(1-x)^2}-y(1-y)\frac{1-T}{T}\cdot\frac{1}{1-x}+\sum_{r=0}^{\infty}(\mu_{r+1}y+\mu'_{r+2}y^2)\left(\frac{T}{1-T}\right)^r(x-1)^r,$$

where μ_r, κ_r, and μ'_r are defined by the generating functions

$$\left(\sum_{r=0}^{\infty}\frac{u^r}{r+1}\right)^{-1}=\sum_{r=0}^{\infty}\mu_r u^r,\quad \log\left(\sum_{r=0}^{\infty}\frac{u^r}{r+1}\right)=\sum_{r=1}^{\infty}\kappa_r u^r,\quad \left(\sum_{r=0}^{\infty}\frac{u^r}{r+1}\right)^{-2}=\sum_{r=0}^{\infty}\mu'_r u^r.$$

Comparing the coefficients of x^n $(n\geqq 1)$ gives

$$\sum_{l=0}^{\infty}\alpha_0(n,l)y^l=y\frac{1-T}{T}-\mu_{n+1}y^{n+1}+O(y^{n+2}),$$

$$\sum_{l=0}^{\infty}\alpha_{-1}(n,l)y^l=\frac{1}{n}-\kappa_n y^n+\left((n+1)\kappa_{n+1}-\frac{n}{2}\kappa_n\right)y^{n+1}+O(y^{n+2}),$$

$$\sum_{l=0}^{\infty}\alpha_1(n,l)y^l=(n+1)\left(y\frac{1-T}{T}\right)^2-y(1-y)\frac{1-T}{T}+\mu_{n+1}y^{n+1}+O(y^{n+2}).$$

The proposition now follows if we note that $\mu_{n+1}=\lambda_{n+1}-\lambda_n$ (from the definitions), $\kappa_n=\frac{1}{n}\lambda_n$ (by differentiation), and

$$y(1-y)\frac{1-T}{T}=\sum_{l=0}^{\infty}(\beta_l-\beta_{l-1})y^l,\ \left(y\frac{1-T}{T}\right)^2=-y^2\left(1+\frac{d}{dy}\right)\left(\frac{1-T}{T}\right)=-\sum_{l=0}^{\infty}((l-1)\beta_l+\beta_{l-1})y^l.$$

§2. PROOF OF THE IDENTITY

Define an operator $f\to f^*$ on polynomials by

$$f(x)=\sum_{n=0}^{N}c_nx^n\to f^*(x)=\sum_{n=0}^{N}c_n\sigma_n\frac{x^{n+1}}{n+1},$$

and for integers $1\leqq r\leqq n$ define a polynomial $\delta_{n,r}(x)$ of degree $2n-1$ by

$$\delta_{n,r}(x)=\frac{n!}{(n-r)!(r-1)!}\left(\frac{1}{x}+\frac{1}{x+r}\right)\binom{x+n}{n}\binom{x+r-1}{n}.$$

We wish to evaluate the expression

$$L(n)=\frac{1}{2}\sum_{r=1}^{n}(-1)^{n-r}\delta^*_{n,r}(-r).$$

We first observe that

$$\delta_{n,r}(x)+\delta_{n-1,r}(x)=\binom{n-1}{r-1}\left[\binom{x+r}{n}\binom{x+n-1}{n-1}+\binom{x+r-1}{n-1}\binom{x+n-1}{n}\right]$$

and hence, denoting the expression in square brackets by $\phi_{n,r}(x)$,

$$L(n)-L(n-1)=\frac{1}{2}\sum_{r=1}^{n}(-1)^{n-r}\binom{n-1}{r-1}\phi^*_{n,r}(-r).$$

Substituting the polynomial expansions of binomial coefficients in terms of Stirling numbers of the first kind, we find

$$\phi_{n,r}(x)=n\sum_{l,m=1}^{n}(-1)^{n-m}\frac{S_1(n,l)S_1(n,m)}{l!\quad m!}[x^{l-1}(x+r)^m+x^l(x+r)^{m-1}].$$

