VOJTA'S CONJECTURE

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§ 1. Nevanlinna theory

Let $f : \mathbb{C}^d \longrightarrow X$ be a holomorphic map, where X is a complex non-singular variety of dimension d. Let D be an effective divisor on X, with associated invertible sheaf $\mathcal{L}$. Let s be a meromorphic section of $\mathcal{L}$, with divisor $(s) = D$. We suppose that f is non-degenerate, in the sense that its Jacobian is not zero somewhere. For positive real r we define

$$m(D,r,f) = \int_{\mathrm{Bd}\,B(r)} -\log|f^*s|^2 \sigma .$$

where σ is the natural normalized differential form invariant under rotations giving spheres area 1. When $d = 1$, then $\sigma = d\theta/2\pi$. Actually, $m(D,r,f)$ should be written $m(s,r,f)$, but two sections with the same divisor differ by multiplication with a constant, so $m(s,r,f)$ is determined modulo an additive constant. One can select this constant such that $m(s,r,f) \geqq 0$, so by abuse of notation, we shall also write $m(D,r,f) \geqq 0$.

We also define

$N(D,r,d)$ = normalized measure of the analytic divisor in the ball of radius r whose image under f is contained in D; (Cf. Griffiths [Gr] for the normalization.)

$$N(D,r,f) = \int_0^r [N(D,r,f)-N(D,0,f)]\,\frac{dr}{r} + N(D,0,f)\log r .$$

$$T(D,r,f) = m(D,r,f) + N(D,r,f) .$$

Remark. If $d = 1$ then $N(D,r,f) = n(D,r,f)$ is the number of points in the disc of radius r whose image under f lies in D .

One formulation of the FIRST MAIN THEOREM (FMT) of Nevanlinna theory runs as follows. *The function* $T(D,r,f)$ *depends only on the linear equivalence class of* D *, modulo bounded functions* $O(1)$.

The first main theorem is relatively easy to prove. More important is the SECOND MAIN THEOREM (SMT), which we state in the following form:

Let D *be a divisor on* X *with simple normal crossings (SNC, meaning that the irreducible components of* D *are non-singular, and intersect transversally). Let* E *be an ample divisor, and* K *the canonical class. Given* ε *, there exists a set of finite measure* $Z(\varepsilon)$ *such that for* r *not in this set,*

$$m(D,r,f) + T(K,r,f) \leqq \varepsilon T(E,r,f) \ .$$

This is an improved formulation of the statement as it is given for instance in Griffiths [Gr] , p. 68, formula 3.5.

§ 2. Weil functions

Let X be a projective variety defined over $\mathbb{C}$ or $\mathbb{C}_p$ (p-adic complex numbers = completion of the algebraic closure of $\mathbb{Q}_p$). Let $\mathcal{L}$ be an invertible sheaf on X and let ρ be a smooth metric on $\mathcal{L}$. If s is a meromorphic section of $\mathcal{L}$ with divisor D , we define the associated Weil function (also called Green's function)

$$\lambda(P) = -\log|s(P)| \quad \text{for} \quad P \notin \operatorname{supp}(D) \ .$$

If we change the metric or s with the same divisor, λ changes by a bounded smooth function, so is determined mod $O(1)$. We denote such a function by λ_D . It has the following properties:

The association $D \longmapsto \lambda_D$ is a homomorphism mod $O(1)$.

If $D = (f)$ on an open set U (Zariski) then there exists a smooth function α on U such that

$$\lambda_D(P) = -\log |f(P)| + \alpha .$$

If D is effective, then $\lambda_D \geqq -O(1)$ (agreeing that values of λ_D on D are then ∞).

If v denotes the absolute value on $\mathbb{C}_v$ then we write

$$v(a) = -\log|a|_v$$

for any element $a \in \mathbb{C}_v$, so we can write

$$\lambda_D = v \circ f + \alpha .$$

In the sequel, metrics will not be used as such; only the associated Weil functions and the above properties will play a role. Note that these Weil-Green functions need not be harmonic. In some cases, they may be, for instance in the case of divisors of degree 0 on a curve. But if the divisor has non-zero degree, then the Green function is not harmonic.

In the sequel, we shall deal with global objects, and then the Weil functions and others must be indexed by v, such as $\lambda_{D,v}$, α_v, etc.

§ 3. Heights (Cf.[La])

Let K be a number field, and let $\{v\}$ be its set of absolute values extending either the ordinary absolute value on $\mathbb{Q}$, or the p-adic absolute values such that $|p|_v = 1/p$. We let K_v be the completion, and K_v^a its algebraic closure. Then we have the product formula

$$\sum_v d_v v(a) = 0$$

where $d_v = \dfrac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}$ and $a \in K$, $a \neq 0$. We let $\|a\|_v = |a|_v^{d_v}$

Let $(x_o, \ldots, x_n) \in \mathbb{P}^n(K)$ be a point in projective space over K. We define its height

$$h(P) = \sum_v \log \max_i \|x_i\|_v$$

If $K = \mathbb{Q}$ and $x_o,\ldots,x_n \in \mathbb{Z}$ are relatively prime, then

$$h(P) = \log \max |x_i|$$

where the absolute value is the ordinary one. From this it is immediate that there is only a finite number of points of bounded height and bounded degree.

