LOOP GROUPS

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§1 General remarks

In this talk a loop group LG will mean the group of smooth maps from the circle S^1 to a compact Lie group G. One reason for studying such groups is that they are the simplest examples of infinite dimensional Lie groups. Thus LG has a Lie algebra $L\mathfrak{g}$ - the loops in the Lie algebra $\mathfrak{g}$ of G - and the exponential map $L\mathfrak{g} \to LG$ is a local diffeomorphism. Furthermore LG has a complexification $LG_{\mathbb{C}}$, the loops in the complexification of G. Neither of these properties is to be expected of infinite dimensional groups: neither holds, for example, for the group of diffeomorphisms of the circle [17].

From this point of view the group Map(X;G) of smooth maps $X \to G$, where X is an arbitrary compact manifold, seems almost as simple as LG. Such groups are of great importance in quantum theory, where they occur as "gauge groups" and "current groups"; the manifold X is physical space. Thus loop groups arise in quantum field theory in two-dimensional space-time. In fact it is not much of an exaggeration to say that the mathematics of two-dimensional quantum field theory is almost the same thing as the representation theory of loop groups.

If $\dim(X) > 1$, however, surprisingly little is known about the group Map(X;G). Essentially only one irreducible representation of it is known - the representation of Vershik, Gelfand and Graev [9] - and that representation does not seem relevant to quantum field theory. For loop groups, in contrast, there is a rich and extensively developed theory. They first became popular because of their connection with the intriguing combinatorial identities of Macdonald [16]. They are the groups whose Lie algebras are the "affine algebras" of Kac-Moody - roughly speaking, the algebras associated to positive-semidefinite Cartan matrices. From that point of view the groups have been discussed in Tits's talk In this talk I shall keep away from the Lie algebra theory, of which there is an excellent exposition in the recent book of Kac [11], and instead shall attempt to survey what is known about the global geometry and analysis connected with the groups.

From any point of view the crucial property of loop groups is the existence of the one-parameter group of automorphisms which simply rotates the loops. It permits one to speak of representations of LG of <u>positive energy</u>. A representation of LG on a topological vector space H has positive energy if there is given a positive action of the circle group $\mathbb{T}$ on H which intertwines with the action of LG so at to provide a representation of the semidirect product $\mathbb{T} \tilde{\times}$ LG, where $\mathbb{T}$ acts on LG by rotation. An action of $\mathbb{T}$ on H is <u>positive</u> if $e^{i\theta} \in \mathbb{T}$ acts as $e^{iA\theta}$, where A is an operator with positive spectrum. It turns out that representations of LG of positive energy are necessarily projective (cf. (4.3) below).

The theory of the positive energy representations of LG (or, more accurately, of $\mathbb{T} \tilde{\times}$ LG) is strikingly simple, and in strikingly close analogy with the representation theory of compact groups.(*) Thus the irreducible representations

(i) are all unitary,

(ii) all extend to holomorphic representations of $LG_{\mathbb{C}}$, and

(iii) form a countable discrete set, parametrized by the points of a positive cone in the lattice of characters of a torus.

None of these properties holds, for example, for the representations of $SL_2(\mathbb{R})$.

The positive energy condition is strongly motivated by quantum field theory: the circle action on H corresponds to the time evolution on the Hilbert space H of states. It would be very interesting if one could formulate an analogous condition for more general groups Map(X;G). Certainly in quantum field theory one might expect such a gauge group to act on a state space on which time evolution was defined by a positive Hamiltonian operator, and

(*) We are thinking of continuous representations on arbitrary complete locally convex topological vector spaces. But we do not distinguish between representations on H and $\tilde{H}$ if there is an injective intertwining operator $H \to \tilde{H}$ with dense image.

the gauge transformations should intertwine in some perhaps complicated way with the time evolution. But there has been no progress on this front, and the attempt may well be misconceived. (Cf. §3 below.)

To conclude these introductory remarks I should say that the material I am going to present is all essentially well-known, and has been worked out independently by many people in slightly different contexts. As representative treatments of various aspects of the subject from standpoints somewhat different from mine let me refer to Garland [8], Lepowsky [15], Kac and Peterson [12], Goodman and Wallach [10], Frenkel [5]. More details of my own approach can be found in [18], [19] and [20].

§2 The fundamental homogeneous space X

In the study of LG the homogeneous space X = LG/G (where G is identified with the constant loops in LG) plays a central role. One can think of X as the space ΩG of based loops in G; but we prefer to regard it as a homogeneous space of LG. I shall list its most important properties.

