A COUNTEREXAMPLE IN 3-SPACE TO A CONJECTURE OF H. HOPF

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In this article we produce a counterexample to the following conjecture of H. Hopf. We shall carefully state the theorems involved in the construction and also provide a geometric description (with suggestive sketches) of the surfaces giving the counterexample. An expanded version complete with proofs is to appear in a paper of the author [8].
Conjecture of Heinz Hopf: If Σ is an immersion of an oriented closed hypersurface in R^n with constant mean curvature $H \neq 0$, then the hypersurface is the standard embedded (n-1)-sphere.

If the immersed surface is known to be embedded then a well-known result of A. D. Alexandroff [1] asserts that the conjecture is true. H. Hopf himself [4] showed that if Σ is an immersion of S^2 into R^3 with constant mean curvature then the conjecture is still true. Recently Wu-Yi Hsiang [5] produced an immersion of S^3 into R^4 with constant mean curvature which is not isometric to the standard sphere. However, his construction does not work in the classical dimension (=3) and the conjecture has remained open in this case. We have the following.

Counterexample Theorem: There exist closed immersed surfaces of genus one in R^3 with constant mean curvature. (In fact, we exhibit a countably infinite number of isometrically distinct examples.)

We shall exhibit the surface by producing a conformal mapping of the plane R^2 into R^3 with constant mean curvature which is doubly periodic with respect to a rectangle in the plane. Let $w = (u,v) = u + iv$ represent a typical point in $R^2 = C$ while $\bar{x} = (x,y,z)$ denotes a point in R^3 so that our immersion is given by a function $\bar{x}(u,v)$. We let

$$d\bar{x}\cdot d\bar{x} = ds^2 = E\ (du^2 + dv^2) = e^{2\omega}\ (du^2 + dv^2) \qquad (1a)$$

$$-d\bar{x}\cdot d\bar{\xi} = L du^2 + 2M du dv + N dv^2 \qquad (1b)$$

be the first and second fundamental forms for the surface. We shall set the mean curvature $H = \frac{1}{2}$. The Gauss and Codazzi-Mainardi equations in this case become (see [4] for details)

$$\Delta\omega + Ke^{2\omega} = 0\ ,\quad K = \text{Gauss curvature} = (LN - M^2)/\ E^2 \qquad (2a)$$

$\Phi(w) = (L - N)/2 - iM$ is a complex analytic function. (2b)

Now suppose that $\omega(u,v)$ is a solution to the differential equation

$$\Delta\omega + \sinh\omega \cosh\omega = 0 . \tag{3}$$

If we set $E = e^{2\omega}$, $L = e^{\omega}\sinh\omega$, $M = 0$, and $N = e^{\omega}\cosh\omega$, then it follows that the Gauss and Codazzi-Mainardi equations are satisfied and by a theorem of Bonnet the system can be integrated to yield a surface $\bar{x}(u,v)$,unique up to a Euclidean motion in R^3, having the given fundamental forms. The equations to be integrated are

$$\begin{aligned}
\bar{x}_{uu} &= \omega_u\bar{x}_u - \omega_v\bar{x}_v + L\bar{\xi} \\
\bar{x}_{uv} &= \omega_v\bar{x}_u + \omega_u\bar{x}_v + M\bar{\xi} \\
\bar{x}_{vv} &= -\omega_u\bar{x}_u + \omega_v\bar{x}_v + N\bar{\xi} \\
\bar{\xi}_u &= -k_1 \bar{x}_u \\
\bar{\xi}_v &= -k_2 \bar{x}_v
\end{aligned} \tag{4}$$

Here $k_1 = L/E = e^{-\omega}\sinh\omega$, $k_2 = e^{-\omega}\cosh\omega$ so we see that the lines of curvature correspond to lines parallel to the coordinate axes in R^2. Furthermore, the surface is free of umbilic points.