Let

$$\varphi : X \longrightarrow \mathbb{P}^n$$

be a morphism of a projective non-singular variety into projective space. We define

$$h_\varphi(P) = h(\varphi(P)) \quad \text{for} \quad P \in V(K^a) \ .$$

The basic theorem about heights states:

There exists a unique homomorphism $c \longmapsto h_c$

$$\mathrm{Pic}(X) \longrightarrow \text{functions from } X(K^a) \text{ to } \mathbb{R} \text{ modulo bounded functions}$$

such that if D *is very ample, and* $\mathcal{O}(D) = \varphi^*\mathcal{O}_{\mathbb{P}}(1)$, *then*

H 1. $\quad h_c = h_\varphi + O(1)$.

In the above statement, we denote by h_c any one of the functions in its class mod bounded functions. Similarly, if D lies in c , we also write h_D instead of h_c . This height function also satisfies the following properties:

H 2. If D is effective, then $h_D \geqq -O(1)$.

<u>H 3</u>. If E is ample and D any divisor, then

$$h_D = O(h_E) \ .$$

In particular, if E_1, E_2 are ample, then

$$h_{E_1} >> << h_{E_2} \ .$$

We are using standard notation concerning orders of magnitude. Since according to our conventions, a given height h_E is defined only mod bounded functions, the notation $h_D = O(h_E)$ or $h_D << h_E$ means that there exists a constant C such that for all points P with $h_E(P)$ sufficiently large, we have $|h_D(P)| \leqq Ch_E(P)$.

Essential to the existence and uniqueness of such height functions h_c is the property of elementary algebraic geometry that given any divisor D , if E is ample, then $D + mE$ is very ample for all $m \geqq m_o$.

A fundamental result also states that one can choose metrics ρ_v "uniformly" such that

$$h_D = \sum_v d_v \lambda_{D,v} + O(1)$$

The right hand side depends on Green-Weil functions $\lambda_{D,v}$, and so is a priori defined only for P outside the support of D . Since h_D depends only on the linear equivalence class of D mod $O(1)$, we can change D by a linear equivalence so as to make the right hand side defined at a given point.

Now let S be a finite set of absolute values on K . We define, relative to a given choice of Weil-Green functions and heights:

$$m(D,S) = \sum_{v \in S} d_v \lambda_{D,v}$$

$$N(D,S) = \sum_{v \notin S} d_v \lambda_{D,v}$$

Then

$$h_D = m(D,S) + N(D,S) \quad ,$$

and one basic property of heights says that h_D depends only on the linear equivalence class of D . This is Vojta's translation of FMT into the number theoretic context, with the height h_D corresponding to the function $T(D)$ of Nevanlinnna theory.

Remark. The properties of heights listed above also hold for T , as well as others listed for instance in [La], e.g. if D is algebraically equivalent to 0 , then $T(D) = O(T(E))$ for E ample. As far as I can tell, in the analytic context, there has been no such systematic listing of the properties of T , similar to the listing of the properties of heights as in number theory.

Vojta's translation of SMT yields his conjecture:

Let X *be a projective non-singular variety defined over a number field* K . *Let* S *be a finite set of absolute values on* K . *Let* D *be a divisor on* X *rational over* K *and with simple normal crossings. Let* E *be ample on* X . *Give* ε . *Then there exists a proper Zariski closed subset* $Z(S,D,E,\varepsilon) = Z(\varepsilon)$ *such that*

$$m(D,S,P) + h_K(P) \leq \varepsilon h_E(P) \quad \textit{for} \quad P \in X(K) - Z(\varepsilon) \ .$$

Or in other words,

$$\sum_{v \in S} d_v \lambda_{D,v} + h_K \leq \varepsilon h_E \quad \text{on} \quad X(K) - Z(\varepsilon)$$

where K is the canonical class.

