Some Recent Results in Complex Manifold Theory Related to Vanishing Theorems for the Semipositive Case

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To put this survey in the proper perspective, let me first make some rather general remarks. To study complex manifolds or in general complex spaces, one works with holomorphic objects like holomorphic maps, holomorphic functions, holomorphic vector bundles and their holomorphic sections. One has to construct such objects. For example, to prove that a complex manifold is biholomorphic to $\mathbb{C}^n$, one tries to produce n suitable holomorphic functions. To prove that a complex manifold is biholomorphic to $\mathbb{P}_n$, one tries to produce n+1 good holomorphic sections of a suitable holomorphic line bundle. To prove that two complex manifolds are biholomorphic, one tries to produce a biholomorphic map. How does one produce such holomorphic objects? So far we have mainly the following methods:

(1) The method of constructing harmonic objects first and then getting holomorphic objects from them. An example is the use of the Dirichlet principle to construct harmonic functions on open Riemann surfaces and then obtaining holomorphic functions from them. Examples of the construction of harmonic objects are the results of Eells-Sampson [10] and Sachs-Uhlenbeck [40] on the existence of harmonic maps. However, unlike the one-dimensional case, in the higher-dimensional case the gap between a harmonic object and a holomorphic object is very wide and, except for some special cases, is impossible to bridge.

(2) The method of using the vanishing theorem of Kodaira to construct holomorphic sections of high powers of positive line bundles [24].

(3) Grauert's bumping technique to construct holomorphic functions on strongly pseudoconvex domains [13].

(4) The method of using L^2 estimates of $\bar{\partial}$ to construct holomorphic functions on strongly pseudoconvex domains (Morrey [30], Andreotti-Vesentini [1], Kohn [25], Hörmander [20]).

These methods produce holomorphic objects from scratch so to speak. There are also other methods like the use of Theorems A and B of Cartan-Serre to construct holomorphic objects, but one has to have a Stein manifold (i.e. a complex submanifold of $\mathbb{C}^n$) or a Stein space to apply Theorems A and B and on such manifolds previously existing global holomorphic functions are essential for the construction.

Let me briefly explain the notions of positive line bundles and strongly pseudoconvex domains and how they are related. A holomorphic line bundle with a Hermitian metric along its fibers is said to be positive if the curvature form associated to the Hermitian metric is a positive-definite quadratic form. A relatively compact domain with smooth boundary in a complex manifold is said to be strongly pseudoconvex if it is defined near its boundary by $r < 0$ for some smooth function r with nonzero gradient such that the complex Hessian of r as a Hermitian form is positive-definite. If L is a Hermitian holomorphic line bundle over a compact complex manifold, then the set Ω of all vectors of the dual bundle L^* of L whose lengths are less than 1 is a strongly pseudoconvex domain in L^* if and only if L is positive. Grauert [14] observed that a holomorphic function on Ω gives rise to holomorphic sections of powers of L because its k^{th} coefficient in the power series expansion along the fibers of L^* is a section of the k^{th} power of L. So producing holomorphic sections of powers of a positive line bundle is a special case of producing holomorphic functions of a strongly pseudoconvex domain.

In the above methods of producing holomorphic objects some positive-definite quadratic form is used, be it the curvature form in the case of a positive line bundle or the complex Hessian of the defining function in the case of a strongly pseudoconvex domain. In the method of using harmonic objects to construct holomorphic objects no positive-definite quadratic form is used. However, in the higher-dimensional case there is a wide gap between harmonic and holomorphic objects and methods known up to now [45, 46, 47, 51, 52] to bridge the gap require positive-definiteness of a certain quadratic form coming from the curvature tensor. This survey talk discusses the situation when the quadratic forms used in producing holomorphic objects are only positive semidefinite instead of strictly positive-definite. In certain cases we may even allow certain benign negativity. One may wonder why one should bother to study the semidefinite case. There are a number of reasons. Let me give two here. One is that some situations are naturally semidefinite, like the seminegativity of the sectional curvature for a bounded symmetric domain. Another is that when limits of holomorphic objects are used in proofs (like in the continuity method), the limit of strictly positive definite objects can only be assumed first to be semidefinite though in the final result it may turn out to be strictly positive definite. The semidefinite case is by far much more complicated than the definite case.

In this talk we will survey some recent results concerning vanishing theorems for the semidefinite case and their applications. More specifically we will discuss the following three topics:

(i) The construction of holomorphic sections for line bundles with curvature form not strictly positive or even with bengin negativity somewhere. An application is a proof of the Grauert-Riemenschneider conjecture characterizing Moishezon manifolds by semipositive line bundles [49, 50].

(ii) The strong rigidity of compact Kähler manifolds with seminegative curvature, in particular the results of Jost-Yau [22] and Mok [29] on the strong rigidity of irreducible compact quotients of polydiscs.

(iii) Subelliptic estimates of Kohn's school [26, 6] and their applications to vanishing theorems for semipositive bundles.