If $\bar{x}(u,v)$ is to be a doubly periodic mapping then so must $\omega(u,v)$. However the converse need not be true. Suppose that $\omega(u,v)$ is a positive solution to the differential equation (3) on a rectangular domain Ω_{AB} lying in the first quadrant with two of its sides on the coordinate axes and the vertex opposite the origin at (A,B). Suppose also that the solution $\omega(u,v)$ vanishes on the boundary of the rectangle. Following the argument used in [3] , one can show that $\omega(u,v)$ satisfies the following symmetry properties.

a) $\omega(u,v)$ is symmetric about the lines $u = A/2$ and $v = B/2$. (5)

b) For a fixed v, $0 < v < B$, $\omega(u,v)$ is an increasing function of u, $0 \leqslant u \leqslant A/2$. For a fixed u, $0 < u < A$, $\omega(u,v)$ is an increasing function of v, $0 \leqslant v \leqslant B/2$.

c) $\omega_u(u,0)$ is strictly increasing for $0 \leqslant u \leqslant A/2$.
$\omega_v(0,v)$ is strictly increasing for $0 \leqslant v \leqslant B/2$.

Furthermore, $\omega(u,v)$ can be extended as a solution of the differential equation (3) on all of R^2 by odd reflections across the grid lines $u = mA$, $v = nB$ (m, n integers).

Theorem 2: Suppose $\omega(u,v)$ is a solution to the differential equation (3) on R^2 which is positive on the fundamental rectangle Ω_{AB}, vanishing on the boundary and satisfying the properties (5). The mapping $\bar{x}(u,v)$ obtained by integrating the system (4) is an immersed surface of constant mean curvature $H = \frac{1}{2}$ and satisfying the following symmetry properties.

(6)

a) The curve $\bar{x}((m + \frac{1}{2})A, v)$ lies in a normal plane Π_m with $\bar{x}_u$ as a normal vector to Π_m. If R_m is the reflection map about Π_m in R^3 then $\bar{x}((m + \frac{1}{2})A + u,v) = R_m \circ \bar{x}((m + \frac{1}{2})A - u,v)$.

b) The curve $\bar{x}(u, (n + \frac{1}{2})B)$ lies in a normal plane Ω_n with $\bar{x}_v$ as a normal vector to Ω_n. If R'_n is the reflection map about Ω_n in R^3 then $\bar{x}(u,(n + \frac{1}{2})B + v) = R'_n \circ \bar{x}(u,(n + \frac{1}{2})B - v)$. Each Ω_n is orthogonal to each Π_m.

c) The curve $\bar{x}(u,0)$ is a planar curve lying in a plane Γ_0 which is a tangent plane to the surface at each point. This curve intersects each plane Π_m orthogonally. $\bar{x}_u(u,0)$ is an even function of u. This allows us to conclude that all of the planes Π_m are parallel.

d) The curve $\bar{x}(0,v)$ satisfies the condition $(\bar{x} + \bar{\xi})(0,v) = \bar{c}_o$ a constant vector. Therefore $\bar{x}(0,v)$ lies on a sphere $S(\bar{c}_o,1)$ with center $\bar{c}_o$ and radius one. Similarly $\bar{x}(kA,v)$ lies on a sphere $S(\bar{c}_k,1)$. The points $\bar{c}_k$ lie in every plane Ω_n.

e) $\bar{x}(u + 2A,v) = \bar{x}(u,v) + \bar{b}$ where $\bar{b} = \bar{c}_2 - \bar{c}_o$ is a vector normal to the planes Π_m carrying Π_0 to Π_2.

f) $\bar{x}(u,v + 2B) = \Theta\,\bar{x}(u,v)$ where Θ is a rotation from Ω_o to Ω_2 about their line of intersection, l.

The surface will close up if we can select the rectangle Ω_{AB} so that the translation $\bar{b} = \bar{0}$ (i.e. all the planes Π_m are identical) and so that the rotation angle Θ is a rational multiple of 2π. We use a continuity argument to show that this is possible. The procedure is as follows. Map (via a homothety) all rectangles of similar shape onto a representative rectangle which we select by the standard Schwartz-Christoffel mapping of rectangles onto the unit disk.

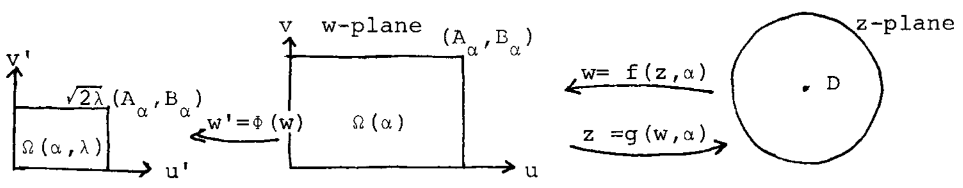


Figure 1: The Fundamental Domain.

