4-MANIFOLDS WITH INDEFINITE INTERSECTION FORM

S.K. Donaldson
All Souls College
Oxford, England

In writing up this lecture I shall not concentrate so much on describing problems of 4-manifold topology; instead I shall explain how a simple topological construction has applications in two different directions. First I will recall that, just as bundles over a single space have homotopy invariants, so do families of bundles, and that these define corresponding invariants in families of connections. Next I will sketch the way in which such a topological invariant, when endowed with a geometric realisation, becomes important for studying holomorphic bundles over algebraic varieties. Last I will indicate how this same homotopy invariant of families of connections, combined with arguments involving moduli spaces of self-dual connections over a Riemannian 4-manifold, gives restrictions on the possible homotopy types of smooth 4-manifolds and I will speculate on possible future progress in this area.

Topology of bundles.

This is standard material that may be found in [2] for example. Consider a fixed manifold X and a family of bundles over X parametrised by some auxiliary space T, so we have a bundle P over the product $X \times T$ with structure group G (compact and connected, say). Take first the case when T is a point so we have a single bundle over X, determined up to equivalence by a homotopy class of maps from X to BG. This may be non-trivial, detected for example by characteristic classes in the cohomology of X. If we choose a connection A on the bundle the real characteristic classes can be represented by explicit differential forms built from the curvature of the connection. Equally if D is an elliptic differential operator over X then using a connection it may be extended to act on objects (functions, forms, spinors etc.) twisted by a vector bundle associated to P. This has an integer valued index:

$$\text{index}\,(D_A) = \dim\ker D_A - \dim\operatorname{coker} D_A$$

which is a rigid invariant of the bundle, independent of the connection. So these are two ways in which the underlying homotopy may be represented geometrically, by curvature and by differential operators. The Chern-Weil and Atiyah-Singer theorems then give formulae relating the three.

In the same way for a general family parametrised by T the bundle P is classified by a homotopy class of maps from T to the mapping space $\mathrm{Maps}(X,BG)$, and at the other extreme from the case T = point we have a universal family parametrised by this mapping space. Again we may always choose a connection over $X \times T$, which we may think of as a family of connections parametrised by T, and conversely any family of equivalence classes of connections on some bundle essentially arises in this way. (This is precisely true if we work with based maps and bundles, removing base points gives small technical differences which can safely be ignored here). Equivalently we have the infinite dimensional space $\mathcal{B}$ of all equivalence classes of connections obtained by dividing the affine space of connections $\mathcal{A}$ by the bundle automorphism group $\mathcal{G}$. $\mathcal{B}$ has the homotopy type of $\mathrm{Maps}(X,BG)$.

Again we may construct topological invariants of such families of bundles. In cohomology we can use the characteristic classes again. There is a slant product:

$$H^{p+q}(X \times T) \otimes H_q(X) \longrightarrow H^p(T)$$

so that characteristic classes of bundles over $X \times T$ contracted with, or integrated over, homology classes in the base manifold X yield cohomology classes in families of connections. In particular if G is, say, a unitary group we obtain in this way a map:

$$\mu : H_2(X) \longrightarrow H^2(T)$$
$$\mu(\alpha) = c_2(P)/\alpha$$

(A simpler example is to take the Jacobian parametrising complex line bundles over a Riemann surface. Operating in the same way with the first Chern class gives the usual correspondence between the 1-dimensional homology of the surface and the cohomology of the Jacobian). We can do the corresponding thing in K-theory and realise the resulting elements in the K-theory of T by using differential operators again. For example if the base manifold X is the 2-sphere then a unitary bundle over $S^2 \times T$ defines an element of $K(S^2 \times T)$ which maps to $K(T)$ by the inverse of the Bott periodicity map. If we take the Dirac operator D over S^2 then a family of connections gives a family of Dirac operators $\{D_t\}$ parametrised by T and, after suitable stabilisation the index of this family [2] defines the required class:

$$\text{index } D_t = [\text{Ker } D_t] - [\text{coker } D_t] \in K(T)$$

Of course we obtain other classes in this way and the Atiyah-Singer index theorem for families gives formulae relating these to the underlying homotopy. In particular we may understand our class above from either point of view via the formula:

$$c_1(\text{index } D_t) = \mu(\text{fundamental class of } S^2) .$$

Stable bundles on algebraic curves and surfaces.

Here I only want to say enough to fit into our overall theme; more details and references may be found in [4], but I learnt the point of view we are adopting now from lectures of Quillen.

