RECENT DEVELOPMENTS IN REPRESENTATION THEORY

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For the purposes of this lecture, "representation theory" means representation theory of Lie groups, and more specifically, of semisimple Lie groups. I am interpreting my assignment to give a survey rather loosely: while I shall touch upon various major advances in the subject, I am concentrating on a single development. Both order and emphasis of my presentation are motivated by expository considerations, and do not reflect my view of the relative importance of various topics.

Initially G shall denote a locally compact topological group which is unimodular — i.e., left and right Haar measure coincide — and $H \subset G$ a closed unimodular subgroup. The quotient space G/H then carries a G-invariant measure, so G acts unitarily on the Hilbert space $L^2(G/H)$. In essence, the fundamental problem of harmonic analysis is to decompose $L^2(G/H)$ into a direct "sum" of irreducibles. The quotation marks allude to the fact that the decomposition typically involves the continuous analogue of a sum, namely a direct integral, as happens already for non-compact Abelian groups. If G is of type I — loosely speaking, if the unitary representations of G behave reasonably — the abstract Plancherel theorem [12] asserts the existence of such a decomposition. This existence theorem raises as many questions as it answers: to make the decomposition useful, one wants to know it explicitly and, most importantly, one wants to understand the structure of the irreducible summands. In principle, any irreducible unitary representation of G can occur as a constituent of $L^2(G/H)$, for some $H \subset G$. The Plancherel problem thus leads naturally to the study of the irreducible unitary representations.

To what extent these problems can be solved depends on one's knowledge of the structure of the group G and on the nature of the subgroup H. Lie groups, p-adic groups, and algebraic groups over finite fields constitute the most interesting and best understood large classes of

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groups. Although the formal similarities are both striking and instructive, the technical aspects of the representation theory for these three classes diverge — hence the limitation to the case of Lie groups.

Semisimple groups play a distinguished role among all Lie groups, since they come up frequently in physical, geometric, and number-theoretic problems. The special emphasis on semisimple groups can also be justified on other grounds: one of the aims of Mackey's theory of induced representations is to reduce the harmonic analysis on general Lie groups to that on semisimple groups; recently Duflo [13] has worked out the reduction step quite concretely, at least for algebraic groups of type I.

From the point of view of harmonic analysis, irreducible unitary representations are the main objects of interest. Nevertheless, there are important reasons for being less restrictive: non-unitary representations not only occur naturally in their own right, for example as solution spaces of linear differential equations invariant under the action of a semisimple group, but they arise even in the context of unitary representations — a hint of this phenomenon will become visible below. Once one leaves the class of unitary representations, one should not insist on irreducibility; various known constructions produce irreducible representations not directly, but as quotients or subspaces of certain larger representations.

After these preliminaries, I let G denote a semisimple Lie group, connected, with finite center, and K a maximal compact subgroup of G. The choice of K does not matter, since any two maximal compact subgroups are conjugate. By a representation of G, I shall mean a continuous representation on a complete, locally convex Hausdorff space, of finite length — every chain of closed, G-invariant subspaces breaks off after finitely many steps — and 'admissible' in the sense of Harish-Chandra: any irreducible K-module occurs only finitely often when the representation is restricted to K. This latter assumption is automatically satisfied by unitary representations, and consequently G is of type I [19]. No examples are known of Banach representations, of finite length, which fail to be admissible.

To study finite dimesional representations of G, one routinely passes to the associated infinitesimal representations of the Lie algebra. Infinite dimensional representations are generally not differentiable in the naive sense, so the notion of infinitesimal representation requires some care. A vector v in the representation space V_π of a representation π is said to be K-finite if its K-translates span a finite dimensional subspace. By definition, $v \in V_\pi$ is a differentiable vector if the assignment $g \longrightarrow \pi(g)v$ maps G into V_π in a C^∞ fashion. Differentiable and K-finite vectors can be constructed readily, by averaging the translates of arbitrary vectors against compactly supported C^∞ or K-finite functions [17]. One may conclude that the K-finite vectors make up a dense subspace $V \subset V_\pi$, which consist entirely of differentiable vectors — at this point the standing assumption of admissibility plays a crucial role. In particular, the complexified Lie algebra $\mathfrak{g}$ of G acts on V by differentiation. The subgroup K also acts, by translation, but G does not. Partly for trivial reasons, and partly as consequence of the original hypotheses on the representation π, the $\mathfrak{g}$- and K-module V satisfies the following conditions:

a) as K-module, V is a direct sum of finite dimensional irreducibles, each occuring only finitely often;

(1) b) the actions of $\mathfrak{g}$ and K are compatible;

c) V is finitely generated over the universal enveloping algebra $U(\mathfrak{g})$.

Here b) simply means that the $\mathfrak{g}$-action, restricted to the complexified Lie algebra $\mathfrak{k}$ of K, coincides with the derivative of the K-action. This definition of the infinitesimal representation, which was introduced by Harish-Chandra [19], has the very desirable feature of associating algebraically irreducible $\mathfrak{g}$-modules to topologically irreducible representations of G; by contrast, $\mathfrak{g}$ acts in a highly reducible fashion on the spaces of all differentiable or analytic vectors of an infinite dimensional representation π.