We have the following identities satisfied by the various functions defined on the domains pictured in Figure 1.

(7)

a) On $\Omega(\alpha,\lambda)$.

$\Delta\omega + \sinh\omega\cosh\omega = 0$

$\Delta\sigma + \sinh\sigma = 0$ where $\sigma = 2\omega$.

b) On $\Omega(\alpha)$.

$\Delta W + 2\lambda\sinh W \cosh W = 0$ where $W = \omega\circ\Phi$.

$\Delta\Sigma + 2\lambda\sinh\Sigma = 0$ where $\Sigma = 2W$.

c) On the disk D.

$\Delta\Psi + \lambda|f'(z,\alpha)|^2(e^{\Psi} - e^{-\Psi}) = 0$, where $\Psi = \Sigma\circ f$.

$$w = f(z,\alpha) = \int_0^z (t^4 + 2(\cos 2\alpha)t^2 + 1)^{-\frac{1}{2}}\,dt$$

The proof of the existence of positive solutions to the system (7c) on D which vanish on the boundary (and such that small values for λ correspond to large solutions Ψ) is based on a method developed by V.K. Weston [7] and R.L. Moseley [6].

<u>Theorem 3</u>: There exists an open set $O \subset (\alpha,\lambda)$-plane where for each α_1,α_2 with $0 < \alpha_1 < \alpha_2 < \pi/2$ there exists $\tilde{\lambda} = \tilde{\lambda}(\alpha_1,\alpha_2) > 0$ so that $[\alpha_1,\alpha_2] \times (0,\tilde{\lambda}] \subset O$, and a mapping from O to $C(\bar{D})$ denoted by $\Psi(z,\alpha,\lambda)$ such that

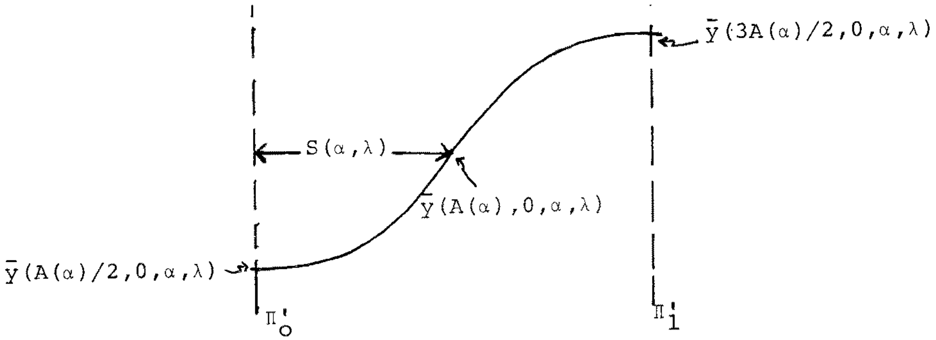
a) $\Sigma(w,\alpha,\lambda) = \Psi(g(w,\alpha),\alpha,\lambda)$ is a positive solution to (7b) which vanishes on the boundary.

b) The functions Σ, Σ_u , Σ_v depend continuously on (α,λ) down to $\lambda = 0$ with $\Sigma(w,\alpha,0) = \Sigma_o(w,\alpha) = 4\log(1/|g(w,\alpha)|)$.

c) For $\lambda > 0$ the mapping $(\alpha,\lambda) \longrightarrow \Psi(z,\alpha,\lambda)$ is a continouosly differentiable mapping of O into $C(\bar{D})$.

<u>Remark on the proof</u>: One first constructs a good approximate solution $U_o(z,\lambda)$ with the correct asymtotic limit as λ approaches 0 by using the Liouville form of the exact solution to the differential equation $\Delta V + \lambda e^V = 0$, namely $\lambda e^V = |F'(z)|^2/(1 + |F(z)|^2)^2$ where $F(z)$ is a complex analytic function with at most simple zeros and poles. Then one applies a modified Newton iteration scheme, starting with $U_o(z,\lambda)$ using the appropriate integral operator, and shows that the resulting sequence converges in $C(\bar{D})$ to the desired solution.

