# EIGENVALUES OF THE DIRAC OPERATOR

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## §1. The Theorems

In recent years mathematicians have learnt a great deal from physicists and in particular from the work of Edward Witten. In a recent preprint [3], Vafa and Witten have proved some striking results about the eigenvalues of the Dirac operator, and this talk will present their results. I shall concentrate entirely on the mathematical parts of their preprint leaving aside the physical interpretation which is their main motivation.

The mathematical context is the following. We fix a compact Riemannian spin manifold $M$ of dimension $d$, and denote by $D$ the Dirac operator of $M$ acting on the spin bundle $S$. In addition if we are given a hermitian vector bundle $V$ with a connection $A$ we can define the extended Dirac operator:

$$D_A : S \otimes V \to S \otimes V.$$

In terms of an orthonormal basis $e_j$ of tangent vectors $D_A$ is given locally by $\sum_{j=1}^{d} e_j \nabla_j$, where $\nabla_j$ is the covariant derivative in the $e_j$-direction and $e_j$ acts on spinors by Clifford multiplication. In particular $D_A$ depends on $A$ only in the 0-order term, i.e. if $B$ is a second connection on $V$, then $D_A - D_B$ is a multiplication operator not involving derivatives.

The operator $D_A$ is self-adjoint and has discrete eigenvalues $\lambda_j$, both positive and negative, which we will suppose indexed by increasing absolute value so that

$$|\lambda_1| \leq |\lambda_2| \leq \ldots .$$

The questions which Vafa and Witten address themselves to concern the way in which the $\lambda_j$ depend on $A$ (and $V$): the metric on $M$ is assumed fixed throughout. More precisely they are interested in getting uniform upper bounds. The simplest and most basic of their results is

THEOREM 1. *There is constant $C$ (depending on $M$ but not on $V$ or $A$) such that $|\lambda_1| \le C$.*

More generally there is a uniform estimate for the n-th eigenvalue:

THEOREM 2. *There is a constant $C'$ (depending on $M$ but not on $V$, $A$ or $n$) such that $|\lambda_n| \le C'n^{1/d}$.*

Remarks. 1) The asymptotic formula $\lambda_n \sim n^{1/d}$ is a very general result for eigenvalues of elliptic operators, but Theorem 2 is much more precise.

2) Theorem 1 does not hold for the Laplace operator $\Delta_A$ of $V$. To see this just consider $d = 2$ and $V$ to be a line-bundle of constant curvature $F$ : then $\lambda_1 = |F| \to \infty$ with the Chern class of $V$. This emphasizes that the uniformity in Theorems 1 and 2 is with respect to the continuous parameter $A$ and also with respect to the discrete parameters describing the topological type of $V$.

3) The inequalities in Theorems 1 and 2 go in the opposite direction to the Kato inequalities for eigenvalues of Laplace type operators. This had, in principle, been conjectured by physicists on the grounds of Fermion-Boson duality.

For odd-dimensional manifolds there are even stronger results, namely:

THEOREM 1*. *If $d$ is odd, there exists a constant $C_*$ so that every interval of length $C_*$ contains an eigenvalue of $D_A$.*

THEOREM 2*. *If* $d$ *is odd, there exists a constant* $C_*'$ *so that every interval of length* $C_*' n^{1/d}$ *contains* $n$ *eigenvalues*.

Note that Theorems 1* and 2* are definitely false in *even* dimensions. To see this recall that, when $d$ is even, $S$ decomposes as $S^+ \oplus S^-$ and $D_A$ is of the form

$$(1.1) \qquad D_A = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix}$$

so that $D_A^2 = D_A^- D_A^+ \oplus D_A^+ D_A^-$, and the non-zero eigenvalues of the two factors $D_A^- D_A^+$ and $D_A^+ D_A^-$ coincide. If $V$ has large positive curvature then typically $D_A^- D_A^+$ will have a zero-eigenvalue of large multiplicity while $D_A^+ D_+^-$ will be a 'large' positive operator. Hence $D_A^2$ will have a large gap between its 0-eigenvalue and its first non-zero eigenvalue. Moreover this gap tends to infinity with the size of the curvature of $V$. When $d = 2$ and $V$ is a line-bundle of constant curvature it is just the first Chern class of $V$ which determines the size of the first gap.

## §2. The even proof

Although the theorems we have just stated appear purely *analytical* results, involving upper bounds on eigenvalues, it is a remarkable feature of the work of Vafa and Witten that the proofs are essentially *topological*. To understand how this comes about I will consider first the case when the dimension $d$ is *even*. Then, as observed at the end of §1, the spinors decompose and $D_A$ takes the form given in (1.1). In particular a 0-eigenvalue of $D_A$ arises whenever either $D_A^+$ or $D_A^-$ has a non-trivial nullspace $N_A^+$ or $N_A^-$ respectively. Next recall that the *index* of $D_A^+$ is defined as

$$\text{index } D_A^+ = \dim N_A^+ - \dim N_A^-$$

so that a non-zero value for index $D_A^+$ forces $D_A$ to have a 0-eigenvalue. On the other hand index $D_A^+$ is a purely topological invariant, given by an explicit formula [1] involving characteristic cohomology classes of $V$ and $M$. Hence, whenever the index, computed topologically, is non-zero we have a 0-eigenvalue for $D_A$ (for all connections $A$ on the given bundle $V$) and so trivially Theorem 1 holds.

