Metrics with Holonomy G_2 or Spin (7)

by

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§1. The Holonomy of Riemannian Manifolds

In this section, all objects are assumed smooth unless stated otherwise, M will denote a connected, simply connected n-manifold and g will denote a Riemannian metric on M. If $\gamma: [0,1] \to M$ is a path in M, then the Levi-Civita connection of g induces a well-defined parallel translation along γ, $P_\gamma: T_{\gamma(0)}M \to T_{\gamma(1)}M$ which is an isometry of vector spaces. For every $x \in M$, we let H_x denote the set of all P_γ where γ ranges over all paths with $\gamma(0) = \gamma(1) = x$. It is well-known, see [1], that the simple connectivity of M implies that H_x is a connected, closed Lie subgroup of $SO(T_xM)$, the group of oriented isometries of T_xM with itself. Moreover $P_\gamma(H_{\gamma(0)}) = H_{\gamma(1)}$ for any path γ. It follows that by choosing an isometry $i: T_xM \simeq \mathbb{R}^n$, we can identify H_x with a subgroup $H \subseteq SO(n)$. The conjugacy class of H in $O(n)$ is independent of the choice of x or i. By abuse of language we speak of H as the holonomy of g.

The holonomy group is a measure of the curvature of g. For example, if H preserves an orthogonal decomposition $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$, then $g = g_1 + g_2$ locally where g_i is a local metric on $\mathbb{R}^{n_i}$. It follows that, in order to determine which subgroups of $SO(n)$ can be holonomy groups of Riemannian metrics, it suffices to determine the subgroups $H \subseteq SO(n)$ which act irreducibly on $\mathbb{R}^n$ and are holonomy groups of Riemannian metrics. By examining the Bianchi identities and making extensive use of representation theory, Berger [2] proved the following classification theorem.

<u>Theorem (Berger)</u>: Let (M^n,g) be a connected, simply connected Riemannian n-manifold and suppose that its holonomy group $H \subseteq SO(n)$ acts irreducibly on $\mathbb{R}^n$. Then either (M,g) is locally symmetric or else H is one of the following subgroups of $SO(n)$

(i) $SO(n)$

(ii) $U(m)$ if $n = 2m > 2$

(iii) $SU(m)$ if $n = 2m > 2$

(iv) $Sp(1)Sp(m)$ if $n = 4m > 4$

(v) $Sp(m)$ if $n = 4m > 4$

(vi) G_2 if $n = 7$

(vii) $Spin(7)$ if $n = 8$

(viii) $Spin\ (9)$ if $n = 16$

After noting that the above list is exactly the list of subgroups of $SO(n)$ which act transitively on $S^{n-1} \subseteq \mathbb{R}^n$, Simons [3] gave a direct proof that the holonomy of an irreducible non-symmetric metric on M^n acts transitively on S^{n-1}.

It is natural to ask which of the possibilities on Berger's list actually do occur. It is easy to show that the "generic" metric on M^n has holonomy $SO(n)$. If $n = 2m$, a matric with holonomy a subgroup of $U(m)$ is, of course, a Kähler metric. Such a metric is given in local coordinates on $\mathbb{C}^m$ in the form

$$g_f = (\ \partial^2 f/\partial z^i\ \partial \bar{z}^j) dz^i \circ d\bar{z}^j$$

where f is a smooth function on $\mathbb{C}^m$ satisfying the condition that its complex hessian $H_f = (\partial^2 f/\partial z^i\ \partial \bar{z}^j)$ be positive definite. For a "generic" f with $H_f > 0$, the metric g_f will have holonomy $U(m)$. Every metric on M^{2m} with holonomy $H \subseteq SU(m)$ can be put in the above form locally where f satisfies the complex Monge-Ampere equation $\det(H_f) = 1$. Again, the "generic" solution of this equation yields a metric whose holonomy is exactly SU(m). Since $Sp(m) \subseteq SU(2m)$, we ca even construct metrics whose holonomy is Sp(m) on M^{4m} $(m > 1)$ locally by selecting a linear map $J: \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ satisfying $J^2 = -I$ and $J = -{}^tJ$ and considering the g_f where f satisfies the system of equations $\bar{H}_f \, J \, H_f = J$. Even though this is an overdetermined system of equations for f, enough solutions can be found to exhibit local metrics with holonomy exactly Sp(m). A similar construction with complex contact structures on $\mathbb{C}^{2m+1}$ allows one to exhibit metrics locally on $\mathbb{R}^{4m}$ with holonomy $Sp(1) \cdot Sp(m)$. It must be emphasized that it is the encoding of holonomy properties into the Cauchy-Riemann equations (which are completely understood locally) that allows the construction of metrics in cases (ii)-(v) to be reduced to a managable partial differential equations problem.