(i) X is a complex manifold, and in fact a homogeneous space of the complex group $LG_{\mathbb{C}}$:

$$X = LG/G \cong LG_{\mathbb{C}}/L^{+}G_{\mathbb{C}} \ . \tag{2.1}$$

Here $L^{+}G_{\mathbb{C}}$ is the group of smooth maps $\gamma : S^1 \to G_{\mathbb{C}}$ which are the boundary values of holomorphic maps

$$\gamma : \{z \in \mathbb{C} : |z| < 1\} \to G_{\mathbb{C}} \ .$$

The isomorphism (2.1) is equivalent to the assertion that any loop γ in $LG_{\mathbb{C}}$ can be factorized

$$\gamma = \gamma_u \cdot \gamma_+$$

with $\gamma_u \in LG$ and $\gamma_+ \in L^{+}G_{\mathbb{C}}$. This is analogous to the factorization of an element of $GL_n(\mathbb{C})$ as (unitary) × (upper triangular).

(ii) For each invariant inner product $< \, , \, >$ on the Lie algebra $\mathfrak{g}$ of G there is an invariant closed 2-form ω on X which makes it a symplectic manifold, and even fits together with the complex structure to make a Kähler manifold. The tangent space to X at its base-point is $L\mathfrak{g}/\mathfrak{g}$, and ω is given there by

$$\omega(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \langle\xi'(\theta), \eta(\theta)\rangle d\theta \ . \tag{2.2}$$

(iii) The energy function $\mathcal{E} : X \to \mathbb{R}_+$ defined by

$$\mathcal{E}(\gamma) = \frac{1}{4\pi}\int_0^{2\pi} \|\gamma'(\theta)\|^2 \, d\theta$$

is the Hamiltonian function corresponding in terms of the symplectic structure to the circle-action on X which rotates loops. The critical points of $\mathcal{E}$ are the loops which are homomorphisms $\mathbb{T} \to G$. Downwards gradient trajectories of $\mathcal{E}$ emanate from every point of X, and travel to critical points of $\mathcal{E}$. The gradient flow of $\mathcal{E}$ and the Hamiltonian circle action fit together to define a holomorphic action on X of the multiplicative semigroup $\mathbb{C}^{\times}_{\leqslant 1} = \{z \in \mathbb{C} : 0 < |z| \leqslant 1\}$.

The connected components $C_{[\lambda]}$ of the critical set of $\mathcal{E}$ are the conjugacy classes of homomorphisms $\lambda : \mathbb{T} \to G$. They correspond to the orbits of the Weyl group W on the lattice $\pi_1(T)$, where T is a maximal torus of G. The gradient flow of $\mathcal{E}$ stratifies the manifold X into locally closed complex submanifolds $X_{[\lambda]}$, where $X_{[\lambda]}$ consists of the points which flow to $C_{[\lambda]}$. Each stratum $X_{[\lambda]}$ is of finite codimension.

Proposition (2.3). The stratification coincides with the decomposition of X into orbits of $L^-G_{\mathbb{C}}$; i.e. $X_{[\lambda]} = L^-G_{\mathbb{C}}.\lambda$.

Here $L^-G_{\mathbb{C}}$ is the group of loops in $G_{\mathbb{C}}$ which are boundary values of holomorphic maps $D_\infty \to G_{\mathbb{C}}$, where $D_\infty = \{z \in S^2 : |z| > 1\}$. Proposition (2.3) is the classical Birkhoff factorization theorem: a loop γ in $G_{\mathbb{C}}$ can be factorized

$$\gamma = \gamma_- . \lambda . \gamma_+ \ ,$$

with $\gamma_\pm \in L^\pm G_{\mathbb{C}}$, and $\lambda : S^1 \to G$ a homomorphism. This is the analogue of factorizing an element of $GL_n(\mathbb{C})$ as

(lower triangular)$\times$(permutation matrix)$\times$(upper triangular).

There is one dense open stratum X_0 in X. It is contractible, and can be identified with the nilpotent group $L_0^- G_{\mathbb{C}} = \{\gamma \in L^- G_{\mathbb{C}} : \gamma(\infty) = 1\}$.

