MANIFOLDS OF NON POSITIVE CURVATURE

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This is mainly a report on recent and rather recent work of the author and others on Riemannian manifolds of nonpositive sectional curvature. The names of the other people involved are M. Brin, K. Burns, P. Eberlein and R. Spatzier.

Denote by M^n a complete connected smooth Riemannian manifold, by K_M the sectional curvature of M and by d the distance on M induced by the Riemannian metric. We always assume $K_M \leq 0$, that is, $K_M(\sigma) \leq 0$ for every tangent plane σ of M.

One of the significant consequences of the assumption $K_M \leq 0$ is as follows. Let γ_1 and γ_2 be unit speed geodesics in the universal covering space $\widetilde{M}$ of M such that $\gamma_1(0) = \gamma_2(0)$. Then for $t,s \geq 0$

$$d^2(\gamma_1(t),\ \gamma_2(s)) \geq t^2 + s^2 - 2ts \cdot \cos(\dot{\gamma}_1(0),\ \dot{\gamma}_2(0))$$

with equality if and only if $\gamma_1|[0,t]$ and $\gamma_2|[0,s]$ belong to the boundary of a totally geodesic and flat triangle. It follows that the exponential map $\exp\colon T_p\widetilde{M} \longrightarrow \widetilde{M}$ is a diffeomorphism for each $p \in \widetilde{M}$. In particular, M is a $K(\pi,1)$; the homotopy type of M is determined by $\Gamma = \pi_1(M)$. As we will see below, there are also strong relations between the structure of Γ and the geometry of M.

One of the principal aims in the study of nonpositively curved manifolds is to specify the circumstances under which assertions about negatively curved manifolds become false - if they become false - under the weaker assumption of nonpositive sectional curvature. For example, a theorem of Milnor [Mi] asserts that Γ has exponential growth if M is compact and negatively curved. As for the weaker assumption $K_M \leq 0$, Avez [Av] showed that Γ has exponential growth if and only if M is not flat.

In general, one expects some kind of flatness in M if some property of negatively curved manifolds is not shared by M. Hence it is only natural to try to measure the flatness of M. In the case of locally symmetric spaces, the rank is such a measure. The question arises, whether such a notion can be introduced in a meaningful way for general manifolds of nonpositive sectional curvature. This is indeed the content of Problem 65 in Yau's list [Y]. We state this problem in a slightly modified form and in two parts.

a) DEFINE THE RANK OF M AND SHOW THAT Γ CONTAINS A FREE ABELIAN SUBGROUP OF RANK k IF M IS COMPACT OF RANK k.

Note that in the case M is compact and locally symmetric, the (usual) rank of M is given by the maximal number k such that Γ contains a free abelian subgroup of rank k. See also Theorem 1 below.

b) SHOW THAT Γ CONTAINS A FREE ABELIAN SUBGROUP OF RANK 2 IF M HAS A 2-FLAT.

Here a k-flat is defined to be a totally geodesic and isometrically immersed Euclidean space of dimension k.

As in the case of locally symmetric spaces, the rank of M should be an integer between 1 and $n = \dim(M)$. Further properties of this notion, which one expects, are as follows.

P1) IF M IS LOCALLY SYMMETRIC, THEN THE RANK OF M SHOULD COINCIDE WITH ITS USUAL RANK.

P2) FLAT MANIFOLDS OF DIMENSION n SHOULD HAVE RANK n. NEGATIVELY CURVED MANIFOLDS SHOULD HAVE RANK ONE.

Vice versa, manifolds of rank one should resemble negatively curved manifolds.

P3) THE RANK OF $\tilde{M}$ SHOULD BE EQUAL TO THE RANK OF M. THE RANK OF A RIEMANNIAN PRODUCT $M_1 \times M_2$ SHOULD BE THE SUM OF THE RANKS OF M_1 AND M_2.

Note that $M_1 \times M_2$ still has nonpositive sectional curvature. If M_1 and M_2 are compact, then $M_1 \times M_2$ does not carry a metric of negative sectional curvature, see Theorem 1 below.

Of course, there may be different satisfactory solutions to problem a). One candidate for the rank of M, and maybe the most obvious one, is the following:

$$\text{Rank}(M) = \max\{k \mid M \text{ contains a } k\text{-flat}\}.$$

At this point it is only conjectural that this notion of rank solves problem a). Also note that with this definition of rank, problem b) is part of problem a). With respect to Rank (M), the following results are known.

Theorem 1 (Gromoll-Wolf [GW], Lawson-Yau [LY]). If M^n is compact, then every abelian subgroup of Γ is free abelian of rank at most n. If Γ contains a free abelian subgroup of rank k, then M contains a totally geodesic and isometrically immersed flat k-torus.

This result is the extension of the theorem of Preissmann [Pr] which states that every abelian subgroup of Γ is infinite cyclic if M is compact and negatively curved. Theorem 1 implies that

$$\text{Rank}(M) \geq \max\{k \mid \Gamma \text{ contains a free abelian subgroup of rank } k\}$$

if M is compact. Problem a) now consists in showing that equality holds.

We say that M satisfies the visibility axiom if any two distinct points in the ideal boundary of $\tilde{M}$ can be joined by a geodesic [E0]. For example, compact negatively curved manifolds satisfy the visibility axiom.

Theorem 2 (Eberlein [E1]). If M is compact, then M satisfies the visibility axiom if and only if M does not contain a 2-flat, that is, Rank (M) = 1.

Thus problem b) can be reformulated as saying that M satisfies the visibility axiom if and only if every abelian subgroup of Γ is infinite cyclic.

