THE TOPOLOGY AND GEOMETRY OF THE MODULI SPACE OF RIEMANN SURFACES

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I would like to describe a sampling of recent results concerning the moduli space M_g of Riemann surfaces. My plan is to present several of the ideas underlying the recent work of John Harer on the topology of M_g and of myself on the Hermitian and symplectic geometry. My purpose is not to give a survey; for instance the reader is referred to the papers [7, 9] for the recent progress on the question of whether $\overline{M}_g$, the moduli space of stable curves, is unirational.

The discussion will be divided into two parts: the topology of M_g and $\overline{M}_g$, especially the homology of the mapping class group Γ_g and the geometry of the Weil-Petersson metric. As background I shall start with the basic definitions and notation.

1. Definitions and Notation .

1.1. Let F be a compact topological surface of genus g with r boundary components and s distinguished points in $F - \partial F$; set $S = \partial F \cup \{\text{points}\}$. I shall always assume that $2g - 2 + s + r > 0$ or equivalently that $F - S$ admits a complete hyperbolic metric. Consider $\text{Homeo}^+(F,S)$, the group of orientation preserving homeomorphisms of F restricting to the identity on S and the normal subgroup $I(F,S)$ of homeomorphisms isotopic to the identity fixing S.

<u>Definition</u> 1.1. $\Gamma^s_{g,r} = \text{Homeo}^+(F,S)/I(F,S)$ is the mapping class group for genus g, r boundary components and s punctures.

I shall use the convention that an omitted index is set equal to zero. For genus g and s punctures the mapping class group Γ^s_g acts properly discontinuously on the Teichmüller space T^s_g via biholomorphisms. The quotient M^s_g, the classical moduli space of Riemann surfaces, is a complex V-manifold. To be more specific start by considering triples (R,f,P), where f is a homeomorphism of the topological surface F to a Riemann surface R with $f(S) = P$. An equivalence relation (the marking) is introduced by defining:

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$(R_0,f_0,P_0) \sim (R_1,f_1,P_1)$ provided there is a conformal map k with

$$\begin{array}{ccc} (F,S) & \xrightarrow{f_0} & (R_0,P_0) \\ & \searrow^{f_1} & \downarrow k \\ & & (R_1,P_1) \end{array}$$

commutative modulo a homotopy fixing S and P.

<u>Definition 1.2</u>. T_g^s, the Teichmüller space for genus g and s punctures, is the set of $\sim$ equivalence classes of triples (R,f,P).

Briefly T_g^s is a complex manifold and is homeomorphic to $\mathbb{R}^{6g-6+2s}$. The mapping class group Γ_g^s acts naturally on T_g^s: to the equivalence classes $\{h\} \in \Gamma_g^s$ and $\{(R,f,P)\} \in T_g^s$ assign the class $\{(R,f\circ h,P)\} \in T_g^s$. The action represents Γ_g^s as biholomorphisms of T_g^s.

<u>Definition 1.3</u>. $M_g^s = T_g^s/\Gamma_g^s$ is the moduli space for genus g and s punctures.

As an example the reader will check that for genus 1 and 1 puncture T_1^1 is the upper half plane $H \subset \mathbb{C}$ and Γ_1^1 is the elliptic modular group $SL(2;\mathbb{Z})$ acting on H by linear fractional transformations.

1.2. Now I shall review the definition of the complex structure on T_g^s. For a Riemann surface R with $2g - 2 + s > 0$ consider the hyperbolic metric $\lambda = ds^2$, of constant curvature -1. Associated to R are the L^2 (relative to λ) tensor spaces $H(R)$ of harmonic Beltrami differentials (tensors of type $\frac{\partial}{\partial z} \otimes d\bar{z}$) and $Q(R)$ of holomorphic quadratic differentials (tensors of type $dz \otimes dz$). Of course harmonic is defined in terms of the Laplace Beltrami operator for the hyperbolic metric. A pairing $H(R) \times Q(R) \to \mathbb{C}$ is defined by integration over R: for $\mu \in H(R)$ and $\phi \in Q(R)$ define $(\mu,\phi) = \int_R \mu\phi$. The Ahflors and Bers description of the complex structure of T_g^s is summarized in the diagram

$$\begin{array}{ccc} T^{1,0}T_g^s \times (T^{1,0})^*T_g^s & \longrightarrow & \mathbb{C} \\ & \nearrow_{(\ ,\ )} & \\ H(R) \times Q(R) & & \end{array}$$

where $T^{1,0}$ is the holomorphic tangent space, $(T^{1,0})^*$ its dual and these spaces are naturally paired [1, 5]. The hyperbolic metric induces a natural inner product on every space of tensors for R. In particular Weil was the first to consider the Hermitian product on $H(R)$ as a metric for T_g^s.

Definition 1.4. Given $\mu,\nu \in H(R)$ the Hermitian product $\langle\mu,\nu\rangle = \int_R \mu\bar{\nu}\lambda^{-1}$ is the Weil-Petersson metric.

The Weil-Petersson metric is Kähler [2,3,10,21] and its Hermitian and symplectic geometry is the subject of Chapter 3.

2. The Homology of the Mapping Class Group.

2.1. In this chapter I shall concentrate on three recent exciting results of John Harer: i) the computation of $H_2(\Gamma^s_{g,r})$, in brief $H_2(\Gamma^s_{g,r}) \approx \mathbf{Z}^{s+1}$, for $g \geq 5$, ii) the stability theorems, in brief $H_k(\Gamma^s_{g,r})$ is independent of g and r when $g \geq 3k+1$ and iii) the virtual cohomological dimension of $\Gamma^s_{g,r}$ is $d = 4g-4+2r+s$ for $r+s > 0$, $4g-5$ for $r = s = 0$, in particular $H_k(M^s_g;\mathbf{Q}) = 0$ for $k > d$, [11, 12, 13]. Of course the reader will consult the references for the complete statements especially for the cases of punctures and boundary components.