(iv) The complex structure of X can be characterized in another way, pointed out by Atiyah [1]. To give a holomorphic map $Z \to X$, where Z is an arbitrary complex manifold, is the same as to give a holomorphic principal $G_{\mathbb{C}}$-bundle on $Z \times S^2$ together with a trivialization over $Z \times D_\infty$. If Z is compact it follows that the space of based maps $Z \to X$ in a given homotopy class is finite dimensional; for the moduli space of $G_{\mathbb{C}}$-bundles of a given topological type is finite dimensional. This is a rather striking fact, showing that X, although a rational variety, is quite unlike, say, an infinite dimensional complex projective space: for in X the set of points which can be joined to the base-point by holomorphic curves of a given degree is only finite dimensional.

§3 The Grassmannian embedding of X

Let us choose a finite dimensional unitary representation V of compact group G, and let H denote the Hilbert space $L^2(S^1;V)$. Evidently $LG_{\mathbb{C}}$ acts on H, and we have a homomorphism $i : LG_{\mathbb{C}} \to GL(H)$ an embedding if V is faithful.

To make a more refined statement we write $H = H_+ \oplus H_-$, where H_+ (resp. H_-) consists of the functions of the form $\sum_{n\geq 0} v_n e^{in\theta}$ (resp. $\sum_{n<0} v_n e^{in\theta}$) with $v_n \in V$. The restricted general linear group $GL_{res}(H)$ is defined as the subgroup of $GL(H)$ consisting of elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.1}$$

whose off-diagonal blocks b,c (with respect to the decomposition $H_+ \oplus H_-$) are Hilbert-Schmidt. The blocks a and d are then automatically Fredholm.

Proposition (3.2). $i(LG_{\mathbb{C}}) \subset GL_{res}(H)$.

The set of closed subspaces of H obtained from H_+ by the action of $GL_{res} = GL_{res}(H)$ will be called the Grassmannian Gr(H). It is naturally a Hilbert manifold, and has the homotopy type of the space known to topologists as $\mathbb{Z} \times BU$. The homomorphism

$$i : LG_{\mathbb{C}} \to GL_{res}(H) \qquad (3.3)$$

induces a smooth map (again an embedding if V is faithful)

$$i : X \to Gr(H) \ .$$

This map is closely connected with the Bott periodicity theorem. In fact Bott's theorem asserts that when $G = U_n$ and $V = \mathbb{C}^n$ the map is a homotopy equivalence up to dimension 2n-2.

It should be remarked that i(X) is far from being a closed submanfold of Gr(H). Indeed it is so highly curved that its closure is not a submanifold of Gr(H).

There is a holomorphic line bundle Det on Gr(H) whose fibre at $W \subset H$ can be thought of as the renormalized "top exterior power" of W. Because of the renormalization needed to define it it is not a homogeneous bundle under GL_{res}, but its group of holomorphic automorphisms is a central extension $\widetilde{GL}_{res}$ of GL_{res} by $\mathbb{C}^\times$. The homomorphism (3.3) then gives us a central extension of $LG_{\mathbb{C}}$ by $\mathbb{C}^\times$; up to finite-sheeted coverings, all extensions of $LG_{\mathbb{C}}$ by $\mathbb{C}^\times$ are obtained in this way. (The Lie algebra cocycle of the extension is given by (2.2), where < , > is the trace form of V.)

The line bundle Det has no holomorphic sections, but its dual Det* has an infinite dimensional space of sections Γ on which $\widetilde{GL}_{res}$ acts irreducibly. Just as the space of sections of the dual of the determinant bundle on the Grassmannian Gr(E) of a finite dimensional vector space E can be identified with the exterior algebra $\Lambda(E^*)$ we find

Proposition 3.4. $\Gamma \cong \Lambda(H_+ \oplus \overline{H}_-)$.

This space is very familiar in quantum field theory as the "fermionic Fock space" got by quantizing a classical state space H (e.g. the space of solutions of the Dirac equation) in which H_+ and H_- are the states of positive and negative energy.

From the point of view of loop groups the importance of Γ is that when $G = U_n$ and $V = \mathbb{C}^n$ the projective action of LU_n on Γ via (3.3) is the "basic" irreducible representation of LU_n (cf. §4 below). It even extends from LU_n to LO_{2n}, for Γ is most correctly regarded as the spin representation of the restricted orthogonal group of the real Hilbert space underlying H.