There remain the "exceptional" cases (vi)-(viii). In a surprising paper, Alekseevski [4] showed that any metric on M^{16} with holonomy Spin(9) was necessarily locally symmetric. Thus, case (viii) can be removed from Berger's list. It is worth remarking that cases (vi) and (vii) do not occur as symmetric spaces [6]. This raises the possibility that these two cases do not occur at all. As of this writing, no examples of cases (vi) or (vii) are known. Nevertheless, there is extensive literature on the properties of these elusive metrics. See [7], [8], and [9] and the bibliographies contained therein.

In this lecture, we shall outline a proof of the existence of local metrics in cases (vi) and (vii). The details, which involve an analysis of a differential system to be constructed below will be published elsewhere. For the appropriate concepts from differential systems and Cartan-Kähler theory, the reader may consult [10].

§2. Linear Algebra, H-structure, and Differential Systems

Our strategy will be to describe a set of differential equations whose solutions will represent metrics on M^n with the desired holonomy. We begin by giving a somewhat non-standard description of G_2.

Let $\omega^1, \omega^2, \ldots, \omega^7$ be an oriented orthonormal coframing of $\mathbb{R}^7$. We define the 3-form

$$\varphi = \omega^{123} + \omega^{145} + \omega^{167} + \omega^{246} - \omega^{257} - \omega^{356} - \omega^{347}$$

where ω^{ijk} is an abbreviation for $\omega^i \wedge \omega^j \wedge \omega^k$.

<u>Proposition 1</u>: $G_2 = \{A \in GL(7) \mid A^*(\varphi) = \varphi\}$ where G_2 is the 14-dimensional simple Lie group of compact type.

We will not prove Proposition 1 here. It is interesting to note that a dimension count shows that the orbit of φ in $\Lambda^3(\mathbb{R}^7)$ under $GL(7)$ is open. (In fact, there are exactly two open $GL(7)$ orbits in $\Lambda^3(\mathbb{R}^7)$. The stabilizer of a form $\tilde{\varphi}$ in the other open orbit is the simple Lie group of non-compact type of dimension 14.) The form φ was discovered by Chevalley [5]. Bonan [7] showed that

$$\Lambda^{G_2} = \mathrm{span}\{1, \varphi, *\varphi, *1 = (1/7)\varphi \wedge *\varphi\}$$

where $\Lambda^{G_2} \subseteq \Lambda(\mathbb{R}^7)$ is the subring of G_2-invariant exterior forms.

If V is a seven dimensional vector space, we will say that $\alpha \in \Lambda^3(V^*)$ is <u>positive</u> if there exists a linear isomorphism $L: V \to \mathbb{R}^7$ so that $\alpha = L^*(\varphi)$. The set $\Lambda^3_+(V^*) \subseteq \Lambda^3(V^*)$ of positive forms is clearly an open subset of $\Lambda^3(V^*)$. If $\alpha \in \Omega^3(M^7)$ we say that α is positive iff $\alpha|_x$ is positive for all $x \in M^7$. We let $E \subseteq \Lambda^3(T^*M)$ denote the open submanifold of positive 3-forms. $\pi: E \to M$ is a smooth fiber bundle with fibers isomorphic to $GL(7)/G_2$. The sections of E are the positive forms on M and are also obviously in 1-1 correspondence with the set of G_2 reductions of the tangent bundle of M, i.e., G_2-structures on M. Since

$G_2 \subseteq SO(7)$, it follows that each G_2-structure on M induces a canonical underlying orientation and Riemannian metric.