LEMMA. *Denote by* $f_{l,m}(x)$ *the polynomial* $x^l(x+r)^m(l,m\geqq 0)$. *Then*

$$f^*_{l,m}(-r)=\frac{(-1)^{l+1}l!\,m!}{(l+m+1)!}(\sigma_l-\sigma_m)r^{l+m+1}.$$

Proof. If $f(x)$ is the derivative of a polynomial $g(x)$ with $g(0)=0$, then

$$f^*(x)=\int_0^1\frac{g(x)-u^{-1}g(ux)}{1-u}\,du.$$

$\left(\text{To see this, take } f(x)=x^n,\ g(x)=\dfrac{x^{n+1}}{n+1}.\right)$ Applying this to $g=f_{l,m}$ $(m\geqq 1)$ gives

$$lf^*_{l-1,m}(-r)+mf^*_{l,m-1}(-r)=(-1)^{l+1}r^{l+m}\int_0^1 u^{l-1}(1-u)^{m-1}du\quad(m\geqq 1).$$

The integral equals $\dfrac{(l-1)!(m-1)!}{(l+m-1)!}$ (beta integral). The lemma now follows by induction on m, the case $m=0$ being trivial.

Now apply the lemma to $\phi_{n,r}(x)$ to get

$$\begin{aligned}\phi^*_{n,r}(-r)&=n\sum_{l,m=1}^{n}\frac{(-1)^{n-m-l}}{(l+m)!}S_1(n,l)S_1(n,m)\left[\frac{1}{l}(\sigma_{l-1}-\sigma_m)-\frac{1}{m}(\sigma_l-\sigma_{m-1})\right]r^{l+m}\\&=2(-1)^n n\sum_{l,m=1}^{n}\frac{S_1(n,l)S_1(n,m)}{(l+m)!}\left[\frac{\sigma_{l-1}}{l}-\frac{\sigma_l}{m}\right](-r)^{l+m},\end{aligned}$$

where to get the second line we have interchanged the roles of l and m in two of the terms. This gives

$$L(n)-L(n-1)=\sum_{l,m=1}^{n}\frac{S_1(n,l)S_1(n,m)}{(l+m)!}\left(\frac{\sigma_l}{m}-\frac{\sigma_{l-1}}{l}\right)\left[\sum_{r=0}^{n}(-1)^r\binom{n}{r}(-r)^{l+m+1}\right].$$

The expression in square brackets is the coefficient of $y^{l+m+1}/(l+m+1)!$ in $(1-e^{-y})^n$, i.e., it equals $(l+m+1)!\,S_2(l+m+1,n)$. Thus

$$L(n)-L(n-1)=\sum_{l=1}^{n}S_1(n,l)\left[-\frac{1}{l}(\sigma_{l-1}-1)\alpha_0(n,l+1)+(l+1)\sigma_l\alpha_{-1}(n,l+1)-\frac{1}{l}\sigma_{l-1}\alpha_1(n,l+1)\right]$$

with $\alpha_\nu(l,n)$ as in §1. The Proposition of §1 now gives

$$\begin{aligned}L(n)-L(n-1)=&\sum_{l=1}^{n}\frac{S_1(n,l)}{l}[\beta_{l+1}+\sigma_{l-1}((n+1)l\beta_{l+1}+n\beta_l)]\\&+\frac{1}{n}(\lambda_n-\lambda_{n+1})-\frac{n-1}{2}\sigma_{n-1}\lambda_n+(n+1)\sigma_n\left(\lambda_{n+1}-\frac{1}{2}\lambda_n\right).\end{aligned}$$

We are now ready to prove the main identity.

THEOREM. *Define rational numbers s_n, $\tilde{Td}_n$, and t_n $(n \geq 1)$ by*

$$\sum_{n\geq 1} \frac{s_n}{n+1} T^n = \frac{1}{1-T} \sum_{k\geq 2} \sigma_{k-1} \beta_k y^{k-1}, \qquad \sum_{n\geq 1} \frac{\tilde{Td}_n}{n+1} T^{n+1} = -\sum_{k\geq 2} \frac{\beta_k y^k}{k(k-1)},$$

$$t_n = (n+1)\lambda_{n+1}(\sigma_{n+1} - 1).$$

Then $L(n) = s_n + \tilde{Td}_n + t_n$. (See Table below.)

