COMMENTARY ON THE ARTICLE OF MANIN

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§1. Mathematics and Physics

Manin's stimulating contribution to the 25th Arbeitstagung which, in his absence, I attempted to present, provided me with an opportunity of adding some further reflections of my own. This commentary, which is therefore a very personal response to Manin's article, consists of very general and speculative remarks about large areas of contemporary mathematics. Such speculations are, for good reason, rarely put down on paper but the record of the 25th Arbeitstagung provides a rather singular occasion where ideas of this type may not be out of place.

In recent years there has been a remarkable resurgence of the traditional links between mathematics and physics. A number of striking ideas and problems from theoretical physics have penetrated into various branches of mathematics, including areas such as algebraic geometry and number theory which are rarely disturbed by such outside influences. Perhaps a few specific examples will illustrate the point. The Kadomtsev-Petiashvilli equation, which arose in plasma physics, has been shown to be extremely relevant to the classical Schottky problem about the characterization of Jacobian varieties of algebraic curves (this was explained in the Arbeitstagung lecture of van der Geer). Witten's analytical approach to the Morse inequalities, based on the physicist's use of stationary-phase approximation, has led Deligne and others to imitate his ideas in number theory with great success. The Yang-Mills equations and their 'instanton' solutions have been brilliantly exploited by Donaldson to solve outstanding problems on 4-manifolds.

All these examples connect physics with various branches of geometry, and it is therefore natural that Manin should have attempted an overview of geometry in the widest sense. The picture he describes is best indicated by the following schematic diagram:

$$\underbrace{\text{Arithmetic} \xleftarrow{\text{Algebra}} \text{Geometry} \xrightarrow{\text{Analysis}} \text{Physics}}$$

$$\begin{tikzcd}[column sep=small] & \text{Topology} \arrow[dl, dash] \arrow[dr, dash] & \\ \text{Homology} & & \text{K-theory} \end{tikzcd}$$

As this suggests, ideas from topology, notably homology and K-theory, provide a common language and underpinning for the whole structure. The bridge between geometry and arithmetic was greatly expanded and developed during the Grothendieck era with the introduction of 'schemes'. The bridge between geometry and physics begins essentially with Einstein's theory of gravitation but has become much stronger with the recent development of gauge theories of elementary particles.

The picture just envisaged is restricted, on the physics side, to classical physics. However, one should be more ambitious and try to fit quantum physics into the picture also. I will have more to say on this aspect later.

§2. Arithmetic manifolds

An algebraic curve defined by equations with integer coefficients can be viewed as a scheme over Spec $\mathbb{Z}$. It is the analogue of a surface mapped onto a curve, the 'fibre' over a prime p being the curve reduced mod p. Such an arithmetic surface can be 'compactified' by adding the Riemann surface of the curve over the 'prime' at ∞. The Arakelov-Faltings theory is then concerned with extending as much as possible of the usual theory of surfaces to this arithmetic case.

For this purpose it turns out that one needs to introduce or find canonical metrics on various objects associated to the Riemann surface. For example, given a line-bundle $\mathcal{L}$ on the Riemann surface one has the one-dimensional complex vector space

$$\det H^{0}(\mathcal{L})/\det H^{1}(\mathcal{L})$$

(where det denotes the highest exterior power), and one wants a natural metric on this space.

This particular problem which was solved in one way by Faltings has been examined in a wider context (e.g. replacing $\mathcal{L}$ by a vector bundle) by Quillen. He has shown that a natural definition arises by using the regularized determinants of Laplace type operators which were introduced into differential geometry by Ray and Singer [3]. Such operator determinants are extensively used by physicists in quantum field theory, and this link between geometry and physics is currently the scene of many investigations. In any case it provides a clear link with quantum and not purely classical physics.

On the Riemann surface itself there are two natural metrics (for $g \geq 2$), one being the Poincaré metric and the other being the metric induced by the holomorphic differentials. In higher dimensions the analogues of the Poincaré metric are the Kähler-Einstein metrics. Similarly for stable vector bundles there are distinguished metrics, and Manin proposes they should be used for a higher dimensional theory of arithmetic manifolds. It is interesting to note that all of these metrics arose in a physical context.

Thus the geometry of Kähler manifolds, and in particular the study of operator determinants on such manifolds, appears as a natural meeting point for arithmetic and physics. In this context it is perhaps worth pointing out that a number of differential-geometric invariants constructed from operator determinants have already been identified with quantities arising in number theory. Thus Ray and

Singer [4] made a connection with the Selberg zeta-function of a Riemann surface while Millson [2] did something similar in higher odd dimensions. Also values of certain L-functions of totally real number fields have been related in [1] to the eta-invariant (essentially the logarithm of a certain operator determinant) : the eta-invariant is also what appears in [2].