2.2. A useful technique for computing the homology of a group G is to construct a cell complex C on which G acts cellularly, i.e. cells are mapped to cells. Then the homology of G may be computed from the homology of the quotient C/G and a description of the cell stabilizers. Now $\Gamma^s_{g,r}$ is comprised of isotopy classes of homeomorphisms; $\Gamma^s_{g,r}$ acts on the isotopy invariants of the surface F. An obvious such invariant is the isotopy class of a union of simple loops. Cell complexes with vertices representing unions of simple loops, satisfying appropriate hypotheses, appeared previously in the work of Hatcher-Thurston [15] and Harvey [14]. I shall describe three such complexes (and simple variants) which are at the center of Harer's considerations.

2.3. The cut system complex CS, [11], A cut system $\langle C_1,\ldots,C_g\rangle$ on F is the isotopy class of a collection of disjoint simple closed curves $C_1,\ldots,C_g$ such that $F - (C_1 \cup \ldots \cup C_g)$ is connected. A *simple move* of cut systems is the replacement of $\langle C_i\rangle$ by $\langle C_i'\rangle$ where $C_j = C_j'$ for $j \neq k$ and C_k intersects C_k' once (all intersections are positive). I shall use the convention below that any loop omitted from the notation remains unchanged. Now consider the following sequences of simple moves (see Figure 1).

$$\begin{tikzcd}[column sep=small] \langle C_i\rangle \arrow[rr, dash] \arrow[dr, dash] & & \langle C'_i\rangle \arrow[dl, dash] \\ & \langle C''_i\rangle & \end{tikzcd} \qquad (R_1)$$

$$\begin{tikzcd} \langle C_i, C_j\rangle \arrow[r, dash] \arrow[d, dash] & \langle C_i, C'_j\rangle \arrow[d, dash] \\ \langle C'_i, C_j\rangle \arrow[r, dash] & \langle C'_i, C'_j\rangle \end{tikzcd} \qquad (R_2)$$

$$\begin{tikzcd}[column sep=small] & \langle C_i, C_j\rangle \arrow[r, dash] \arrow[dl, dash] & \langle C_j, C'_j\rangle \arrow[dr, dash] & \\ \langle C_i, C'_i\rangle \arrow[dr, dash] & & & \langle C_j, C\rangle . \\ & \langle C_i, C\rangle \arrow[r, dash] & \langle C, C'\rangle \arrow[ur, dash] & \end{tikzcd} \qquad (R_3)$$

Figure 1

The three sequences can be described in terms of the relevant loops (see Figure 2)

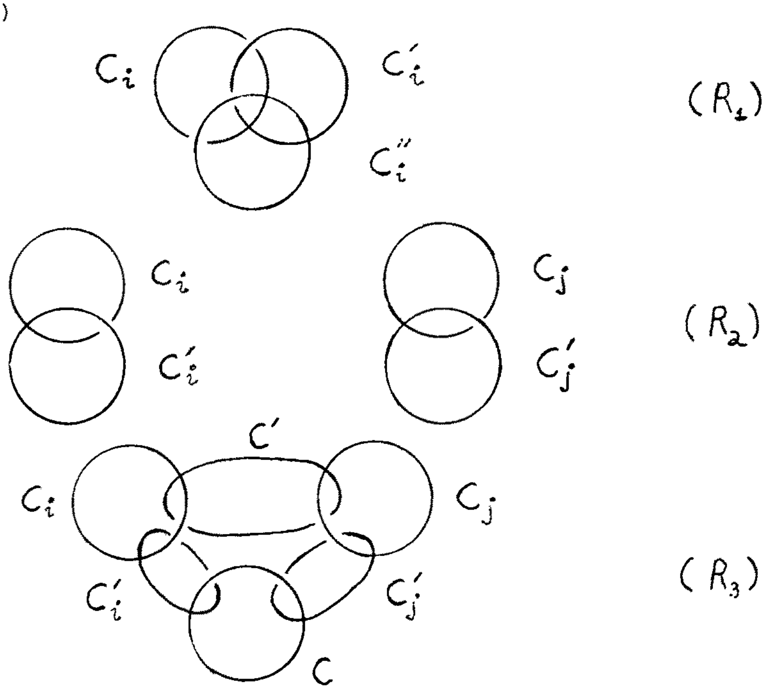

Figure 2

Now define a 0-complex CS_0 with one vertex for each cut system on F; a 1-complex CS_1 by attaching a 1-cell to CS_0 for each simple move; and a 2-complex CS_2 by attaching a 2-cell to CS_1 for each occurence of the cycles R_1, R_2 and R_3. Hatcher and Thurston prove that CS_2 is connected and simply connected. Harer simplifies the description of CS_2 and then attaches 3 cells to obtain a 2-connected 3 complex CS_3, [11]. The stabilizer of a cut system (a vertex of CS_3) is essentially a braid group and Harer analyzes the cell stabilizers of CS_3. Combining this information with an analysis of the homology of the quotient CS_3/Γ_g he obtains the following theorem, [11].

Theorem 2.1. $H_2(\Gamma^s_{g,r};\mathbb{Z}) = \mathbb{Z}^{s+1}$, $g \geq 5$.

The reader will find a slight difference between the above statement and that found in [11]. John Harer has assured me that the above is indeed the correct statement.

The homology group $H_2(\Gamma)$ also admits an interpretation as bordism classes of fibre bundles $F \to W^4 \to T$ with T a closed oriented surface. In particular two bundles are bordant if they cobound a 5-manifold fibering over a 3-manifold with fibre F. The bundle $F \to W^4 \to T$ has s canonical sections $\sigma_1,\ldots,\sigma_s$ defined by the distinguished points of F. In this setting H_2 is spanned by the $s + 1$ natural invariants of W: $\sigma(W)/4$, σ the signature and $\sigma_j \# \sigma_j$, $j = 1,\ldots,s$ the self-intersection numbers of the sections.