Proof. Let $R(n)$ denote $s_n + \tilde{Td}_n + t_n$; we will write $R(n)$ in terms of Stirling numbers and then show that $R(n) - R(n-1)$ agrees with the above expression for $L(n) - L(n-1)$, establishing the result by induction. The generating function for s_n is equivalent to $\sum_{n\geq 1} \frac{s_n}{(n+1)^2} T^{n+1} = \sum_{k\geq 2} \sigma_{k-1} \beta_k \frac{y^k}{k}$, as we see by integration. Hence from the definition of $S_1(n, k)$ we get

$$s_n = (n+1)^2 \sum_{k=2}^{n+1} \sigma_{k-1} \frac{\beta_k}{k} S_1(n+1, k),$$

$$\tilde{Td}_n = -(n+1) \sum_{k=2}^{n+1} \frac{\beta_k}{k(k-1)} S_1(n+1, k)$$

and therefore, using the recursion satisfied by $S_1(n, k)$,

$$s_n - s_{n-1} = \sum_{l=1}^{n} \left[\sigma_{l-1} \frac{\beta_l}{l} n + \sigma_l \beta_{l+1}(n+1) \right] S_1(n, l), \qquad \tilde{Td}_n - \tilde{Td}_{n-1} = -\sum_{l=1}^{n} \frac{\beta_{l+1}}{l} S_1(n, l).$$

Also,

$$\lambda_n = \text{coefficient of } T^n \text{ in } \frac{e^y - 1}{y} = \sum_{l=1}^{n} \frac{1}{(l+1)!} S_1(n, l),$$

so — using the recursion of $S_1(n, l)$ again —

$$(n+1)\lambda_{n+1} = \sum_{l=1}^{n} \left[\frac{n+1}{(l+1)!} - \frac{1}{(l+2)!} \right] S_1(n, l).$$

Combining these formulas and the formula for $L(n) - L(n-1)$, we find after some work

$$R(n) - R(n-1) - L(n) + L(n-1) = \frac{n-1}{n} \sum_{l=1}^{n} \left[\frac{n}{l} \beta_{l+1} - \frac{l}{2(l+2)!} \right] S_1(n, l).$$

But this is zero because

$$\sum_{n=1}^{\infty} \sum_{l=1}^{n} \left[\frac{n}{l} \beta_{l+1} S_1(n, l) - \frac{l}{2(l+2)!} S_1(n, l) \right] T^n = T \frac{d}{dT} \left(\sum_{l=1}^{\infty} \beta_{l+1} \frac{y^l}{l} \right) - \frac{1}{2} \sum_{l=0}^{\infty} \frac{l}{(l+2)!} y^l$$

$$= \frac{T}{1-T} \sum_{l=1}^{\infty} \beta_{l+1} y^{l-1} - \frac{y}{2} \frac{d}{dy} \left(\frac{e^y - 1 - y}{y^2} \right) = \frac{e^y - 1}{y} \left(\frac{1}{e^y - 1} - \frac{1}{y} + \frac{1}{2} \right) - \frac{y}{2} \frac{d}{dy} \left(\frac{e^y - 1 - y}{y^2} \right) = 0.$$

This completes the proof of the theorem.

n	1	2	3	4	5	6
s_n	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{649}{1080}$	$\frac{1445}{1728}$	$\frac{162871}{151200}$	$\frac{171311}{129600}$
$\tilde{Td}_n$	$-\frac{1}{12}$	$-\frac{1}{8}$	$-\frac{329}{2160}$	$-\frac{149}{864}$	$-\frac{56947}{302400}$	$-\frac{1933}{9600}$
t_n	$\frac{5}{12}$	$\frac{15}{16}$	$\frac{3263}{2160}$	$\frac{7315}{3456}$	$\frac{553523}{201600}$	$\frac{1172311}{345600}$
$L(n)$	$\frac{1}{2}$	$\frac{19}{16}$	$\frac{529}{270}$	$\frac{3203}{1152}$	$\frac{2198159}{604800}$	$\frac{4678657}{1036800}$

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