THE SCHOTTKY PROBLEM

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By associating to a (smooth irreducible) curve C of genus $g > 0$ its Jacobian $\mathrm{Jac}(C)$ one obtains a morphism $M_g \to A_g$ from the moduli space of curves of genus g to the moduli space of principally polarized Abelian varieties of dimension g. A well-known theorem of Torelli says that this morphism is injective. The image of M_g in A_g is not closed, it is only closed inside A_g^o, the set of points of A_g that correspond to indecomposable principally polarized abelian varieties (i.e. that are not products). For $g=1,2,3$ the closure of the image of M_g equals A_g. Since $\dim A_g = g(g+1)/2$, $\dim M_g = 3g-3$ (for $g > 1$) one sees that for $g > 3$ $\dim A_g > \dim M_g$, and so the question arises how we can characterize the image of M_g in A_g. This question goes back to Riemann, but is usually called Schottky's problem.

In "Curves and their Jacobians" Mumford treats the Schottky problem and the closely related question how to distinguish Jacobians from general principally polarized abelian varieties. In his review of the situation at that moment (1975) he describes four approaches and their merits. He concludes that none of these seems to him a definitive solution. In the meantime the situation has changed a lot. Some of the approaches have been worked out more completely, while new and successfull approaches have appeared. This paper deals with them. I hope to convince the reader that Mumford's statement that problems in this corner of nature are subtle and worthy of his time still very much holds true.

The ingredients.

To begin with, some standard notations.

$\mathbb{H}_g$: the Siegel upper half space of degree g,

$\Gamma_g = Sp(2g, \mathbb{Z})$ the symplectic group acting on $\mathbb{H}_g$,

$\Gamma_g(n,2n) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g : \begin{matrix} A \equiv D \equiv 1 \pmod n \\ C \equiv B \equiv 0 \pmod n \end{matrix}, \mathrm{diag}\,{}^tAC \equiv \mathrm{diag}\,{}^tBD \equiv 0 \pmod{2n}\}$

$A_g = \Gamma_g \backslash \mathbb{H}_g$, the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{C}$,

$A_g(n,2n) = \Gamma_g(n,2n)\backslash\mathbb{H}_g$, a Galois cover of A_g.

If X is a principally polarized abelian variety over $\mathbb{C}$ we denote by L_X (or simply L) a symmetric invertible ample sheaf of degree 1 defining the polarization and by Θ the divisor of a non-zero section of L_X. We put $X_n = \{x \in X : nx = 0\}$. If $X = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ $(\tau \in \mathbb{H}_g)$ as a complex torus then we write $X = X_\tau$.

The space $\Gamma(X, L_X^{\otimes 2})$ has dimension 2^g. A basis is defined by the functions

$$\Theta_2[\sigma](\tau,z) = \sum_{m \in \mathbb{Z}^g} \exp 2\pi i ({}^t(m+\tfrac{\sigma}{2})\tau(m+\tfrac{\sigma}{2}) + 2(m+\tfrac{\sigma}{2})z)$$

$$z \in \mathbb{C}^g, \ \sigma \in (\mathbb{Z}^g/2\mathbb{Z}^g)$$

Here σ is viewed as a vector of length g with zeroes and ones as entries. A different set of generators of $\Gamma(X, L_X^{\otimes 2})$ is given by the squares of

$$\theta[{}^{\varepsilon}_{\varepsilon'}](\tau,z) = \sum_{m \in \mathbb{Z}^g} \exp \pi i ({}^t(m+\tfrac{\varepsilon}{2})\tau(m+\tfrac{\varepsilon}{2}) + 2(m+\tfrac{\varepsilon}{2})(z+\tfrac{\varepsilon'}{2}))$$

with $\varepsilon, \varepsilon' \in (\mathbb{Z}/2\mathbb{Z})^g$, ${}^t\varepsilon\varepsilon' \equiv 0 \pmod 2$. These are related by

$$\theta^2[{}^{\varepsilon}_{\varepsilon'}](\tau,z) = \sum_{\sigma} \langle\sigma,\varepsilon'\rangle \Theta_2[\sigma+\varepsilon](\tau,0)\, \Theta_2[\sigma](\tau,z) \qquad (1)$$

$$\langle\sigma,\varepsilon'\rangle = \exp\pi i\, {}^t\sigma\varepsilon'$$

We call a principally polarized abelian variety indecomposable if it is not a product of two principally polarized abelian varieties, i.e. if its theta divisor is irreducible.

The functions $\Theta_2[\sigma](\tau,z)$ define for $X = X_\tau$ a morphism

$$\Phi_X : X \to \mathbb{P}^N \qquad N = 2^g - 1$$

$$z \to (\ldots, \Theta_2[\sigma](\tau,z), \ldots) = \vec{\Theta}_2(\tau,z)$$

which factors through $z \to -z$ and is of degree 2 for indecomposable X. The image is the Kummer variety of X. By taking $z=0$ and varying X we get a morphism

$$\Phi: A_g(2,4) \to \mathbb{P}^N$$
$$\tau \to (\ldots, \Theta_2[\sigma](\tau,0), \ldots) = \vec{\Theta}_2(\tau,0)$$

which is generically of degree 1. We also define

$$\Psi: A_g(2,4) \to \mathbb{P}^M \qquad M = 2^{g-1}(2^g+1)-1$$
$$\tau \to (\ldots, \theta^2[{}^{\varepsilon}_{\varepsilon}](\tau,0), \ldots).$$

Φ and Ψ are connected by the special Veronese V defined by (1) :

$$\begin{tikzcd} & \mathbb{P}^N \arrow[d, "V"] \\ A_g(2,4) \arrow[ur, "\Phi"] \arrow[r, "\Psi"'] & \mathbb{P}^M \end{tikzcd}$$

The morphisms Φ and Ψ can be extended to morphisms of the Satake compactification $\overline{A}_g(2,4)$ of $A_g(2,4)$.

The functions $\Theta_2[\sigma]$ satisfy the differential equations

$$\frac{\partial^2}{\partial z_i \partial z_j} \Theta_2[\sigma] = 4\pi i\,(1+\delta_{ij}) \frac{\partial}{\partial \tau_{ij}} \Theta_2[\sigma], \qquad 1 \le i,j \le g.$$

(δ_{ij}: Kronecker δ)

which are called the Heat Equations.

If $\mathcal{M}_g$ is the moduli space of curves of genus g then the map $\mathcal{M}_g \to A_g$ defined by $C \to \mathrm{Jac}(C)$ is injective. The closure of the image in A_g (or $\overline{A}_g$) is called the Jacobian locus. Notation : J_g.