A simultaneous $\mathfrak{g}$- and K-module V with the properties (1a-c) is called a Harish-Chandra module. All Harish-Chandra modules can be lifted to representations of G [10,41], not uniquely, but the range of possible topologies is now well-understood [45,54]. If V arises from an irreducible unitary representation π, it inherits an inner product which makes the action of $\mathfrak{g}$ skew-hermitian. An irreducible Harish-Chandra module admits at most one such inner product, up to a positive factor; if it does, the completion becomes the representation space of a unitary representation of G [19,39]. In other words, there is a one-to-one correspondence between irreducible unitary representations of G and Harish-Chandra modules which carry an inner product of the appropriate type. The problem of describing the irreducible unitary representations thus separates naturally into two sub-problems: the description of all irreducible Harish-Chandra modules, and secondly, the determination of those which are "unitarizable". Of the two, the latter seems considerably more difficult, and has not yet been solved, except in special cases — more on this below.

The irreducible Harish-Chandra modules of a general semisimple Lie group were classified by Langlands [33] and Vogan [48]; one of the ingredients of Langlands' classification is due to Knapp-Zuckerman [31]. To describe the classification in geometric terms, I introduce the flag variety of $\mathfrak{g}$,

$$X = \text{set of Borel subalgebras of } \mathfrak{g} \ . \tag{2}$$

It is a complex projective variety and a homogeneous space for the complex Lie group

$$G_{\mathbf{C}} = \text{identity component of } \operatorname{Aut}(\mathfrak{g}) \ . \tag{3}$$

In the case of the prototypical example $G = Sl(n,\mathbf{R})$, X can be identified with the variety of all 'flags' in $\mathbf{C}^n$, i.e. chains of subspaces $0 \subset V_1 \subset V_2 \subset \dots \subset V_n = \mathbf{C}^n$, with $\dim V_k = k$: every Borel subalgebra of the complexified Lie algebra $\mathfrak{g} = \mathbf{sl}(n,\mathbf{C})$ stabilizes a unique flag $\{V_n\}$. The group G acts on the flag variety via the adjoint homomorphism. There are finitely many G-orbits — for $G = Sl(n,\mathbf{R})$, for example, these are characterized by the position of flags relative to the real structure $\mathbf{R}^n \subset \mathbf{C}^n$. Now let $D \subset X$ be a G-orbit, and $L \longrightarrow D$ a homogeneous line bundle — a line bundle with a G-action compatible with that on the base D. A cohomological construction, which I shall describe next, if only in rough outline, associates a family of Harish-Chandra modules to the pair (D,L).

At one extreme, if G contains a compact Cartan subgroup[1], and if D is an open orbit, the homogeneous line bundles over D are parametrized by a lattice. As an open subset of X, D has the structure of complex manifold. Every homogeneous line bundle $L \longrightarrow D$ can be turned into a holomorphic line bundle, so that G acts as a group of holomorphic bundle maps. Thus G acts also on the sheaf cohomology groups of L. The differentiated action of the Lie algebra $\mathfrak{g}$ turns

$$H^*(D,\mathcal{O}(L))_{(K)} = \{ \omega \in H^*(D,\mathcal{O}(L)) \mid \omega \text{ is } K\text{-finite} \} \ , \tag{4}$$

into Harish-Chandra modules. Whenever the line bundle L is negative in the appropriate sense — for example, if L extends to a line bundle over the projective variety X whose inverse is ample — the cohomology appears in only one degree and is irreducible:

$$\begin{aligned} &H^p(D,\mathcal{O}(L)) = 0 \qquad \text{if } p \neq s \ , \\ &H^s(D,\mathcal{O}(L))_{(K)} \text{ is a non-zero, irreducible Harish-Chandra module} \end{aligned} \tag{5}$$

[43]; here s denotes the largest dimension of compact subvarieties of

[1] equivalently, a torus which is a maximal Abelian subgroup.

D. These modules can be mapped $(\mathfrak{g},K)$-equivariantly into $L^2(G) \cap C^\infty(G)$, and are consequently unitarizable. The unitary structure is visible also in terms of the geometric realization: the L^2-cohomology of L injects into the Dolbeault cohomology, its image is dense, has a natural Hilbert space structure, and contains all K-finite cohomology classes [3,44]. The resulting unitary representations make up the discrete series, which was originally constructed by Harish-Chandra via character theory [21].

The opposite extreme, of a totally real G-orbit $D \subset X$, occurs only when $\mathfrak{g}$ contains Borel subalgebras defined over $\mathbb{R}$, as is the case for $G = Sl(n,\mathbb{R})$. In this situation D is necessarily compact, and the cohomological construction collapses to that of the single Harish-Chandra module

(6) $\quad C^\infty(D,L)_{(K)}$ = space of K-finite, C^∞ sections of L .

The module (6) need not be irreducible, but it has a unique irreducible quotient, provided L satisfies a suitable negativity condition. Harish-Chandra modules of this type are induced from a Borel subgroup of G ; they belong to the principal series.

The construction for a general G-orbit D combines elements of "complex induction", as in (4), and ordinary induction, as in (6). It can be viewed as a cohomological form of geometric quantization. There is completely parallel, algebraic version of the construction, due to Zuckerman, which offers certain technical advantages. It is this version that has been studied and used extensively [49]. Subject to certain hypotheses on the pair (L,D) , Zuckerman's "derived functor construction" — equivalently, the geometric construction — produces cohomology in only one degree, a Harish-Chandra module that arises also by induction from a discrete series module of a subgroup of G. Under more stringent assumptions, the Harish-Chandra module corresponding to (L,D) has a unique irreducible quotient [38]. Every irreducible Harish-Chandra module can be realized as such a quotient in a distinguished manner — this, in effect, is Langlands' classification [33]. The problem of understanding the irreducible Harish-Chandra modules does not end here, however. The irreducible quotient may be all of the original module, or may be much smaller. In principle, the Kazhdan-Lusztig conjectures for Harish-Chandra modules, proved by Vogan [51] in the generic case, provide this type of information, but not as explicitly or concretely as one might wish.

