ON RIEMANNIAN METRICS ADAPTED TO THREE-DIMENSIONAL CONTACT MANIFOLDS

by

S.S. Chern
Mathematical Sciences Research Institute
1000 Centennial Drive
Berkeley, California 94720

and

R.S. Hamilton
Department of Mathematics
University of California, San Diego
La Jolla, California 92093

0. Introduction It was proved by R. Lutz and J. Martinet [8] that every compact orientable three-dimensional manifold M has a contact structure. The latter can be given by a one-form ω, the contact form, such that $\omega\wedge d\omega$ never vanishes; ω is defined up to a non-zero factor. A Riemannian metric on M is said to be adapted to the contact form ω if: 1) ω has the length 1; and 2) $d\omega=2{*}\omega$, $*$ being the Hodge operator. The Webster curvature W, defined below in [9], is a linear combination of the sectional curvature of the plane ω and the Ricci curvature in the direction perpendicular to ω.

Adapted Riemannian metrics have interesting properties. The main result of the paper is the theorem:

Every contact structure on a compact orientable three-dimensional manifold has a contact form and an adapted Riemannian metric whose Webster curvature is either a constant ≤ 0 or is everywhere strictly positive.

The problem is analogous to Yamabe's problem on the conformal transformation of Riemannian manifolds Most recently, R. Schoen has proved Yamabe's conjecture in all cases, including that of positive scalar curvature [9]. It is thus an interesting question whether in the second case of our theorem the Webster curvature can be made a positive constant.

---

1),2) Research supported in part by NSF grants DMS84-03201 and DMS84-01959.

After our theorem was proved, we learned that a similar theorem on CR-manifolds of any odd dimension has been proved by Jerison and Lee. [7] As a result, our curvature was identified with the Webster curvature. We feel that our viewpoint is sufficiently different from Jerison-Lee and that the three-dimensional case has so many special features to merit a separate treatment.

In an appendix, Alan Weinstein gives a topological implication of the vanishing of the second fundamental form in (54). For an interesting account of three-dimensional contact manifolds, cf. [2].

1. **Contact Structures.** Let M be a manifold and B a subbundle of the tangent bundle TM. There is a naturally defined anti-symmetric bilinear form Λ on B with values in the quotient bundle TM/B

(1) $\Lambda\colon B \times B \longrightarrow TM/B$

defined by the Lie bracket;

(2) $\Lambda(V,W) \equiv [V,W] \bmod B.$

It is easy to verify that the value of $\Lambda(V,W)$ at a point $p \in M$ depends only on the values of V and W at p. The bundle B defines a foliation if and only if it satisfies the Frobenius integrability condition $\Lambda = 0$. Conversely, a <u>contact</u> <u>structure</u> on M is a subbundle B of the tangent bundle of codimension 1 such that Λ is non-singular at each point $p \in M$. This can only occur when the dimension of M is odd.

It is an interesting problem to find some geometric structure which can be put on every three-manifold, since this would be helpful in studying its topology. Along these lines we have the following remarkable theorem of Lutz and Martinent (see [8], [10]).

1.1 **Theorem.** *Every compact orientable three-manifold possesses a contact structure.*

There are many different contact structures possible, since the set of B with $\Lambda \neq 0$ is open. Even on S^3 there are contact structures for which the bundles B_1 and B_2 are topologically distinct. Nevertheless the notion of a contact structure is rather flabby, in the following sense. We say B is conjugate to B_* if there is a diffeomorphism $\varphi:M \longrightarrow M$ which has $\varphi(B) = B_*$. Then we have the following result due to Gray (see [4]).

1.2 Theorem. *Given a contact structure* B, *any other contact structure* B_* *close enough to* B *is conjugate to it.*

2. Metrics adapted to contact structures. A contact form ω is a 1-form on M which is nowhere zero and has the contact bundle B for its null space. In a three-manifold a non-zero 1-form ω is a contact form for the contact structure B = Null ω if and only if $\omega \wedge d\omega \neq 0$ at every point. The contact structure B determines the contact form up to a scalar multiple. The choice of a contact form ω also determines a vector field V in the following way.

2.1 Lemma. *There exists a unique vector field* V *such that* $\omega(V) = 1$ *and* $d\omega(V,W) = 0$ *for all* $W \in TM$.

Proof. Choose V_0 with $\omega(V_0) = 1$. Since $d\omega \wedge \omega \neq 0$, the form $d\omega$ is non-singular on B. Therefore there exists a unique $V_1 \in B$ with

$$d\omega(V_1,W) = d\omega(V_0,W)$$

for all $W \in B$. Let $V = V_0 - V_1$. Then $\omega(V) = \omega(V_0) - \omega(V_1) = 1$, and $d\omega(V,W) = 0$ for all $W \in B$. Since V is transverse to B and $d\omega(V,V) = 0$, we have $d\omega(V,W) = 0$ for all $W \in TM$.

Locally any two non-zero vector fields are conjugate by a diffeomorphism. However, this fails globally, since a vector field may have closed orbits while a nearby vector field does not. It is a classical result that locally any two contact forms are conjugate by a diffeomorphism. But globally two nearby contact forms may not be conjugate, since the vector fields they determine may not be.

A choice of a Riemannian metric on a contact manifold determines a choice of the contact form ω up to sign by the condition that ω have length 1. Let $*$ denote the Hodge star operator. We make the following definition.

2.2 Definition. A Riemannian metric on a contact three-manifold is said to be adapted to the contact form ω if ω is of length one and satisfies the structural equation

$$d\omega = 2 *\omega. \tag{3}$$

Such metrics have nice properties with respect to the contact structure. For example, we have the following results.

2.3 Lemma. *If the metric is adapted to the form* ω, *then the vector field* V *determined by* ω *is the unit vector field perpendicular to* B.

Proof. Let V be the unit vector field perpendicular to B. Then $\omega(V) = 1$, and for all vectors W in B we have $d\omega(V,W)=2*\omega(V,W)=0$. Hence V is the vector field determined by the contact form ω.

2.4 Lemma. *If the metric is adapted to the contact form* ω, *then the area form on* B *is given by* $\frac{1}{2}\, d\omega$.

Proof. The area form on B is $*\omega$.