I. Producing Sections for Semipositive Bundles

We want to discuss how one can produce holomorphic sections for a Hermitian line bundle whose curvature form is only semipositive or may even be negative somewhere. The original motivation for this kind of study is to prove the so-called Grauert-Riemenschneider conjecture[15, p.277]. Kodaira[24] characterized projective algebraic manifolds by the existence of a Hermitian holomorphic line bundle whose curvature form is positive definite. The conjecture of Grauert-Riemenschneider attempts to generalize Kodaira's result to the case of Moishezon manifolds. A Moishzon manifold is a compact complex manifold with the property that the transcendence degree of its meromorphic function field equals its complex dimension. Moishezon showed [28] that such manifolds are precisely those which can be transformed into a projective algebraic manifold by proper modification. The concept of a Moishezon space is similarly defined.

The conjecture of Grauert-Riemenschneider asserts that a compact complex space is Moishezon if there exists on it a torsion-free coherent analytic sheaf of rank one with a Hermitian metric whose curvature form is positive definite on an open dense subset. Here a Hermitian metric for a sheaf is defined by going to the linear space associated to the sheaf and the curvature form is defined only on the set of points where the sheaf is locally free and the space is regular. The difficulty with the proof of the conjecture is how to prove the following special case.

Conjecture of Grauert-Riemenschneider. Let M be a compact complex manifold which admits a Hermitian holomorphic line bundle L whose curvature form is positive definite on an open dense subset G of M. Then M is Moishezon.

Since the conjecture of Grauert-Riemenschneider was introduced, a number of other characterizations of Moishezon spaces have been obtained [38,57,53,12,35] which circumvent the difficulty of proving the Grauert-Riemenschneider conjecture by stating the characterizations in such a way that a proof can be obtained by using blow-ups, Kodaira's vanishing and embedding theorems, or L^2 estimates of $\bar\partial$ for complete Kähler manifolds. If the manifold M is assumed to be Kähler, then Riemenschneider [39] observed that Kodaira's original proof of his vanishing and embedding theorems together with the identity theorem for solutions of second-order elliptic partial differential equations [2] already yields right away the conjecture of Grauert-Riemenschneider. If the set of points where the curvature form of L is not positive definite is of complex dimension zero [38] or one

[44] or if some additional assumptions are imposed on the eigenvalues of the curvature form of L [47], the conjecture of Grauert-Riemenschneider can rather easily be proved. Recently Peternell [33] used degenerate Kähler metrics to obtain some partial results about the Grauert-Riemenschneider conjecture. However, all the above results fail to deal with the fundamental question of how to produce in general holomorphic sections for a line bundle not strictly positive definite.

Recently a new method of obtaining holomorphic sections for nonstrictly positive line bundles was introduced [49]. There it was used to give a proof of the conjecture of Grauert-Riemenschneider in the special case where M-G is of measure zero in M. It was later refined to give a proof of the general case and a stronger version of the conjecture of Grauert-Riemenschneider [50]. The method imitates the familiar technique in analytic number theory of using the Schwarz lemma to prove the identical vanishing of a function by estimating its order and making it vanish to high order at a sufficient number of points. Such a technique applied to the holomorphic sections of a holomorphic line bundle was used by Serre [41] and also later by Siegel [43] to obtain an alternative proof of Thimm's theorem [54] that the transcendence degree of the meromorphic function field of a compact complex manifold cannot exceed its complex dimension. In [49, 50] the technique was applied to harmonic forms with coefficients in a holomorphic line bundle and its use was coupled with the theorem of Hirzebruch-Riemann-Roch [19, 3].