2.4. Certainly a basic question is whether or not the homology of the mapping class group falls into any pattern. As an example consider the analogous question for A_n, the coarse moduli space of principally polarized n dimensional abelian varieties. Borel in a fundamental paper computed the rational cohomology of A_n and found that $H^i(A_n;\mathbb{Q})$ is independent of n for n large relative to i, [31]. Recently Charney and Lee have extended these results to $\bar{A}_n$, the Satake compactification of A_n, [8]. Harer establishes the analogous result for $H_k(\Gamma^s_{g,r})$: the answer is independent of g and r provided $g \geq 3k + 1$. In fact the reader will find in Theorem 2.2 that Harer establishes much more but first I would like to mention the work of E. Miller, [18]. Starting with the work of Harer as a basis Miller observes that the boundary connected sum for surfaces $F_{g,1} \# F_{h,1} = F_{g+h,1}$ induces the structure on $A = \lim\limits_{\to} H_*(\Gamma_{g,1};\mathbb{Q})$ of a polynomial algebra on even generators with an exterior algebra on odd generators. Furthermore by generalizing an example of Atiyah for the generator of $H_2(\Gamma_g)$ Miller is able to find a generator in each even dimension.

To give a precise statement of Harer's stability theorem it is necessary to consider the following three maps of surfaces

$$\Phi\colon\ F^s_{g,r} \to F^s_{g,r+1}\ ,\quad r \geq 1$$

$$\Psi\colon\ F^s_{g,r} \to F^s_{g+1,r-1},\quad r \geq 2 \quad \text{and}$$

$$\Xi\colon\ F^s_{g,r} \to F^s_{g+1,r-2},\quad r \geq 2$$

where Φ and Ψ are given by sewing on a pair of pants (a copy of $F^0_{0,3}$) (see Figure 3).

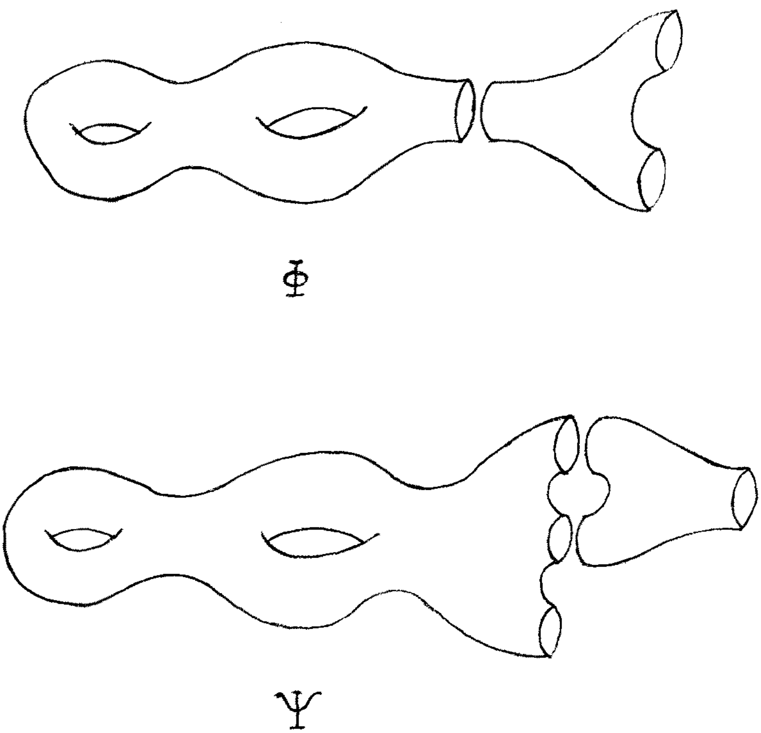

Figure 3

along one boundary for Φ, two for Ψ and Ξ is given by sewing together two boundary components (see Figure 4)

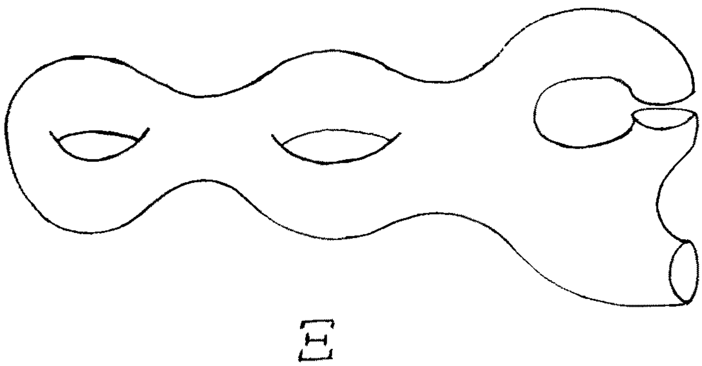

Figure 4

Certainly the maps induce homomorphisms of mapping class groups. The following results for the induced maps on homology with integer coefficients can be found in [12].

Theorem 2.2.

$\Phi_*: \; H_k(\Gamma^s_{g,r}) \to H_k(\Gamma^s_{g,r+1})$ is an isomorphism

for $k > 1$, $g \geq 3k - 2$, $r \geq 1$

$\Psi_*: \; H_k(\Gamma^s_{g,r}) \to H_k(\Gamma^s_{g+1,r+1})$ is an isomorphism

for $k > 1$, $g \geq 3k - 1$, $r \geq 2$

$\Xi_*: \; H_k(\Gamma^s_{g,r}) \to H_k(\Gamma^s_{g+1,r-2})$ is an isomorphism

for $g \geq 3k$, $r \geq 2$.

Corollary 2.3. $H_k(\Gamma^s_{g,r})$ is independent of g and r for $g \geq 3k + 1$.