There is a general algebraic theory dealing with the action of a complex reductive group $G^{\mathbb{C}}$ on a vector space $\mathbb{C}^{n+1}$ via a linear representation. Equivalently we may take the induced action on $\mathbb{CP}^n$ and the hyperplane bundle H over it. In that theory there is a definition of a "stable" point. Now suppose that $\mathbb{C}^{n+1}$ has a fixed Hermitian metric, inducing metrics on H and on $\mathbb{CP}^n$, and picking out a maximal compact subgroup $G \subset G^{\mathbb{C}}$ whose action preserves these metrics. There is a general theory dealing with the metrical properties of these actions and relating them to the purely complex algebraic properties. Roughly speaking if we restrict to the stable points then a transversal to the $G^{\mathbb{C}}$-action on $\mathbb{CP}^n$ is induced by taking the points in $\mathbb{C}^{n+1}$, or equivalently H^{-1}, which minimise the norm in their $G^{\mathbb{C}}$ orbits. The corresponding variational equations cutting out the transversal take a simple form and are the zeros of a map:

$$m : \mathbb{CP}^n \longrightarrow J^* \qquad [7] , [8] .$$

Large parts of this theory can be developed abstractly from general properties of Lie groups and the fact that the curvature form of the Hermitian line bundle H gives the Kähler symplectic form on $\mathbb{CP}^n$.

Atiyah and Bott [1] observed that the theory of holomorphic structures on a vector bundle E over an algebraic curve C could be cast in the same form, except with an infinite dimensional affine space in place of a projective space. For a holomorphic structure on E is given by a $\bar{\partial}$-operator and these are parametrised by a complex affine space A . The infinite dimensional group $G^{\mathbb{C}}$ of complex linear automorphisms of E acts by conjugation and the quotient set is by definition the set of equivalence classes of holomorphic (or algebraic) bundles, topologically equivalent to E . Independently, and from another point of view, stability of algebraic bundles had been defined in algebraic geometry; the definition uses the notion of the degree of a bundle - the integer obtained by evaluating the first Chern class on the fundamental cycle.

If now E has a fixed Hermitian metric then a $\bar{\partial}$-operator induces a unique unitary connection. Regarded as connections the symmetry group of the affine space A is reduced to the subgroup $G \subset G^{\mathbb{C}}$ of unitary automorphisms, and this subgroup preserves the natural metric form on the space of connections A derived from integration over C . We would have all the ingredients for the abstract theory described above if we had a Hermitian line bundle L over A with curvature generating this metric form, and acted on by $G^{\mathbb{C}}$.

It was explained above that over a space of connections we obtain virtual bundles from the associated elliptic operators. In particular we can take the Dirac operator over the algebraic curve C , which is the same as the $\bar{\partial}$-operator after tensoring with a square root $K_C^{1/2}$ of the canonical bundle, so the kernel and cokernel form the usual sheaf cohomology. Moreover we get a genuine line bundle if we take the highest exterior power or determinant of the relevant vector spaces. Thus we get a complex line bundle L_C over A :

$$\mathcal{L}_C = \chi(E \otimes K_C^{1/2}) = \det H^0(E \otimes K_C^{1/2}) \otimes \det H^1(E \otimes K_C^{1/2})^{-1}$$

acted upon by $G^{\mathbb{C}}$, and realising via the first Chern class the cohomology class obtained under our map μ from the fundamental cycle of the curve C, as in Section I.

Now Quillen has defined Hermitian metrics [9] on such determinant line bundles and computed the associated curvature to be precisely the metric form above. Thus all the ingredients for applying the general theory are present - the map m cutting out a transversal to the stable orbits is given by the curvature of a connection and the preferred points, minimising Quillens analytic torsion norm, are given by the projectively flat unitary connections.

We can study algebraic bundles over any projective variety; in particular over an algebraic surface X. Now the definition of stability requires the choice of a polarisation - the first chern class of an ample line bundle L over X. This means that the degree of a bundle is defined, in the normal way. We can represent this polarising class by a Kähler form ω, the curvature of some metric on L. Then the same theory holds; we do not find flat connections on stable bundles but connections whose curvature is orthogonal to the Kähler metric at each point. The relation with metrics on cohomology is less well established but the relevant line bundle should probably be of a form such as:

$$\mathcal{L}_X = \chi(E \otimes K_X^{1/2} \otimes L^{1/2}) \otimes \chi(E \otimes K_X^{1/2} \otimes L^{-1/2})^{-1} .$$