Let us briefly consider generalizing the foregoing discussion to the group Map(X;G), where X is a compact odd-dimensional Riemannian manifold. If H is the space of spinor fields on X then $\mathrm{Map}(X;U_n)$ acts naturally on $H \otimes \mathbb{C}^n$. We can decompose

$$H \otimes \mathbb{C}^n = (H_+ \otimes \mathbb{C}^n) \oplus (H_- \otimes \mathbb{C}^n) \quad ,$$

where $H_\pm$ are the positive and negative eigenspaces of the Dirac operator. We get an embedding

$$\mathrm{Map}(X;U_n) \to GL_{(m)}(H \otimes \mathbb{C}^n) \quad , \tag{3.5}$$

where $GL_{(m)}$ denotes the group of operators of the form (3.1) in which the off-diagonal blocks belong to the Schatten ideal $\mathcal{J}_m$ with $m-1 = \dim(X)$. (Cf. [21],[4].)

The homomorphism (3.5) is very interesting: topologically it represents the index map in K-theory [4]. On the other hand no representations of $GL_{(m)}$ are known, and one even feels that representations are not the natural thing to look for, as the two-dimensional cohomology class which forces $GL_{res} = GL_{(2)}$ to have a projective rather than a genuine representation is replaced by an m-dimensional class for $GL_{(m)}$. Alternatively expressed, on the Grassmannian $Gr_{(m)}(H)$ associated with $GL_{(m)}$ there is a tautological infinite dimensional bundle with a connection. The "determinant" line bundle of this - i.e. its first Chern class - cannot be defined, but nevertheless the higher components of its Chern character do make geometric sense.

§4 The Borel-Weil theory

(i) The basic representation

To simplify the discussion we shall assume from now on that the compact group G is simply connected and simple. Then $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$, and so the complex line bundles L on X are classified by an integer invariant $c_1(L)$. In fact each bundle has a unique holomorphic structure, and has non-zero holomorphic sections if and only if $c_1(L) \geqslant 0$. The space of holomorphic sections of the bundle L_1 with $c_1(L_1) = 1$ is called the basic representation of $LG_{\mathbb{C}}$: we have remarked that when $G = SU_n$ the bundle L_1 is the restriction of Det* on Gr(H). As we saw in that case, L_1 is not quite homogeneous under $LG_{\mathbb{C}}$. The holomorphic automorphisms of L_1 which cover the action of $LG_{\mathbb{C}}$ on X form a group $\widetilde{L}G_{\mathbb{C}}$ which is a central extension of $LG_{\mathbb{C}}$ by $\mathbb{C}^{\times}$ - in fact its universal central extension. It corresponds to the Lie algebra cocycle (2.2) for an inner product $\langle\ ,\ \rangle$ on which I shall also call "basic".

One reason for the name "basic" is provided by

Proposition (4.1). If G is a simply-laced group and Γ is the basic representation of $LG_{\mathbb{C}}$ then any irreducible representation of positive energy is a discrete summand in $\rho^*\Gamma$, where $\rho : LG_{\mathbb{C}} \to LG_{\mathbb{C}}$ is an endomorphism.

(ii) The Borel-Weil theorem

To describe all the positive energy irreducible representations of LG we must consider the larger complex homogeneous space $Y = LG/T$, where T is a maximal torus of G. This manifold Y is fibred over X with the finite dimensional complex homogeneous space G/T as fibre. Complex line bundles on Y are classified topologically by

$$H^2(Y;\mathbb{Z}) \cong \mathbb{Z} \oplus H^2(G/T;\mathbb{Z}) \cong \mathbb{Z} \oplus \hat{T} ,$$

where $\hat{T}$ is the character group of T. Once again each bundle has a unique holomorphic structure, and is homogeneous under $\widetilde{L}G_{\mathbb{C}}$. If we denote the bundle corresponding to $(n,\lambda) \in \mathbb{Z} \oplus \hat{T}$ by $L_{n,\lambda}$ then we have the following "Borel-Weil" theorem.

Proposition (4.2).

(a) The space $\Gamma(L_{n,\lambda})$ of holomorphic sections of $L_{n,\lambda}$ is either zero or an irreducible representation of $LG_{\mathbb{C}}$ of positive energy.

(b) Every projective irreducible representation of LG of positive energy arises in this way.

(c) $\Gamma(L_{n,\lambda}) \neq 0$ if and only if (n,λ) is positive in the sense that

$$0 \leqslant \lambda(h_\alpha) \leqslant n\langle h_\alpha, h_\alpha\rangle$$

for each positive coroot h_α of G, where $\langle\ ,\ \rangle$ is the basic inner product on $\mathfrak{g}$.