The proof of a stability theorem for $\Gamma^s_{g,r}$ requires suitable

Γ complexes whose connectivity increases with g. Harer starts with the complex X a variant of the basic complex Z. In order to define X consider the *sub cut system* of rank k i.e. the isotopy class of $k+1$ disjoint simple loops $C = \{C_0, \ldots, C_k\}$ such that $F - \{C_0, \ldots, C_k\}$ is connected. Define the simplicial complex X of dimension $g-1$ by taking a k-simplex for every rank k sub cut system of F and identifying C as a face of C' if $C \subset C'$. The first theorem is that X has the homotopy type of a wedge of $g-1$ dimensional spheres. This is proven by first enlarging X to Z, the analogous complex where $F - C$ is now allowed to be disconnected but each component must have negative Euler characteristic. The complex Z, the second of the three basic complexes (CS being the first), has dimension $3g - 4 + r + s$ and is $2g - 3$ connected. After studying several additional complexes the proof follows a standard outline, [12].

2.5. For the sake of simplicity I shall only discuss one part of Harer's results on the virtual cohomological dimension: that M_g^1 has the homotopy type of a $4g - 3$ dimensional spine. The discussion starts with the description of a Γ_g^1 invariant ideal triangulation of T_g^1. The triangulation arises from the following result of Strebel, [19].

<u>Theorem 2.4</u>. Given a compact Riemann surface R and a point p there exists a unique meromorphic quadratic differential ϕ on R such that

i) ϕ has exactly one pole,

ii) in terms of an appropriate complex coordinate z in a neighborhood of p $\phi = \dfrac{dz^2}{z^2}$ and

iii) the real trajectories of ϕ are closed.

The differential ϕ may also be described by starting with the differential $\dfrac{dz^2}{z^2}$ on the disc $D = \{|z| \leq 1\}$ and identifying arcs on ∂D (linearly in radian measure) to obtain the pair (R,ϕ). To see this consider the following simple example (see Figure 5) where one obtains a surface of genus 2; in general any pattern for a genus g surface will occur.

The data for a pair (R,ϕ) is merely the combinatorial pattern for identifying arcs on ∂D, as well as their lengths. In order to record this information for each pair of arcs consider a loop γ

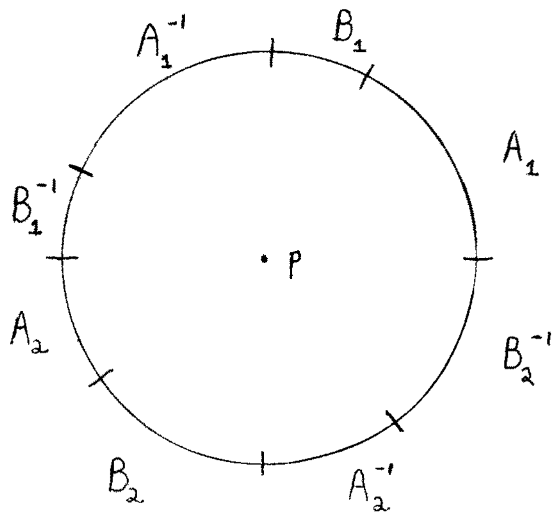

Figure 5

based at p, formed by rays from the origin with endpoints on ∂D identified under the arc pairing and assign to γ a weight w equal to the length of the arcs on ∂D; to (R,ϕ) assign the tuple (γ_j, w_j). Specifically for the loop intersecting A_1 in the above example the picture is (see Figure 6)

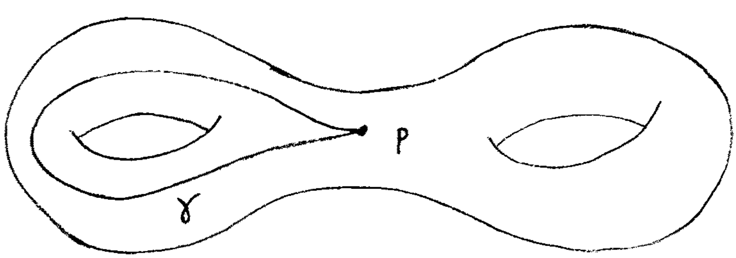

Figure 6

To interpret this situation the third basic complex A is introduced. A *rank* k *arc system* $\{\alpha_0,\ldots,\alpha_k\}$ is the isotopy class of $k+1$ simple loops based at p, intersecting only at p, and representing distinct, nontrivial homotopy classes. A is the cell complex with a k simplex $\alpha = \langle\alpha_0,\ldots,\alpha_k\rangle$ for each rank k arc system and α is identified as a face of α' if $\alpha \subset \alpha'$. Strebel's theorem provides a Γ equivariant map $T_g^1 \to A$. To the pair (R,p) associate the quadratic differential ϕ and to ϕ associate the arc system $\{\alpha_0,\ldots,\alpha_k\}$ with weights, choosing one arc for each pair of segments of ∂D occurring in the construction of ϕ. Simplicial coordinates on A are given simply by the weights. The map $T_g^1 \to A$ is in fact a homeomorphism onto the complement of a subcomplex A_∞; the ideal triangulation is the pull-back of the cell structure of A.

Recalling the construction of R by identifying segments on ∂D, observe that every pattern must have at least $4g$ segments. In particular the rank k arc systems arising from Strebel's theorem have rank $\geq 2g-1$; the $2g-2$ skeleton of A is contained in A_∞. Now introduce the dual complex Y of $A - A_\infty$. Accordingly Y will not have cells for the $2g-2$ skeleton of A. Thus the dimension of Y is $\dim T_g^1 - (2g-1) = 4g-3$. Harer shows that T_g^1 may be Γ equivariantly retracted onto Y.

The case of T_1^1 provides a helpful example. A is the standard $SL(2;\mathbb{Z})$ tessellation of H, A_∞ is the rational points of $\mathbb{R} = \partial H$ and Y is Serre's tree for $SL(2;\mathbb{Z})$.