APPROACH 1 : ALGEBRAIC EQUATIONS.

This is Schottky's original approach for characterizing the Jacobian locus. It is based on the construction of Prym varieties. For an excellent treatment of Prym varieties, see Mumford [13].

Suppose we start with a curve C of genus g and a non-zero point η of order 2 on $J = \mathrm{Jac}(C)$. This determines an unramified covering

$\pi: \tilde{C} \to C$ of degree 2 and an induced map $\mathrm{Nm} : \tilde{J} = \mathrm{Jac}(\tilde{C}) \to J$ and gives us a diagram

$$\begin{array}{ccc} \tilde{J} & \xrightarrow{\tilde{\lambda}} & \tilde{J} \\ \phi \uparrow & & \downarrow \hat{\phi} = \mathrm{Nm} \\ J & \xrightarrow[2\lambda]{} & J \end{array} \tag{2}$$

where $\phi = \pi^*$, $\hat{}$ denotes transpose and $\tilde{\lambda},\lambda$ are the principal polarizations. One defines the Prym variety of $\pi: \tilde{C} \to C$ as the identity component of the kernel of Nm : $P = (\ker \mathrm{Nm})^{o}$. It is an abelian variety of dimension g. Mumford showed that from a diagram (2) it follows that there exist a symplectic isomorphism $H_1/\{0,\eta\} \to P_2$ with $H_1 = \{\alpha \in J_2 : e_2(\alpha,\eta) = 1\}$ (e_2: Weil-pairing) such that

$$\tilde{J} = J\times P/\{(\alpha,\psi(\alpha)) : \alpha \in H_1\} .$$

Let $\sigma: J\times P \to \tilde{J}$ be the natural isogeny. Then the polarization $\begin{pmatrix} 2\lambda & 0 \\ 0 & \rho \end{pmatrix}$ is the pull back under σ of the polarization $\tilde{\lambda}$ and this implies that ρ is twice a principal polarization. So P carries a principal polarization ! For these facts, see [13],§2.

Now use the elementary

(1.1) <u>Lemma</u>. If D is a divisor of degree $g-1$ on C, then $h^{o}(\pi^*(D)) \neq 0$ if and only if $h^{o}(D) \neq 0$ or $h^{o}(D+\eta) \neq 0$.

One finds (using that for Jacobians the theta divisor in Jac^{g-1} consists of the effective divisor classes of degree $g-1$)

$$\sigma^{-1}(\tilde{\Theta}_o) \cap (\mathrm{Jac}(C) \times (0)) = \Theta_o + \Theta_{o,\eta} .$$

Here $\tilde{\Theta}_o$ (resp. $\tilde{\Theta}_o$) denotes the theta divisor on $\mathrm{Jac}^{g-1}(C)$ (resp. $\mathrm{Jac}^{2g-2}(C)$, i.e, $\Theta_o = \{x \in \mathrm{Jac}^{g-1}(C) : h^{o}(x) > 0\}$. If one now chooses $\alpha \in \mathrm{Jac}(C)_4$ such that $2\alpha = \eta$ and a theta characteristic ζ on C, then $\tilde{\zeta} = \pi^{-1}(\zeta+\alpha)$ is a theta characteristic on $\tilde{C}$. If $\tilde{\Theta} = \{x \in \tilde{J} : h^{o}(x+\tilde{\zeta}) > 0\}$, $\Theta = \{x \in J : h^{o}(x+\zeta) > 0\}$ are the theta divisors on $\tilde{J}$ and on J then (since $\sigma|J\times(0) = \pi^*$):

$$(\pi^*)^{-1}\tilde{\Theta} = \Theta_{\alpha} + \Theta_{-\alpha}. \tag{3}$$

(1.2) The link between the Kummer variety of P and that of J is obtained as follows. There is a morphism

$$\begin{aligned} \delta: P &\to |2\Theta_J| \\ p &\to (\pi^*)^{-1}(\tilde{\Theta}_{\tilde{J},-p}). \end{aligned}$$

Mumford shows in [M1] that δ is the usual Kummer map followed by an inclusion

$$\begin{tikzcd} & \mathbb{P}(H^0(L_P^{\otimes 2})^\vee) \arrow[d, "i"] \\ P \arrow[ur, "\Phi_P"] \arrow[r, "\delta"'] & |2\Theta_J| \end{tikzcd}$$

For any principally polarized abelian variety X the Riemann theta formula

$$\theta(u+v)\theta(u-v) = \sum c_{\alpha\beta} s_\alpha(u) s_\beta(v)$$

with θ a non-zero section of L_X and $\{s_\alpha\}$ a basis of $\Gamma(X, L_X^{\otimes 2})$ gives us a non-degenerate form B on $\Gamma(X, L_X^{\otimes 2})$ via the $(c_{\alpha\beta})$ and gives rise to a diagram

$$\begin{tikzcd} & \mathbb{P}(H^0(X, L_X^{\otimes 2})^\vee) \arrow[d, "\wr"', "B'"] \\ X \arrow[ur, "\Phi_X"] \arrow[r, "\Phi'_X"'] \arrow[ur, phantom, "\sim"] & |2\Theta_X| \end{tikzcd}$$

where $\Phi'_X(x) = \Theta_{X,x} + \Theta_{X,-x}$ and B' is induced by B.

Formula (3) thus implies the fundamental relation

$$i(\Phi_P(0)) = B'(\Phi_X(\alpha)). \tag{4}$$

(1.3) For any indecomposable principally polarized X the theta group $G(L_X^{\otimes 2})$ acts on $\Gamma(X, L_X^{\otimes 2})$ and this defines an action of $G(L_X^{\otimes 2})$ modulo scalars $\cong X_2$ on $\mathbb{P}(\Gamma(X, L_X^{\otimes 2})^\vee)$. If $\alpha \in X_2$, $\alpha \neq 0$, then α defines a projective involution i_α of $\mathbb{P}^N$ with

$$i_\alpha(\Phi_X(x)) = \Phi_X(x+\alpha). \tag{5}$$

It is a classical fact that the involution i_α on $\mathbb{P}^N = \mathbb{P}(\Gamma(X, L_X^{\otimes 2})^\vee))$ has as its fixed point set two linear subspaces V_α^+, V_α^-, each of dimension $2^{g-1}-1$ and each intersecting the Kummer variety of X in $2^{2(g-1)}$ points; moreover,

$$(V_\alpha^+ \cup V_\alpha^-) \cap \Phi_X(X) = \Phi_X(\{x \in X_4: \ 2x = \alpha\}).$$

The linear spaces $V_\alpha^\pm$ cut out on the modular variety $\Phi(\overline{A}_g(2,4))$ the boundary components. To be precise,

(1.4) Proposition. Let $A_{g-1}(2,4)$ be one of the $2(2^{2g}-1)$ boundary components of $\overline{A}_g(2,4)$ of maximal dimension. The image $\Phi(\overline{A}_{g-1}(2,4))$ in $\mathbb{P}^{2^g-1}$ is the intersection of $\Phi(\overline{A}_g(2,4))$ with one of the linear spaces $V_\alpha^\pm$.