We give a more precise brief description of the method of [49,50]. To make the description easier to understand, we first impose the condition that M-G is of measure zero in M. By the theorem of Hirzebruch-Riemann-Roch (which for the case of a general compact complex manifold is a consequence of the index theorem of Atiyah-Singer [3]), $\Sigma_{q=0}^{n}(-1)^q \dim H^q(M,L^k) \geq ck^n$ for some positive constant c when k is sufficiently large, where n is the complex dimension of M. To prove that L^k admits enough holomorphic sections to give sufficiently many meromorphic functions to make M Moishezon, it suffices to show that $\dim H^0(M,L^k) \geq ck^n/2$ for k sufficiently large. Thus the problem is reduced to proving that for any given positive number ε and for $q \geq 1$ one has $\dim H^q(M, L^k) \leq \varepsilon k^n$ for k sufficiently large. Give M a Hermitian metric and represent elements of $H^q(M,L^k)$ by L^k-valued harmonic forms. By using the L^2 estimates of $\bar\partial$ one obtains a linear map from the space of harmonic forms to the space of cocycles. Take a lattice of points with distances $k^{-1/2}$ apart in a small neighborhood W of M-G. Then one uses the usual technique of Bochner-Kodaira for the case of a compact Hermitian (not necessarily Kähler) manifold [16, p.429, (7.14)] and uses the Schwarz lemma to show that any cocycle coming from a harmonic form via the linear map and vanishing at all the lattice points to an appropriate fixed order must vanish identically, otherwise its norm is so small that the $\bar\partial$-closed form constructed from it by using a partition of unity would have a norm smaller than that of the harmonic form in its cohomology class, contradicting the minimality of the norm of a harmonic form in its cohomology class. It follows that dim $H^q(M,L^k)$ is dominated by a fixed constant times the number of lattice points (which is comparable to the volume of W times k^n), otherwise there is a nonzero combination of cocycles coming from a basis of harmonic forms via the linear map and having the required vanishing orders. Since M-G is of measure zero in M, we can make the volume of W as small as we please and therefore can choose ε smaller than any prescribed positive number after making k sufficiently large. The reason why such a lattice of points is chosen is that the pointwise square norm of a local holomorphic section of L^k is of the form $|f|^2e^{-k\phi}$, where f is holormorphic function and ϕ is a plurisubharmonic function corresponding to the Hermitian metric of L. The factor $e^{-k\phi}$ is an obstacle to applying the Schwarz lemma. To overcome this obstacle, one chooses a local trivialization of L so that ϕ as well as $d\phi$ vanishes at a point. The on the ball of radius $k^{-1/2}$ centered at that point, $e^{-k\phi}$ is bounded below from zero and from above by constants independent of k. The reason why one uses cocycles instead of dealing directly with harmonic forms is that the Schwarz lemma is a consequence of the log plurisubharmonic property of the absolute value of holomorphic functions and there is no corresponding Schwarz lemma for harmonic forms.

The method outlined above can be refined in the following way so that it works in the general case where G is only assumed to be nonempty. Let R be the set of points of M where the smallest eigenvalue of the curvature form $\partial\bar\partial\phi$ of L does not exceed some positive number λ. For every point O in R one can choose a coordinate polydisc D with coordinates $z_1,\ldots,z_n$ centered at O and can choose a global trivialization of L over D such that for some constant $C > 0$

$$|\phi(P_1) - \phi(P_2)| \leq C\ (\lambda|z_1(P_1) - z_1(P_2)|^2 + \sum_{i=2}^{n} |z_i(P_1) - z_i(P_2|^2)$$

for P_1, P_2 in D. Moreover, both C and the polyradius of D can be chosen to be the same for all points O of R. Cover R by a finite number of such coordinate polydiscs so that for some constant m depending only on n no more than m of them intersect. Then one chooses the lattice points so that they are $(\lambda k)^{-1/2}$ apart along the z_1 direction but are $k^{-1/2}$ apart along the directions of $z_2,\ldots,z_n$. Now the total number of lattice points is no more than a constant times λk^n times the volume of R. By choosing λ sufficiently small, we conclude that for any given positive number ε and for $q \geq 1$ one has $\dim H^q(M,L^k) \leq \varepsilon k^n$ and therefore $\dim H^0(M,L^k)$ is no less than ck^n for some positive number c when k is sufficiently large. Thus we have the following theorem [49, 50].

Theorem 1. *Let M be a compact complex manifold and L be a Hermitian holomorphic line bundle over M whose curvature form is everywhere semipositive and is strictly positive at some point. Then M is a Moishezon manifold.*

By the result of Grauert-Riemenschneider, one has as a corollary the vanishing of $H^q(M,LK_M)$ for $q \geq 1$, where K_M is the canonical line bundle of M.

In conjunction with the characterization of Moishezon manifolds, I would like to mention the recent result of Peternell [34] that a 3-dimensional Moishezon manifold is projective algebraic if in it no positive integral linear combination of irreducible curves is homologous to zero. Together with Hironaka's example [18 and17, p.443] of a 3-dimensional non-projective-algebraic Moishezon manifold Peternell's result gives us the complete picture of the difference between projective-algebraic threefolds and Moishezon threefolds.

The noncompact analog of Theorem 1 is the following conjecture which is still open.

Conjecture. Let Ω be a relatively compact open subset of a complex manifold such that its boundary is weakly pseudoconvex at every point and is strictly pseudoconvex at some point P. Then there exists a holomorphic function on Ω going to infinity along some sequence in Ω approaching P.

Theorem 1 corresponds to the case where Ω is the set of vectors in the dual bundle of L with length < 1.

The method used in the proof of Theorem 1 can be further refined to yield results about the existence of holomorphic sections for line bundles whose curvature form is allowed to be negative somewhere [50]. An example of such results is the following.

Theorem 2. *For every positive integer n there exists a constant C_n depending only on n with the following property: Let M be a compact Kähler manifold of complex dimension n and L be a Hermitian line bundle over M. Let G be an open subset of M and a, b be positive numbers such that the curvature form of L admits a as a lower bound at every point of G and admits -b as a lower bound at every point of M-G. Assume that*

$$C_n \ (1 + \log^+(b/a))^n \ (b^2/a)^n(\text{volume of } M-G) \leq c_1(L)^n$$

where $c_1(L)$ *is the first Chern class of* L. *Then* $\dim H^0(M, L^k)$ *is* $\geq c_1(L)^n \ k^n/2(n!)$ *for* k *sufficiently large.*

When the metric of the manifold is Hermitian instead of Kähler, there is a corresponding theorem with the constant C_n depending on the torsion of the Hermitian metric. The inequality in the assumption of Theorem 2 is not natural. There should be better and more natural formulations of this kind of results.