For $d$ even Theorem 1 therefore has significant content only for those bundles $V$ for which the index formula gives zero. To treat these the key idea is now the following. Suppose we can find a connection $A_o$ on $V$ so that

(i) $D_{A_o}$ has a 0-eigenvalue.

(ii) $\|D_A - D_{A_o}\| \leq C$

then it will follow that the smallest eigenvalue of $D_A$ does not exceed $C$. Now we cannot actually find such a connection on $V$ itself but we can find one on some multiple $NV = V \otimes C^N$ of $V$, and this will do equally well since the only effect of taking multiple copies of $D_A$ is to increase the multiplicity of each eigenvalue.

We now proceed as follows. First choose a bundle $W'$ so that the index of $D^+$ on $S^+ \otimes V \otimes W'$ is non-zero. From the index formula (and the assumption that the index of $D^+$ on $S \otimes V$ is zero) it is enough to take $W'$ to be the pull-back to $M$ of a generating bundle on $S^{2d}$ (i.e. with $c_{2d} = (d-1)!$) by a map $M \to S^{2d}$ of degree 1: this makes the index equal to dim $V$. Thus for any connection $B'$ on $W'$ (which combines with $A$ to give a connection say $A'$ on $V \otimes W'$) the operator $D_{A'}^+$ has a non-zero index. Hence $D_{A'}$ has a zero-eigenvalue.

Next choose an orthogonal complement $W''$ to $W'$, i.e. a bundle so that

$$(2.1) \quad W' \oplus W'' \cong M \times C^N$$

and fix a connection $B''$ on $W''$ (defining, together with A, a connection $A''$ on $V \otimes W''$). The operator

$$D_{A'\oplus A''} = D_{A'} \oplus D_{A''}$$

still of course has a zero-eigenvalue (since $D_{A'}$ has). On the other hand $A_o = A' \oplus A''$ is a connection on

$$V \otimes (W' \oplus W'') \cong V \otimes C^N = NV$$

and so can be compared with the connection NA (once we have fixed the isomorphism (2.1)). Comparing the corresponding Dirac operators we find

$$(2.2) \quad D_{A_o} - D_{NA} = B$$

where B is the matrix valued 1-form which describes the connection $B' \oplus B''$ in the trivialization given by (2.1). Since B is quite independent of V and A we get a uniform constant $C = \|B\|$ and this completes the proof of Theorem 1 in the even case.

Note that the simple formula (2.2), which is essential for the proof, depends on the fact that the highest order part of $D_A$ is independent of A.

To prove Theorem 2 (for even d) we proceed in a similar manner but this time we pull back the bundle $W'$ (and its complement $W''$) from $S^d$ by using maps of degree n. The index formula then shows that $D^+_{A'}$ has index n dim V. Theorem 2 then follows easily if one can show that the constants $C = \|B\|$ grow like $n^{1/d}$. When M is a torus T and $n = r^d$ (with r an integer) this follows by using the covering map $T^d \to T^d$ given by $x \to rx$ : since $B$ is a 1-form (with matrix values) it picks up a factor $r$. For general $M$ one applies this construction to a small box in $M$ and the case of general $n$ follows by interpolation.

## §3. The odd proof

If we replace $M$ by $M \times S^1$, where $S^1$ is the circle, the eigenvalues $\lambda_j$ of $D_A$ get replaced by $\pm\sqrt{\lambda_j^2 + m^2}$ where $m$ runs over the integers. The smallest eigenvalues are therefore the same on $M$ and on $M \times S^1$. This means that theorem 1 for $d$ even, when applied to $M \times S^1$, immediately yields Theorem 1 for $d$ odd. A similar but more careful count of eigenvalues shows that Theorem 2 for $d$ even also implies Theorem 2 for $d$ odd.

Notice also that conversely, if we first establish Theorems 1 and 2 for $d$ odd, they then follow for $d$ even. In fact for $d$ odd we want to establish directly the much stronger results given by Theorems 1* and 2*. The reason why the odd case yields stronger results is roughly the following. In §2, for $d$ even, we used the index theorem, together with a deformation argument relating a connection $A$ to another connection $A_o$. In the odd case the analogue of the index theorem is itself concerned with 1-parameter families, as we shall now recall.

Suppose that $D_t$ is a periodic one-parameter family of self-adjoint elliptic operators, with the parameter $t \in S^1$. The eigenvalues $\lambda_j$ are now functions of $t$ and when $t$ goes once round the circle the $\lambda_j$ have, as a set, to return to their original position. However $\lambda_j$ need not return to $\lambda_j$: we may get a shift, e.g. $\lambda_j$ might return to $\lambda_{j+n}$ for some integer $n$. This integer $n$ is called the <u>spectral flow</u> of the family and it is a topological invariant of the family. It represents the number of negative eigenvalues which have become positive (less the number of positive eigenvalues which have become negative).