On the other hand, if (M^7,g) is an oriented Riemannian manifold with holonomy G_2, it is easy to see that there is a unique parallel positive 3-form α_g on M whose underlying orientation and metric are the given ones.

Proposition 2: Let α be a positive 3-form on M, and let $*\alpha$ be the dual 4-form with respect to the underlying metric and orientation. Then α is parallel with respect to the underlying metric's Levi-Civita connection iff $d\alpha = d{*\alpha} = 0$.

Proposition 2 is due to Gray [8] in the context of vector cross products. It follows from this that every positive 3-form α which satisfies the system of partial differential equations $d\alpha = d{*\alpha} = 0$ has an underlying metric whose holonomy is a subgroup of G_2 and conversely every metric whose holonomy is a subgroup of G_2 arises from such an α.

The conditions $d\alpha = d{*\alpha} = 0$ form a quasi-linear first order system for the 35 (= dim $\Lambda^3_+(V^*)$) unknown coefficients of α. The system is quasi-linear because coefficients of $*\alpha$ are algebraic functions of the coefficients of α. A priori, this appears to be 56 (= $\dim(\Lambda^4(\mathbb{R}^7) \oplus \Lambda^5(\mathbb{R}^7))$ equations for the 35 unknowns. However, there is a (miraculous) identity

$$(*d\beta) \wedge \beta + (*d{*\beta}) \wedge {*\beta} = 0$$

valid for any positive β where the $*$ is the Hodge star of the underlying SO(7) structure. It can be shown that the remaining 49 = 56 - 7 equations are independent.

This overdetermined system is invariant under the diffeomorphism group of M and hence cannot be elliptic. However, it can be shown that it is <u>transversely elliptic</u>, i.e., elliptic when restricted to a local slice of the action of Diff(M) on $\Omega^3_+(M)$.

Our first main result is

<u>Theorem 1</u>: The system $d\alpha = d{*}\alpha = 0$ for $\alpha \in \Omega^3_+(M)$ is involutive with Cartan characters $(s_1, s_2, \ldots, s_7) = (0,0,1,4,10,13,7)$. In particular, the "generic" solution has the property that its underlying metric has holonomy exactly G_2.

We remark that Theorem 1 is essentially a calculation. One describes the appropriate differential system with independence condition on $E \subseteq \Lambda^3(T^*M)$ and calculates both the integral elements and the Cartan characters to arrive at the result. Note that this system is real analytic in local coordinates. The transversality property actually implies that any solution is real analytic in some coordinate system anyway, so the application of Cartan-Kähler theory is vindicated. Details will appear elsewhere.

We now turn to the analogous case $H = \mathrm{Spin}(7)$. Write $\mathbb{R}^8 = \mathbb{R}^1 \oplus \mathbb{R}^7$ and augment the given coframing of $\mathbb{R}^7$ by an ω^0. We then define the 4-form on $\mathbb{R}^8$

$$\Phi = \omega^0 \wedge \varphi + \psi = {*}\Phi$$

where $\psi = {*}\varphi \in \Lambda^4(\mathbb{R}^7)$.

<u>Proposition 3</u>: $\mathrm{Spin}(7) = \{A \in GL(8) \mid A^*(\Phi) = \Phi\}$ where $\mathrm{Spin}(7) \subseteq SO(8)$ is isomorphic to the universal cover of $SO(7)$.

Proposition 3 is not difficult to prove assuming Proposition 1. The form Φ was discovered by Bonan [7] who showed that