3. The Weil-Petersson Geometry.

3.1. Ideally the purpose of introducing an invariant metric on the Teichmüller space T_g^s is to provide information on the intrinsic, i.e. independent of the metric, properties of the space. I shall try to indicate the extent that the Weil-Petersson metric has successfully filled this role. In brief a sketch will be given of the results in [2, 3, 17, 22-30]. Recently Fischer and Tromba have independently undertaken an investigation of Teichmüller space and the W-P metric, substituting the viewpoint of Riemannian geometry for the classical theory of quasiconformal maps, [20, 21].

As background the reader may check [2, 3, 10, 21] for proofs that the metric is Kähler. Recall that the W-P metric is invariant for the action of Γ_g^s on T_g^s and thus descends to the moduli space of Riemann surfaces M_g^s. On both T_g^s and M_g^s the metric is not complete, [22]. Now this result is best understood in terms of the latter results [17, 26] that the metric has an extension to $\overline{M_g^s}$, the Deligne-Mumford stable curve compactification of M_g^s.

The study of the W-P geometry can be divided into three major topics: curvature considerations, extension of the metric to the compactification $\overline{M_g^s}$, and the symplectic geometry of the Kähler form. As a sample of the results I shall start by sketching three applications. The Kähler form ω_{WP} extends to the compactification $\overline{M_g^s}$; the extension $\overline{\omega}_{WP}$ defines a cohomology class in $H^2(\overline{M_g^s})$, [26, 27, 28]. In [27] it is found that $\frac{1}{2\pi^2}\overline{\omega}_{WP}$ is actually the first Chern class of a known line bundle $\overline{\kappa}_1$ (discussed below) on $\overline{M_g^s}$. In particular $\frac{1}{2\pi^2}\overline{\omega}_{WP}$ is a rational class and the line bundle $\overline{\kappa}_1$ is positive. At this point the Kodaira theorem may be quoted to obtain a purely analytic proof that $\overline{M_g^s}$ is projective algebraic. As a second application consider the Nielsen conjecture: every finite subgroup of Γ_g^s fixes a point of T_g^s. An important ingredient of Kerckhoff's proof of the conjecture is that the *geodesic length functions* ℓ_* are convex along Thurston's earthquake paths, [16]. I have recently found that the length functions ℓ_* are strictly convex along the W-P geodesics [30]. This result provides the basis of an independent but similar proof of the conjecture. The proof starts with an observation of Fricke-Klein: that a suitable sum $S = \sum_j \ell_j$ of length functions will be a proper function on T_g^s. Given $\tilde{\Gamma} \subset \Gamma_g^s$, a finite subgroup, then the sum $S_0 = \sum_{\gamma\in\tilde{\Gamma}} S(\gamma)$ is $\tilde{\Gamma}$ invariant and is also a sum of length functions. Now S_0 is strictly convex along W-P geodesics and thus a critical point is necessarily a relative minimum (S_0 is an index 0 Morse function). Since S_0 is proper it follows that it has a unique minimum and finally the $\tilde{\Gamma}$ invariance of S_0 guarantees that the minimum is fixed by $\tilde{\Gamma}$, the desired conclusion. And finally since the W-P metric is Kähler it follows immediately that the length functions ℓ_* are in fact plurisubharmonic; this observation leads to a new proof that T_g^s is a Stein manifold, [30].

3.2. Ahlfors was the first to consider the curvature of the metric; he obtained singular integral formulas for the Riemann curvature tensor, [3]. As an application he found that the Ricci, holomorphic sectional and scalar curvatures are all negative. Royden later showed that the holomorphic sectional curvature is bounded away from zero and more recently Tromba has found that the general sectional curvature is indeed negative. I shall now present a simple formula for the curvature tensor, [29]. Recall first that the holomorphic tangent space $T^{1,0}\, T_g^s$ at $\hat{R} \in T_g^s$ is identified with the space of harmonic Beltrami differentials $H(R)$ and that dA denotes the hyperbolic area element on R and D the hyperbolic Laplacian. The Riemann curvature is a 4-tensor in particular for $\mu_\alpha, \mu_\beta, \mu_\gamma, \mu_\delta \in H(R)$

$$
\begin{aligned}
R_{\alpha\bar{\beta}\gamma\bar{\delta}} &= -2 \int_R (D-2)^{-1}(\mu_\alpha \bar{\mu}_\beta)(\mu_\gamma \bar{\mu}_\delta)\,dA \\
&= -2 \int_R (D-2)^{-1}(\mu_\alpha \bar{\mu}_\delta)(\mu_\gamma \bar{\mu}_\beta)\,dA
\end{aligned}
$$

where $(D-2)^{-1}$ is the indicated self adjoint operator and observe for $\mu, \nu \in H$ that the product $\mu\bar{\nu}$ is a function. Starting with the above formula it follows that the metric has negative sectional curvature, that the holomorphic sectional and Ricci curvatures are bounded above by $\frac{-1}{2\pi(g-1)}$ and that the scalar curvature is bounded above by $\frac{-3(3g-2)}{4\pi}$. In fact the arguments show that the curvatures are governed by the spectrum of the Laplacian: the negative curvature is a manifestation of the nonpositivity of the Laplacian. These last results have also been obtained by Royden.

As a further application of the techniques I wish to consider the characteristic classes of the Teichmüller curve $\mathcal{T}_g$. $\mathcal{T}_g$ is the natural fibre space over T_g; the fibre above $\hat{R} \in T_g$ is a compact submanifold isomorphic to R. If $\pi : \mathcal{T}_g \to T_g$ is the projection then the kernel of the differential $d\pi : T^{1,0}\mathcal{T}_g \to T^{1,0}T_g$ defines a line bundle (v) on $\mathcal{T}_g$, the vertical bundle of the fibration. The restriction of (v) to a fibre of π is simply the tangent bundle of the fibre; by the uniformization theorem the hyperbolic metric induces a metric on (v). I have computed the curvature 2-form for this metric and found that it is negative, a pointwise version of Arakelov's result that the dual $(v)^*$ is numerically effective, [4]. Once again the curvature is governed by the spectrum of the Laplacian. By integrating powers of the Chern class $c_1(v)$ over the fibers of π one obtains Γ_g invariant characteristic classes $\kappa_n(p) = \int_{\pi^{-1}(p)} c_1(v)^{n+1}$ defined on T_g. Mumford has many intriguing results on the behavior of these classes in the cohomology ring: in particular the κ_n, $n \le 3g - 3$ are nontrivial and many geometrically interesting cycles may be written in terms of the κ_n. Mumford guesses that the low dimensional part of the cohomology ring may actually be polynomial in the κ_n. As an example in [29] I find that in fact $\kappa_1 = \frac{1}{2\pi^2}\omega_{WP}$, ω_{WP} the W-P Kähler form.