A CR structure on a mainfold is a contact structure together with a complex structure on the contact bundle B; that is, an involution $J: B \longrightarrow B$ with $J^2 = -I$ where I is the identity. If M has dimension 3 then B has dimension 2, and a complex structure on B is equivalent to a conformal structure; that is knowing how to rotate by 90°. Hence, a Riemannian metric on a contact three-manifold also produces a CR structure. CR structures have been extensively studied since they arise naturally on the boundaries of complex manifolds. The following observation will be basic to our study.

2.5 Theorem. *Let* M *be an oriented three-manifold with contact structure* B. *For every choice of contact form* ω *and a* CR *structure* J *there exists a unique Riemannian metric* g *adapted to the contact form* ω *and inducing the* CR *structure* J.

Proof. The form ω determines the unit vector field V perpendicular to B. The metric on B is determined by the conformal structure J and the volume form $*\omega|B = \frac{1}{2}\, d\omega|B$.

3. **Structural equations.** We begin with a review of the structural equations of Riemannian geometry. Let $\omega_\alpha, 1 \leq \alpha, \beta \leq \dim M$, be an orthonormal basis of 1-forms on a Riemannian manifold M. Then there exists a unique anti-symmetric matrix of 1-forms $\varphi_{\alpha\beta}$ such that the structural equations

(4) $$d\omega_\alpha + \varphi_{\alpha\beta} \wedge \omega_\beta = 0.$$

hold on M. The forms $\varphi_{\alpha\beta}$ describe the Levi-Civita connection of the metric in the moving frame ω_α. We can also view the ω_α as intrinsically defined 1-forms on the principal bundle of orthonormal bases. Then the $\varphi_{\alpha\beta}$ are also intrinsically defined as 1-forms on this principal bundle, and the collection $\{\omega_\alpha, \varphi_{\alpha\beta}\}$ forms an orthonormal basis of 1-forms in the induced metric on the principal bundle. The curvature tensor $R_{\alpha\beta\gamma\delta}$ is defined by the structural equation

(5) $$d\varphi_{\alpha\beta} = -\varphi_{\alpha\gamma} \wedge \varphi_{\gamma\beta} + R_{\alpha\beta\gamma\delta}\, \omega_\gamma \wedge \omega_\delta, \quad 1 \leq \alpha, \beta, \gamma, \delta \leq \dim M,$$

where the summation convention applies.

In three-dimensions it is natural to replace a pair of indices in an anti-symmetric tensor by the third index. Thus we will write φ_{12}

$= \varphi_3$ and $R_{1212} = K_{33}$, etc. Here $K_{\alpha\beta}$ are the components of the Einstein tensor

$$K_{\alpha\beta} = \frac{1}{2} Rg_{\alpha\beta} - R_{\alpha\beta}, \tag{6}$$

which has the property that, for any unit vector V, K(V,V) is the Riemannian sectional curvature of the plane $V^{\perp}$. The structural equations then take the following form.

3.1 Structural equations in three dimensions.

$$\begin{aligned} d\omega_1 &= \varphi_2\wedge\omega_3 - \varphi_3\wedge\omega_2, \\ d\omega_2 &= \varphi_3\wedge\omega_1 - \varphi_1\wedge\omega_3, \\ d\omega_3 &= \varphi_1\wedge\omega_2 - \varphi_2\wedge\omega_1, \end{aligned} \tag{7}$$

and

$$\begin{aligned} d\varphi_1 &= \varphi_2\wedge\varphi_3+K_{11}\omega_2\wedge\omega_3+K_{12}\omega_3\wedge\omega_1+K_{13}\omega_1\wedge\omega_2, \\ d\varphi_2 &= \varphi_3\wedge\varphi_1+K_{21}\omega_2\wedge\omega_3+K_{22}\omega_3\wedge\omega_1+K_{23}\omega_1\wedge\omega_2, \\ d\varphi_3 &= \varphi_1\wedge\varphi_2+K_{31}\omega_2\wedge\omega_3+K_{32}\omega_3\wedge\omega_1+K_{33}\omega_1\wedge\omega_2, K_{\alpha\beta}=K_{\beta\alpha}. \end{aligned} \tag{8}$$

If the metric is adapted to the contact from ω, we choose the frames such that $\omega_3 = \omega$. As a consequence K_{33} is the sectional curvature of the plane $V^{\perp}$ and $\frac{1}{2}(K_{11}+K_{22})$ is the Ricci curvature in the direction V. The Webster curvature is defined by

$$W = \frac{1}{8}(K_{11}+K_{22}+2K_{33}+4) \tag{9}$$

and has remarkable properties.

We proceed to illustrate these equations with three examples which are very relevant to our discussion, the sphere S^3, the unit tangent bundle of a compact orientable surface of genus > 1, and the

Heisenberg group H^3.

3.2 **Example.** The sphere S^3 is defined by the equation

$$x^2+y^2+z^2+w^2 = 1 \tag{10}$$

in R^4. Differentiating we get

$$\omega_0 = xdx + y\,dy + z\,dz + w\,dw = 0. \tag{11}$$

A specific choice of an orthonormal basis in the induced metric is

$$\begin{aligned}
\omega_1 &= x\,dy - y\,dx + z\,dw - w\,dz,\\
\omega_2 &= x\,dz - z\,dx + y\,dw - w\,dy,\\
\omega_3 &= x\,dw - w\,dx + y\,dz - z\,dy.
\end{aligned} \tag{12}$$

The reader can verify that if $\langle dx,dx\rangle = 1$, $\langle dx,dy\rangle = 0$, etc., then $\langle\omega_1,\omega_1\rangle = 1$, $\langle\omega_1,\omega_2\rangle = 0$, etc., and that $\langle\omega_0,\omega_0\rangle = 1$, $\langle\omega_0,\omega_1\rangle = 0$, etc. Taking exterior derivative we have

$$d\omega_1 = 2\omega_2\wedge\omega_3, \quad d\omega_2 = 2\omega_3\wedge\omega_1, \quad d\omega_3 = 2\omega_1\wedge\omega_2. \tag{13}$$

and hence in this basis

$$\varphi_1 = \omega_1, \quad \varphi_2 = \omega_2, \quad \varphi_3 = \omega_3. \tag{14}$$

which makes

$$K_{11} = 1, \quad K_{22} = 1, \quad K_{33} = 1, \tag{15}$$

and the other entries are zero. The Webster curvature $W = 1$.

3.3 **Example.** The unit tangent bundle of a compact orientable surface of genus $\neq 1$.