It follows from (4) and proposition (1.4) that for a Jacobian X the intersection $\Phi_X(X) \cap \Phi(A_{g-1}(2,4))$ is not empty (Here we view $A_{g-1}(2,4)$ as a boundary component of $A_g(2,4)$.) : the intersection contains the image of a point of order 4 of X.

(1.5) Definition. The Schottky locus $S_g \subseteq \overline{A}_g$ is the smallest closed subset of $\overline{A}_g$ containing the points $[X]$ with X indecomposable for which $\Phi_X(X) \cap \Phi(A_{g-1}(2,4)) \neq \emptyset$ for all boundary components $A_{g-1}(2,4)$.

By construction S_g contains J_g, the Jacobian locus (cf. (4)). S_g can be described in terms of theta constants as well. The point is that P can be written as

$$P = \mathbb{C}^{g-1}/\mathbb{Z}^{g-1} + \rho_{g-1}\mathbb{Z}^{g-1} \qquad \text{for some } \rho_{g-1} \in \mathbb{H}_{g-1}$$

and that after suitable normalizations

$$\theta^2[{\varepsilon \atop \varepsilon'}](\rho_{g-1},0) = c\,\theta[{\varepsilon\ 0 \atop \varepsilon'\ 0}](\tau_g,0)\,\theta[{\varepsilon\ 0 \atop \varepsilon'\ 1}](\tau_g,0) \tag{6}$$

with a constant $c \in \mathbb{C}^*$ independent of $\varepsilon,\varepsilon' \in (\mathbb{Z}/2\mathbb{Z})^{g-1}$. Thus (6) is a translation of (4).

Let

$$T_g \subset \mathbb{C}[\, X_{[{\varepsilon \atop \varepsilon'}]} : \ \varepsilon,\varepsilon' \in (\mathbb{Z}/2\mathbb{Z})^g, \ t_{\varepsilon\varepsilon'} = 0 \,]$$

be the ideal of $\Psi(\overline{A}_g(2,4))$. To an element $f \in T_{g-1}$ we associate

$$\sigma(f) = f(\ldots,\theta[{\varepsilon \atop \varepsilon'}{0 \atop 0}]\theta[{\varepsilon \atop \varepsilon'}{0 \atop 1}](\tau_g,0),\ldots)$$

by substituting $\theta[{\varepsilon \atop \varepsilon'}{0 \atop 0}]\theta[{\varepsilon \atop \varepsilon'}{0 \atop 1}](\tau_g,0)$ for $X_{[{\varepsilon \atop \varepsilon'}]}$. The group $\Gamma_g/\Gamma_g(4,8)$ acts on $\mathbb{C}[\ \theta[{\varepsilon \atop \varepsilon'}](\tau_g,0) : {}^t\varepsilon\varepsilon' = 0]$. Let Σ_g be the smallest $\Gamma_g/\Gamma_g(4,8)$-invariant ideal of this ring containing all $\sigma(f)$ with f in T_{g-1}. Then S_g is the zero-locus of Σ_g in $\overline{A}_g$.

Of course, this description is explicit only if we know T_{g-1} and in general the structure of this ideal is not known.

For $g=4$ one finds that Σ_g is the ideal generated by a Siegel modular form of weight 8 as Schottky showed.

The important question about S_g is whether $S_g = J_g$ and if not, what the components of S_g are. For $g=4$ Igusa proved that S_4 is irreducible. This implies $S_4 = J_4$. Recently van Geemen proved

(1.6) Theorem. (van Geemen [6]) J_g is an irreducible component of S_g.

His proof uses an induction argument and an analysis of the intersection of the Schottky locus with blow-up of a boundary component of $\overline{A}_g(4,8)$.

It is a recurring phenomenon in the history of the Schottky problem that one finds algebraic subsets of A_g that contain J_g as as irreducible component but that may have other components as well. Another example is the Andreotti-Mayer approach. Since it is known that for a Jacobian one has $\dim \mathrm{Sing}\,\Theta \geq g-4$ one looks at

$$N_g^m = \{[X] \in A_g : \mathrm{Sing}\,\Theta \neq \emptyset, \dim \mathrm{Sing}\,\Theta \geq m\}$$

Andreotti and Mayer proved that J_g is an irreducible component of N_g^{g-4}, $g \geq 4$. However, N_g^{g-4} contains other components.

APPROACH 2 : TRISECANTS.

One of the remarkable features of Jacobians is that their Kummer varieties possess trisecants :

(2.1) Proposition. Let C be a non-singular curve and let a,b,c,d be points of C. If $r \in X = \mathrm{Jac}(C)$ is such that $2r = a+b-c-d$, then $\Phi_X(r)$, $\Phi_X(r-b+c)$ and $\Phi_X(r-b+d)$ are collinear.

Fay's trisecant identity [3] implies this fact. Gunning [9] has generalized this identity. The idea behind it is essentially the following.

Let N be a line bundle on $X\times X$ such that

$$N_{|X\times t} \cong T_{-t}^{*}(\mathcal{O}(2\Theta)) \qquad (\,T_t\ :\ \text{translation by } t).$$

Fix a point p of C. This defines $\phi : C \to X = \mathrm{Jac}(C)$ by $c \mapsto c-p$. Let Δ be a divisor of degree g on C such that $\phi^{*}\mathcal{O}(\Theta) \cong \mathcal{O}(\Delta)$. We let M be the vector bundle on X whose fibre at t is $H^{0}(C,\mathcal{O}(2\Delta+2t))$. Pull back of sections via

$$H^{0}(X,T_{-t}^{*}(\mathcal{O}(2\Theta)) \xrightarrow{\phi_t^{*}} H^{0}(C,\mathcal{O}(2\Delta+2t)) \qquad (7)$$

gives rise to a bundle map

$$\psi :\ (p_1)_{*}N \longrightarrow M.$$

(2.2) Lemma. The map ψ is surjective.