I now turn to a different, more recent construction of Harish-Chandra modules, that of Beilinson-Bernstein [6]; similar ideas, in the context of Verma modules, can be found also in the work of Brylinski and

Kashiwara [9]. Some preliminary remarks are necessary. The flag variety X may be thought of as a quotient $G_{\mathbb{C}}/B$; here B is a particular Borel subgroup of $G_{\mathbb{C}}$, i.e. the normalizer of a Borel subalgebra $\mathbf{b} \subset \mathbf{g}$. The differentials of the algebraic characters of B constitute a lattice Λ. Each $\chi \in \Lambda$ — or more precisely, the corresponding character e^{χ} of B — associates a $G_{\mathbb{C}}$-homogeneous, holomorphic line bundle $L_{\chi} \longrightarrow X$ to the principal bundle $B \longrightarrow G_{\mathbb{C}} \longrightarrow X$. Its cohomology groups are finite dimensional $G_{\mathbb{C}}$-modules, which are described by the Borel-Weil-Bott theorem [8]. In particular,

$$
(7) \qquad \begin{array}{l} H^{*}(X,\mathbf{O}(L_{\chi})) \text{ vanishes except in one degree } p = p(\chi) , \\ H^{p}(X,\mathbf{O}(L_{\chi})) \text{ is irreducible .} \end{array}
$$

The center $Z(\mathbf{g})$ of the universal enveloping algebra $U(\mathbf{g})$ acts on the cohomology by scalars (Schur's lemma!), so

$$
(8) \qquad I_{\chi} = \text{annihilator of } H^{*}(X,\mathbf{O}(L_{\chi})) \text{ in } Z(\mathbf{g})
$$

is a maximal ideal in $Z(\mathbf{g})$. As a result of its construction, the line bundle L_{χ} carries an algebraic structure, and it makes sense to define

$$
(9) \qquad \begin{array}{l} \mathbf{D}_{\chi} = \text{sheaf of linear differential operators, with} \\ \text{algebraic coefficients, acting on the sections of } L_{\chi} \text{ ;} \end{array}
$$

the notion of sheaf is taken with respect to the Zariski topology, as befits the algebraic setting. To picture $\mathbf{D}_{\chi}$, one should note that it is locally isomorphic to the sheaf of scalar differential operators on X. The Lie algebra $\mathbf{g}$ operates on sections of L_{χ} by infinitesimal translation. This operation extends to a homomorphism from $U(\mathbf{g})$ into $\Gamma\mathbf{D}_{\chi}$ (= algebra of global sections of $\mathbf{D}_{\chi}$), which in turn drops to an isomorphism

$$
(10) \qquad \Gamma\mathbf{D}_{\chi} \cong U_{\chi} =_{\text{def}} U(\mathbf{g})/I_{\chi}U(\mathbf{g})
$$

[6]. This is the point of departure of the Beilinson-Bernstein construction.

The passage from $U(\mathbf{g})$ to the sheaf $\mathbf{D}_{\chi}$ has a counterpart on the level of $U(\mathbf{g})$-modules: a pair of functors between

$$
(11) \qquad M(U_{\chi}) = \text{category of } U_{\chi}\text{-modules}
$$

— equivalently, the category of $U(\mathbf{g})$-modules on which the center $Z(\mathbf{g})$ acts as it does on the cohomology groups (7) — and

$$
(12) \qquad M(\mathbf{D}_{\chi}) = \text{category of quasi-coherent sheaves of } \mathbf{D}_{\chi}\text{-modules .}
$$

Quasi-coherence means simply that the sheaves admit local presentations in terms of generators and relations, though not necessarily finite presentations. In one direction, the global section functor

$$(13)\qquad \Gamma : M(\mathbf{D}_\chi) \longrightarrow M(U_\chi)$$

maps sheaves of $\mathbf{D}_\chi$-modules to modules over $\Gamma\mathbf{D}_\chi \cong U_\chi$. Extension of scalars from the algebra of global sections U_χ to the stalks of $\mathbf{D}_\chi$ determines a functor in the opposite direction,

$$(14)\qquad \begin{aligned} \Delta &: M(U_\chi) \longrightarrow M(\mathbf{D}_\chi)\ , \\ \Delta V &= \mathbf{D}_\chi \otimes_{U_\chi} V\ ; \end{aligned}$$

the sheaves ΔV are quasi-coherent because every $V \in M(U_\chi)$ can be described by generators and relations.