Let N be a compact orientable surface of genus g. If N is equipped with a Riemannian metric, its orthonormal coframe θ_1, θ_2, and the connection form θ_{12} satisfy the structural equations

$$d\theta_1 = \theta_{12}\wedge\theta_2,\ d\theta_2 = \theta_1\wedge\theta_{12},\ d\theta_{12} = -K\theta_1\wedge\theta_2, \tag{16}$$

where K is the Gaussian curvature. Suppose $g\neq 1$. We can choose the metric such that

$$K = \epsilon = \begin{cases} +1, & \text{when } g=0, \\ -1, & \text{when } g>1. \end{cases} \tag{17}$$

The unit tangent bundle T_1N of N, as a three-dimensional manifold, has the metric

$$\frac{1}{4}\,(\theta_1^2 + \theta_2^2 + \theta_{12}^{\,2}). \tag{18}$$

Putting

$$\omega_1 = \frac{1}{2}\,\theta_1,\ \omega_2 = \frac{1}{2}\,\theta_2,\ \omega_3 = -\frac{1}{2}\,\epsilon\theta_{12}, \tag{19}$$

we find

$$d\omega_1 = 2\epsilon\omega_2\wedge\omega_3,\ d\omega_2 = 2\epsilon\omega_3\wedge\omega_1,\ d\omega_3 = 2\omega_1\wedge\omega_2, \tag{20}$$

and

$$\varphi_1 = \omega_1,\ \varphi_2 = \omega_2,\ \varphi_3 = (2\epsilon-1)\omega_3. \tag{21}$$

It follows that

$$K_{11} = K_{22} = 1,\quad K_{33} = 4\epsilon-3, \tag{22}$$

all other $K_{\alpha\beta}$'s being zero. By (9) we get

$$W = \epsilon.$$

This includes the example in §3.2 when $g = 0$, for the unit tangent bundle of S^2 is the real projective space RP^3, which is covered by S^3, and our calculation is local. On the other hand, T_1N, for $g > 1$, has a contact structure and an adapted Riemannian metric with $W = -1$.

3.4. Example. The Heisenberg group.

We can make C^2 into a Lie group by identifying (z,w) with the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ w & -\bar{z} & 1 \end{bmatrix}. \tag{23}$$

The subgroup given by the variety

$$z\bar{z} + w + \bar{w} = 0 \tag{24}$$

is the Heisenberg group H^3. The group acts on itself by the translations

$$\begin{aligned} z &\longrightarrow z + a, \\ w &\longrightarrow w - \bar{a}z + b, \end{aligned} \tag{25}$$

which leave invariant the complex forms

$$dz \text{ and } dw + \bar{z}\, dz. \tag{26}$$

Hence an invariant metric is given by

$$ds^2 = \left|dz\right|^2 + \left|dw + \bar{z}\, dz\right|^2. \tag{27}$$

Introduce the real coordinates

$$z = x + iy \qquad w = u + iv. \tag{28}$$

Then the variety (24) is

(29) $$x^2 + y^2 + 2u = 0$$

and differentiation gives

(30) $$du + x\,dx + y\,dy = 0.$$

Then an orthonormal basis of 1-forms in the metric above is given by

(31) $$\omega_1 = dx, \quad \omega_2 = dy, \quad \omega_3 = dv + x\,dy - y\,dx,$$

and we compute

(32) $$\begin{cases} d\omega_1 = 0, & d\omega_2 = 0, & d\omega_3 = 2\omega_1 \wedge \omega_2, \\ \varphi_1 = \omega_1, & \varphi_2 = \omega_2, & \varphi_3 = -\omega_3, \\ K_{11} = 1, & K_{22} = 1, & K_{33} = -3, \end{cases}$$

and the other entries are zero. By (9) we have $W=0$. All these examples give metrics adapted to a contact form $\omega = \omega_3$, since in an orthonormal basis $*\omega_3 = \omega_1 \wedge \omega_2$.

In general, given a metric adapted to a contact form ω, we shall restrict our attention to orthonormal bases of 1-forms ω_1, ω_2, ω_3 with $\omega_3 = \omega$. Considering the dual basis of vectors, we only need to choose a unit vector in B. These form a principal circle bundle, and all of our structural equations will live naturally on this circle bundle. It turns out to be advantageous to compare the general situation to that on the Heisenberg group. Therefore, we introduce the forms ψ_1, ψ_2, ψ_3 and the matrix $L_{11}, L_{12}, \ldots, L_{33}$ defined by

(33) $$\begin{cases} \varphi_1 = \psi_1 + \omega_1, & \varphi_2 = \psi_2 + \omega_2, & \varphi_3 = \psi_3 - \omega_3, \\ K_{11} = L_{11} + 1, & K_{22} = L_{22} + 1, & K_{33} = L_{33} - 3, \\ K_{12} = L_{12}, & K_{13} = L_{13}, & K_{23} = L_{23}. \end{cases}$$

Thus the ψ and L all vanish on the Heisenberg group. We then compute the following.

3.5. Structure equations for an adapted metric. They are:

$$
(34)\qquad \begin{cases} d\omega_1 = \psi_2\wedge\omega_3-\psi_3\wedge\omega_2, \\ d\omega_2 = \psi_3\wedge\omega_1-\psi_1\wedge\omega_3, \\ d\omega_3 = 2\omega_1\wedge\omega_2, \end{cases}
$$

and

$$
(35)\qquad \begin{cases} \psi_1\wedge\omega_2 - \psi_2\wedge\omega_1 = 0, \\ \psi_1\wedge\omega_1 + \psi_2\wedge\omega_2 = 0, \end{cases}
$$

and

$$
(36)\qquad \begin{cases} d\psi_1 = \psi_2\wedge\psi_3+L_{11}\omega_2\wedge\omega_3+L_{12}\omega_3\wedge\omega_1+L_{13}\omega_1\wedge\omega_2, \\ d\psi_2 = \psi_3\wedge\psi_1+L_{21}\omega_2\wedge\omega_3+L_{22}\omega_3\wedge\omega_1+L_{23}\omega_1\wedge\omega_2, \\ d\psi_3 = \psi_1\wedge\psi_2+L_{31}\omega_2\wedge\omega_3+L_{32}\omega_3\wedge\omega_1+L_{33}\omega_1\wedge\omega_2. \end{cases}
$$

Proof. The equation $d\omega_3 = 2\omega_1\wedge\omega_2$ comes from the condition $d\omega = 2{*}\omega$ that the metric is adapted to the contact form ω. Then the corresponding structural equation yields $\psi_1\wedge\omega_2 - \psi_2\wedge\omega_1 = 0$. Using $dd\omega_3 = 0$ we compute $\psi_1\wedge\omega_1 + \psi_2\wedge\omega_2 = 0$ also.