3.3. Masur was the first to consider the extension of the W-P metric from M_g to $\overline{M}_g$, $\overline{M}_g$ the moduli space of stable curves. In the paper [17] Masur develops the foundations of this topic. Recall now that the compactification locus $\mathcal{D} = \overline{M}_g - M_g$ is a divisor with normal crossings. Briefly Masur shows if $\mathcal{D}$ is given locally as $z_1 = 0$ in a coordinate chart $z = (z_1,\ldots,z_n)$ then

$$ds^2_{WP} = \frac{dz_1 d\bar{z}_1}{|z_1|^2(\log 1/|z_1|)^3} + O\left(\frac{1}{|z_1|(\log 1/|z_1|)^3}\right), \quad [17].$$

In particular the W-P length of a differentiable curve in $\overline{M}_g$ is finite and the metric extends to a complete metric on $\overline{M}_g$. Starting from Masur's results it follows that the Kähler form ω_{WP} extends to a closed, positive, (1,1) current $\bar{\omega}_{WP}$ on $\overline{M}_g$, [26]. In fact the above singularity is sufficiently mild for $\bar{\omega}_{WP}$ to be written as $\partial\bar{\partial}F$, F a *continuous* function (as an example consider $F = \frac{1}{\log 1/|z|}$), [28]. Standard approximation techniques may then be applied to conclude that $\bar{\omega}_{WP}$ is the limit of smooth Kähler forms in its cohomology class. Thus even though the Kähler form is not smooth it is suitable for applications in particular the Kodaira theorem may be quoted to conclude that there is a projective embedding. Now the discussion will continue with the results on describing the cohomology class of $\bar{\omega}_{WP}$.

3.4. The divisor $\mathcal{D} \subset \overline{M}_g$ is reducible $\mathcal{D} = \mathcal{D}_0 \cup \ldots \cup \mathcal{D}_{[\frac{g}{2}]}$, where the generic surface represented in $\mathcal{D}_k$ has one node separating it into components of genus k and genus $g - k$ (for $k = 0$ the node is nonseparating). Certainly the divisors $\mathcal{D}_k$ define cohomology classes in $H_{6g-8}(\overline{M}_g)$ and by Poincaré duality (over $\mathbb{Q}$, $\overline{M}_g$ is a V-manifold) $\bar{\omega}_{WP}$ also defines a class in H_{6g-8}. The first result is contained in the following, [27].

Theorem 3.1. $\{\mathcal{D}_0,\ldots,\mathcal{D}_{[\frac{g}{2}]},(\bar{\omega}_{WP})\}$ is a basis for $H_{6g-8}(\overline{M}_g;\mathbb{Q})$.

The sketch of the proof is simple enough. By the result of Harer on $H_2(M_g)$ and an application of Mayer-Vietoris one verifies at the outset that $H_2(\overline{M}_g;\mathbb{Q})$ has rank $2 + [\frac{g}{2}]$. A candidate basis is then presented for each of H_2 and H_{6g-8} and the intersection pairing is evaluated. The pairing is found to be nonsingular and the proof is complete. The trick for evaluating the integrals of $\overline{\omega}_{WP}$ is to perform a single integration [27] and then deduce the remaining integrals by formal properties.

As an example a 2-cycle E for $\overline{M}_g$ is obtained by considering the family of (stable) curves given as the one point sum of a *fixed* surface S_0 of genus $g - 1$ and an elliptic curve E, which will vary over all (even degenerate) structures represented in its moduli space $\overline{M_1^1}$ (see Figure 7).

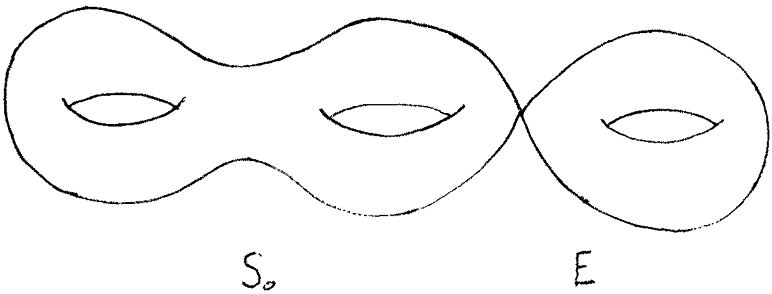

Figure 7

The 2-cycle E is parametrized by $\overline{M_1^1}$. Now to state the desired formal property of ω_{WP} let ω_g^s be the W-P Kähler form for $\overline{M_g^s}$. Then briefly $\omega_g^0|_E = \omega_1^1|_{\overline{M_1^1}}$; the restriction of ω_g^0 to E is naturally identified with the Kähler form for $\overline{M_1^1}$. And so the integral $\int_E \overline{\omega}_{WP}$ is reduced to the $\overline{M_1^1}$ case. This last integral may be evaluated directly; the value is $\frac{\pi^2}{6}$, [25]. The formal properties of the Kähler form will be discussed further in section 3.6 as consequences of the Fenchel-Nielsen coordinates.