We describe below the refinement needed to get a proof of Theorem 2. The method described above can readily yield Theorem 2 if we allow the constant C_n to depend on M, but then Theorem 2 would be far less interesting. The reason why the above method can only yield a

C_n depending on M is that in constructing a correspondence from the space of harmonic forms to the space of cocycles, besides solving the $\bar\partial$ equations, one has to use a partition of unity and the constants obtained in the process depend very heavily on the manifold M. To solve this problem, we make use of the estimate of the (0,1)-covariant derivative of the harmonic form from the Bochner-Kodaira formula. We locally solve with estimates the inhomogeneous $\bar\partial$ equations with $\bar\partial$ of the coefficients of harmonic form on one side so that the differences between the coefficients of the harmonic form and the solutions are holomorphic and then apply the Schwarz lemma to the differences. This way we avoid passing from the Dolbeault cohomology to the Cech cohomology and can make the constant C_n independent of M.

II. Strong Rigidity of Seminegatively Curved Compact Kähler Manifolds

A compact Kähler manifold is said to be strongly rigid if any other Kähler manifold homotopic to it is biholomorphic or antibiholomorphic to it. Strong rigidity can be regarded as the complex analog of Mostow's strong rigidity [31]. Compact Kähler manifolds M with curvature tensor negative in a suitable sense are known to be strongly rigid [45, 46, 47]. The way to obtain the strong rigidity of M is to consider a harmonic map f to M from the compact Kähler manifold N homotopic to M which is a homotopy equivalence. The existence of such a harmonic map is guaranteed by the result of Eells-Sampson[10] because of the nonpositivity of the sectional curvature of M. As a section of the tensor product of the bundle of (0,1)-forms of N and the pullback under f of the (1,0)-tangent bundle of M, $\bar\partial f$ is harmonic. By using the technique of Bochner-Kodaira we conclude that either ∂f or $\bar\partial f$ vanishes because of the curvature condition of M. The reason why we can only conclude the vanishing of either ∂f or $\bar\partial f$ is that the curvature term from the Bochner-Kodaira formula is homogeneous of degree two in ∂f and of degree two in $\bar\partial f$ because it comes from pulling back of the curvature tensor of M under f. Actually the Bochner-Kodaira technique is applied to the image of $\bar\partial f$ under the complexified version of the Hodge star operator. In other words we are applying the Bochner-Kodaira technique to the dual of the bundle. That is the reason why the curvature tensor of M has to be assumed negative instead of positive and also that is the reason why the Ricci tensor of N does not enter the picture. The most general formulation of this kind of results on strong rigidity is the following theorem [47].

Theorem 3. A compact Kähler manifold M of complex dimension n is strongly rigid if there exists a positive number p less than n with the following properties: (i) The bundle of (1,0)-forms on M is positive semidefinite in the sense of Nakano [32] and the bundle of (p,0)-forms on M is positive definite in the sense of Nakano [32]. (ii) At any point of M the complex tangent space of M does not contain two nontrivial orthogonal subspaces with combined dimension exceeding p such that the bisectional curvature of M in the direction of two vectors one from each subspace vanishes.

As a corollary any compact quotient of an irreducible bounded symmetric domain of complex dimension at least twc is stronglg rigid, because we have the following table giving the complex dimension and the smallest p satisfying the assumptions of Theorem 3 for each bounded symmetric domain.

Type	Complex Dimension	Smallest p
$I_{m,n}$	mn	(m-1)(n-1)+1
II_n	n(n-1)/2	(n-2)(n-3)/2 +1
III_n	n(n+1)/2	n(n-1)/2 +1
IV_n	n	2
V	16	6
VI	27	11

The values of the smallest p for the two exceptional domains were computed by Zhong [58].

This method also yields the holomorphicity or antiholomrorphicity of any harmonic map from a compact Kähler manifold into M whose rank over $\mathbb{R}$ is $\geq 2p+1$ at some point [47].

This method can be regarded as an application of the quasilinear version of Kodaira's vanishing theorem. Though strict negativity of the curvature tensor is not needed for this method, this method should be considered as corresponding to the strictly definite case rather than the semidefinite case of the vanishing theorem, because through the use of the complexified Hodge star operator the vanishing required is in codimension one rather than in dimension one.