3.6. Corollary. We can find functions a and b on the principal circle bundle so that

$$
(37)\qquad \begin{cases} \psi_1 = a\omega_1+b\omega_2, \\ \psi_2 = b\omega_1-a\omega_2. \end{cases}
$$

Proof. This follows algebraically from the equations (35).

It is even more convenient to write these equations in complex form. We make the following substitutions.

3.7. Complex substitutions.

On account of the complex structure in B it is convenient to use the complex notation. We shall set:

$$
(38)\quad \left\{\begin{array}{l}
\Omega = \omega_1 + i\omega_2, \qquad \omega = \omega_3, \\
\Psi = \psi_1 + i\psi_2, \qquad \psi = \psi_3, \\
\iota = a + ib, \\
p = \frac{1}{2}(L_{11}+L_{22}), \quad q = \frac{1}{2}(L_{11}-L_{22}), \quad r = L_{12}, \\
s = q + ir, \\
z = \frac{1}{2}(L_{13}+iL_{23}), \\
t = L_{33}, \\
W = \frac{1}{4}(t - a^2 - b^2),
\end{array}\right.
$$

where W is the Webster curvature, to be verified below. Note that $\Psi = \iota\bar{\Omega}$. Thus Ω and ω give a basis for the 1-forms on M, while ι and ψ define the connection.

3.8 Complex structural equations.

$$
(39)\quad \begin{array}{l}
d\Omega = i(\psi\wedge\Omega - \iota\bar{\Omega}\wedge\omega), \\
d\omega = i\Omega\wedge\bar{\Omega},
\end{array}
$$

and

$$
(40)\quad \left\{\begin{array}{l}
d\psi = i[2W\Omega\wedge\bar{\Omega} + (z\bar{\Omega} - \bar{z}\Omega)\wedge\omega], \\
d\iota \equiv i(2\iota\psi + z\Omega - s\omega) \mod \bar{\Omega}, \\
p + |\iota|^2 = 0.
\end{array}\right.
$$

Proof. This is a direct computation. Note that the real functions p,W and the complex functions z,s give the curvature of the metric.

The equation $p+|\iota|^2 = 0$ has the important consequence that we can compute the Webster curvature W from the $K_{\alpha\beta}$. The result is the expression for W in (9).

The following notation will be useful. If f is a function on a Riemannian manifold with frame ω_α, then

$$df = D_\alpha f \cdot \omega_\alpha, \tag{41}$$

where $D_\alpha f$ is the derivative of f in the direction of the dual vector field V_α. If f is a function on the principal bundle then we can still define $D_\alpha f$ as the derivative in the direction of the horizontal lifting of V_α. In this case we will have

$$df \equiv D_\alpha f \cdot \omega_\alpha \mod \varphi_{\alpha\beta}. \tag{42}$$

If the function f represents a tensor then $D_\alpha f$ are its covariant derivatives, and the extra terms in $\varphi_{\alpha\beta}$ depend on what kind of tensor is represented. In the example if T is a covariant 1-tensor and

$$f = T(V_\gamma), \tag{43}$$

then,

$$df = D_\alpha f \cdot \omega_\alpha + T(V_\beta)\ \varphi_{\beta\gamma}, \tag{44}$$

while if T is a covariant 2-tensor and

$$f = T(V_\gamma,\ V_\delta), \tag{45}$$

then

$$df = D_\alpha f \cdot \omega_\alpha + T(V_\beta,\ V_\delta)\ \varphi_{\beta\gamma} + T(V_\gamma,\ V_\beta)\ \varphi_{\beta\delta}, \tag{46}$$

and so on. In the complex notation we write

$$df = \partial f \cdot \Omega + \bar{\partial} f \cdot \bar{\Omega} + D_V f \cdot \omega \tag{47}$$

as the definition of the differential operators ∂f, $\bar{\partial} f$, and $D_v f$. As usual

$$\text{(48)} \qquad \begin{cases} \partial f = \frac{1}{2}(D_1 f - iD_2 f), \\ \bar{\partial} f = \frac{1}{2}(D_1 f + iD_2 f), \\ D_v f = D_3 f, \end{cases}$$

reflecting the transition from real to complex notation. If f is a function on the principal circle bundle coming from a symmetric k-tensor on B then

$$\text{(49)} \qquad df = \partial f \cdot \Omega + \bar{\partial} f \cdot \bar{\Omega} + D_v f \cdot \omega + ikf\psi.$$

For example, the function ι represents a trace-free symmetric 2-form on B, and the structural equation for ι tells us

3.9. Lemma.

$$\text{(50)} \qquad \bar{\partial}\iota = iz \text{ and } D_v \iota = -is.$$

4. Change of basis. We start with the simplest change of basis, namely rotation through an angle θ. We take θ to be a function on M and study what happens on the principal circle bundle. The new basis ω_1^*, ω_2^*, ω_3^* is given by $\omega_3^*=\omega_3=\omega$ and

$$\text{(51)} \qquad \begin{aligned} \omega_1^* &= \cos.\theta\ \omega_1 - \sin.\theta \omega_2, \\ \omega_2^* &= \sin.\theta\ \omega_1 + \cos.\theta\ \omega_2 \end{aligned}$$

or in complex terms $\omega^* = \omega$ and

$$\text{(52)} \qquad \Omega^* = e^{i\theta}\ \Omega.$$

Then from the structural equations we immediately find that

4.1. Lemma.

(53)
$$\psi^* = \psi + d\theta,$$
$$\iota^* = \iota\ e^{2i\theta}.$$

Now a function or tensor on the principal circle bundle comes from one on M by the pull-back if and only if it is invariant under rotation by θ. Thus we see that the curvature form $d\psi^* = d\psi$ is invariant and hence lives on M. The form $\Omega\wedge\bar{\Omega}$ is also invariant, so $W = W^*$ is invariant and W is a function on M. This W is the scalar curvature introduced by Webster (see [11]). Likewise $|\iota|^2$ is invariant and hence a function on M. The function ι defines a tensor $\iota\bar{\Omega}^2$ which is invariant. Hence its real and imaginary parts

(54)
$$a(\omega_1^2-\omega_2^2) + 2b\ \omega_1\omega_2,$$
$$b(\omega_1^2-\omega_2^2) - 2a\ \omega_1\omega_2$$

define trace-free symmetric bilinear forms on B (they differ by rotation). This form is called the torsion tensor by Webster (see [11]); it is analogous to the second fundamental form for a surface.