Proof. The map $H^{0}(X,T_{-t}^{*}\mathcal{O}(\Theta)) \to H^{0}(C,\mathcal{O}(\Delta+t))$ is surjective if the divisor $\Delta+t$ is non-special. Therefore, if $D \in |2\Delta+t|$ can be written as $D = D_1 + D_2$, $D_i \in |\Delta+t_i|$ with $\Delta+t_i$ non-special, then D is the zero divisor of a section in the image of (7). Define a non-empty open set in the symmetric product $C^{(g)}$ by (κ = canonical divisor)

$$U = \{z_1+\ldots+z_g \in C^{(g)}\ :\ h^{0}(D-\textstyle\sum z_i)=1,\ h^{0}(\kappa-\sum z_i)=0\ \}.$$

If $D = z_1+\ldots+z_g+z_1'+\ldots+z_g'$ with $\sum z_i \in U$, then $h^{0}(\sum z_i') = 1$, hence $\sum z_i$ and $\sum z_i'$ are both non-special. This shows that the image of

$\mathbb{P}(H^o(X,T^*_{-t}(\mathcal{O}(2\Theta)) \to \mathbb{P}(H^o(C,\mathcal{O}(2\Delta+t)))$ contains a non-empty open set.

We put as usual

$$W^r_d = \{ x \in \mathrm{Jac}^{(d)}(C) \; : \; h^o(x) \geq r+1 \}.$$

(2.3) Theorem. (Gunning [10]) If $z_1,\ldots,z_n$ are distinct points of C, then

$$W^{n-\mu}_{n-2} - \sum_{i=1}^{n} z_i + 2p = \{2t \in \mathrm{Jac}(C) \; : \; \mathrm{rank}\; \vec{\theta}_2(t+\phi(z_i))^n_{i=1} < \mu \} \; .$$

Proof. By the lemma, the rank of this $2^g{\times}n$-matrix is less than $\mu \iff h^o(2\Delta+2t-\sum z_i) > g+1-\mu \iff 2t+2\Delta-\sum z_i \in W^{g+1-\mu}_{2g-n}$. Applying Serre duality $\kappa - W^{g+1-\mu}_{2g-n} = W^{n-\mu}_{n-2}$ and the fact that $\phi(\kappa)=\phi(2\Delta)$ gives the result.

The special case $n=\mu=3$ gives proposition (2.1).

We can generalize this by allowing the points z_i to coincide. If $z_1+\ldots+z_n = m_1x_1+\ldots+m_ex_e$ with $x_i \neq x_j$ if $i \neq j$ then in the rank condition the m_j vectors $\vec{\theta}_2(t+\phi(x_j))$ have to be replaced by

$$\vec{\theta}_2(t+\phi(x_j)) \qquad \Delta_1\vec{\theta}_2(t+\phi(x_j)) \qquad \ldots \qquad \Delta_{m_j-1}\vec{\theta}_2(t+\phi(x_j)),$$

where the Δ_k are differential operators defined as follows. The curve $\phi(C)$ contains at $\phi(x_j)$ an artinian subscheme $\mathrm{Spec}\;\mathbb{C}[\varepsilon]/(\varepsilon^{m_j})$ and this is given by a local homomorphism

$$\begin{aligned} \mathcal{O}_{X,\phi(x_j)} &\to \mathbb{C}[\varepsilon]/(\varepsilon^{m_j}) \\ f &\to f(y_j) + \Delta_1 f(y_j)\varepsilon + \ldots + \Delta_{m_j-1}f(y_j)\varepsilon^{m_j-1}, \qquad y=\phi(x_j) \end{aligned}$$

The special case $n=\mu=3$ is important since it gives us back the curve C : Note that $W^o_1 \cong C$ and

$$W^o_1 - \sum_{i=1}^{3} z_i + 2p = \{2t \in \mathrm{Jac}(C) \; : \; \mathrm{rank}\; (\vec{\theta}_2(t+\phi(z_i))(t) \leq 2 \} \quad !$$

Gunning's idea in [8] was to use this property to characterize Jacobians. Gunning used distinct points z_i but Welters has infinitesimalized Gunning's case to include the case of coinciding points and transformed it into the following beautiful criterion :

(2.4) Theorem. (Gunning-Welters [19]) Let X be an indecomposable principally polarized abelian variety and let $Y \subset X$ be an artinian subscheme of length 3. Assume that

$$V = \{2t \in X : t+Y \subset \Phi_X^{-1}(\ell) \text{ for some line } \ell \subset \mathbb{P}^N\}.$$

has positive dimension at some point. Then V is a smooth irreducible curve and X is its Jacobian.

(2.5) The property of having flexes is closely related to the Kadomčev-Petviashvili equation (K-P-equation), a fourth order partial differential equation satisfied by the theta functions of Jacobians. In [16] Mumford noticed that if the points a,b,c,d in proposition (2.1) coincide, Fay's trisecant identity leads to the K-P-equation.

To get the link, note that an inclusion $\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^{N+1}) \to (X,0)$ is given by a local homomorphism

$$\begin{array}{ccc} \mathcal{O}_{X,0} & \longrightarrow & \mathbb{C}[\varepsilon]/(\varepsilon^{N+1}) \\ f & \to & \sum_{i=1}^{N} \Delta_i(f)\varepsilon^i \end{array}$$

where the Δ_i are differential operators satisfying

$$\Delta_o = \mathrm{id}, \quad \Delta_i(gh) = \sum_{k+\ell=i} \Delta_k(g)\Delta_\ell(h)$$

One can show that this is equivalent to the existence of translation invariant vector fields $D_1,\dots,D_N$ on X such that

$$\Delta_\nu = \sum_{h_1+2h_2+\dots+\nu h_\nu > 0} (h_1!\dots h_\nu!)^{-1} D_1^{h_1}\dots D_\nu^{h_\nu},$$

or formally

$$e^{\sum_{j=1} D_j\varepsilon^j} \equiv \sum_{k=0} \Delta_k \varepsilon^k \pmod{\varepsilon^{N+1}}.$$

We apply this to criterion (2.4). Note that V is defined by the vanishing of the 3×3 minors f_ν, $\nu \in (\mathbb{Z}^g/2\mathbb{Z}^g)^3$, of $(\vec{\theta}_2\ \ \Delta_1\vec{\theta}_2\ \ \Delta_2\vec{\theta}_2)$ at some point. If we assume that this point is the origin and that $Y =$

$\mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon^3) \to (X,0)$ is given by D_1, D_2 one finds (using the fact that the rank of $(\vec{\theta}_2\ (\partial_i\partial_j\vec{\theta}_2)_{i,j})$ equals $g(g+1)/2+1$ at $(\tau,0)$; $\partial_i = \partial/\partial z_i$)) that $(V)_2 = Y$. Then, as Welters noticed, one has

$$(V)_3 = \mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon^4) \iff \exists\ D_3 \text{ such that } (\tfrac{1}{3}D_1^3 + D_1D_2 + D_3)f_\nu = 0 \ (\text{all } \nu)$$

$$\iff \mathrm{rank}((1\ \ D_1^2\ \ D_1^4 + 3D_2^2 - 3D_1D_3)\vec{\theta}_2)(\tau,0) \le 2.$$