The spectral flow, like the index, is given by an explicit topological formula [2]. Moreover, for the first order differential operators (e.g. Dirac operators) this formula is actually related to an index formula as follows. If $D_t$ is the family, defined on a manifold $M$, consider the single operator

$$\mathcal{D} = \frac{\partial}{\partial t} + D_t$$

defined on $M \times S^1$. Note that

$$\mathcal{D}^* = -\frac{\partial}{\partial t} + D_t$$

so that $\mathcal{D}$ is not self-adjoint. Then one has [2]

(3.1) spectral flow of $\{D_t\}$ = index of $\mathcal{D}$.

As an illustrative example consider the case when $M$ is also a circle with angular variable $x$ and take

$$D_t = -i\frac{\partial}{\partial x} + t.$$

The eigenvalues are $n + t$ with $n$ integral and so, as $t$ increases from 0 to 1, we get a spectral flow of precisely one. The periodicity of $D_t$ is expressed by the conjugation property:

$$D_{t+1} = e^{-ix} D_t e^{ix} .$$

The operator $\mathcal{D}$ acts naturally on the functions $f(x,t)$ such that

$$\begin{aligned} f(x+1,t) &= f(x,t) \\ f(x,t+1) &= e^{-ix} f(x,t). \end{aligned} \tag{3.2}$$

In fact these equations describe sections of a certain line-bundle on the torus $S^1 \times S^1$.

Functions satisfying (3.2) have a Fourier series expansion

$$(3.3) \quad f(x,t) = \Sigma f_n(t) e^{inx}$$

where $f_n(t+1) = f_{n+1}(t)$.

Solving the equation $\mathcal{D} f = 0$ leads to the relations

$$f_n'(t) + (n+t) f_n(t) = 0$$

and so

$$f_n(t) = C_n \exp\{-\frac{(n+t)^2}{2}\} .$$

In view of the conditions (3.3) $C_n$ is independent of $n$. Thus $\mathcal{D}$ has a one-dimensional null-space spanned by the theta function

$$f(x,t) = \exp\left(\frac{-t^2}{2}\right) \sum_n \exp(inz - n^2/2)$$

where $z = x - it$. A similar calculation shows that $\mathcal{D}^* f = 0$ has no $L^2$-solution, so that index $\mathcal{D} = 1$ which checks with the spectral flow.

After this digression about spectral flow we return to consider the Dirac operators $D_A$ on a manifold $M$ of odd dimension $d$. Let $S^d \to U(N)$ be a generator of $\pi_d(U(N))$, where we take $N$ in the stable range, i.e. $N \geq \frac{d+1}{2}$, and now compose with a map $M \to S^d$ of degree one to give a map $F : M \to U(N)$. Consider $F$ as a multiplication operator on the bundle $S \otimes NV = S \otimes V \otimes C^N$, on which the Dirac operator $D_{NA}$ is defined. Since the matrix parts of $F$ and $D_{NA}$ act on different factors in the tensor product they commute, and so

$$[D_{NA}, F] = X$$

is independent of $A$. This multiplication operator $X$ acts essentially on $S \otimes C^N$ (trivially extended to $S \otimes V \otimes C^N$), and is locally given by

$$X = \Sigma e_i\, F^{-1}\, \partial_i F.$$

In particular $\|X\| = C$ is a uniform constant independent of $V$ and $A$.

Consider now the linear family of connections

$$A_t = (1 - t)A + t\, F(A)$$

joining $A$ to its gauge transform $F(A)$. The corresponding family of Dirac operators is

$$(3.4) \quad D_t = D_A + tX.$$

By construction $D_o = D_A$ and $D_1 = F^{-1}D_A F$ is unitarily equivalent to $D_o$. Thus we have a periodic family of self-adjoint operators with a spectral flow. Moreover the general formula for the spectral flow (e.g. via the index formula on $M \times S^1$) shows that in our case, because of the construction of $F$, we have spectral flow equal to one. It follows that, for some value of $t$, the operator $D_t$ has a zero-eigenvalue. Hence as before (3.4) shows that the smallest eigenvalue of $D_A$ does not exceed $C$.

The use of spectral flow to prove Theorem 1 for odd $d$ is so far quite similar to the use of the index to prove Theorem 1 for even $d$. However, spectral flow has the advantage that 0 is not a distinguished point of the spectrum, i.e. the spectral flow of a family is unchanged by adding a constant. Replacing 0 by some other value $\mu$ and repeating our argument then shows that there is an eigenvalue of $D_A$ within $C$ of $\mu$, and this is the content of Theorem 1*.

Theorem 2* follows by extending the argument using maps $F : M \to U(N)$ of higher degree, on the same lines as Theorem 2 was proved in the even case.

Finally it is worth pointing out that the upper bounds on the eigenvalues of Dirac operators given by those methods are fairly sharp. In fact Vafa and Witten actually determine the best bound when $M$ is a flat torus. For this they use the index theorem for multi-parameter families of elliptic operators - not just the spectral flow of a one-parameter family.

## References

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