We now consider more interesting changes of basis. First we change the CR structure while leaving the contact form ω fixed. In order to keep the metric adapted to the contact form we must leave $\omega_1\wedge\omega_2$ invariant. This gives a new basis

(55)
$$\omega_1^* = A\omega_1+B\omega_2,$$
$$\omega_2^* = C\omega_1+D\omega_2,$$
$$\omega_3^* = \omega_3$$

with $AD-BC = 1$. An infinitesimal change of basis is given by the tangent to a path at $t = 0$. Thus an infinitesimal change of the basis which changes CR structure but leaves the contact form invariant and keeps the metric adapted is given by

$$\omega_1' = g\omega_1 + h\omega_2,$$
$$\omega_2' = k\omega_1 + l\omega_2,$$
$$\omega_3' = 0$$

with $g+l = 0$. Since the rotations are trivial we may as well take $h=k$. This gives

$$(56) \qquad \begin{aligned} \omega_1' &= g\omega_1 + h\omega_2, \\ \omega_2' &= h\omega_1 - g\omega_2, \\ \omega_3' &= 0. \end{aligned}$$

In complex notation if $f = g+ih$ then

$$(57) \qquad \Omega' = f\bar{\Omega} \text{ and } \omega' = 0.$$

For future use we compute the infinitesimal change ψ' in ψ and ι' in ι from the structural equations (39), (40). We find that f transforms as a 2-tensor

$$(58) \qquad df = \partial f \cdot \Omega + \bar{\partial} f \cdot \bar{\Omega} + D_V f \cdot \omega + 2if\psi$$

and that

4.2. Lemma.

$$(59) \qquad \begin{aligned} \iota' &= -i\ D_V f, \\ \psi' &= i(\partial f \cdot \bar{\Omega} - \bar{\partial}\bar{f} \cdot \Omega) - (\iota\bar{f} + \bar{\iota} f)\omega \end{aligned}$$

using the fact that we know $\psi \wedge \Omega$ and ψ is real.

On the other hand we may wish to fix the CR structure and change the contact form while keeping the metric adapted. In this case let $\omega_3^* = f^2\omega_3$ where f is a positive real function. Excluding rotation we find that to keep the metric adapted we need

$$\begin{aligned} \omega_1^* &= f\cdot\omega_1 - D_2 f\cdot\omega_3, \\ \omega_2^* &= f\cdot\omega_2 + D_1 f\cdot\omega_3, \\ \omega_3^* &= f^2\omega_3, \end{aligned} \tag{60}$$

In complex notation

$$\begin{aligned} \Omega^* &= f\Omega + 2i\ \bar{\partial} f\cdot\omega, \\ \omega^* &= f^2\omega. \end{aligned} \tag{61}$$

For an infinitesimal variation we differentiate to obtain

$$\begin{aligned} \Omega' &= f'\ \Omega + 2i\ \bar{\partial} f'\cdot\omega, \\ \omega' &= 2f'\ \omega. \end{aligned} \tag{62}$$

Hence changes of metric fixing the CR structure are given by a potential function f, much the same way as changes of metric fixing a conformal structure. The main difference is that the derivatives of f enter the formula for the new basis.

As a consequence of $ddf = 0$ we have

$$\partial\bar{\partial} f - \bar{\partial}\partial f + iD_V f = 0. \tag{63}$$

We also define the sub-Laplace operator

$$\Box f = 2(\partial\bar{\partial} f + \bar{\partial}\partial f) = (D_1 D_1 f + D_2 D_2 f). \tag{64}$$

Then a straightforward computation substituting in the structural equations yields

(4.3. **Lemma.**)

4.3. Lemma.

$$\psi^* = \psi + 3i\left[\frac{\partial f}{f}\,\Omega - \frac{\bar\partial f}{f}\,\bar\Omega\right] - \left[\frac{\Box f}{2f} + 6\,\frac{\partial f\,\bar\partial f}{f^2}\right], \tag{65}$$

$$\iota^* = \iota - 2\,\frac{\bar\partial\bar\partial f}{f} - 6\left[\frac{\bar\partial f}{f}\right]^2.$$

Differentiating the first we get

$$d\psi^* \equiv d\psi - 2i\,\frac{\Box f}{f}\,\Omega\wedge\bar\Omega \mod \omega,$$

which shows the remarkable relation given by

4.4 Lemma.

$$f^3 W^* = fW - \Box f. \tag{66}$$

4.5. Corollary. In an infinitesimal variation

$$W' = -\Box f' - 2\,f'\,W. \tag{67}$$

5. Energies. Let μ be the measure on M

$$\mu = \omega_1\wedge\omega_2\wedge\omega_3 = \frac{i}{2}\,\Omega\wedge\bar\Omega\wedge\omega \tag{68}$$

induced by the metric. Here are two interesting energies which we may form. The first is

$$E_W = \int_M W\,\mu, \tag{69}$$

which is analogous to the energy

$$E = \int_M R\,\mu \tag{70}$$

in the Yamabe problem. The second is

(71) $$E_\iota = \int_M |\iota|^2 \mu,$$

which is a kind of Dirichlet energy.

In this section we shall study the critical points of these energies.

First we observe that for computational reasons it is easier to integrate over the principal circle bundle P. The measure there is

(72) $$\nu = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \psi_3 = \frac{i}{2} \Omega \wedge \bar{\Omega} \wedge \omega \wedge \psi.$$

If f is a function on the base **M** then

(73) $$\int_P f \; \nu = 2\pi \int_M f \; \mu,$$

so nothing is lost.

Next we observe that we can integrate by parts.

5.1. **Lemma.** For any f on P

(74) $$\int_P \partial f \cdot \nu = 0 \text{ and } \int_P D_V f \cdot \nu = 0.$$

Proof. The first follows from

$$\int_P d(f \bar{\Omega} \wedge \omega \wedge \psi) = 0$$

and the second follows from

$$\int_P d(f \Omega \wedge \bar{\Omega} \wedge \psi) = 0,$$

since $d\Omega \equiv 0 \mod \bar{\Omega}$, ω and $d\omega \equiv 0 \mod \Omega$, $\bar{\Omega}$ and $d\psi \equiv 0 \mod \Omega$, $\bar{\Omega}$, ω.