Those parameters $\chi \in \Lambda$ which correspond to ample line bundles L_χ span an open cone $C \subset \mathbf{R}\otimes_{\mathbf{Z}}\Lambda$. One calls χ dominant if it lies in the closure of C, dominant nonsingular if χ lies in C itself. The inverse of the canonical bundle is ample, and is therefore indexed by a particular dominant nonsingular quantity, customarily denoted by 2ρ. With these conventions it possible to state the following remarkable analogue of Cartan's theorems A and B:

<u>(15) Theorem (Beilinson-Bernstein [6])</u> A) The global sections of any quasi-coherent sheaf of $\mathbf{D}_\chi$-modules generate its stalks, provided $\chi + \rho$ is dominant and nonsingular. B) If $\chi + \rho$ is dominant, the sheaf cohomology groups $H^p(X,\mathbf{V})$ vanish, for every $\mathbf{V} \in M(\mathbf{D}_\chi)$ and $p > 0$.

As a direct consequence, Beilinson-Bernstein deduce:

<u>(16) Corollary</u> In the situation of a dominant nonsingular $\chi + \rho$, the functor Γ defines an equivalence of categories $M(U_\chi) \cong M(\mathbf{D}_\chi)$, with inverse Δ.

Perhaps surprisingly, the equivalence of categories implies properties of general U_χ-modules that were previously unknown. The most fruitful applications, however, occur in the context of certain smaller categories, in particular the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand [7] and the category of Harish-Chandra modules.

Irreducible modules in either of these categories are annihilated by maximal ideals in $Z(\mathfrak{g})$, but not every maximal ideal is of the form (8), with $\chi \in \Lambda$. According to Harish-Chandra [20], the correspondence $\chi \longrightarrow I_\chi$ extends naturally to a surjective map from the vector space $\mathbf{C}\otimes_{\mathbf{Z}}\Lambda$ onto the set of all maximal ideals; any two of the ideals I_χ, I_ψ, for χ, $\psi \in \mathbf{C}\otimes_{\mathbf{Z}}\Lambda$, coincide precisely when $\chi + \rho$ and $\psi + \rho$ belong to the same orbit of the Weyl group W — a finite group which acts linearly on $\mathbb{C}\otimes_{\mathbb{Z}}\Lambda$. I want to emphasize one consequence of Harish-Chandra's description of the maximal ideal space:

(17) every maximal ideal in $Z(\mathfrak{g})$ can be realized as I_χ , with $\chi \in \mathbb{C}\otimes_{\mathbb{Z}}\Lambda$ having the property that $\mathrm{Re}(\chi + \rho)$ is dominant.

Although the bundle L_χ ceases to exist as soon as the parameter χ leaves the lattice Λ , there are "phantom line bundles" attached to all $\chi \in \mathbb{C}\otimes_{\mathbb{Z}}\Lambda$, locally defined line bundles to which the action of $\mathfrak{g}$ on X lifts. The construction (9) of the sheaf of algebras $\mathbf{D}_\chi$ continues to make sense in this wider setting, as do the isomorpism (10), the categories $M(\mathbf{D}_\chi)$, $M(U_\chi)$, and the functors Γ, Δ . Most importantly, theorem (15) and its corollary remain valid [6], with one minor adjustment: the phrase "$\chi + \rho$ is dominant" should be replaced by "$\mathrm{Re}(\chi + \rho)$ is dominant". Different values of χ may correspond to identical maximal ideals I_χ and quotients $U_\chi = U(\mathfrak{g})/I_\chi$, but an appropriate choice of χ will bring any such quotient within the scope of part B of the theorem — this follows from (17). The same χ makes part A and the corollary apply at least generically, for parameters outside a finite number of hyperplanes. The equivalence of categories breaks down in the remaining, singular cases only because certain sheaves fail to have global sections.

The maximal compact subgroup $K \subset G$ possesses a complexification, a complex algebraic group $K_{\mathbb{C}}$, defined over $\mathbb{R}$, which contains K as the group of real points. If $V \in M(U_\chi)$ is a Harish-Chandra module, the K-action induces an algebraic $K_{\mathbb{C}}$-action on the sheaf ΔV . The differential of this action agrees with the multiplication action of $\mathfrak{k}$, viewed as Lie subalgebra of $\Gamma\mathbf{D}_\chi \cong U_\chi$ — in short, $K_{\mathbb{C}}$ and $\mathfrak{k}$ act compatibly. For the purpose of the preceeding discussion, the finiteness condition (1a) in the definition of Harish-Chandra module becomes irrelevant. It is necessary only that K act locally finitely, i.e. the K-translates of any $v \in V$ must span a finite dimensional subspace. The passage from locally finite K-actions on U_χ-modules to algebraic $K_{\mathbb{C}}$-actions on sheaves of $\mathbf{D}_\chi$-modules can be reversed; in other words, both Γ and Δ restrict to functors between

(18)
$M(U_\chi, K)$ = category of U_χ-modules with a compatible, locally finite K-action , and

$M(\mathbf{D}_\chi, K_{\mathbb{C}})$ = category of sheaves of quasi-coherent $\mathbf{D}_\chi$-modules with a compatible, algebraic $K_{\mathbb{C}}$-action

Whenever $\chi + \rho$ is nonsingular and $\mathrm{Re}(\chi + \rho)$ is dominant, the equivalence of categories (16) identifies these two subcategories,

(19) $\Gamma \ : \ M(D_\chi, K_{\mathbb{C}}) \ \cong \ M(U_\chi, K)$.

A theorem of Harish-Chandra asserts, in effect, that finitely generated modules in the category $M(U_\chi, K)$ satisfy the finiteness condition (1a) [19], hence

(20) the irreducible objects in the categories $M(U_\chi, K)$, $\chi \in \mathbb{C}\otimes_{\mathbb{Z}}\Lambda$, exhaust the class of irreducible Harish-Chandra modules .