The only case of compact quotients of bounded symmetric domains which are expected to enjoy the property of strong rigidity as suggested by Mostow's result [31] and which are not covered by the results of [47] is the case of an irreducible compact quotient of a polydisc of complex dimension at least two. This remaining case corresponds to the semidefinite case of the vanishing theorem. Jost-Yau [21] first considered this remaining case and obtained some partial results. Recently Jost-Yau [22] and Mok [29] completely solved this case. We would like to sketch a slightly more streamlined version of the proof in [29]. First we make some general observations about the application of the Bochner-Kodaira technique to the case of a compact quotient of a polydisc and discuss a simple but rather surprising theorem about the existence of holomorphic maps from compact Kähler manifolds into compact hyperbolic Riemann surfaces.

Let f be a harmonic map from a compact Kähler manifold M to a compact quotient Q of a polydisc D^n of complex dimension n. The following conclusions are immediate from the Bochner-Kodaira technique.

(i) f is pluriharmonic in the sense that the restriction of f to any local complex curve in M is harmonic.

(ii) $\partial f^i \wedge \partial \overline{f^i}$ is zero for $1 \leq i \leq n$, where f^i is the i^{th} component of f when it is expressed in terms of local coordinates along the n component discs.

From conclusion (ii) above it follows that the pullback $f^*T_Q^{1,0}$ under f of the (1,0)-tangent bundle $T_Q^{1,0}$ of Q can be endowed with the structure of a holomorphic vector bundle in the following way. A local section is defined to be holomorphic if its covariant derivative in the (0,1) direction is identically zero. This can be done because (ii) implies that the (0,1) covariant exterior differentiation composed with itself is identically zero, which is the integrability condition for such a holomorphic vector bundle structure. The same argument can be applied to the pullback $f^*T_Q^{0,1}$ under f of the (0,1)-tangent bundle $T_Q^{0,1}$ of Q to give it a holomorphic vector bundle structure. Moreover, if every element of the fundamental group of Q maps each individual component disc of D^n to itelf, then each of these two holomorphic vector bundles are the direct sum of the n holomorphic line bundles which are the pullbacks of the line subbundles of the tangent bundle of Q defined by the directions of the individual component discs. In such a case let L_i be the line subbundles of $f^*T_Q^{1,0}$ and L_i' be the line subbundles of $f^*T_Q^{0,1}$.

Because of conclusion (i) ∂f is a holomorphic section of $f^*T_Q^{1,0} \times \Omega_M^1$ and $\partial \bar{f}$ is a holomorphic section of $f^*T_Q^{0,1} \times \Omega_M^1$, where Ω_M^1 is the bundle of holomorphic 1-forms on M. Assume that every element of the fundamental group of Q maps each individual component disc of D^n to itself. Then for each fixed $1 \le i \le n$,

(iii) ∂f^i is a holomorphic section of $L_i \otimes \Omega_M^1$ and $\partial \overline{f^i}$ is a holomorphic section of $L_i' \otimes \Omega_M^1$.

For any local holomorphic section s_i of the dual bundle of L_i, $s_i \partial f^i$ is a (local) holomorphic 1-form on M whose exterior derivative equals its product with some 1-form. By the theorem of Frobenius, near points where ∂f^i does not vanish we have a holomorphic family of local complex submanifolds of complex codimension one whose tangent spaces annihilate ∂f^i. Such a holomorphic foliation of codimension one defined by the kernel of ∂f^i exists also at points where ∂f^i can be divided by a local holomorphic function to give a nowhere zero holomorphic local section of $L_i \otimes \Omega_M^1$. If in addition the rank of df^i is two over $\mathbb{R}$ at the points under consideration, because of (ii) the local leaves of the holomorphic foliation agree with the fibers of the locally defined map f^i. The same consideration can be applied to $\partial \overline{f^i}$. Also because of (ii) the holomorphic foliation defined by the kernel of ∂f^i agrees with the holomorphic foliation of $\partial \overline{f^i}$ at points where both ∂f^i and $\partial \overline{f^i}$ can be divided by local holomorphic functions to give nowhere zero holomorphic local sections of $L_i \otimes \Omega_M^1$ and $L_i' \otimes \Omega_M^1$ respectively. These rather straightforward discussions lead us immediately to the following theorem [48].

Theorem 4. Let M be a compact Kähler manifold and R be a compact hyperbolic Riemann surface such that there exists a continuous map f from M to R which is nonzero on the second homology. Then there exists a holomorphic map g from M into a compact hyperbolic Riemann surface S and a harmonic map h from S to R such that h∘g is homotopic to f.