5.2. **Theorem.** *The energy* E_W *is critical over all contact forms with a fixed CR structure and fixed volume if and only if* **W** *is constant. It is critical* *over all CR structures with a fixed contact form if and only if* $\iota = 0$.

Proof. We compute the infinitesimal variation E_W'. Fixing the CR structure and varying the potential f of the contact form with $\omega^* = f^2\omega$ gives $\nu' = 4f'\nu$ and

$$E_W' = \int_P (-\Box f' + 2f' W)\,\nu = 2\int f' W\,\nu,$$

since $\Box$ integrates away. The volume is fixed when $\int f'\,\nu = 0$. Thus, $E_W' = 0$ precisely when **W** is constant.

Fixing the contact form and varying the CR structure we use the following.

5.3 Lemma.

(75) $$E_W = \frac{1}{2}\int_P d\psi\wedge\omega\wedge\psi.$$

Proof. We use the structural equation to see

$$d\psi\wedge\omega = 2iW\Omega\wedge\bar{\Omega}\wedge\omega$$

and integrate by parts to get the result. Then we have

$$E_W' = \frac{1}{2}\int_P d\psi'\wedge\omega\wedge\psi + d\psi\wedge\omega\wedge\psi'$$

(using $\omega' = 0$), and this gives

$$E_W' = \frac{i}{2}\int_P \psi'\wedge\Omega\wedge\bar{\Omega}\wedge\psi.$$

Then using Lemma 4.2 we get

$$E_W' = -\frac{1}{2}\int_P (\iota\bar{f}+\bar{\iota}f)\,\nu.$$

so that the CR structure is critical for fixed ω precisely when $\iota=0$.

Next we consider the energy E_ι.

5.4. **Theorem.** *The energy* E_ι *is critical over all* CR *structures with fixed contact form if and only if* $D_V\iota = 0$, *which is equivalent to* $s = 0$, *or* $K_{11} = K_{22}$ *and* $K_{12} = 0$. *The energy* E_ι *is critical over all contact forms with fixed* CR *structure and fixed volume if and only if*

$$2i(\partial z - \bar{\partial}\bar{z}) + 3p = \text{constant}. \tag{76}$$

Proof. The energy E_ι is given by

$$E_\iota = \int_P |\iota|^2 \, \nu,$$

so its first variation is

$$E_\iota' = \int_P (\iota' \bar{\iota} + \iota \bar{\iota}')\nu + |\iota|^2 \nu'.$$

When ω is fixed, $\omega' = 0$ and $\nu' = 0$. By Lemma 4.2 we have the result that if $\Omega' = f\,\bar{\Omega}$ then $\iota' = -iD_V f$, and this gives

5.5. **Lemma.**

$$E_\iota' = 2 \,\mathrm{Im} \int_P \bar{f}\, D_V \iota \ \nu.$$

Since f is any real function on M, we see $E_\iota' = 0$ when $D_V\iota = 0$. Then $s = 0$ by Lemma 3.9 and $K_{11} = K_{22}$ and $K_{12} = 0$ by substitution (38).

This condition says that, at each point of M, the sectional curvature of all planes perpendicular to the contact plane B are equal.

If, on the other hand, we fix the CR structure and vary the contact form by a potential f, we have from Lemma 4.3

$$\iota^* = \iota - 2\,\frac{\bar\partial\bar\partial f}{f} - 6\left[\frac{\bar\partial f}{f}\right]^2.$$

Taking an infinitesimal variation

$$\iota' = -2\bar\partial\bar\partial f',\ \nu' = 3f'\ \nu$$

Then the variation in E_ι is

$$E_\iota{}' = \int_P \{-2(\iota\ \partial\partial f' + \bar\iota\bar\partial\bar\partial f') + 3\left|\iota\right|^2 f'\}\ \nu,$$

from which we see that $E_\iota{}' = 0$ precisely when

$$2(\partial\partial\iota + \bar\partial\bar\partial\bar\iota) - 3\left|\iota\right|^2$$

is constant. Since $\partial\iota = iz$ by Lemma 3.9, and $\left|\iota\right|^2 + p = 0$, this gives the equation (76).

6. Changing Webster Scalar Curvature. The problem of fixing the CR structure and changing the Webster scalar curvature is precisely analogous to the Yamabe problem of fixing the conformal structure and changing the scalar curvature, except the problem is subelliptic, and the estimates and constants for the 3-dimensional CR case look like the 4-dimensional conformal case. The first result is the following.

6.1. Theorem. *Let* M *be a compact orientable three-manifold with fixed* CR *structure. Then we can change the contact form so that the Webster scalar curvature* W *of the adapted Riemannian metric is either positive or zero or negative everywhere.*

Proof. We have $f^3W^* = fW - \Box f$ from Lemma 4.4. We take f to be the eigenfunction of $W-\Box$ with lowest eigenvalue λ_1. By the strict maximum principle for subelliptic equations (see Bony [1]) we conclude that f is strictly positive. Since $Wf - \Box f = \lambda_1 f$ we have

$f^2W^* = \lambda_1$. Hence W^* always has the same sign as λ_1.

Next we show that in the negative curvature case we can make W whatever we want, in particular, a negative constant.

6.2. Theorem. *Let* M *be a compact orientable three-manifold with a fixed* CR *structure. If some contact form has negative Webster scalar curvature, then every negative function* $W<0$ *is the Webster scalar curvature of one and only one contact form* ω.

Proof. Let C be the space of all contact forms and let $\mathcal{F}$ be the space of functions. We define the operator P by

$$P{:}C \longrightarrow \mathcal{F}, \qquad P(\omega) = W.$$

Let $\mathcal{F}^-$ be the space of negative functions and let C^- be the space of contact forms with negative Webster curvature. Then

$$P{:}C^- \longrightarrow \mathcal{F}^- \tag{77}$$

is also defined. We claim the P in (77) is a global diffeomorphism. This follows from the following observations.

a) C^- is not empty.

b) P is locally invertible.

c) P is proper (the inverse image of a compact set is compact).

d) $\mathcal{F}^-$ is simply connected.

We then argue that (a) allows us to start inverting somewhere, (b) allows us to continue the inverse along paths, (c) says that the inverse doesn't stop until we run out of $\mathcal{F}^-$, and (d) tells us that the inverse is independent of the path and hence unique.