In particular, the identification (19) relates irreducible Harish-Chandra modules to irreducible sheaves $\mathbf{V} \in M(D_\chi, K_{\mathbb{C}})$.

Geometric considerations suggest how to find such sheaves. The support of any $\mathbf{V} \in M(D_\chi, K_{\mathbb{C}})$ is invariant under the translation action of $K_{\mathbb{C}}$ on X , via the adjoint homomorphism. If $\mathbf{V}$ is irreducible, the support must be an irreducible variety — necessarily the closure of an orbit, since

(21) $K_{\mathbb{C}}$ acts on X with finitely many orbits

[36]. Now let $Y \subset X$ be a $K_{\mathbb{C}}$-orbit, $\bar{Y}$ its closure. The operation of pushforward yields irreducible sheaves with support in $\bar{Y}$, as I shall explain next.

Ordinarily the $\mathbf{D}$-module pushforward of a sheaf exists only as an object of the derived category. In the situation at hand it can be described quite explicitely. The analogy with the C^∞ case is instructive. Linear differential operators on a C^∞ manifold M cannot be applied naturally to the C^∞ functions on a closed submanifold $N \subset M$. However, after the choice of a smooth measure, functions in $C^\infty(N)$ may be viewed as distributions on M , and the sheaf of differential operators $\mathbf{D}_M$ — here in the C^∞ sense — does act on these. The $\mathbf{D}$-module pushforward of the sheaf of smooth measures on N is the sheaf generated by that action; in other words, the sheaf of distributions on M , with support in N , which are smooth along N . Formally, measures and distributions must be treated as sections not of the trivial bundle, but the top exterior power of the cotangent bundle. For this reason the pushforward from N to M involves a twist by the quotient of the two determinant bundles, i.e., a twist by the top exterior power of the conormal bundle. The preceeding discussion can be expressed in terms of local coordinates, and then makes sense equally in the algebraic setting.

Back to the $K_{\mathbb{C}}$-orbit $Y \subset X$! Under an appropriate integrality condition on the parameter χ , the 'phantom line bundle' corresponding to χ extends to the orbit as a true $K_{\mathbb{C}}$-equivariant line bundle, possibly in several different ways. I let $L_{Y,\chi}$ denote a particular such extension, tensored by the top exterior power of the normal bundle. Its sheaf of sections $\mathbf{O}_Y(L_{Y,\chi})$ is a module for a twisted sheaf of differential operators $\mathbf{D}_{Y,\chi}$ on Y. The complement of the boundary ∂Y in X contains Y as a smooth, closed subvariety. Because of the shift built into the definition, the $\mathbf{D}$-module pushforward of $\mathbf{O}_Y(L_{Y,\chi})$ from Y to $X - \partial Y$ is a sheaf of modules over the sheaf $\mathbf{D}_\chi$, restricted to $X - \partial Y$. It becomes a sheaf of $\mathbf{D}_\chi$-modules over all of X when pushed forward once more — naively, simply as a sheaf — from the open subset $X - \partial Y$ to X. The resulting sheaf, which I denote by $\mathbf{V}_{Y,\chi}$, belongs to the category $M(\mathbf{D}_\chi, K_{\mathbf{C}})$, since $K_{\mathbf{C}}$ operates at each step of its construction. A basic result of Kashiwara, on sheaves of $\mathbf{D}$-modules supported by smooth subvarieties, implies

(22)
a) the sheaf of $\mathbf{D}_\chi$-modules $\mathbf{V}_{Y,\chi}$ has a finite composition series and contains a unique irreducible subsheaf ;
b) every irreducible sheaf in the category $M(\mathbf{D}_\chi, K_{\mathbf{C}})$ arises in this manner, for some $K_{\mathbf{C}}$-orbit Y and line bundle $L_{Y,\chi}$.

Under the hypotheses of the equivalence of categories, this statement translates immediately into a classification of the irreducible Harish-Chandra modules which are annihilated by the maximal ideal $I_\chi \subset Z(\mathfrak{g})$:

(23) $\Gamma\mathbf{V}_{Y,\chi}$ has a unique irreducible submodule; the assignment of that module to the datum of the orbit Y and line bundle $L_{Y,\chi}$ establishes a bijection between such pairs $(Y, L_{Y,\chi})$ and irreducible Harish-Chandra modules in $M(U_\chi, K)$.

When $\chi + \rho$ is singular, the situation becomes more complicated, as it does also from the point of view of the Langlands classification. Irreducible Harish-Chandra modules in $M(U_\chi, K)$ can still be realized as submodules of some $\Gamma\mathbf{V}_{Y,\chi}$, but not always as an only irreducible submodule, nor in a unique manner.

The reducibility or irreduciblity of $\mathbf{V}_{Y,\chi}$, in the category of sheaves of $\mathbf{D}_\chi$-modules, is a local phenomenon. All stalks at points of the complement of $\bar{Y}$ vanish, and a small calculation shows those over points of Y to be automatically irreducible. If a non-trivial quotient of $\mathbf{V}_{Y,\chi}$ exists, it also belongs to the category $M(\mathbf{D}_\chi, K_{\mathbf{C}})$ and has support in the boundary. In particular, the sheaf $\mathbf{V}_{Y,\chi}$ cannot possibly reduce unless there is a non-empty boundary: subject to the usual positivity condition on χ,

(24) the Harish-Chandra modules $\Gamma\mathbf{V}_{Y,\chi}$ associated to closed $K_{\mathbf{C}}$-orbits are irreducible .