The Riemann surface S is constructed from the holomorphic foliation described above in the following way. By the result of Eells-Sampson we can assume without loss of generality that f is harmonic and therefore real-analytic. Let Z be the complex subvariety of complex codimension ≥ 2 in M consisting of all points where either ∂f or $\partial \bar{f}$ cannot be divided by any local holomorphic function to give a nowhere zero holomorphic local section of the tensor product of Ω^1_M with the pullback under f of the (1,0) or (0,1) tangent bundle of R. Let V be the set of points of M where the rank of df over $\mathbb{R}$ is $<$ 2. On M-Z we have a holomorphic foliation described above with the property that whenever a leaf of the foliation has a point in common with M-V, the leaf agrees with the real-codimension-two branch of the fiber of f passing through that point and therefore can be extended to a complex-analytic subvariety of codimension one in M. Because of the Kähler metric of M, by using Bishop's theorem [4] on the limit of subvarieties of bounded volume and by passing to limit, we conclude that every leaf of the holomorphic foliation can be extended to a complex-analytic subvariety of codimension one in M. Since Z is of complex codimension ≥ 2 in M, by using the theorem of Remmert-Stein on extending subvarieties [37] we conclude that M is covered by the holomorphic family of subvarieties consisting of the extensions of the leaves of the holomorphic foliation. The Riemann surface S is now obtained as the nonsingular model of the quotient of M whose points are the branches of the extensions of the leaves of the holomorphic foliation.

The rather surprising aspect of Theorem 4 is that from the existence of a continuous map from a compact Kähler manifold to a compact hyperbolic Riemann surface nonzero on the second homology we can conclude the existence of a nontrivial holomorphic map from the Kähler manifold to a compact hyperbolic Riemann surface. In particular by going to the respective universal covers we obtain a nontrivial bounded holomorphic function on the univeral cover of the Kähler manifold. So far there is no known general method of constructing bounded holomorphic functions on complex manifolds which are expected to admit a large number of bounded holomorphic functions, such as the universal cover of compact Kähler manifolds of negative curvature. Here to conclude the existence of a nontrivial bounded holomorphic function we do not use any curvature property of the compact Kähler manifold. Instead the existence of a continuous map to a compact hyperbolic Riemann surface is used. Since until now there is no general way of constructing nontrivial bounded holomorphic functions, this could only mean that the existence of the kind of continuous map we want is rather rare and if such a continuous map exists, its existence would be rather difficult to establish. Even for negatively curved compact Kähler manifolds in general we do not expect such continuous maps to exist. As a matter of fact, for compact quotients of a ball of complex dimension at least two the only known examples so far that admit nontrivial holomorphic maps into any compact hyperbolic Riemann surface are the ones constructed by Livné [27] by taking branched covers of certain elliptic surfaces. It is not known whether in other dimensions there are similar examples of maps between compact quotients of balls of different dimensions besides the obvious ones.

Problem. Suppose $1 < m < n$ are integers. Let M and N be respectively compact quotients of the balls of complex dimensions m and n.
(a) Is it true that there exists no surjective holomorphic map from N to M?
(b) Is it true that every holomorphic embedding of M in N must have a totally geodesic image?

Yau conjectured that Problem (b) should be a consequence of uniquenss results for proper holomorphic maps between balls of different dimensions. For $n \geq 3$ Webster [56] showed that the only proper holomorphic maps from the n-ball to the (n+1)-ball C^3 up to the boundary are the obvious ones. Faran [11] showed that, up to automorphisms of the two balls, there are only four proper holomorphic maps from the 2-ball to the 3-ball C^3 up to the boundary . Unfortunately until now there are no general results about proper holomorphic maps between balls of different dimensions without any known boundary regularity. In our case the proper map, though without any known boundary regularity, has the additional property that it comes from maps between compact quotients. Hopefully this additional property may be used instead of boundary regularity.

We now introduce the theorem on the strong rigidity of irreducible compact quotients of polydiscs and sketch its proof.

Theorem 5 (Jost-Yau [22] and Mok [29]). Suppose Q is an irreducible compact quotient of an n-disc D^n with $n \geq 2$, M is a compact Kähler manifold, and f is a harmonic map from M to Q which is a homotopy equivalence. Let $\tilde{M}$ be the universal cover of M and F: $\tilde{M} \to D^n$ with components $(F^1,\ldots,F^n)$ be induced by F. Then for each $1 \leq i \leq n$, F^i is either holomorphic or antiholomorphic.