We want to measure the distance between the parallel planes Π_0 and Π_1 and wish to show that for certain (α,λ) the distance is zero. It is better to look at the surfaces $\bar{y}(w,\alpha,\lambda) = \bar{x}\circ\Phi(w,\alpha,\lambda)/\sqrt{2\lambda}$ defined relative to the fundamental domain $\Omega(\alpha)$ and to measure the distance between the parallel planes Π_0' and Π_1' which correspond to the mapping $\bar{y}$. We do this by looking at the curve $\bar{y}(u,0,\alpha,\lambda)$, a planar curve which cuts through the planes Π_m' orthogonally and has the symmetry indicated in Figure 2.



<u>Figure</u> <u>2</u>:Measuring the Distance between the Parallel Planes Π_0' and Π_1'

The functions $\bar{y}(u.v,\alpha,\lambda)$ are conformal immersions into R^3 with constant mean curvature $H = \sqrt{2\lambda}$, so that as λ approaches 0 the mean curvature approaches 0 and the mapping tends to a planar map. The functions $\bar{y}$ satisfy a system just like (4) with ω replaced by $W = \Sigma/2$, L is replaced by $\tilde{L} = \sqrt{2\lambda}\ L$ and so on. Since by Theorem 3b the function $W(u,v,\alpha,\lambda)$ approaches $W(u,v,\alpha,0) = 2\log(1/|g(w,\alpha)|)$ as λ approaches 0, the curve $\bar{y}(u,0,\alpha,\lambda)$ approaches a limit curve $\bar{y}(u,0,\alpha,0)$ as λ approaches 0. It follows that the distance function $S = S(\alpha,\lambda)$, as indicated in Figure 2, is continuous down to $\lambda = 0$ and differentiable if λ is positive. Since $W(u,v,\alpha,0)$ is known explicitly one can calculate $S(\alpha,0)$, obtaining

$$S(\alpha,0) = \int_0^\beta (\cos 2\theta/(2\cos 2\theta - 2\cos 2\beta)^{\frac{1}{2}})\, d\theta \ , \ \beta = (\pi/2)-\alpha. \tag{8}$$

We immediately have the following conclusions.

a) $S(\alpha,0)$ is strictly increasing for $0 < \alpha < \pi/2$.

b) $S(\alpha,0)$ approaches $-\infty$ as α approaches 0.

c) $S(\alpha,0)$ is positive for α greater than $\pi/4$.

It follows that there is exactly one value α^*, $0 < \alpha^* < \pi/4$, for which $S(\alpha^*,0) = 0$. We have the following picture (see Figure 3). There is a small rectangle $[\alpha_1,\alpha_2] \times [0,\tilde{\lambda}]$ with $S(\alpha_1,\lambda)$ negative, $S(\alpha_2,\lambda)$ positive, and $S(\alpha^*,0) = 0$. There is a connected set X included in this small rectangle on which S vanishes and which separates the left side of the rectangle from the right side. In particular $(\alpha^*,0)$ is in the set X and every line λ = constant slices into X.

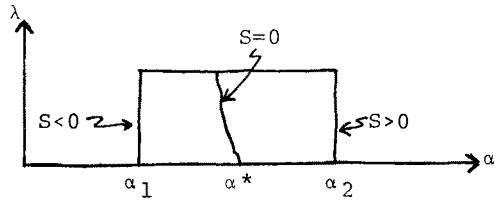


Figure 3: The Set S=0: All the Planes Π_m are Identical.

Now we measure the rotation angle between the planes Ω_o and Ω_1 by looking at the image of the curve $\bar{y}(0,v,\alpha,\lambda)$, $B(\alpha)/2 < v < 3B(\alpha)/2$. From Theorem 2d it follows that this curve lies on the sphere with center $\tilde{c}_o$ and radius $(2\lambda)^{-\frac{1}{2}}$, connecting the planes Ω_o to Ω_1 and intersecting them orthogonally. Let $T(\alpha,\lambda)$ be the distance between these planes as measured on a great circle of the sphere whose radius is $(2\lambda)^{-\frac{1}{2}}$. By repeating the calculation used to compute $S(\alpha,\lambda)$, one finds that for small λ, and α less than $\pi/4$, $T(\alpha,\lambda)$ is positive down to the limit $\lambda = 0$ with the expression for $T(\alpha,0)$ being similar to that for $S(\alpha,0)$. However, for the angle function $\Theta(\alpha,\lambda)$ we have the identity $\Theta(\alpha,\lambda) = (2\lambda)^{\frac{1}{2}} T(\alpha,\lambda)$. This gives us the following:

a) $\Theta(\alpha,\lambda)$ is positive for λ positive.

b) $\Theta(\alpha,\lambda)$ approaches 0 as λ approaches 0.