Non-trivial quotients of $\mathbf{V}_{Y,\chi}$ do exist whenever the line bundle $L_{Y,\chi}$ extends, $\mathfrak{g}$-equivariantly, across some $K_{\mathbf{C}}$-orbit in ∂Y. Matsuki [37] and Springer [47] have worked out the closure relations between $K_{\mathbb{C}}$-orbits; their results make it possible to interpret this geometric irreducibility criterion quite explicitely.

The proof of the original Kazhdan-Lusztig conjectures was the first triumph of $\mathbf{D}$-modules in representation theory. Irreducible modules in the category $\mathcal{O}$ arise from orbits of a Borel subgroup $B \subset G_{\mathbb{C}}$, via the same process of pushforward, taking sections, and passing to the unique irreducible submodule. Kazhdan and Lusztig [29] had already related their conjectured composition multiplicities for Verma modules to the intersection cohomology of closures of B-orbits, i.e., of Schubert varieties. Both Brylinski-Kashiwara [9] and Beilinson-Bernstein [6] saw the connection with the theory of $\mathbf{D}$-modules; they independently established the conjectures, by relating the intersection cohomology to the composition multiplicities of the appropriate sheaves. This second step carries over, essentially unchanged, to the setting of Harish-Chandra modules. The paper [35] of Lusztig and Vogan contains the analogue of the first ingredient, namely the combinatorics of the intersection cohomology of closures of $K_{\mathbb{C}}$-orbits. Vogan [51], finally, deduces multiplicity formulas for the Langlands classification, which he had conjectured earlier [50]. I should point out that the original Kazhdan-Lusztig conjectures cover only U_χ-modules with $\chi \in \Lambda$, as does the known version for Harish-Chandra modules; Vogan's conjectures, by contrast, apply to the general case.

At first glance, the Beilinson-Bernstein construction appears far removed from the construction of Harish-Chandra modules in terms of line bundles on G-orbits. The former leads quickly to geometric reducibility criteria, as we just saw, and opens a path towards the Kazhdan-Lusztig conjectures for Harish-Chandra modules. It also has points of contact with Vogan's classification via K-types [48]; indeed, it probably implies the results of [48]. The G-orbit construction, on the other hand, is closely tied to Langlands' classification, which in turn relates it to certain analytic invariants of Harish-Chandra modules: the asymptotic behavior of matrix coefficients, for example, and the global character [10,11,24]. Since the two constructions complement each other, the possible connections between them merit attention.

Results of Matsuki [36], on orbits in flag varieties, provide an important clue. For each G-orbit $D \subset X$, there exists a unique $K_{\mathbb{C}}$-orbit Y, such that K acts transitively on the intersection $D \cap Y$; conversely every $K_{\mathbb{C}}$-orbit intersects a unique G-orbit in this manner. The correspondence $D \longleftrightarrow Y$ reverses the relative sizes of orbits, as measured by their dimensions — I shall therefore call D "dual" to the orbit Y. Once the parameter χ has been fixed, the duality $D \longleftrightarrow Y$ extends to the line bundles which enter the two constructions: a homogeneous line bundle $L \to D$ is dual to $L_{Y,\chi} \to Y$ if the tensor product $L \otimes L_{Y,\chi}$ restricts to a trivial K-homogeneous vector bundle over $D \cap Y$.

It is instructive to examine the special case of $Sl(2,\mathbf{R})$, or equivalently, its conjugate $SU(1,1)$. The diagonal matrices in $G = SU(1,1)$ constitute a maximal compact subgroup $K \cong U(1)$. Both G and $K_{\mathbf{C}} \cong \mathbf{C}^*$ act on the flag variety $X \cong \mathbf{CP}^1 \cong \mathbf{C} \cup \{\infty\}$, as groups of Möbius transformations. The duality relates the three $K_{\mathbf{C}}$-orbits $\{0\}$, $\{\infty\}$, $\mathbf{C}^*$, in the given order, to the G-orbits Δ (= unit disc), Δ' (= complement of the closure of Δ), S^1. A homogeneous line bundle over Δ is determined by a character of the isotropy subgroup at 0 , i.e. a character e^λ of K. Dually, a $K_{\mathbf{C}}$-homogeneous line bundle over the one-point space $\{0\}$ has a single fibre, on which $K_{\mathbf{C}}$ acts by an algebraic character e^χ. The differentials λ, χ may be viewed as linear functions on $\mathbf{k} \cong \mathbf{C}$, whose values on $\mathbf{Z} \subset \mathbf{C}$ are integral multiples of $2\pi i$; here the duality reduces to $\lambda \longleftrightarrow \chi = -\lambda$. The situation for the orbits Δ', $\{\infty\}$, is entirely analogous. At points $z \in S^1$, the isotropy subgroup $G_z \subset G$ has two connected components. Its characters are parametrized by pairs (λ, ζ) , consisting of a complex number λ and a character ζ of the center $\{\pm 1\} \subset G$, which meets both connected components of G_z. The corresponding G-equivariant line bundle over S^1 extends holomorphically at least to the $K_{\mathbf{C}}$-orbit $\mathbf{C}^*$, as does the dual, or inverse line bundle. If the line bundle is to extend even across $\{0\}$ or $\{\infty\}$, the pair (λ, ζ) must lift to a character of the complexification of G_z — this happens precisely when $\lambda/2\pi i$ is integral and ζ trivial or non-trivial, depending on the parity of $\lambda/2\pi i$.