Before we start the proof we remark on a few technical details. There are two possible approaches to the proof. One is to work with C^∞ functions and quote the Nash-Moser theorem (see [5] for an exposition) using the ideas in [6] to handle the subelliptic estimates. The other is to work with the Folland-Stein spaces S_k^p (see [3]) which measure k derivatives in the direction of the contact structure in L^p norm. We can take $\omega \epsilon S_{k+2}^p$ and $W \epsilon S_k^p$ provided pk>8 so that $W \epsilon C^\circ$ by the appropriate Sobolev inclusion. The easiest case analytically is to take p = 2, which necessitates $k \geqslant 5$.

We proceed with the proof. Observation (a) follows from the hypothesis. To see (b) we compute the derivative of P, and apply the inverse function theorem.

In fact, from Corollary 4.5 we write

$$\bar{\Box} f' + 2\bar{W} f' = -W',$$

by putting dashes on the original metric. The operator $\bar{\Box} + 2\bar{W}$ has zero null space by the maximum principle, since $\bar{W} < 0$. Since it is self-adjoint, it must also be onto and hence invertible. This proves that DP is invertible when $\bar{W} < 0$, and so P is locally invertible on all of C^-.

To see assertion (c) that P is proper, we apply the maximum principle to the equation

$$f^3 W = f\bar{W} - \bar{\Box} f.$$

Where f is a maximum $\bar{\Box} f \leqslant 0$, and where f is a minimum $\bar{\Box} f \geqslant 0$. Since W and $\bar{W}$ are both negative we get the estimate

$$\text{(78)} \qquad \left[(\bar{W}/W)_{\min}\right]^{\frac{1}{2}} \leqslant f_{\min} \leqslant f_{\max} \leqslant \left[(\bar{W}/W)_{\max}\right]^{\frac{1}{2}}.$$

Notice that the estimate fails if W and $\bar{W}$ are positive. Having control of the maximum and minimum of f, it is easy to control the higher derivatives using the equation and the subelliptic Garding's inequality

(79) $$\|f\|_{S^p_{k+2}} \leqslant C\,(\|\bar{\Box} f\|_{S^p_k} + \|f\|_{L^p}).$$

In the C^∞ case this shows P is proper. For given any compact set of M, we have uniform bounds on W_{max} and W_{min} and all $\|W\|_{S^p_k}$. This gives bounds on f_{max} and f_{min} and all $\|f\|_{S^p_k}$ for all f in the preimage, so the preimage is compact since C^∞ is a Montel space. To work in the Banach space S^p_k we also need the following observation. Suppose we have a sequence of contact forms ω_n with $W_n \longrightarrow \bar{W} < 0$ in S^p_k. The previous estimates give bounds on ω_n in S^p_{k+2}, which implies convergence of a subsequence in S^p_k. Let $\omega_n \longrightarrow \bar{\omega}$, and write $\omega_n = f_n^2\bar{\omega}$. The maximum principle estimate shows $f_n \longrightarrow 1$ in C°. Then using the equation we get the estimate

(80) $$\|f_n - 1\|_{S^p_{k+2}} \leqslant C\,\|W_n - \bar{W}\|_{S^p_k};$$

this shows $\omega_n \longrightarrow \bar{\omega}$ in S^p_{k+2}, and proves P is proper.

The assertion (d) that $\mathcal{T}^-$ is simply connected follows by shrinking along straight line paths to $W = -1$. This completes the proof of the theorem.

7. **Minimizing Torsion.** We consider finally the problem of minimizing the energy

(81) $$E_\iota = \int_P |\iota|^2\,\nu$$

representing the L^2 norm of the torsion by the heat equation with the contact form ω fixed. From Lemma 5.5 we have the result that if we take a path of Ω's depending on t with $\Omega' = f\bar{\Omega}$ then

$$E_\iota' = 2\,\mathrm{Im}\int_P \bar{f}\, D_V\iota\ \nu.$$

Following the gradient flow of E_ι we let $f = i\, D_V\iota$. This gives heat equation for E_ι. Since $D_V \iota = -$ is by Lemma 3.9 we get the following results.

7.1. Heat Equation Formulas.

$$\Omega' = i\, D_V \iota \cdot \bar{\Omega},$$

$$E_\iota{}' = -2\int_P |s|^2\ \nu, \tag{82}$$

$$\iota' = D_V^2 \iota.$$

These equations show that if the solution exists for all time then the energy E_ι decreases and the curvature $s \longrightarrow 0$. The equation $\iota' = D_V^2 \iota$ is a highly degenerate parabolic equation, since the right hand side involves only the second derivative in the one direction V. Nevertheless, it is not a bad equation, since the maximum principle applies. This shows that the maximum absolute value of ι decreases. The equation is in fact just the ordinary heat equation restricted to each orbit in the flow of V. Physically we can imagine the manifold P to be made of a bundle of wires insulated from each other, with the heat flowing only along the wires. When the orbits of V are closed, the analysis should be fairly straightforward. When the orbits of V are dense, things are much more complicated, and probably lead to small divisor problems.

A regular foliation is one where each leaf is compact and the space of leaves is Hausdorff. In this case we always have a Seifert foliation, one where each leaf has a neighborhood which is a finite quotient of a bundle. In three dimensions the Seifert foliated manifolds are well-understood by the topologists, and provide many of the nice examples. We conjecture the following result.

7.2. Conjecture. Let M be a compact three-manifold with a fixed contact form ω whose vector field V induces a Seifert foliation. There there exists a CR structure on M such that the associated metric has $s = 0$, i.e., the sectional curvature of all planes at a given point perpendicular to the contact bundle $B = \text{Null}\ \omega$ are equal. The metric is obtained as the limit of the heat equation flow as $t \longrightarrow \infty$.

REFERENCES

[1] J.M. Bony, Principe du maximum, inégalité de Harnack, et unicité du problème de Cauchy pour les opérateurs elliptiques dégénerés, Ann. Inst. Fourier **19**(1969), 277-304.

[2] A. Douady, Noeuds et structures de contact en dimension 3, d'après Daniel Bennequin, Séminaire Bourbaki, 1982/83, no.° 604.

[3] G.B. Folland and E.M. Stein, Estimates for the $\bar{\partial}_b$-complex and analysis on the Heisenberg group, Comm. Pure and App. Math **27**(1974), 429-522.

[4] J.W. Gray, Some global properties of contact structures, Annals of Math **69**(1959), 421-450.