Since X is a connected set with more than one point (see Figure 3), it follows by continuity that on the set X the function $\Theta(\alpha,\lambda)$ takes on a continuum of values $[0,\varepsilon]$ where ε is positive. Whenever $\Theta(\alpha,\lambda)$ is a rational multiple of 2π the surface will close up. This establishes the existence of a countable number of isometrically distinct immersions of a torus into R^3 with constant mean curvature.

A View of the Immersed Tori.

Let $\Omega = \Omega_{AB}$ be a representative rectangle chosen so that the smallest eigenvalue of the Laplace differential equation

$$\Delta v + \gamma v = 0 \text{ on } \Omega, \quad v = 0 \text{ on boundary } \Omega \tag{9}$$

is $\gamma_1 = 1$. This means that $1 = \gamma_1 = \pi^2((1/A^2)+(1/B^2))$ and in particular A and B are both greater than π. We are to solve the differential equation

$$\Delta W + 2\lambda \sinh W \cosh W = 0 \text{ on } \Omega, \quad W = 0 \text{ on } \delta\Omega. \tag{10}$$

We have the following facts regarding solutions to the differential equation (10).

a) There exists a branch of positive solutions to (10) which bifurcate from the zero solution at $2\lambda = \gamma_1 = 1$ or $\lambda = 1/2$.

b) For any positive solution (W,λ) we must have $0 < \lambda < 1/2$, and for any λ in this interval there exists at least one positive solution.

c) As λ approaches 0 there is a curve of large positive solutions (W,λ) obtained by applying Theorem 3.

It is tempting (but not yet proven) to conjecture that the branch bifurcating from the zero solution at $\lambda = 1/2$ connects up with the branch of large solutions established in Theorem 3. Even more tempting is the following conjecture.

Conjecture: Let (W_1,λ_1) and (W_2,λ_2) be two positive solutions to the system (10). If $0 < \lambda_1 < \lambda_2 < 1/2$ then W_1 is greater than W_2 at every point inside Ω.

For each solution of the system (10) we may apply (7) to get a solution $\omega(u,v)$ to the differential equation (3) and then apply our recipe to construct an immersion $\bar{x}(u,v)$ with constant mean curvature. In the limit case where $W = 0$ the resulting immersion is simply a conformal mapping of the plane onto a circular cylinder whose cross section is a circle of radius one.

In the figures that follow we shall sketch the image $\bar{x}(u,v)$ of a portion of the fundamental rectangle $\sqrt{2\lambda}\ \Omega_{AB}$ as indicated in the first figure and labeled $\{1,2,3,4,5,6\}$. A + sign indicates that $\omega(u,v)$ is positive and hence the Gauss curvature of the image surface $K = e^{-2\omega}\sinh\omega\cosh\omega$ is positive, while a − sign indicates that both functions are negative. The rest of the surface is obtained by rotating the surface 180° about the normal line at the image of 2 followed by a series of reflections about the appropriate planes.

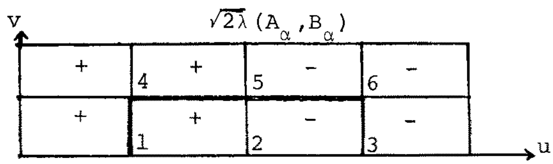


Figure 4: The Fundamental Domain $\sqrt{2\lambda}\ \Omega(\alpha) = \Omega(\alpha,\lambda)$.