To a $K_{\mathbf{C}}$-homogeneous line bundle $L_\chi \to \{0\}$, the Beilinson-Bernstein construction assigns the Harish-Chandra module of 'holomorphic distributions' supported at 0 , with values in the bundle L_χ — in other words, the $U(\mathfrak{g})$-submodule generated by 'evaluation at 0' in the algebraic dual of the stalk $\mathbf{O}_{\{0\}}(L_\chi^* \otimes T_X^*)$; the formal duality between functions and differentials accounts for the appearance of the cotangent bundle T_X^*. By its very definition, this module is dual, in the sense of Harish-Chandra modules, to $H^0(\Delta, \mathbf{O}(L_\chi^* \otimes T_X^*))_{(K)}$, the module associated to the G-orbit Δ and the line bundle $L_\chi^* \otimes T_X^*$. For non-negative values of the integral parameter $\chi/2\pi i$, the resulting Harish-Chandra modules are irreducible and belong to the discrete series. They become reducible if $\chi/2\pi i < -1$; in this situation the equivalence of categories (19) no longer applies. The preceeding discussion carries over, word-for-word, to the pair of orbits $\{\infty\}$, Δ'. As for the orbits S^1 and $\mathbf{C}^*$, the two constructions start with the choice of a G-homogeneous line bundle $L_{\lambda,\zeta} \longrightarrow S^1$. Its extension to $\mathbf{C}^*$, which I denote by the same symbol, comes equipped with an action of $\mathfrak{g}$ and an algebraic structure. Since $\mathbf{C}^*$ is open in X, the pushforward construction attaches the space of algebraic sections $H^0(\mathbf{C}^*, \mathbf{O}(L_{\lambda,\zeta}))$ to the datum of the $K_{\mathbf{C}}$-orbit $\mathbf{C}^*$ and line bundle $L_{\lambda,\zeta}$ Integration over S^1 pairs this Harish-Chandra module nondegenerately with $C^\infty(S^1, L^*_{\lambda,\zeta} \otimes T^*_X)_{(K)}$, the module corresponding to the G-orbit S^1 and line bundle $L^*_{\lambda,\zeta} \otimes T^*_X$. The hypothesis of the equivalence of categories translates into the inequality $\operatorname{Re}\lambda > -1$. On the Beilinson-Bernstein side, this implies the existence of a unique irreducible submodule: the entire module generically, when $L_{\lambda,\zeta}$ cannot be continued across 0 and ∞, otherwise the finite dimensional submodule consisting of sections regular at the two punctures. The realization of the dual module $C^\infty(S^1, L^*_{\lambda,\zeta} \otimes T^*_X)_{(K)}$ exhibits both Harish-Chandra modules as members of the principal series.

One phenomenon that does not show up in the case of $G = SU(1,1)$ is the occurence of higher cohomology. For general groups, without any positivity assumption on the parameter χ, the sheaves $\mathbf{V}_{Y,\chi}$ can have non-zero cohomology groups above degree zero, but these are still Harish-Chandra modules. Zuckerman's derived functor construction also produces a family of Harish-Chandra modules $I^p(D,L)$, for each G-orbit D and G-equivariant line bundle $L \longrightarrow D$, indexed by an integer $p \geq 0$. The example of $SU(1,1)$ suggest a duality between the two constructions, and indeed this is the case. I fix pairs of data $(Y, \mathbf{V}_{Y,\chi})$, (D,L), which are dual in the sense described above, and define $s = \dim_{\mathbf{R}}(Y \cap D) - \dim_{\mathbf{C}} Y$, $d = \dim_{\mathbf{C}} X$. Then

(25) there exists a natural, nondegenerate pairing between the Harish-Chandra modules $H^p(Y, \mathbf{V}_{Y,\chi})$ and $I^{s-p}(D, L \otimes \Lambda^d T^*_X)$,

for all $p \in \mathbf{Z}$, with no restriction on χ (Hecht-Miličić-Schmid-Wolf [23]). In both constructions homogeneous vector bundles can be substituted for line bundles. The duality carries over to this wider setting, and then becomes compatible with the coboundary operators. Earlier, partial results in the direction of (25) appear in Vogan's proof of the Kazhdan-Lusztig conjectures for Harish-Chandra modules [51]; there Vogan identifies certain Beilinson-Bernstein modules with induced modules, by explicit calculation.

The duality does not directly relate the Beilinson-Bernstein classification to that of Langlands: in the language of geometric quantization, the latter uses partially real polarizations, whereas the former works with arbitrary, mixed polarizations. This problem can be dealt with on the level of Euler characteristics, and the known vanishing theorems for the two constructions are sufficiently complementary to permit a comparison after all. In particular, it is possible to carry techniques and results back and forth between the two constructions [23].