It should be emphasized that except for the "if" part of (c) this proposition is quite elementary, amounting to little more than the observations that (i) any representation of positive energy contains a ray invariant under $L^-G_{\mathbb{C}}$, and (ii) $L^-G_{\mathbb{C}}$ acts on Y with a dense orbit. Thus the elementary part already yields

Corollary (4.3). For positive energy representations of LG:

(a) each representation is necessarily projective,

(b) each representation extends to a holomorphic representation of $LG_{\mathbb{C}}$, and

(c) each irreducible representation is of finite type, i.e. if it is decomposed into energy levels $H = \bigoplus H_q$, where H_q is the part where the rotation $e^{i\theta} \in \mathbb{T}$ acts as $e^{iq\theta}$, then each H_q has finite dimension.

Assertion (c) holds because a holomorphic section of $L_{n,\lambda}$ is determined by its Taylor series at the base-point. That gives one an injection

$$\Gamma(L_{n,\lambda}) \to \hat{S}(T_Y^*) \quad , \tag{4.4}$$

where T_Y is the tangent space to Y at the base-point, and $\hat{S}$ denotes the completed symmetric algebra. The injection is compatible with the action of $\mathbb{T}$, and the right hand side of (4.4) is of finite type.

(iii) Unitarity

We have mentioned that all positive energy representations of LG are unitary. In fact a simple formal argument shows that each irreducible representation has a non-degenerate invariant sesquilinear form, but it is not so simple to show that it is positive definite. By (4.1) it is enough to consider the basic representation. When $G = SU_n$ or SO_{2n} the unitarity is then clear from the description (3.4) of the basic representation; and one can deal similarly with all simply laced groups by the method of §5 below. The only proof known in the general case is an inductive argument in terms of generators and relations, due to Garland [7].

It would obviously be very attractive to prove the unitarity directly by putting an invariant measure on the infinite dimensional manifold Y and using the standard L^2 inner product. That has not yet been done, though it seems to be possible. The measure will be supported not on Y but on a thickening Y*, to which the holomorphic line bundles L extend. One expects to have an LG-invariant measure on sections of $\bar{L} \otimes L$ for each positive bundle L. There is no difficulty in finding a candidate for Y*: the manifold Y is modelled on the Lie algebra $N^-\mathfrak{g}_{\mathbb{C}}$ of holomorphic maps $\xi : D_\infty \to \mathfrak{g}_{\mathbb{C}}$ (with $\xi(\infty)$ lower triangular) which extend smoothly to the boundary of D_∞; the thickening is modelled on the dual space, i.e. the holomorphic maps with distributional boundary values on S^1. (*)

(iv) The Kac character formula and the Bernstein-Gelfand-Gelfand resolution

Because each irreducible representation of $\mathbb{T} \tilde{\times} LG$ is of finite type it makes sense to speak of its formal character, i.e. of its decomposition under the torus $\mathbb{T} \times T$. This is given by the Kac character formula, an exact analogue of the classical Weyl character formula.

Thinking of $Y = LG/T$ as $\mathbb{T} \tilde{\times} LG/\mathbb{T} \times T$, we observe that the torus $\mathbb{T} \times T$ acts on Y with a discrete set of fixed points. This set is the affine Weyl group $W_{aff} = N(\mathbb{T} \times T)/(\mathbb{T} \times T)$. If one ignores the infinite dimensionality of Y and writes down formally

(*) An interesting family of measures on Y is constructed in [5], but it does not include the measure needed to prove unitarity.

the Lefschetz fixed-point formula of Atiyah-Bott [2] for the character of the torus action on the holomorphic sections of a positive line bundle L on Y then one obtains the Kac formula, at least if one assumes that the cohomology groups $H^q(Y; \mathcal{O}(L))$ vanish for $q > 0$. (Here $\mathcal{O}(L)$ is the sheaf of holomorphic sections of L.) Unfortunately it does not seem possible at present to prove the formula this way.

One can do better by using more information about the geometry of the space Y. It possesses a stratification just like that of X described in §2. The strata $\{\Sigma_w\}$ are complex affine spaces of finite codimension, and are indexed by the elements w of the group W_{aff}: indeed Σ_w is the orbit of w under $N^- G_{\mathbb{C}} = \{\gamma \in L^- G_{\mathbb{C}} : \gamma(\infty)$ is lower triangular$\}$.