After evaluating the intersection pairing one finds that

$$\frac{1}{2\pi^2}\,\overline{\omega}_{WP} = \overline{\kappa}_1$$

where $\overline{\kappa}_1$ is the extension of the class κ_1 discussed in section 3.2. Indeed the above is the generalization to $\overline{M}_g$ of the earlier result $\frac{1}{2\pi^2}\,\omega_{WP} = \kappa_1$ for M_g. Finally the basic techniques for constructing cycles and computing intersections may be applied to the higher homology groups. For instance in [27] it is shown that $H_{2k}(\overline{M}_g)$, $k < g$ has rank at least $\frac{1}{2}\binom{g-1}{k}$.

3.5. The last two sections will be devoted to the symplectic geometry of the Kähler form ω_{WP}. The symplectic geometry of the triple $(\omega_{WP}, t_*, \ell_*)$, t_* the Fenchel-Nielsen vector fields and ℓ_* the geodesic length functions, is *dual* to the trigonometry, as will be described below, of geodesics in the hyperbolic metric of a surface.

A construction of Fenchel-Nielsen provides for natural flows on Teichmüller space. Fix the free homotopy class of a nontrivial simple loop $\{\alpha\}$ on F and an increment δ of time. If $\hat{R} = \{(R,f)\} \in T_g$ then $\{f(\alpha)\}$ is represented on $\hat{R}$ by a unique geodesic $\alpha_{\hat{R}}$. Cut R open along $\hat{\alpha}_R$, rotate one *side* of the cut relative to the other (by a distance of δ) and then glue the sides in their new position. The hyperbolic structure in the complement of the cut extends naturally to define a hyperbolic structure on the new surface. A geodesic γ intersecting $\alpha_{\hat{R}}$ is deformed to a broken geodesic γ_b with endpoints separated δ units along α. As δ varies a flow, the F-N flow, is defined on the Teichmüller space T_g. The infinitesimal generator of the flow is the F-N vector field t_α. The free homotopy class $\{\alpha\}$ also determines a function ℓ_α, the geodesic length function, on T_g. In brief define $\ell_\alpha(\hat{R})$, $\hat{R} \in T_g$ to be the length of $\alpha_{\hat{R}}$ and the exterior derivative $d\ell_\alpha$ will also be discussed. The basic formula of the symplectic geometry is a duality formula

$$\omega_{WP}(t_\alpha, \ ) = -d\ell_\alpha, \qquad [23, 24, 26].$$

An immediate consequence is that the symplectic form is invariant under the F-N flows on T_g, in particular the flows are W-P volume preserving. There are also formulas for the Lie derivatives $t_\alpha \ell_\beta$ and $t_\alpha t_\beta \ell_\gamma$, [24].

$$\omega(t_\alpha, t_\beta) = t_\alpha \ell_\beta = \sum_{p \in \alpha \# \beta} \cos \theta_p \tag{3.1}$$

$$t_\alpha t_\beta \ell_\gamma \;=\; \sum_{(p,q)\in\alpha\#\gamma\times\beta\#\gamma} \frac{e^{\ell_1}+e^{\ell_2}}{2(e^{\ell_\gamma}-1)} \sin\theta_p \sin\theta_q \tag{3.2}$$

$$-\sum_{(r,s)\in\alpha\#\beta\times\beta\#\gamma} \frac{e^{m_1}+e^{m_2}}{2(e^{\ell_\beta}-1)} \sin\theta_r \sin\theta_s .$$

The right hand side of (3.1) evaluated at $\hat{R} \in T_g$ is the sum of cosines of the angles at the intersections of the geodesics $\alpha_{\hat{R}}$ and $\beta_{\hat{R}}$. Similarly the right hand side of (3.2) is a sum of trigonometric invariants for pairs of intersections; ℓ_1 and ℓ_2 are the lengths of the segments of γ defined by p, q and likewise, for m_1 and m_2 relative to β. Recently W. Goldman has generalized these formulas to the representation space $\mathrm{Hom}(\pi_1(F),G)/G$, G a Lie group with nondegenerate symmetric bilinear form on its Lie algebra, [10]. Consequences of the above formulas are considered in [25, 26]. In particular if $\alpha \# \beta \neq \emptyset$ then $t_\alpha t_\alpha \ell_\beta > 0$ and (3.2) represents a quantitative version of Kerckhoff's observation that the geodesic length functions are convex along earthquake paths, [16]. Finally note that the infinitesimal generators of Thurston's earthquake flows form the completion (in the compact-open topology) of the F-N vector fields.

3.6. Introducing coordinates on Teichmüller space is a question of parametrizing Riemann surfaces. Fenchel-Nielsen suggested a particularly simple solution to this problem. It starts with the observation that the lengths of alternating sides of a right hexagon in the hyperbolic plane may be chosen arbitrarily. Given such a hexagon, form its metric double across the remaining sides to obtain the basic object P, a pair of pants (see Figure 8).

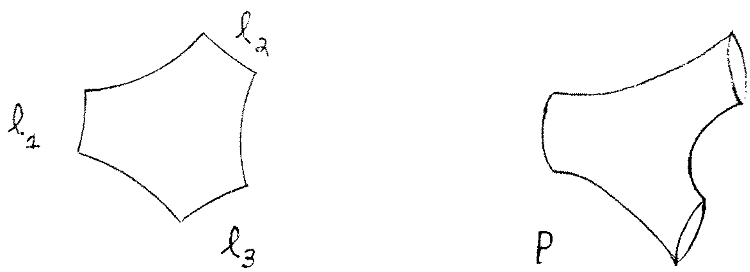

Figure 8

Topologically P is the complement of three disjoint discs in S^2; metrically P is a hyperbolic surface with geodesic boundary. The key observation is that pants P and P' may be metrically summed along their boundaries provided merely that the boundaries are of equal length. Now fixing a combinatorial pattern, then summing the pants $P_1,\ldots,P_{2g-2}$, one obtains the general genus g surface (see Figure 9).