$$\Lambda^{\mathrm{Spin}(7)} = \{1, \Phi = {*}\Phi, {*}1 = (1/14)\Phi^2\}$$

where $\Lambda^{Spin(7)} \subseteq \Lambda(\mathbb{R}^8)$ is the subring of Spin(7)-invariant exterior forms on $\mathbb{R}^8$. The GL(8)-orbit of $\phi \in \Lambda^4(\mathbb{R}^8)$ is not open but is, of course, a smooth submanifold of $\Lambda^4(\mathbb{R}^8)$. We shall say that an $\alpha \in \Lambda^4(V^*)$ is admissible if there exists a linear isomorphism $L: V \to \mathbb{R}^8$ so that $\alpha = L^*(\phi)$. If $\alpha \in \Omega^4(M^8)$, we shall say that α is admissible if $\alpha|_x$ is admissible for all $x \in M^8$. We let $F \subseteq \Lambda^4(T^*M)$ denote the submanifold of admissible 4-forms. $\pi: F \to M^8$ is a smooth fiber bundle with fibers isomorphic to GL(8)/Spin(7). Clearly the space of sections of F, i.e. the space of admissible 4-forms on M, is in 1-1 correspondence with the space of Spin(7)-structures on M. Since $Spin(7) \subseteq SO(8)$, we see that each admissible α on M canonically induces an orientation and metric on M.

On the other hand, if (M^8,g) is an oriented Riemannian manifold with holonomy Spin(7), it is easy to see that there is a unique parallel admissible 4-form α_g on M whose underlying orientation and metric are the given ones.

Proposition 4: Let α be an admissible 4-form on M. Then α is parallel with respect to the Levi-Civita connection of the underlying metric iff $d\alpha = 0$.

Proposition 4 is actually more elementary than the corresponding Proposition 2, but seems to have been overlooked. It follows from this that every admissible 4-form α which satisfies $d\alpha = 0$ has an underlying metric whose holonomy is a subgroup of Spin(7) and conversely every metric whose holonomy is a subgroup of Spin(7) arises from such an α.

Since F is not an open subset of a vector bundle over M, the condition $d\alpha = 0$ is only a quasi-linear first order system of 56 (= dim $\Lambda^5(\mathbb{R}^7)$) equations for the 43 (= dim(GL(8)/Spin(7)) unknown coefficents of the section $\alpha: M \to F$. It can be shown that these 56 equations are algebraically independent. Again, this over-determined system is invariant under the diffeomorphism group of M and can be shown to be transversely elliptic.

The analogue of Theorem 1 for Spin(7) is

Theorem 2: The system $d\alpha = 0$ for sections $\alpha: M \to F$ is involutive with Cartan characters $(s_1, s_2, \ldots, s_8) = (0,0,0,1,4,10,20,8)$. In particular, the "generic" solution has the property that its underlying metric has holonomy exactly Spin(7).

Theorem 2 is also a calculation with the appropriate differential system with independence condition on $F \subseteq \Lambda^4(T^*M)$. Details will appear elsewhere.

§3. Closing Remarks

The methods of §2 only yield the weakest positive result. Namely, that there exist local metrics on $\mathbb{R}^7$ and $\mathbb{R}^8$ which are not locally symmetric and have holonomy equal to G_2 and Spin(7) respectively. This at least shows that Berger's list cannot be shortened any further. Of course, in many respects this is quite unsatisfactory.

In the first place, we do not know a single example of such a metric in either case. The search for such metrics is led by Gray [8] but has so far proved fruitless.

In the second place, we do not know if there exists a complete metric even on $\mathbb{R}^7$ or $\mathbb{R}^8$ with holonomy G_2 or Spin(7). This problem reminds us, in some respects, of the conjecture that a complete Kähler metric on $\mathbb{C}^m$ which has holonomy a subgroup of $SU(m)$ is actually flat [11].

Finally, we do not know if there exists a compact example of either kind. Nevertheless, the descriptions of such metrics afforded by Theorems 1 and 2 allow one to prove a good number of theorems about possible examples. In a forthcoming joint work by the author and Reese Harvey it is shown that a compact (M^7, g) with holonomy G_2 must be orientable, spin, and have finite fundamental group. The first Pontriagin class of M^7 must be non-zero and the deformation theory of the solutions of $d\alpha = d^*\alpha = 0$ is unobstructed, the dimension of the local moduli space being $b_3 > 0$ where b_3 is the third Betti number of M. Similar results are obtained for 8-manifolds with holonomy Spin(7). The difficulty of explicitly writing down such a metric can be appreciated by contemplating the fact that no explicit example of a Calabi-Yau metric on a K-3 surface is known as of this writing.

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