Suppose that L has a section s cutting out a curve $C \subset X$, we can think in the sense of currents of C as a degenerate form of a metric. There is an exact sequence:

$$0 \longrightarrow E \otimes K_X^{1/2} \otimes L^{1/2} \xrightarrow{s} E \otimes K_X^{1/2} \otimes L^{1/2} \longrightarrow E|_C \otimes K_C \longrightarrow 0$$

whose long exact sequence in cohomology gives an isomorphism $\mathcal{L}_X \cong \mathcal{L}_C$; moreover one can compute formulae for the difference in norms, one defined relative to C and one to the metric ω on X , compared under this isomorphism, with an explicit difference term given by integrals involving Chern-Weil polynomials in the curvature. These are useful for throwing problems back to the curve from the surface.

All this should probably be understood in the following way. Topologically we have a map μ from $H_2(X)$ to the cohomology of any family of connections over X . If we wish to define stable bundles then we need a polarisation $[\omega]$ of X which via this map μ and Poincaré Duality induces a corresponding "polarising class" in the infinite dimensional space of connections. We may represent the original class in various explicit ways; by a metric or by a line bundle or by a curve, and to each such representation on X we get a corresponding representation in the space of connections. The usual formulae for homologies between the representations on X go over to corresponding formulae on the connections which we can use in our arguments involving stable bundles. But the existence of these formulae underlines that the basic correspondence between the geometry of the base manifold and its stable bundles is generated by our simple construction of Section I.

Connections over smooth 4-manifolds.

Self dual connections are solutions to a differential equation which is special to 4-dimensions. On an oriented Riemannian 4-manifold the 2-forms decompose into the ± 1 eigenspaces of the star operator; the same is true for bundle valued forms, and a connection is self-dual if its curvature lies in the $+1$ eigenspace. If the manifold is an algebraic surface with the standard orientation reversed these are the connections whose existence characterised stable bundles in the previous section. (For on a Kähler surface the self dual 2-forms are made up of the (0,2) and (2,0) forms and the span of the Kähler form). Correspondingly these solutions of differential equations in Riemannian geometry behave rather like objects in algebraic geometry; in particular the solutions, up to equivalence by bundle automorphisms, are parametrised by finite dimensional moduli spaces rather as the Jacobian parametrises line bundles over a Riemann surface. Moreover these moduli spaces have applications in differential topology.

At present there is no general theory of smooth 4-manifolds. A central problem is to understand the relationship between homotopy and differentiable structures and to quantify the gap between them. For simply connected 4-manifolds the homotopy type is easily understood - the sole invariant is the intersection form on the integral 2-dimensional homology. Likewise the classification of topological 4-manifolds up to homeomorphism has been established by Freedman [6], and is virtually the same as that up to homotopy. Now while there are many integral definite forms, and so corresponding topological 4-manifolds, it was proved by methods similar to those described below [5] that none of these arise from smooth manifolds beyond the obvious examples given by diagonalisable forms. The interesting remaining class of forms are the even (which corresponds to spin manifolds) indefinite forms which are all of the shape:

$$n\, E_8 + m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For smooth manifolds n must be even by Rohlin's Theorem and the simplest known example, beyond $S^2 \times S^2$ which has form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, is the smooth 4-manifold underlying a complex K3-surface, having 2 E_8's and 3 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$'s in the intersection form. By taking connected sums with $S^2 \times S^2$ one can always increase m so that the problem of realisation of these forms is to discover the minimal value of m for each given n. It is hoped that a proof that for positive n the value of m must be at least 3 (implying in particular that the K3 surface is smoothly indecomposable, hence genuinely the simplest "non-obvious" smooth 4-manifold) using the methods described below, will appear very shortly.

First a word on the formal structure of these proofs. We need some way of distinguishing the forms which are obviously realised when n is zero from the case when n is positive. The relevant property that emerges is that a direct sum $H_1 \oplus H_2 \oplus \ldots \oplus H_k$ of copies of the "hyperbolic" form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is distinguished by the fact that the symmetric power:

$$(H_1 \oplus \ldots \oplus H_k)^{k+1}$$

is identically zero mod 2 . For example when $k = 1$ this says that for any four integral elements $\alpha_1, \ldots \alpha_4$:

$$(\alpha_1 \cdot \alpha_2)(\alpha_3 \cdot \alpha_4) + (\alpha_1 \cdot \alpha_3)(\alpha_2 \cdot \alpha_4) + (\alpha_1 \cdot \alpha_4)(\alpha_2 \cdot \alpha_3) = 0 \mod 2$$

So our proofs are really to establish such identities when α_i are integral homology classes and $(\cdot)$ is the intersection pairing.