Without changing $\mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon^4) \to (X,0)$ one may effect the change $D_1 \to aD_1$, $D_2 \to a^2D_2 + bD_1$, $D_3 \to a^3D_3 + a^2bD_2 + cD_1$, $a \neq 0, b, c$, hence we can rewrite this as

$$((D_1^4 - D_1D_3 + \tfrac{3}{4}D_2^2 + d)\vec{\theta}_2)(\tau,0) = 0 \tag{8}$$

This is the K-P equation. By (2.3) the theta functions $\Theta_2[\sigma]$ of a Jacobian yield solutions. (Usually, the K-P equation is written $u_{yy} + (u_t + u_{xxx} + uu_x)_x = 0$. It is satisfied on a Jacobian by $u = D_1^2 \log\theta(z + xa_1 + ya_2 + ta_3) + c$ for some $a_1, a_2, a_3 \in \mathbb{C}^g$, $c \in \mathbb{C}$, see [15]. Dubrovin formulated the equivalent form (8).) That theta functions yield solutions was noticed by Krichever, who arrived at it in a completely different way. Novikov conjectured then that this should characterize Jacobians :

(2.6) Novikov's Conjecture. An indecomposable principally polarized abelian variety X is a Jacobian if and only if there exist constant vector fields D_1, D_2, D_3 on X and a constant d such that

$$((D_1^4 - D_1D_3 + \tfrac{3}{4}D_2^2 + d)\vec{\theta}_2)(\tau,0) = 0. \tag{9}$$

Dubrovin proved in [2] that the locus of $[X]$ in A_g for which (9) holds for some D_1, D_2, D_3 and d contains the Jacobian locus as an irreducible component.

(2.7) Soon after a weaker version of (2.4) had appeared Arbarello and De Concini realized that one does not need the positive dimensionality of V, but only the fact that $\mathcal{O}_{V,0}$ contains an artinian subscheme of sufficiently big length, i.e. the condition is that there exist constant vector fields $D_1,\ldots,D_M$ for some big M such that

$$e^{\sum_{j=1}^{M} D_j\varepsilon^j}(\vec{\theta}_2\wedge\Delta_1\vec{\theta}_2\wedge\Delta_2\vec{\theta}_2)\equiv 0 \pmod{\varepsilon^M} \quad \text{at} \quad (\tau,0).$$

In this way they were the first to write down equations that characterize the Jacobian locus, see [1]. Using the version of (2.4) given here one can take $M=6^g g!+1$.

Recently, Shiota showed that if one makes a minor technical assumption on X then Novikov's conjecture is true, see section 4.

APPROACH 3 : THE GEOMETRY OF THE MODULI SPACE.

The approach here, worked out in joint work with van Geemen [6], is based on the observation that under Φ and Φ_X both the moduli space $A_g(2,4)$ and the Kummer variety of X are mapped to the same projective space, so that we can compare their positions in this space. It was motivated by the special case $g=2$ studied in [7] and a paper of Frobenius dealing with $g=3$,[4].

(3.1) We first look at the tangent space to $\Phi(A_g(2,4))$ at $\Phi([X])$. i.e. we look at the hyperplanes

$$\sum_\sigma \alpha_\sigma\Theta_2[\sigma](\tau,0)=0 \qquad (X=X_\tau) \qquad (10)$$

satisfying

$$\frac{\partial}{\partial\tau_{ij}}\Big(\sum_\sigma \alpha_\sigma\Theta_2[\sigma]\Big)(\tau,0)=0 \qquad \text{for all } i,j.$$

By applying the Heat Equations this is transformed into

$$\Big(\sum_\sigma \alpha_\sigma\frac{\partial^2}{\partial z_i\partial z_j}\Theta_2[\sigma]\Big)(\tau,0)=0. \qquad (11)$$

So let us look at the sections of $\Gamma(X,L_X^{\otimes 2})$ satisfying (10) and (11), i.e. define

$$\Gamma_{00}(X,L_X^{\otimes 2})=\{\, s\in\Gamma(X,L_X^{\otimes 2}) \;:\; m_0(s)\geq 4\,\}\,.$$

with m_0 the multiplicity of a section at zero. Note that for $s\neq 0$

$m_o(s)$ is even). If X is indecomposable, then

$$\mathrm{rk}\ (\ \vec{\theta}_2\ \ \frac{\partial^2}{\partial z_1 \partial z_1}\vec{\theta}_2, \ldots, \frac{\partial^2}{\partial z_g \partial z_g}\vec{\theta}_2\ )$$

in $(\tau,0)$ equals $\frac{1}{2}g(g+1)+1$, so the codimension of Γ_{oo} in $\Gamma(X,L_X^{\otimes 2})$ equals $\frac{1}{2}g(g+1)+1$. If $\Phi(A_g(2,4))$ is non-singular at $\Phi([X])$ and if $I_g \subset \mathbb{C}[\ldots,X_\sigma,\ldots]$ is the ideal of $\Phi(A_g(2,4))$, then $\Gamma_{oo}(X,L_X^{\otimes 2})$ equals

$$\{\ \sum \frac{\partial f}{\partial X_\sigma}(\ldots,\Theta_2[\sigma](\tau,0),\ldots)\ \Theta_2[\sigma](\tau,z)\ :\ f \in I_g\ \}\ .$$

(3.2) As an example we take g=3. The theory of theta functions gives us a relation

$$\theta\begin{bmatrix}000\\000\end{bmatrix}\theta\begin{bmatrix}000\\100\end{bmatrix}\theta\begin{bmatrix}000\\010\end{bmatrix}\theta\begin{bmatrix}000\\110\end{bmatrix} - \theta\begin{bmatrix}001\\000\end{bmatrix}\theta\begin{bmatrix}001\\100\end{bmatrix}\theta\begin{bmatrix}001\\010\end{bmatrix}\theta\begin{bmatrix}001\\110\end{bmatrix} + $$
$$- \theta\begin{bmatrix}000\\001\end{bmatrix}\theta\begin{bmatrix}000\\101\end{bmatrix}\theta\begin{bmatrix}000\\011\end{bmatrix}\theta\begin{bmatrix}000\\111\end{bmatrix} = 0$$

between the $\theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(\tau,0)$. We write this as $r_1 - r_2 - r_3 = 0$. This implies the relation

$$r_1^4 + r_2^4 + r_3^4 - 2r_1^2r_2^2 - 2r_1^2r_3^2 - 2r_2^2r_3^2 = 0$$

between the squares of the even thetas. Using (1) this gives an equation

$$F(\ldots,\Theta_2[\sigma](\tau,0),\ldots) = 0$$

of degree 16 defining a hypersurface in $\mathbb{P}^7$. Hence

$$\phi = \sum \frac{\partial F}{\partial X_\sigma}(\ldots,\Theta_2[\sigma](\tau,0),\ldots)\ \Theta_2[\sigma](\tau,z)$$

belongs to Γ_{oo} and one can check that for indecomposable X_τ it is non-zero. It generates Γ_{oo}. In fact, when expressed in the theta squares this is the function studied by Frobenius in [4].