Here an irreducible quotient means one that cannot be decomposed as a product of two lower-dimensional quotients of polydiscs. For the proof of this theorem, by replacing both M and Q by finite covers, we can assume without loss of generality that the fundamental group of Q is a product of n groups $G_1,\ldots,G_n$, each of which is a (nondiscrete) subgroup of the automorphism group of the 1-dimensional disc D. We regard ∂F^i and $\partial \overline{F^i}$ as holomorphic sections of the holomorphic vector bundles on M described above (rather as (1,0)-forms on $\tilde{M}$). A consequence of the irreducibility of Q is that for each $1 \leq i \leq n$ every orbit of G_i is dense in D. We have to show that for every $1 \leq i \leq n$ either ∂F^i or $\partial \overline{F^i}$ vanishes identically on M. Without loss of generality we assume that the assertion fails for i = 1 and try to get a contradiction. Since f is a homotopy equivalence, the length of ∂F^1 and the length of $\partial \overline{F^1}$ cannot agree at every point. Without loss of generality we can assume that the length of ∂F^1 is greater than the length of $\partial \overline{F^1}$ at some point. By (ii) and (iii) we can write $\partial F^1 = g \; \partial \overline{F^1}$ so that locally g is the product of a nowhere zero smooth function and a meromorphic function. Thus the pole-set V of g is a complex-analytic hypersurface in M if it is nonempty. The pole-set V cannot be empty, otherwise by considering the Laplacian of the log of the absolute value h of g we get a contradiction at a maximum point of h. Since f is a homotopy equivalence, the real rank of f on the regular points of V must be precisely 2n - 2, otherwise the homology class represented by V would be mapped to zero by f. Let $p:\tilde{M} \to M$ be the projection of the universal cover and $q:D^n \to D$ be the projection onto the first component. Because of the holomorphic foliation discussed above the function $h \circ p$ on $\tilde{M}$ must be constant along the components of the fibers of $F^1 \colon M \to D$. The proper closed subset $F(p^{-1}(V))$ of D^n contains an entire fiber of q whenever it contains one of its points. It follows that $q(F(p^{-1}(V)))$ is a proper closed subset of D which is invariant under the group G_1. This contradicts the density of every orbit of G_1 in D.

Mok [29] also showed that for any harmonic map from a compact Kähler manifold to an irreducible compact quotient of the n-disc ($n \geq 2$) with real rank 2n somewhere, each of the n components of the map between the universal covers induced by it is either holomorphic or antiholomorphic.

III. Vanishing Theorems Obtained by Subelliptic Estimates

So far all the vanishing theorems for bundles with curvature conditons make use of the pointwise property of the curvature form. The recent theory of subelliptic multipliers developed by Kohn, Catlin, and others [26, 5, 6] makes it possible to get vanishing thoerems based on the local property of the curvature form when the curvature form is semidefinite. Kohn developed his theory to deal with the question of boundary regularity for solutions of the $\bar{\partial}$ equation in the case of a weakly pseudoconvex boundary.

Let Ω be an open subset of $\mathbb{C}^n$ whose boundary is smooth and weakly pseudoconvex at a boundary point x_0. Let $1 \leq q \leq n$ be an integer. We say that a subelliptic estimate holds for (0,q)-forms at x_0 if there exists a neighborhood U of x_0 and constants $\varepsilon > 0$ and $C > 0$ such that

$$\|\phi\|_\varepsilon^2 \leq C(\|\bar{\partial}\phi\|^2 + \|\bar{\partial}^*\phi\|^2 + \|\phi\|^2)$$

for all smooth (0,q)-form ϕ on $U \cap \bar{\Omega}$ with compact support belonging to the domain of $\bar{\partial}^*$, where $\| \ \|$ means the L^2 norm and $\| \ \|_\varepsilon$ means the Sobolev ε-norm. In order to obtain subelliptic estimates Kohn introduced the concept of a subelliptic multiplier. A smooth function f on U is said to be a subelliptic multiplier if there exist positive ε

and C so that

$$\| f\phi \|_{\varepsilon}^{2} \leq C(\| \bar{\partial}\phi \|^{2} + \| \bar{\partial}^{*}\phi \|^{2} + \| \phi \|^{2})$$

for all ϕ. The subelliptic multipliers form an ideal I_q. Let c_{ij} $(1 \leq i,j \leq n-1)$ be the Levi form of the boundary of Ω near x in terms of an orthonormal frame field of (1,0) vectors tangential to the boundary of Ω. The starting point of Kohn's theory is the following results concerning the ideal I_q of subelliptic multipliers. For notational simplicity we describe the case q = 1 and the general case is similar.

(i) A smooth function r with nonzero gradient whose zero-set is the boundary of Ω belongs to I_1.

(ii) The determinant of the matrix $(c_{ij})_{1 \leq i,j \leq n-1}$ belongs to I_1.

(iii) Whenever $f_1,\dots,f_k$ belong to I_1, the determinant formed in the following way belongs to I_1. The i^{th} column consists of the components of ∂f_i in terms of the frame field of (1,0) vectors tangential to the boundary of Ω. The other n-1-k columns are any n-1-k columns of the matrix $(c_{ij})_{1 \leq i,j \leq n-1}$.

(iv) I_1 equals to its real radical in the sense that if f belongs to I_1 and g is a smooth function with $|g|^m \leq |f|$ for some positive integer m, then g also belongs to I_1.