These identities are obtained by pairing two kinds of information and, since we are interested in the differences between homotopy and differentiable structures, it is probably important to stress the contrast in the ways that these arise. By definition our moduli space M of self dual connections on some bundle parametrises a family of connections and we have seen in the first section above that we can produce cohomology classes in such parameter spaces. Alternatively we can think of the moduli space as a subset of the infinite dimensional space $\mathcal{B}$ of all equivalence classes of connections, cut out by the non-linear differential equations giving the self duality condition. Since we regard the homotopy type of the base manifold X^4 as known we may regard the homotopy type of this infinite dimensional parameter space of connections as known. For example we have our map; defined in an elementary way:

$$\mu : H_2(X^4) \longrightarrow H^2(\mathcal{B}_X)$$

and in fact this generates a copy of the polynomial algebra on $H_2(X^4)$ within $H^*(\mathcal{B}_X)$.

Our moduli space M sits within this infinite dimensional space. At present we may regard this as largely unknown and mysterious, except for properties that can be understood by linearisation, for example the dimension of the space. What we do know is that the moduli space carries a fundamental class in homology; or rather, as we shall see, that it may be truncated, typically, to a manifold with boundary ∂M so we may assert:

$$< \phi , [\partial M] > = 0 \text{ for any } \phi \text{ in } H^*(\mathcal{B}_X)$$

To produce a suitable cohomology class ϕ we may use our map μ - this builds in the two dimensional homology that we wish to study; likewise we may produce more subtle cohomology classes coming, from our present point of view, from the index of the 4-dimensional Dirac operator on a spin 4-manifold. But all this is homotopy, the smooth structure and the difference between differentiable and topological manifolds enters by the existence of the relative homology class carried by the moduli space of solutions to the differential equation.

Here is an explicit example, directly relevant to the case when we study 4-manifolds with one negative eigenvalue in their intersection form. Take the complex projective plane with its standard orientation reversed; then we may study the self-dual connections via the stable holomorphic bundles as above, and in particular if we consider rank 2 bundles with $c_1 = 0$; $c_2 = -2$ then the appropriate moduli space has been described by Barth [3] as follows. To the original projective plane P we may associate the dual plane P^* , so points of one plane are lines in the other. The conic curves in P^* are parametrised by a copy of $\mathbb{CP}^5$; the non-singular conics form an open subset, the complement of a divisor which is naturally identified with the symmetric product $\mathrm{sym}^2(P)$ (since a singular conic is made up of two lines). According to Barth the moduli space of algebraic bundles may be identified with these non-singular conics, which we may obviously truncate by removing an open neighbourhood of $\mathrm{sym}^2(P)$ to get a manifold M with boundary ∂M made up; loosely speaking, of a circle bundle over $\mathrm{sym}^2(P)$ with fibre L say.

We can understand our map μ very easily in this example, and doing so explicitly will illustrate the general case. Let ℓ be a line in P (so representing a generator of $H_2(P)$). Then it follows essentially immediately from our discussion of the previous sections and the description by Barth of the "jumping lines" of a bundle that a representative for the cohomology class $\mu[\ell]$ is given by the hyperplane V_ℓ in $\mathbb{CP}^5$ consisting of conics through the point ℓ in P^*. Consider four general lines $\ell_1, \ell_2, \ell_3, \ell_4$ in P . The eight dimensional cohomology class $\mu(\ell_1)\cdot\mu(\ell_2)\cdot\mu(\ell_3)\cdot\mu(\ell_4)$ is represented by the projective line $V_{\ell_1} \cap V_{\ell_2} \cap V_{\ell_3} \cap V_{\ell_4}$. The intersection of this with our truncated moduli space M is a surface with boundary three copies of the loop L , corresponding to the three point-pairs:

$$((\ell_1 \cap \ell_2),(\ell_3 \cap \ell_4)),((\ell_1 \cap \ell_3),(\ell_2 \cap \ell_4)),((\ell_1 \cap \ell_4),(\ell_2 \cap \ell_3))$$