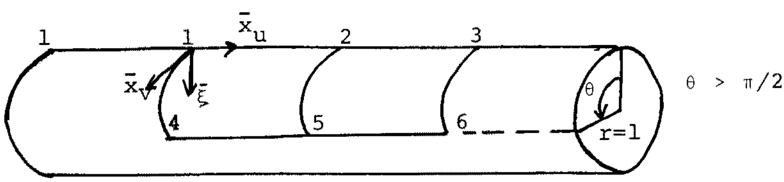
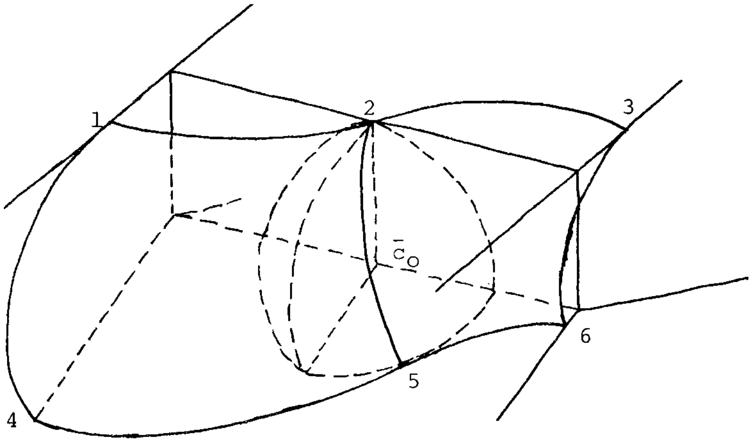
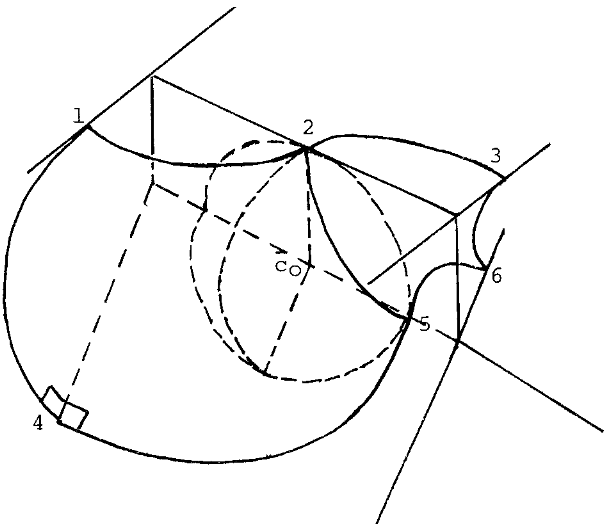


Figure 5: Case 1, W=0, A Pure Cylinder.



<u>Figure</u> <u>6</u>: Case 2. W is positive on $\Omega(\alpha,\lambda)$ but not too Large.



<u>Figure</u> <u>7</u>: Case 3. W somewhat larger, the Planes Π_o, Π_1 still separated.

If one keeps α fixed and lets λ approach 0, then one can easily show the following.

1) $\int_{\bar{x}(\Omega^+)} K\, dA =$ area of the Gauss map $\longrightarrow 4\pi$ as λ approaches 0.

2) $\int_{\Omega^+} e^{2\omega}\, dudv =$ Area of $\bar{x}(\Omega^+) \longrightarrow 4\pi(2)^2$ as λ approaches 0.

3) $\int_{\Omega^-} e^{2\omega}\, dudv =$ Area of $\bar{x}(\Omega^-) \longrightarrow 0$ as λ approaches 0.

These calculations suggest that as λ approaches 0, $\bar{x}(\Omega^+)$ takes on the shape of a sphere of radius 2.

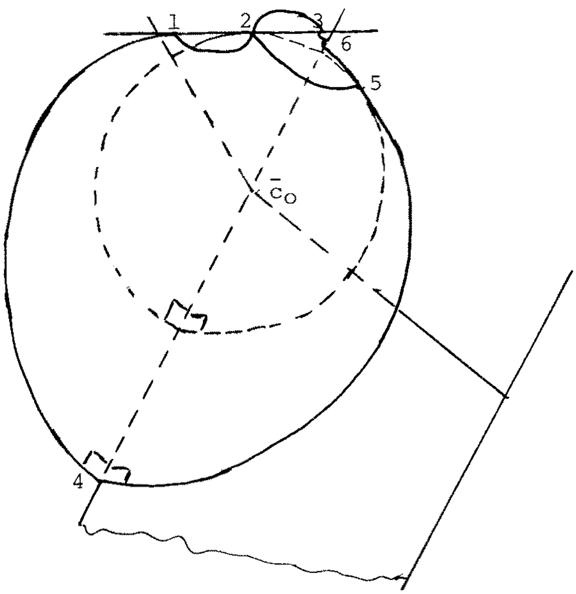


Figure 8: Case 4. The Parallel Planes Π_o, Π_1 are Identical.

If one reflects the sketched Figure 8 about the plane of the paper ($\Pi_o = \Pi_1$) you obtain a surface which resembles a clam shell. Upon rotating this shall 180^o about the vertical line c_o-(2) one obtains the other shell. The combined figure is now a clam with the shells opened a bit.

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