Kohn [26] showed that if the boundary of Ω is real-analytic near x_0 and contains no local complex-analytic subvariety of complex dimension q, then the constant function 1 belongs to the ideal I_q of subelliptic multipliers and as a consequence a subelliptic estimate for (0,q)-forms holds at x_0. (Diederich-Fornaess [9] contributed to the formulation of the assumptions in Kohn's result.) Recently Catlin [5,6] carried out the investigation for the case of smooth boundary and showed that a subelliptic estimate for (0,1)-forms holds at x_0 if and only if the boundary of Ω is of finite type at x_0 in the sense of D'Angelo [7,8]. (Similar statements hold for subelliptic estimates for (0,q)-forms.) D'Angelo's definition of finite type is as follows. The boundary of Ω is of type $\leq$ t at x_0 if for every holomorphic map $h = (h_1,\dots,h_n)$ from the open 1-dimensional disc D to $\mathbb{C}^n$ with h(0) =

x_0 the vanishing order of $r \cdot h$ at 0 does not exceed t times the minimum of the vanishing orders of $h_1, \dots, h_n$ at 0. At every point x of the boundary of Ω let $t(x)$ be the smallest number such that the boundary of Ω at x is of type $\leq t(x)$. D'Angelo showed that $t(x)$ in general is not upper semicontinuous, but satisfies $t(x) \leq t(x_0)^{n-1}/2^{n-2}$ for x near x_0. The order ε in the subelliptic estimate at x_0 is expected to be the reciprocal of the maximum of $t(x)$ for x near x_0. Catlin's result showed that ε cannot be bigger than the expected number but he can so far only show that subelliptic estimates hold for an ε of the order of $t(x_0)$ raised to the power $-t(x_0)^{n^2}$.

We now study how the subelliptic estimates can be used to get vanishing theorems. We follow Grauert's approach to vanishing theorems [14]. A number of vanishing theorems can be formulated from the method of subelliptic estimates. Some of them can readily be derived by other means. We illustrate here by an example of such vanishing theorems. Let M be a compact complex manifold and L be a Hermitian holomorphic line bundle over M whose curvature form is semipositive. Let V be a holomorphic vector bundle over M. Let $p: L^* \to M$ be the dual bundle of L. Let Ω be the open subset of L^* consisting of all vectors of L^* of length < 1. If subelliptic estimates for (0,q)-forms hold for the boundary of Ω at every one of its points, then one concludes that $H^q(\Omega, p^*V)$ is finite-dimensional by representing the cohomology by harmonic forms. It follows that $H^q(M, V \otimes L^k)$ vanishes for k sufficiently large, because the k^{th} coefficient in the power series expansion in the fiber coordinate of L^* of a local holomorphic function defined near a point in the zero-section of L^* is a local section of L^k.

When the Hermitian metric of L is real-analytic, by Kohn's result subelliptic estimates for (0,q)-forms hold if the boundary of Ω contains no local q-dimensional complex-analytic subvariety. If there is such a subvariety, its projection under p is a local q-dimensional subvariety W with the property that with respect to some local trivialization of L the Hermitian metric of L is represented by a function which is constant on W. If we give M a Hermitian metric, then all covariant derivatives of the curvature form of L along the directions of W must vanish. Hence we have the following theorem.

Theorem 6. Let M be a compact complex manifold with a Hermitian metric, L a holomorphic line bundle over M with a real-analytic Hermitian metric, and V a holomorphic vector bundle over M. Let θ be the curvature form of L. Let q be a positive integer. Suppose θ is positive semidefinite and suppose at every point x of M the following is true. If E is a q-dimensional complex linear subspace of the space all (1,0)-vectors at x such that the restriction of θ to E × $\overline{E}$ is zero (where $\overline{E}$ is the complex conjugate of E), then for some positive integer m the m^{th} covariant derivative of θ evaluated at some m+2 vectors from E and $\overline{E}$ is not zero. Then $H^q(M,V \otimes L^k) = 0$ for k sufficiently large.

By using Catlin's result [6] for weakly pseudoconvex smooth boundary, one can drop the real-analytic assumption on the Hermitian metric of L. This kind of result tells us that in the case of a semipositive line bundle we can still get vanishing of the cohomology if the derivatives of the curvature form satisfy certain nondegeneracy conditions. Similar theorems can be formulated for holomorphic vector bundles and noncompact pseudoconvex manifolds. When q = 1, Theorem 5 can be proved by using the method of producing holomorphic sections for semipositive line bundles described above and Grauert's criterion of ampleness [14, p.347, Lemma] to show that the line bundle L must be ample. Though for lack of known examples there is no application yet for the kind of vanishing theorems derived from subelliptic estimates, hopefully in the future this approach may turn out to be fruitful.

We would like to remark that on compact projective algebraic manifolds there is another kind of vanishing theorems motivated by Seshadri's criterion of ampleness [42, p.549] and obtained by Ramanujam [36], Kawamata [23], and Viehweg [55] for line bundles satisfying conditions weaker than ampleness. An example of such a kind of vanishing theorems is the following. If L is a holomorphic line bundle over a compact projective algebraic manifold M of complex dimension n such that $c_1(L)^n > 0$ and $c_1(L|C) \geq 0$ for every complex-analytic curve C in M, then $H^q(M,L^k K_M) = 0$ for $q \geq 1$, where $c_1(\cdot)$ denotes the first Chern class and K_M denotes the canonical line bundle of M. The assumptions involved are weaker than local curvature conditions. However, such results apply only to the projective algebraic case.

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