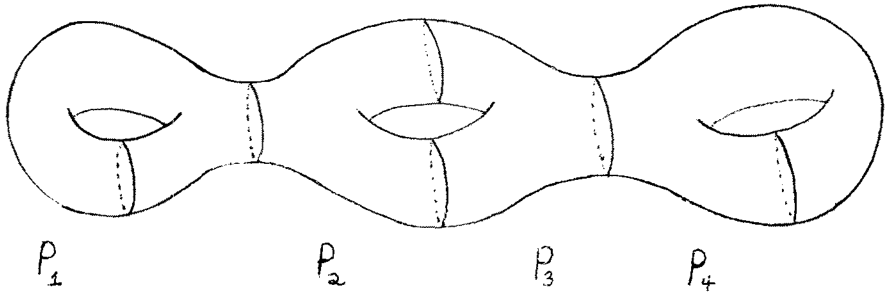

Figure 9

The coordinates for T_g are simply the free parameters for this construction. There are exactly two parameters at each summing locus. Of course the first is simply the length ℓ of the locus, this varies freely in $\mathbb{R}^+$. The second, the twist parameter τ, measures the net displacement between the boundaries. The parameter τ is defined to be the hyperbolic distance between the feet of perpendiculars dropped from appropriate boundaries (see Figure 10).

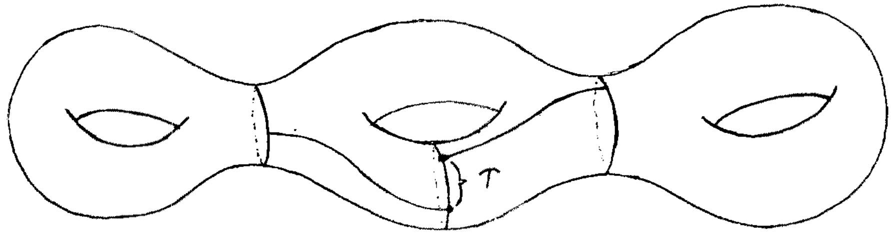

Figure 10

After an initial choice τ is determined by analytic continuation and varies freely in $\mathbb{R}$. In this way Fenchel-Nielsen established the following result, [1, 26].

Theorem 3.2. The map $T_g \to (\mathbb{R}^+ \times \mathbb{R})^{3g-3}$ given by $\hat{R} \to (\ell_j, \tau_j)_{j=1}^{3g-3}$ is a homeomorphism.

In particular T_g is a cell. Furthermore the Deligne-Mumford compactification $\overline{M}_g$ can be constructed from M_g by simply allowing the length parameters ℓ_* to vanish, [1,26]. The discussion of the previous section already suggests a relationship between the Kähler form ω_{WP} and the F-N coordinates. Recall that ω_{WP} is invariant under the F-N vector fields and observe that the coordinate vector fields $\frac{\partial}{\partial \tau_j}$ are indeed F-N vector fields. Consequently the coefficients of ω_{WP} in F-N coordinates (τ_j, ℓ_j) must be independent of the twist variables τ_j. In fact much more is true, [26].

Theorem 3.3. $\omega_{WP} = \sum_j d\ell_j \wedge d\tau_j$.

Briefly (ω, τ_j, ℓ_j) is a completely integrable Hamiltonian system. The Kähler form ω_{WP} is Γ_g invariant in particular the 2-form $\sum_j d\ell_j \wedge d\tau_j$ is independent of the combinatorial pattern for combining pants. Finally the discussion will be concluded with two applications of the above formula. Set a length parameter ℓ_k equal to zero to obtain a degenerate surface S (see Figure 11)

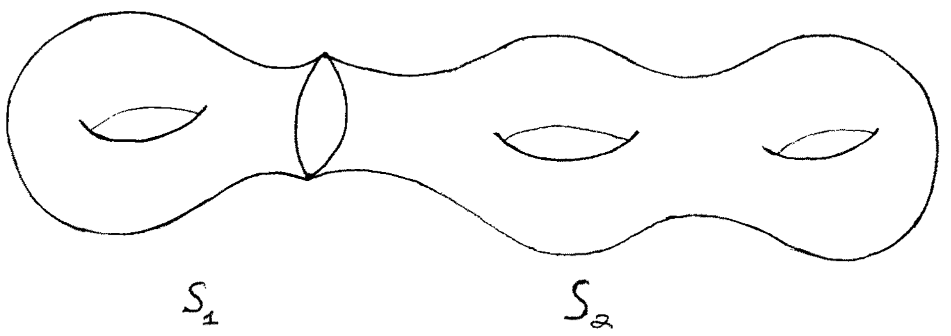

Figure 11

the sum at punctures of surfaces S_1 and S_2. It follows immediately from the above formula that ω_S converges to the sum $\omega_{S_1} + \omega_{S_2}$ as $\ell_k \to 0$. Briefly stated the Kähler form for a sum of surfaces is the sum of the component forms, [27]. To demonstrate the second formal property, behavior for an unramified covering $R \to S$ of surfaces, consider the following example (see Figure 12).

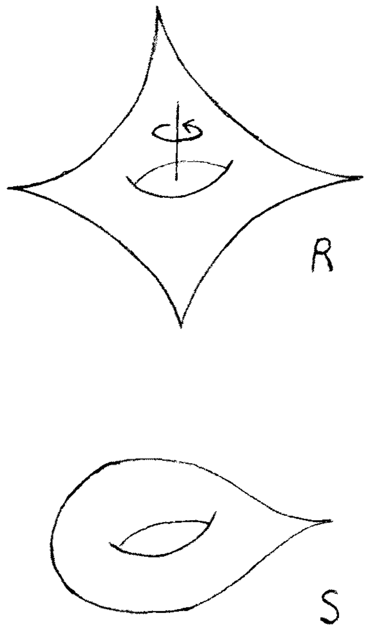

Figure 12

R is a 4 punctured torus, S is a one punctured torus and the covering transformation is the order 4 rotation about the axis of the hole. Introducing F-N coordinates relative to the indicated loops the reader will easily check that $\omega_R = 4\omega_S$; the Kähler form multiplies under unramified coverings, [27].

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