We now discuss a different notion of rank which was introduced in [BBE]. We need some definitions. Denote by SM the unit tangent bundle of M . For $v \in SM$, let γ_v be the geodesic which has v as initial velocity vector. Along γ_v consider the space $J^P(v)$ of all parallel Jacobi fields. Note that by the assumption $K_M \leq 0$, a parallel field X along γ_v , which is linearily independent of $\dot{\gamma}_v$, is such a parallel Jacobi field if and only if $K_M(\dot{\gamma}_v(t) \wedge X(t)) = 0$ for all t . Now set

$$\text{rank } (v) = \dim (J^P(v)) \quad \text{and}$$

$$\text{rank } (M) = \min \{ \text{rank } (v) \mid v \in SM \} .$$

Note that rank (M) = 1 if M has a point p such that the sectional curvatures of all tangent planes at p are negative. In particular, rank (M) = 1 if M is a compact surface of negative Euler characteristic.

The above definition of rank was motivated by the results in the papers [B1], [B2], and [BB] which deal primarily with geodesic flows on manifolds of rank one. (Formally, the general assumption in [B1] and [B2] is that M has a geodesic which does not bound a flat half plane, but in view of Theorem 4 below this is equivalent to rank (M) = 1.) The geodesic flow g^t operates on SM, and by definition $g^t(v) = \dot{\gamma}_v(t)$. The geodesic flow leaves invariant the Liouville measure of SM.

We now state some of the properties of manifolds of rank one.

Theorem 3. Suppose rank (M) = 1.

i) [BB] If M is compact, then g^t is ergodic.
ii) [B1] If M has finite volume, then g^t has a dense orbit.
iii) (Eberlein [B2]) If M has finite volume, then tangent vectors to closed geodesics are dense in SM.

Part i) of this theorem generalizes, at the same time, the celebrated theorem of Anosov that the geodesic flow on a compact negatively curved manifold is ergodic [An] and the result of Pesin that the geodesic flow on a compact surface of negative Euler characteristic is ergodic [Pe]. The proof of part i) makes essential use of the results of Pesin [Pe] and of the results in [B1].

As for manifolds of higher rank, the following result is one of the basic ingredients in all the further developments.

Theorem 4 [BBE]. If the volume of M is finite or if M is analytic, then

$$\operatorname{rank}(M) = \max\{k \mid \text{each geodesic of } M \text{ is contained in a } k\text{-flat}\}.$$

In particular, $\operatorname{rank}(M) \leq \operatorname{Rank}(M)$. There are examples where this inequality is strict, see the introduction of [BBE]. In an earlier version of Theorem 4, Burns proved that each geodesic in M bounds a flat half plane if $\operatorname{rank}(M) \geq 2$, see [Bu].

The counterpart to Theorem 3 in the higher rank case is as follows.

Theorem 5. Suppose that $\operatorname{rank}(M) = k \geq 2$ and that K_M has a lower bound $-a^2$.

i) [BBE] If M has finite volume, then g^t is not ergodic.
ii) [BBS] If M has finite volume, then g^t has $k-1$ independent differentiable first integrals on an open, dense, and g^t-invariant subset of SM.
iii) [BBS] If M is compact, then tangent vectors to totally geodesic and isometrically immersed flat k-tori are dense in SM.

It follows from iii) that Γ contains free abelian subgroups of rank k if $\operatorname{rank}(M) = k$. In particular, problem a) is solved with this notion of rank.

There are some immediate questions related to the assumptions in Theorems 3 and 5. Namely, is it possible to delete the assumption that K has a lower bound in Theorem 5 and the assumption that M is compact in part i) of Theorem 3? I believe that the answer is yes in both cases. That the compactness assumption can be deleted in part iii) of Theorem 5 is a consequence of the following result.

Theorem 6 [B3, BS]. Suppose that rank $(M) \geq 2$, K_M has a lower bound $-a^2$ and M has finite volume. If $\tilde{M}$ is irreducible, then M is a locally symmetric space of noncompact type.

Actually, Burns-Spatzier [BS] need the stronger assumption that M is compact. Under the further assumptions M compact and $\dim(M) \leq 4$, Theorem 6 was proved earlier by the author in joint work with Heintze [BH]. All these proofs are along completely different lines, up to the fact that they are based on the results in [BBE] and [BBS].

The use of Theorem 6 lies in the fact that, for many purposes, it will be sufficient to prove a given assertion in the rank one case and the symmetric space case separately in order to get a conclusion in the general case. Using this device and results of Prasad-Raghunathan [PR], the author in collaboration with Eberlein defined algebraically a number rank (Γ), the rank of the fundamental group Γ of M, and showed that rank (Γ) = rank (M). Using various other previous results of Eberlein and a recent result of Schroeder one obtaines the following conclusion.

Theorem 7 [BE]. Suppose that $K_M \geq -a^2$ and M has finite volume. Then M is an irreducible locally symmetric space of noncompact type of rank $k \geq 2$ if and only if the following three conditions are satisfied:

i) Γ does not contain a normal abelian subgroup (except $\{e\}$)
ii) no finite index subgroup of Γ is a product
iii) rank $(\Gamma) = k$.

Here a Riemannian manifold N is called irreducible if no finite covering of N is a Riemannian product. Theorem 7 can be used to extend the rigidity results of Mostow [Mo] and Margulis [Ma]. Namely, using their results and Theorem 7 we obtain:

Theorem 8. Suppose that $K_M \geq -a^2$ and M has finite volume. Suppose M^* is an irreducible locally symmetric space of noncompact type and higher rank with finite volume. If the fundamental groups of M and M^* are isomorphic, then M and M^* are isometric up to normalizing constants.

Under the stronger assumption that M is compact, Theorem 8 was proved earlier by Gromov [GS] and, in a special case, by Eberlein [E2].

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