The first question about Γ_{oo} is its zero locus. Define

$$F_X = \{\ x \in X\ :\ s(x) = 0 \ \text{ for all }\ s \in \Gamma_{oo}(X,L_X^{\otimes 2})\ \}\ .$$

(3.3) <u>Proposition</u>. If $X = \mathrm{Jac}(C)$ then $F_X \supseteq \{(x-y) \in \mathrm{Jac}(C) : x,y \in C\}$.

Proof. Use (2.1) and put $a=b$, $c=d$ there. One finds a relation

$$\vec{\theta}_2(a-b) = \lambda\vec{\theta}_2(0) + \sum \mu_{ij}\partial_i\partial_j\vec{\theta}_2(0).$$

(3.4) For a Jacobian one can use the geometry of C to construct elements of Γ_{oo}. Let $|2\Theta|_{oo} = \{ D \in |2\Theta| : m_o(D) \geq 4 \}$. If $x \in \mathrm{Sing}\,\Theta$ then $\Theta_x \cup \Theta_{-x} \in |2\Theta|_{oo}$. Define

$$\Theta_o = \{ \alpha \in \mathrm{Jac}^{g-1}(C) : h^o(\alpha) > 0 \}$$

$$\mathrm{Sing}\ \Theta_o = \{ \alpha \in \Theta_o : h^o(\alpha) > 1 \}$$

and define for $\alpha \in \mathrm{Jac}^{g-1}(C)$:

$$\Theta_\alpha = \{x \in \mathrm{Jac}(C) : \alpha - x \in \Theta_o \}.$$

Then obviously,

$$F_X \subseteq \bigcap_{\alpha \in \mathrm{Sing}\,\Theta} (\Theta_\alpha \cup \Theta_{\kappa-\alpha}). \qquad (\kappa\text{: canonical divisor})$$

If C is hyperelliptic then

$$\mathrm{Sing}\ \Theta_o = g^1_2 + W^o_{g-3},$$

hence if $f \in F_X$ one has $\pm f + g^1_2 + W^o_{g-3} \subset W^o_{g-1}$, so $\pm f + g^1_2 \in W^o_2$ and this implies $f = (a-b)$ for some $a,b \in C$. So for hyperelliptic C one finds

$$F_X = \{ (x-y) \in \mathrm{Jac}(C) : x,y \in C \}.$$

By semi-continuity it follows that for general X $\dim F_X \leq 2$ and for a general Jacobian $\dim F_X = 2$. We conjectured in [6] that for every C

$$F_{\mathrm{Jac}(C)} = \{ (x-y) \in \mathrm{Jac}(C) : x,y \in C \}$$

and provided a lot of evidence for it. Independently, the conjecture was formulated by Mumford [15] (in a dual form) and by Gunning [10]. This conjecture has now been proved by Welters. However, there is one exceptional case, namely $g=4$, where

$$F_{\mathrm{Jac}(C)} = \{ (x-y) \in \mathrm{Jac}(C) : x,y \in C \} \cup \{ \pm(f-f') \},$$

with f, f' the two g^1_3 's on C, see [20] .

(3.5) <u>Conjecture</u>. Let X be a principally polarized abelian variety of dimension $g \geq 2$. Then X is a Jacobian if and only if $\dim F_X \geq 2$.

An infinitesimal form of this conjecture is related to the Novikov conjecture. Note that

$$x \in F_X \iff \exists\, \lambda, \mu_{ij} \in \mathbb{C} \text{ such that } \vec{\theta}_2(x) = \lambda \vec{\theta}_2(0) + \sum \mu_{ij} \partial_i \partial_j \vec{\theta}_2(0)$$

Now, if $Y = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^{N+1})$ is contained in X at 0 via the local homomorphism $\mathcal{O}_{X,0} \to \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^{N+1})$, $f \to \sum_{i=0}^{N} \Delta_i(f)\varepsilon^i$, we have

$$Y \subset F_X \iff \exists\, \lambda, \mu_{ij} \in \mathbb{C}[\varepsilon] \text{ such that}$$
$$\sum_{k=0}^{N} \Delta_k \vec{\theta}_2(\tau,0)\varepsilon^k = \lambda \vec{\theta}_2(\tau,0) + \sum \mu_{ij} \partial_i \partial_j \vec{\theta}_2(\tau,0).$$

Working out the condition for $N=4$ gives

$$\left(\left(\tfrac{1}{24}D_1^4 + \tfrac{1}{2}D_2^2 - D_1 D_3\right)\vec{\theta}_2\right)(\tau,0) = d\,\vec{\theta}\,(\tau,0) + \sum e_{ij} \partial_i \partial_j \vec{\theta}_2(\tau,0),$$

where e_{ij} is the coefficient of ε^4 in μ_{ij}. If $\sum e_{ij}\partial_i\partial_j \vec{\theta}_2(\tau,0)$ is a multiple of $D_1^2\vec{\theta}_2(\tau,0)$, then we can change coordinates such that this relation becomes the K-P equation.

Gunning has studied the generalizations of Fay's identity. This leads to interesting analogues of (3.3) involving higher derivatives at zero, cf [11] .

Instead of intersecting the Kummer variety with the tangent space of the moduli space we can also intersect the Kummer variety with the moduli space itself. As an analogue to (3.3) we find

(3.6) <u>Proposition</u>. If $X = \operatorname{Jac}(C)$ then $\Phi_X(X) \cap \Phi(A_g(2,4))$ contains $\Phi_X(\{\tfrac{1}{4}(x-y) : x, y \in C\})$.

Here $\frac{1}{4}$ means the inverse image under multiplication by 4.