§3. Fermions

There is a basic distinction in physics between two types of particles, namely bosons and fermions. Bosons involve commuting variables and so are easily understood on a geometric level, but fermions involve anti-commuting variables and so are more mysterious geometrically. On a purely algebraic level of course there is no mystery: polynomial and exterior algebras are both well understood and extensively studied. However, the development of gauge theories in physics where geometric insight and interpretation greatly assist the purely formal algebraic aspect has naturally led to the attempt to develop a 'super-geometry' in which both sets of variables are incorporated.

The development of super-manifolds as outlined by Manin appears to be an elegant extension of classical geometric ideas and it should throw light on the algebraic computation of physicists who build 'super-symmetric' theories. Nevertheless the theory still appears to lack some essential ingredients and Manin asks whether the fermionic coordinates can somehow be 'compactified' so as to make them topologically more interesting. In this context I would like to make a tentative suggestion concerning the right geometric way to interpret fermionic variables.

Consider first a smooth manifold M, its de Rham complex $\Omega^*(M)$ and in particular the ring $\Omega^O(X)$ of smooth functions. For a super-symmetric analogue, suppose now that M is a closed sub-manifold of another manifold N, and let $A = \Omega^*_M(N)$ denote the complex of currents on N which are supported on M and smooth in the M-directions (recall that a current is just a differential form with distributional coefficients). Locally an element of A can be expressed in the form

$$\Sigma\, f_{\alpha\beta\gamma}(x)\,\partial_y^\gamma \delta\, dx^\alpha \wedge dy^\beta$$

where $x = (x_1 \ldots x_m)$ are coordinates on M, $y = (y_1, \ldots, y_r)$ are normal coordinates to M in N, δ is the Dirac δ-function of M, α, β are skew-symmetric multi-indices and γ is a symmetric multi-index (so that ∂_y^γ represents derivatives in y). If we take α, γ to be empty and β to be a single index we get a subspace $R\delta$ of A where R is the super-ring of the super-manifold given by M and its normal bundle in N. On the other hand A itself should be viewed as the super de Rham complex of this super-manifold.

The advantage of this point of view is that approximating the δ-function by suitable smooth functions (e.g. Gaussians) we can try to interpret fermions as bosons on N which are very sharply peaked along M. More precisely the fermions should appear as 'leading terms' of such sharply peaked bosons. Geometrically this might correspond to putting a metric on N which is very sharply curved along M, so that M is an 'edge of regression' in the language of classical differential geometry.

I am trying to suggest that super-geometry should be some kind of limit of ordinary geometry and not an entirely different kind of entity constructed simply by formal analogy.

§4. The quantum level

Quantum theory is characterized by infinite-dimensionality and by non-commutativity. When trying to understand the possible geometric counterpart of some aspect of quantum-theory this must be borne in mind.

As I have already mentioned the study of linear elliptic operators provides one bridge between geometry and quantum field theory. For example ideas from supersymmetric field theories have cast new light on the index theorem.

In a different direction it is I think not inappropriate to consider Connes' non-commutative differential geometry (see the survey talk by Connes in this volume) as a version of quantized geometry. Recall that Connes studies situations such as the ergodic action of a discrete group on a manifold where the geometric quotient does not exist in any way as a reasonable space. However, a non-commutative algebra exists with which various geometric constructions can still be made.

In the lecture of Lang he explained a conjecture of Vojta based on an interesting analogy between arithmetic surfaces and Nevanlinna theory. It is perhaps interesting in this connection that John Roe in his Oxford D.Phil. thesis shows how the Nevanlinna theory fits into Connes' framework. Analysing this situation might shed light on the analogy between Connes' theory and questions in Arithmetic.

If one asks for the analogue of quantum theory in Arithmetic one can hardly avoid considering the whole Langlands programme. Adelic groups are obvious analogues of gauge groups and Hilbert space representations are the basic objects of the theory. This analogy deserves closer scrutiny, particularly in view of the fact that non-abelian dualities, generalizing class-field theory on the one hand and electric-magnetic Maxwell duality on the other, seem to be a main objective in both number theory and physics. Perhaps our classical diagram should be enlarged to a quantum diagram in the following way:

$$\begin{tikzcd}
\text{Langlands} \arrow[d, dash] & \text{Connes} \arrow[l] \arrow[r] \arrow[d, dash] & \text{Quantum Field Theory} \arrow[d, dash] & \text{Quantum} \\
\text{Arithmetic} & \text{Geometry} \arrow[l] \arrow[r] & \text{Physics} & \text{Classical}
\end{tikzcd}$$

References

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