By definition, the discrete series is the family of irreducible unitary representations which occur discretely in $L^2(G)$. It was remarked earlier that G has a non-empty discrete series if it contains a compact Cartan subgroup; these representations then correspond to open G-orbits, and their unitary structures are related to the geometric realization. Open G-orbits are dual to closed $K_{\mathbb{C}}$-orbits, so the observation (24) "explains" the irreducibility statement (5). The discrete series lies at one extreme of the various non-degenerate series of irreducible unitary representations — the other series consist of representations unitarily induced from discrete series representations of proper subgroups. These are precisely the representations which occur in the decomposition of $L^2(G)$ [22]. Roughly speaking, they are parametrized by hermitian line bundles over G-orbits. Here, too, the inner products have geometric meaning [56]. As for the rest of the unitary dual, the picture remains murky, though substantial progress has been made during the past few years. I shall limit myself to some brief remarks, since more detailed summaries can be found in the articles [30,52] of Knapp-Speh and Vogan.

A unitarizable Harish-Chandra module V is necessarily conjugate isomorphic to its own dual, a property which translates readily into a condition on the Harish-Chandra character, or on the position of V in the Langlands classification. If the condition holds, V admits a nontrivial $\mathfrak{g}$-invariant hermitian form — only one, up to scalar multiple, provided V is irreducible. The real difficulty lies in deciding whether the hermitian form has a definite sign. For a one parameter family V_t of irreducible Harish-Chandra modules of this type, the form stays definite if it is definite anywhere: not until the family reduces at some $t = t_0$ can the hermitian form become indefinite; even the composition factors at the first reduction point are unitarizable. In the case of $Sl(2,\mathbb{R})$, such deformation techniques generate the complementary series and the trivial representation — in other words, all of the unitary dual outside the discrete series and the unitary principal series [5]. Examples of Knapp and Speh [30] suggest that the analogous phenemenon for general groups can become quite complicated.

Neither induction nor deformation techniques account for isolated points in the unitary dual. Typically isolated unitary representations do exist, beyond those of the discrete series, but with certain formal similarities to the discrete series. Zuckerman's derived functor construction, and the Beilinson-Bernstein construction as well, extends to orbits in generalized flag vartieties, i.e., quotients $G_{\mathbb{C}}/P$ by parabolic subgroups $P \subset G_{\mathbb{C}}$. The G-invariant hermitian line bundles over an open G-orbit $D \subset G_{\mathbb{C}}/P$ are parametrized by the character group of the center of the isotropy subgroup $G_z \subset G$ at some reference point $z \in D$. Whenever that center is compact, the derived functor construction produces a discrete family of irreducible Harish-Chandra modules. According to a conjecture of Zuckerman, which was recently proved by Vogan [53], these modules are unitarizable. Vogan actually proves more; in geometric language, the cohomology of G-invariant vector bundles, modeled on irreducible unitary representations of the isotropy group G_z, vanishes in all but one degree and can be made unitary, again under an appropriate negativity assumption on the bundles. The proof consists of an algebraic reduction to the case of the non-degenerate series: Vogan introduces a notion of signature for $\mathfrak{g}$-invariant hermitian forms on Harish-Chandra modules, formal sums of irreducible characters of K with integral coefficients; he then calculates these signatures for the derived functor modules, in terms of the K-multiplicities of induced modules. Because of the origin of Zuckerman's conjecture, one might hope for a geometric proof. Earlier attempts in this direction were only marginally successful, but give a hint of a possible strategy [42].

A complete description of the unitary dual exists for groups of low dimension, for groups of real rank one [4,27,32], and the family SO(n,2) [1]. Vogan has just announced a classification also for the special linear groups over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ — a big step, since there is no bound on real rank. In effect, the methods of unitary induction, degeneration, and Vogan's proof of the Zuckerman conjecture generate all irreducible unitary representations of the special linear groups. One feature that makes these groups more tractable is a hereditary property of their parabolic subgroups: all simple factors of the Levi component are again of type Sl_n. In the general case, conjectures of Arthur [2] and Vogan [52] predict the unitarity of certain highly singular representations. There are also results about particular types of unitary representations [14, 15,28], but a definite common pattern has yet to emerge.

I close my lecture by returning to its starting point, the decomposition of $L^2(G/H)$. The solution of this problem for $H = \{e\}$ — the explicit Plancherel formula [22] — was aim and crowning achievement of Harish-Chandra's work on real groups. A discussion of his proof would lead too far afield. However, I should mention a recent elementary", though not simple, argument of Herb and Wolf [26]. It is based on Herb's formulas for the discrete series characters [25], and emulates Harish-

Chandra's original proof in the case of $Sl(2,\mathbf{R})$, by integration by parts [18].

The decomposition problem has been studied systematically for two classes of subgroups H, besides the identity group: arithmetically defined subgroups, and symmetric subgroups, i.e. groups of fixed points of involutive automorphisms. The symmetric case contains the case of the trivial group, since G can be identified with $G\times G$/diagonal. Oshima and Matsuki [40], building on a remarkable idea of Flensted-Jensen [16], have determined the discrete summands of $L^2(G/H)$, for any symmetric $H \subset G$; these representations are parametrized by homogeneous line bundles over certain orbits in generalized flag varieties. Oshima has also described a notion of induction in the context of symmetric quotients. Presumably $L^2(G/H)$ is made up of representations which are induced in this sense, from discrete summands belonging to smaller quotients, but the explicit decomposition remains to be worked out. The case of arithmetically defined subgroups is the most interesting from many points of view, and the most difficult. Again the discrete summands constitute the "atoms" of the theory, as was shown by Langlands [34] — the Eisenstein integral takes the place of induction. There is an extensive literature on the discrete spectrum, too extensive to be summarized here, yet a full understanding does not seem within reach.

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