in P . If we proceed analogously on any (simply connected) 4-manifold with one negative part of the intersection form then we have a broadly similar moduli space - a non-compact manifold of real dimension 10 . If we consider a cup product $\mu(\alpha_1)\mu(\alpha_2)\mu(\alpha_3)\mu(\alpha_4)$ for any 4 surfaces α_i then we are led in the same way to consider a set of point pairs of the form:

$$((\alpha_i \cap \alpha_j),(\alpha_k \cap \alpha_l))$$

and the number of such pairs, modulo 2, is just the expression in terms of the intersection pairing given above. The key additional fact is that for a manifold with a spin structure (unlike $\mathbb{CP}^2$) the corresponding loop L is essential in the space of connections, detected by a mod 2 cohomology class w_1 , thus we argue in the manner above with the cohomology class $\phi_q = w_1 \cdot \mu(\alpha_1) \cdot \mu(\alpha_2) \cdot \mu(\alpha_3) \cdot \mu(\alpha_4)$.

Finally I will make two general remarks. Following Taubes [10] the structure of these boundaries to moduli spaces can be understood reasonably explicitly in terms of a number of "instantons" - connections concentrated near a finite set of points on the manifold. In the complex algebraic version we should probably think of these as being bundles obtained from deformations of ideal sheaves, rather in the way that the symmetric products of an algebraic curve map into the Jacobian. The ways that these instantons can be oriented relative to each other give the structure of the "link" L in the moduli space itself and this depends upon the values of the anti self-dual harmonic 2-forms at the points. The possibilities become rapidly more complicated as the number m of negative parts of the intersection form grows larger, and roughly speaking what distinguishes the cases $m = 0,1,2$ is that the codimension of the "special divisors", on which the forms are aligned in exceptional ways, is sufficiently high. It would seem to be possible that the behaviour of these harmonic forms (which of course globally reflect the cohomology, via Hodge Theory) contains differential topological information about the 4-manifold. In the complex case these anti self-dual forms are made up of the Kähler symplectic form and the holomorphic 2-forms and these are well known to carry a lot of information about the complex structure. Rather similarly the "periods" of the harmonic forms, the relation with the integral structure, also enter into the Riemannian theory via line bundles and Hodge Theory.

I have emphasised here that no global properties of these moduli spaces beyond existence are really used in these arguments, and indeed the number of explicit examples that are known is rather small. On the other hand we have seen that we may easily construct cohomology classes over these moduli spaces and that we have obtained information by pairing these with the relative homology class carried by the manifold. It seems that the moduli spaces should carry an absolute homology class with respect to cohomology with sufficiently small support, which can then be paired to give integer valued invariants. Moreover these should be independent of the Riemannian metric on the 4-manifold in the usual way that the homology class carried by the fibre of a map is a deformation invariant.

Of course there are many ways in which rigid integer valued invariants can be produced by analytic methods - integration of forms or indices of operators; but as I recalled in the first section these can all be understood entirely from homotopy, via the usual formulae. This is not obviously the case for our moduli spaces. For example if we take the case described above on the projective plane then we see that $(\mu[\ell])^5\,[M] = 1$, given by the intersection of five hyperplanes. It is not clear that this could be predicted from the homotopy type of $\mathbb{CP}^2$ alone. Again, the fact that these cohomology classes appear so naturally in the complex algebraic theory gives extra motivation in this direction.

References:

[1] Atiyah, M.F. and Bott, R. "The Yang-Mills equations over Riemann surfaces". Trans. Roy. Soc. London A 308 (1982) 523-615.

[2] Atiyah, M.F. and Singer, I.M. "The index of elliptic operators IV". Annals of Math. 93 (1971) 119-138.

[3] Barth, W. "Moduli of vector bundles on the projective plane". Inventiones math. 42 (1977) 63-91.

[4] Donaldson, S.K. "Anti self dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles". To appear in Proc. Lond. Math. Soc.

[5] Donaldson, S.K. "An application of gauge theory to four dimensional topology". Journal Diff. Geom. 18 (1983) 279-315.

[6] Freedman, M.H. "The topology of four dimensional manifolds". Journal Diff. Geom. 17 (1982) 357-453.

[7] Kirwan, F.C. "The cohomology of quotients in symplectic and algebraic geometry". Princeton U.P. to appear.

[8] Ness, L. "A stratification of the null cone by the moment map". To appear in the Amer. Journal Math.

[9] Quillen, D. Lecture at 1982 Arbeitstagung, Bonn.

[10] Taubes, C.H. "Self-dual connections on manifolds with indefinite intersection form". To appear in Journal of Diff. Geom.