Proof. A divisor class $\underline{a}$ with $2\underline{a} = x+y$ defines a (2:1)-covering $\pi: \tilde{C} \to C$. The Prym variety $P = \ker\{ Nm: Jac(\tilde{C}) \to Jac(C)\}$ is a principally polarized abelian variety of dimension g for general $x,y \in C$. There exist theta structures on P and X such that

$$\Phi(\tfrac{1}{4}(x-y)) = \Phi([P]),$$

cf. [13],p.340, where $\frac{1}{4}(x-y) \in Jac(C)$ is such that $2(\frac{1}{4}(x-y)) = \frac{1}{2}(x-\underline{a})$. By symmetry it then follows that all of $\Phi_X(\frac{1}{4}(x-y))$ lies in $\Phi(\overline{A}_g(2,4))$ for general x,y, hence for all x,y.

We made two conjectures in relation to this. First, for a Jacobian we conjectured that

$$\Phi_X(X) \cap \Phi(A_g(2,4)) = \Phi_X(\{ \alpha \in X : 4\alpha = x-y,\ x,y \in C \})$$

and we proved this for $g=3$. Secondly, we hope that this characterizes Jacobians :

(3.7) Conjecture. Let X be an indecomposable principally polarized abelian variety of dimension $g \geq 2$. Then X is a Jacobian if and only if $\dim \Phi_X(X) \cap \Phi(\overline{A}_g(2,4)) \geq 2$.

(3.8) The preceding sections suggest to look at the morphism

$$\begin{aligned} \Xi: \Gamma_g(2,4) \ltimes \mathbb{Z}^{2g} \backslash \mathbb{H}^g \times \mathbb{C}^g = U_g(2,4) &\to \mathbb{P}^N \\ (\tau,z) &\to (\ldots,\vec{\theta}_2[\sigma](\tau,z),\ldots) \end{aligned}$$

Is it everywhere of maximal rank ? Since the Kummer variety of an indecomposable X is singular at the images of the points of order 2 of X the rank is certainly not maximal at those (τ,z) for which $2z \in \mathbb{Z}^g + \tau\mathbb{Z}^g$. Using the Heat Equations the question becomes whether the rank of

$$(\partial_i\partial_j \vec{\theta}_2 \quad 1 \leq i \leq j \leq g \quad \partial_k \vec{\theta}_2 \quad 1 \leq k \leq g \quad \vec{\theta}_2)$$

at (τ,z) is maximal.

Suppose that $2^g \geq \frac{1}{2}g(g+1)+g+1$, i.e. $g \geq 4$ and that $X = X_\tau$ is indecomposable. If there exist a relation

$$((\sum \alpha_{ij}\partial_i\partial_j + \sum \beta_k\partial_k + \gamma)\vec{\theta}_2)(\tau,z) = 0$$

for z such that $2z \notin \mathbb{Z}^g + \tau\mathbb{Z}^g$ then $\phi_X(X)$ possesses a flex at $\Phi_X(z)$.

A Jacobian is known to possess a lot of such flexes : if $X = \mathrm{Jac}(C)$, then applying (2.3) with $z_1=z_2=z_3=p$ we find that all points of $\{\frac{1}{2}(x-y) \in X : x,y \in C\}$ are flex points. Hence the rank of Ξ is not maximal at these points.

(3.9) Question. In view of (2.4) we can ask whether for an indecomposable Jacobian $X = X_\tau$ with $g \geq 4$ the only points (τ,z) where Ξ is not of maximal rank are those corresponding to $\{\frac{1}{2}(x-y) : x,y \in C\}$ and whether one could use this to characterize Jacobians.

APPROACH 4 : RINGS OF DIFFERENTIAL OPERATORS

As mentioned above Shiota has settled Novikov's conjecture up to a technical assumption.

(4.1) Theorem. (Shiota [17]) An indecomposable principally polarized abelian variety X of dimension g is the Jacobian of a complete smooth non-singular curve C over $\mathbb{C}$ of genus g if and only if

i) the vector $\vec{\theta}_2(\tau,z)$ satisfies the K-P equation (8) for some D_1,D_2,D_3 and d, and

ii) no translate of the theta divisor of X contains an abelian subvariety of X which is tangent to $D_1(0)$.

Shiota's approach incorporates ideas of Mulase and is based on Krichever's dictionary. Let D be the non-commutative ring which as an additive group equals $\mathbb{C}[[x]][\partial]$ with $\partial = \frac{d}{dx}$ and with multiplication such that

$$\partial \cdot f = f\partial + f' \qquad \text{for} \quad f \in \mathbb{C}[[x]][\partial]. \tag{12}$$

If R is a commutative subring of D containing $\mathbb{C}$ and two elements A,B with $A = \partial^n + \dots$ ($\dots$ = lower order terms), $B = \partial^m + \dots$ with $(n,m) = 1$, then any element of R can be written as $C = \alpha\partial^r + \dots$ with $\alpha \in \mathbb{C}$, $r \in \mathbb{Z}_{\geq 0}$ (Proof: work out the commutator $[A,C]$). If $R_n = \{ C \in R : C = \alpha\partial^r + \dots \text{ with } r \leq n \}$ one has $\dim R_n/R_{n-1} \leq 1$ and $=1$ for $n >> 0$.

(4.2) Theorem. (Krichever) There is a natural bijection between the following two sets of data:

1) C an irreducible curve, P a smooth point of C, a tangent vector at P and a torsion free rank 1 $\mathcal{O}_C$-module F with $h^0(F) = h^1(F) = 1$.

2) $R \subset D$ a commutative subring containing $\mathbb{C}$ and two elements A, B as above.

Let us sketch how to go from 1) to 2). Choose a neighbourhood U of P such that the local coordinate z at P is a unit on $U-P$. Let x be the standard coordinate on $\mathbb{C}$. We now glue $F \otimes \mathcal{O}_{\mathbb{C}}$ on $U \times \mathbb{C}$ and $F \otimes \mathcal{O}_{\mathbb{C}}$ on $(C-P) \times \mathbb{C}$ by multiplication with $e^{x/z}$. This defines a sheaf F^* on $C \times \mathbb{C}$. If V is a suitably chosen neighbourhood of $0 \in \mathbb{C}$ then $H^i(C \times V, F^*) = 0$ $i=0,1$. Define now

$$\nabla : F^*(\ell P) \to F^*((\ell+1)P)$$

by taking $\frac{d}{dx}$ on $C-P$ and $\frac{1}{z} + \frac{d}{dx}$ on U. A non-zero section s_o $H^o(F^*(P))$ generates $H^o(F^*(P))$ as a $H^o(V,\mathcal{O}_{\mathbb{C}})$-module. We normalize s_o such that $s_o = 1 + O(z)$ at $p \times V$, i.e. $\frac{d}{dx}s_o = (z^{-1} + O(z))s_o$. Put $s_n = \nabla^n s$. The sections $s_o,\dots,s_n$ generate $H^o(F^*(n+1)P)$.