Let Y_p denote the union of the strata of complex codimension p. The cohomology groups $H^*(Y; \mathcal{O}(L))$ are those of the cochain complex $K^{\cdot}$ formed by the sections of a flabby resolution of $\mathcal{O}(L)$. Filtering $K^{\cdot}$ by defining $K^{\cdot}_p$ as the subcomplex of sections with support in $\overline{Y}_p$ gives us a spectral sequence converging to $H^*(Y; \mathcal{O}(L))$ with $E_1^{pq} = H^{p+q}(K^{\cdot}_p/K^{\cdot}_{p+1})$. Because Y_p is affine and has an open neighbourhood U_p isomorphic to $Y_p \times \mathbb{C}^p$ the spectral sequence collapses, and its E_1-term reduces to

$$E_1^{p0} = H^p_{Y_p}(U_p; \mathcal{O}(L)) ,$$

$$E_1^{pq} = 0 \qquad \text{if} \qquad q \neq 0 .$$

In other words $H^*(Y; \mathcal{O}(L))$ can be calculated from the cochain complex $\{H^p_{Y_p}(U_p; \mathcal{O}(L))\}$. Here $H^p_{Y_p}(U_p; \mathcal{O}(L))$ means the cohomology of the sheaf $\mathcal{O}(L)|U_p$ with supports in Y_p. It is simply the space of holomorphic sections of the bundle on Y_p whose fibre at y is

$$L_y \otimes H^p_{\{0\}}(N_y; \mathcal{O}) ,$$

where $N_y \cong \mathbb{C}^p$ is the normal space to Y_p at y; furthermore, $H^p_{\{0\}}(N_y; \mathcal{O})$ is the dual of the space of holomorphic p-forms on N_y. Thus as a representation of $\mathbb{T} \times T$

$$E_1^{p0} \cong \bigoplus_w S(T^*_w \oplus N_w) \otimes \det(N_w) \otimes L_w ,$$

where w runs through the elements of W_{aff} of codimension p, and T_w and N_w are the tangent and normal spaces to Σ_w at w. If we know that $H^q(Y; \mathcal{O}(L)) = 0$ for $q > 0$ then we can read off the Kac formula.

The cochain complex $E_1^{\cdot 0}$ is the dual of the Bernstein-Gelfand-Gelfand resolution, described in the finite dimensional case in [3] (cf. also [13]). Its exactness can be proved by standard algebraic arguments, and one can deduce the vanishing of the higher cohomology groups $H^q(Y; \mathcal{O}(L))$. But it would be attractive to reverse the argument by proving the vanishing theorem analytically.

§5 "Blips" or "vertex operators"

The Borel-Weil construction of representations is quite inexplicit. I shall conclude with a very brief description of an interesting explicit construction of the basic representation of LG, for simply laced G, which was independently extracted from the physics literature in [14], [6] and [19].

The idea is to start with a standard irreducible projective representation H of LT, and to extend the action from LT to LG. The abelian group LT is essentially a vector space, and for H we take its "Heisenberg" representation. To make the Lie algebra $L\mathfrak{g}_{\mathbb{C}}$ act on H amounts to defining, for each basis element ξ_i of $\mathfrak{g}_{\mathbb{C}}$, an "operator-valued distribution" B_i on S^1: for then an element $\Sigma f_i \xi_i$ of $L\mathfrak{g}_{\mathbb{C}}$ will act on H by

$$\sum_i \int_{S^1} f_i(\theta) B_i(\theta) d\theta \ .$$

We must construct B_i for each basis element of $\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$. These are indexed by the <u>roots</u> of G, and the remarkable fact about simply-laced groups (i.e. those for which all the roots have the same length) is that the roots correspond precisely to the set of all homomorphisms $\alpha : \mathbb{T} \to T$ of minimal length. Now for each $\theta \in S^1$ and each small positive ε let us consider the blip-like element $B_{\alpha,\theta,\varepsilon}$ of LT such that

$$B_{\alpha,\theta,\varepsilon}(\theta') = 1 \quad \text{if} \quad |\theta' - \theta| > \varepsilon \ ,$$

while on the interval $(\theta - \varepsilon,\ \theta + \varepsilon)$ of the circle $B_{\alpha,\theta,\varepsilon}$ describes the loop α in T. When $B_{\alpha,\theta,\varepsilon}$ is regarded as an operator on H it turns out that the renormalized limit

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} B_{\alpha,\theta,\varepsilon}$$

exists in an appropriate sense, and is the desired $B_\alpha(\theta)$. Such operators have been called "vertex operators" in the physics literature.

Extending the representation from theLie algebra to LG presents no problems.

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