EXAMPLES

Example 1. Let $X = \mathbb{P}^1$, $K = \mathbb{Q}$, $E = (\infty)$. Let α be algebraic, and let

$$f(t) = \prod_{\sigma} (\sigma\alpha - t)$$

where the product is taken over all conjugates $\sigma\alpha$ of α over $\mathbb{Q}$. Let D be the divisor of zeros of f. The canonical class $\mathbb{K}$ is just $-2(\infty)$. A rational point P corresponds to a rational value $t = p/q$ with $p,q \in \mathbb{Z}$, $q > 0$, and p,q relatively prime. We let S consist of the absolute value at infinity. If $|f(p/q)|$ is small, then p/q is close to some root of f. If p/q is close to α, then it has to be far away from the other conjugates of α. Consequently Vojta's inequality yields from the definitions:

$$-\log |\alpha - p/q| - 2h_{\infty}(p/q) \leqq \varepsilon h_{\infty}(p/q)$$

with a finite number of exceptional fractions. Exponentiating, this reads

$$\left| \alpha - \frac{p}{q} \right| \geqq \frac{1}{q^{2+\varepsilon}} ,$$

which is Roth's theorem.

Remark. Some time ago, I conjectured that instead of the q^{ε} in Roth's theorem, one could take a power of $\log q$ (even possibly $(\log q)^{1+\varepsilon}$). Similarly, in Vojta's conjecture, the right hand side should be replaced conjecturally by $O(\log h_E)$. If one looks back at the Nevanlinna theory, one then sees that the analogous statement is true, and relies on an extra analytic argument which is called the lemma on logarithmic derivatives. Cf. Griffiths [Gr].

Example 2. Let $X = \mathbb{P}^n$ and let $D = L_o+\ldots+L_n$ be the formal sum of the hyperplane coordinate sections, with L_o at infinity, and $E = L_o$. Let φ_i be a rational function such that

$$(\varphi_i) = L_i - L_o .$$

Let S be a finite set of absolute values. Note that in the case of $\mathbb{P}^n$, the canonical class K contains $-(n+1)L_o$. Consequently, Vojta's inequality in this case yields

$$\prod_i \prod_{v \in S} \| \varphi_i(P) \|_v \geqq \frac{1}{H(P)^{n+1+\varepsilon}}$$

for all P outside the closed set $Z(\varepsilon)$. This is Schmidt's theorem, except that Schmidt arrives at the conclusion that the exceptional set is a finite union of hyperplanes. In order to make Vojta's conjecture imply Schmidt strictly, one would have to refine it so as to give a bound on the degrees of the components of the exceptional set, which should turn out to be 1 if the original data is linear.

Example 3. Let X be a curve of genus $\geqq 2$. Take S empty. The canonical class has degree $2g-2$ where g is the genus, and so is ample. Then Vojta's inequality now reads

$$h_K \leqq \varepsilon\, h_E \quad \text{on} \quad X(K) ,$$

except for a finite set of points. Since K is ample, such an inequality holds only if $X(K)$ is finite, which is Falting's theorem.

Example 4. This is a higher dimensional version of the preceding example. Instead of assuming that X is a curve, we let X have any dimension, but assume that the canonical class is ample. The same inequality shows that the set of rational points is not Zariski dense.

This goes toward an old conjecture of mine, that if a variety is hyperbolic, then it has only a finite number of rational points. The effect of hyperbolicity should be to eliminate the exceptional Zariski set in Vojta's conjecture. For progress concerning this conjecture in the function field case, cf. Noguchi [No], under the related assumption that the cotangent bundle is ample, and that the rational points are Zariski dense.

To apply the argument of Vojta's inequality it is not necessary to assume that the canonical invertible sheaf is ample, it suffices to be in a situation when for any ample divisor E, $h_E = O(h_K)$. This is the case for varieties of general type, which means that the rational map of X defined by a sufficiently high multiple of the canonical class gives a rational map of dimension $d = \dim X$. Then we have $h^0(mK) \gg m^d$ for m sufficiently large, and we use the following lemma.

Lemma. *Let X be a non-singular variety. Let E be very ample on X, and let D be a divisor on X such that $h^0(mD) \gg m^d$ for $m \geq m_0$. Then there exists m_1 such that $h^0(mD-E) \gg m^d$, and in particular, $mD-E$ is linearly equivalent to an effective divisor, for all $m \geq m_1$.*

Proof. First a remark for any divisor D. Let E' be ample, and such that $D + E'$ is ample. Then we have an inclusion

$$H^0(mD) \subset H^0(mD + mE'),$$

which shows that $h^0(mD) \leq h^0(mD + mE') = \chi(m(D + E'))$ for m large because the higher cohomology groups vanish for m large, so $h^0(mD) \gg m^d$.

Now for the lemma, without loss of generality we can replace E

by any divisor in its class, and thus without loss of generality we may assume that E is an irreducible non-singular subvariety of X. We have the exact sequence

$$0 \longrightarrow \mathcal{O}(mD-E) \longrightarrow \mathcal{O}(mD) \longrightarrow \mathcal{O}(mD)|E \longrightarrow 0 \quad ,$$

whence the exact cohomology sequence

$$0 \longrightarrow H^0(X,mD-E) \longrightarrow H^0(X,mD) \longrightarrow H^0(E,(\mathcal{O}(D)|E)^{\otimes m})$$

noting that $\mathcal{O}(mD)|E = (\mathcal{O}(D)|E)^{\otimes m}$. Applying the first remark to this invertible sheaf on E we conclude that the dimension of the term on the right is $\ll m^{d-1}$, so $h^0(X,mD-E) \gg m^d$ for m large, and in particular is positive for m large, whence the lemma follows.