If $a \in \Gamma(C-P,\mathcal{O}_C)$ then $as_o \in H^o(F^*(nP))$, hence

$$as_o = \sum_{i=0}^{n-1} a_i(x)\nabla^i s_o.$$

This gives us a map

$$\begin{array}{ccc} \Gamma(C-P,\mathcal{O}_C) & \to & D \\ a & \to & \sum_{i=0}^{n-1} a_i(x)\partial^i . \end{array}$$

The image is a commutative subring R of D.

(4.3) In order to obtain Jacobians one observes that F defines a point of $\mathrm{Jac}(C)$. So let us deform F. Choose variables $t_1,\dots,t_N$ and consider instead of $F\otimes\mathcal{O}_{\mathbb{C}}$ now $F\otimes\mathcal{O}_{\mathbb{C}^N}$ on $U\times\mathbb{C}^N$ and $(C-P)\times\mathbb{C}^N$ and glue now by $\exp(\sum_{j=1}^{N} t_j z^{-j})$. We introduce formally a variable x by replacing t_1 by t_1+x. This now gives us F^* as above. Define ∇ as above and define

$$\nabla_n : F^* \to F^*(nP)$$

by taking $\frac{\partial}{\partial t_n}$ on $C-P$. We choose a normalized s_o again as above. We now obtain

$$\Gamma(C-P,\mathcal{O}_C) \to D$$

and the image is a commutative subring R_t depending on $t=(t_1,\dots,t_n)$. The question arises : how does R_t deform with t ? If T denotes the the tangent space we get a map

$$\begin{array}{ccc} \Phi: T_t\mathbb{C}^N & \to & D \\ \frac{\partial}{\partial t_n} & \to & B_n(t) \end{array}$$

where $B_n(t)$ is defined as follows. By the normalization $\frac{\partial}{\partial t_n} s_o = (z^{-n}+O(z))s_o$, so $\frac{\partial}{\partial t_n} s_o = \sum b_i \nabla^i s_o$. We put $B_n(t) = \sum b_i \partial^i$.

We need some notation. Let Ψ be the non-commutative $\mathbb{C}$-algebra whose elements are formal Laurent series in $(\frac{d}{dx})^{-1}$ with coefficients from $\mathbb{C}[[x]]\otimes\mathcal{O}_{\mathbb{C}^N}$. The multiplicative structure is defined by extending the rule (12). Let

$$\Psi^- = \{P\in\Psi : \ \mathrm{ord}\, P \le -1\} .$$

So the elements of Ψ^- are expressions $\sum_{j=-\infty}^{-1} a_i(x)\partial^i$, where we suppress the dependence of t in the notation. By extending the map $\Gamma(C-P,\mathcal{O}_C) \to D$ to $Q(\Gamma(C-P,\mathcal{O}_C) \to \Psi$ (Q : quotient field) we see that $1/z$ corresponds to an element of $\frac{d}{dx} + \Psi^-$ which we call L. From the normalization we obtain $B_n = (L^n)_+$, where $(\ )_+$ means taking the differential operator part (non-negative powers of ∂).

The dependence of R_t on t is now expressed by the following deformation equations for $L \in \frac{d}{dx} + \Psi^-$:

$$(\frac{\partial}{\partial t_n}) L = [(L^n)_+, L] \qquad n=1,\ldots,N$$

Take now infinitely many variables $t_1, t_2, \ldots$, i.e. $t \in \mathbb{C}^\infty = \varinjlim \mathbb{C}^N$ and consider the equations for $L \in \frac{d}{dx} + \Psi^-$:

$$(\frac{\partial}{\partial t_n}) L = [(L^n)_+, L] \qquad n=1,2,\ldots$$

This set of equations is called the K-P hierarchy. We do not explain here the translation of solutions to this hierarchy of equations into differential equations satisfied by theta functions, but we refer to Shiota's paper and the references there.

If L is a solution to the K-P hierarchy then consider

$$dL : T_t\mathbb{C} \to \Psi^- \qquad \sum c_n \frac{\partial}{\partial t_n} \to [\sum c_n B_n, L] ,$$

the tangent map of the map $t \to L(t)$ at t. We call L a finite dimensional solution if dL is of finite rank. Shiota considers for a finite dimensional L

$$R_L = \Phi(\ker dL) \oplus \mathbb{C}$$

with $\Phi: T_o\mathbb{C} \to D$, $\frac{\partial}{\partial t_n} \to B_n$. He proves

$$R_L = \{P \in D : [P,L] = 0\} .$$

and that R_L is a maximal commutative subring of D if $R_L \neq \mathbb{C}$. Thus a finite dimensional solution to the K-P hierarchy yields a curve by (4.2). Moreover, it turns out that $\ker dL$ can be identified with the tangent space of the Jacobian of this curve at a certain point.

Basically, this is the way Mulase arrived at his theorem which states that the whole K-P-hierarchy characterizes Jacobians. Both Mulase and Shiota then noticed that in fact finitely many equations from this hierarchy suffice,arriving thus at a theorem very similar to the result of Arbarello and De Concini. Shiota then continued by showingthat under condition 2) of (4.1) one can extend a solution to the K-P-equation (the first of the K-P-hierarchy) to a solution of the whole hierarchy, see [17]. Namikawa informed me that Mulase now also obtained such a reduction.

A FINAL REMARK.

Our summaryof recent attacks on the Schottky problem is not intended to be complete. One of the approaches that should be mentioned also is the approach that uses the reducubility of $\Theta \cap \Theta_a$. It is closely related to approach 2 and was suggested by Mumford in [14]. For a Jacobian X with theta divisor Θ one has : if $x \in X$ and $x \neq 0$ then there exist u,v in X with $\{0,x\} \cap \{u,v\} = \emptyset$ such that $\Theta \cap \Theta_x \subset \Theta_u \cup \Theta_v$ if and only if x belongs to $\{(a-b) \in X = \mathrm{Jac}(C) : a,b \in C\}$. (Note that one implication follows from (2.1) by using $X \to \mathbb{P}^N$, $x \to \Theta_x \cup \Theta_{-x} \in |2\Theta|$.) Welters proved the following theorem : Let X be a complex principally polarized abelian variety of dimension g. Assume 1) dim $\mathrm{Sing}\Theta \leq g-4$, 2) there exist a one-dimensional subset $Y \subset X$ such that for generic $y \in Y$ one has : $\Theta \cap \Theta_y \subset \Theta_u \cap \Theta_v$ for some $u,v \in X$ with $\{0,y\} \cap \{u,v\} = \emptyset$. Then X is the polarized Jacobian of a non-hyper-elliptic curve, see [21]

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