[5] R. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. **7**(1982), 65-222.

[6] R. Hamilton, Three-manifolds with positive Ricci curvature, J. Diff. Geom. **17**(1982), 255-306.

[7] D. Jerison and J. Lee, A subelliptic, non-linear eigenvalue problem and scalar curvature on CR manifolds, Microlocal Analysis, Amer. Math Soc. Contemporary Math Series, **27**(1984), 57-63.

[8] J. Martinet, Formes de contact sur les variétés de dimension 3, Proc. Liverpool Singularities Symp II, Springer Lecture Notes in Math **209**(1971), 142-163.

[9] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, preprint 1984.

[10] W. Thurston and H.E. Winkelnkemper, On the existence of contact forms. Proc. Amer. Math. Soc. **52**(1975), 345-347.

[11] S.M. Webster, Pseudohermitian structures on a real hypersurface, J. Diff. Geom. **13**(1978), 25-41.

APPENDIX

by

Alan Weinstein

THREE-DIMENSIONAL CONTACT MANIFOLDS WITH VANISHING TORSION TENSOR

In a lecture on some of the material in the preceding paper, Professor Chern raised the question of determining those 3-manifolds admitting a contact structure and adpated Riemannian metric for which the torsion invariant $c^2=a^2+b^2$ is identically zero. (See §3. A variational characterization of such structures is given in Theorem 5.2.) The purpose of this note is to show that the class of manifolds in question consists of certain Seifert fiber manifolds over orientable surfaces, and that the first real Betti number $b_1(M)$ of each such manifold M is even. These results are not new; see our closing remarks.

By a simple computation, it may be seen that the matrix $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ (see Corollary 3.5) represents the Lie derivative of the induced metric on the contact bundle B with respect to the contact vector field V. We thus have:

A.1. Lemma. *The invariant* c^2 *is identically zero if and only if* V *is a Killing vector field. (In other words,* M *is a "K-contact manifold"; see* [1].)

We would like the flow generated by V to be periodic. If this is not the case, we can make it so by changing the structures in the following way. Let G be the closure, in the automorphism group of M with its contact and metric structures, of the 1-parameter group generated by V. G must be a torus, so in its Lie algebra we can find Killing vector fields V' arbitrarily close to V and having periodic flow. Let ω' be the 1-form which annihilates the subbundle B' perpendicular to V' and which satisfies $\omega'(V') \equiv 1$. For V' sufficiently close to V, ω' will be so close to the original contact form ω that it is itself a contact form. Since the flow of V' leaves the metric invariant, it leaves the invariant the form ω', from which it follows that V' is the contact vector field associated with ω'.

Having made the changes described in the previous paragraph, we may revert to our original notation, dropping primes, and assume that the flow of V is periodic. A rescaling of ω will even permit us to assume that the least period of V is 1. (Note that, by Gray's theorem [2], we could actually assume that the new contact structure equals the one which was originally given.)

Suppose for the moment that the action of $S^1 = \mathbb{R}/\mathbb{Z}$ generated by V is free. Then M is a principal S^1 bundle over the surface M/S^1. The form ω is a connection on this bundle; since ω is a contact form, the corresponding curvature form on M/S^1 is nowhere vanishing. Thus M/S^1 is an orientable surface, and the Chern class of the fibration $M \longrightarrow M/S^1$ is non-zero. By the classification of surfaces, $b_1(M/S^1)$ is even; by the Gysin sequence, $b_1(M) = b_1(M/S^1)$ and is therefore even as well.

We are left to consider the case where the action of S^1, although locally free, is not free. The procedure which we will follow is that of [8]. Let $\Gamma \subsetneq S^1$ be the (finite) subgroup generated by the isotropy groups of all the elements of M. Then M is a branched cover of M/Γ, and M/Γ is a principal bundle over M/S^1 with fiber the circle S^1/Γ. The branched covering map $M \longrightarrow M/\Gamma$ induces isomorphisms on real cohomology, so it suffices to show that $b_1(M/\Gamma)$ is even. To see this, we consider the fibration $S^1/\Gamma \longrightarrow M/\Gamma \longrightarrow M/S^1$. The quotient spaces M/Γ and M/S^1 are V-manifolds in the sense of [4], and we have a fibre bundle in that category. The base M/S^1 is actually a topological surface which is orientable since it carries a nowhere-zero 2-form on the complement of its singular points. Now the contact form may once again be considered as a connection on our V-fibration, and so, just as in the preceding paragraph, we may conclude that $b_1(M/\Gamma)$ is even.

Remarks. A K-contact manifold is locally a 1-dimensional bundle over an almost-Kähler manifold. When the base is Kähler, the contact manifold is called <u>Sasakian.</u> Using harmonic forms, Tachibana

[5] has shown that the first Betti number of a compact Sasakian manifold is even. On the other hand, since every almost complex structure on a surface is integrable, every 3-dimensional K-contact mainfold is Sasakian, and hence our result follows from Tachibana's theorem. In higher dimensions, compact symplectic manifolds with odd Betti numbers in even dimension are known to exist [3] [7], and circle bundles over them will carry K-contact structures, while having odd Betti numbers in even dimension.

The paper [6] contains a study of which Seifert fiber manifolds over surfaces actually admit S^1-invariant contact structures.

Acknowledgments. This research was partially supported by NSF Grant DMS84-03201. I would like to thank Geoff Mess for his helpful advice.

REFERENCES

1. D. Blair, Contact Manifolds in Riemannian Geometry, Lecture Notes in Math., vol. 59 (1976).
2. J. Gray, Some global properties of contact structures, Ann. of Math. **69** (1959), 421-450.
3. D.McDuff, Examples of simply connected symplectic manifolds which are not Kähler, preprint, Stony Brook, 1984.
4. I. Satake, The Gauss-Bonnet theorem for V-manifolds, J. Math. Soc. Japan **9** (1957), 464-492.
5. S. Tachibana, On harmonic tensors in compact Sasakian spaces, Tohoku Math. J. **17** (1965), 271-284.
6. C.B. Thomas, Almost regular contact manifolds, J. Diff. Geom. **11** (1976), 521-533.
7. W. Thurston, Some simple examples of symplectic manifolds. Proc. A.M.S. **55** (1976), 467-468.
8. A. Weinstein, Symplectic V-manifolds, periodic orbits of hamiltonian systems, and the volume of certain riemannian manifolds, Comm. Pure Appl. Math. **30** (1977), 265-271.