For $mD - E$ effective, we get $h_E \leq h_{mD} + O(1)$ as desired.

<u>Example 5</u>. Let A be an abelian variety, and let D be a very ample divisor with SNC. Let S be a finite set of absolute values of K containing the archimedean ones. Let $\varphi_1,\ldots,\varphi_n$ be a set of generators for the space of sections of $\mathcal{O}(D)$. Let $\mathfrak{o}_S$ be the ring of S-integers in K (elements of K which are integral at all $v \notin S$). A point $P \in A(K)$ is said to be <u>S-integral relative to these generators</u> if $\varphi_i(P) \in \mathfrak{o}_S$ for $i = 1,\ldots,n$. On the set of such S-integral points, we have

$$\sum_{v\in S} d_v \lambda_{D,v} = h_D + O(1)$$

immediately from the definitions. The canonical class is 0. Then again Vojta's inequality shows that the set of S-integral points as above is not Zariski dense.

This is in the direction of my old conjecture that on any affine open subset of an abelian variety, the set of S-integral points is finite. However, in this stronger conjecture, we again see the difference between finiteness and the property of not being Zariski dense.

Example 6. Hall's conjecture Marshall Hall conjectured that if x,y are integers, and $x^3-y^2 \neq 0$ then

$$|x^3-y^2| \geq \max(|x^3|, |y^2|)^{\frac{1}{6}-\varepsilon}$$

with a finite number of exceptions. Actually, Hall omitted the ε, but Stark and Trotter for probabilistic reasons have pointed out that it is almost certainly needed, so we put it in.

Vojta has shown that his conjecture implies Hall's. We sketch the argument. Let

$$f : \mathbb{P}_1^2 \longrightarrow \mathbb{P}_2^2$$

be the rational map defined on projective coordinates by

$$f(x,y,z) = (x^3, y^2z, z^3) .$$

Then f is a morphism except at $(0,1,0)$. We have indexed projective 2-space by indices 1 and 2 to distinguish the space of departure and the space of arrival. We let $L = L_1$ be the hyperplane at infinity on $\mathbb{P}_1^2$, and L_2 the hyperplane at infinity on $\mathbb{P}_2^2$.

Let C be the curve in $\mathbb{P}_1^2$ defined by $x^3-y^2 = 0$. Let φ be the rational function defined by

$$\varphi(x,y) = x^3-y^2 .$$

Then the divisor of φ is given by

$$(\varphi) = C - 3L \ .$$

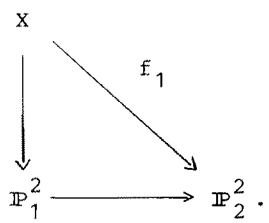
In terms of heights, Hall's conjecture can be formulated in the form

$$\log |\varphi(x,y)| \geq \frac{1}{6} h_{L_2} \circ f(x,y) + \text{error term},$$

or if v denotes the ordinary absolute value on $\mathbb{Q}$,

$$\text{(1)} \qquad v \circ \varphi (x,y) \leq -\frac{1}{6} h_{L_2} \circ f(x,y) + \text{error term.}$$

Note that $v \circ \varphi = \lambda_{(\varphi)}$ is a Weil function associated with the divisor (φ) . Thus Hall's conjecture amounts to an inequality on Weil functions. By blowing up the point of indeterminacy of f and the singularity of C at $(0,0)$, one obtains a variety X and a corresponding morphism $f_1 : X \longrightarrow \mathbb{P}^2$ making the following diagram commutative:



The blow ups are chosen so that the exceptional divisor and C have simple normal crossings. By taking D to be their sum together with the hyperplane at infinity, Vojta shows that his conjecture implies Hall's. By a similar technique, Vojta shows that his conjecture implies several other classical diophantine conjectures. I refer the reader to his forthcoming paper on the subject.

BIBLIOGRAPHY

[Gr] P. Griffiths, *Entire holomorphic mappings in one and several variables:* Hermann Weyl Lectures, Institute for Advanced Study, Institute for Advanced Study, Princeton Univ. Press, Princeton NJ, 1976.

[La] S. Lang, *Fundamentals of Diophantine Geometry*, Springer Verlag, 1984.

[No] J. Noguchi, *A higher dimensional analogue of Mordell's conjecture over function fields*, Math. Ann. 258 (1981) pp. 207-212.

[Vo] P. Vojta, *Integral points on varieties*, Thesis, Harvard, 1983.