AN INTRODUCTION TO INFINITESIMAL VARIATIONS OF HODGE STRUCTURES

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The purpose of this note is to give a simple introduction to the notion of infinitesimal variation of Hodge structure. This is an object first defined and used in [1] (though the underlying ideas had been in the air for a while) and more recently the subject of an excellent monograph by Peters and Steenbrink [2]. Unfortunately, this theory, which in fact should make life easier for mathematicians trying to apply Hodge theory to geometry, gives at first the impression of being complicated and technical. It is my hope here to avoid this impression by presenting the basic ideas of the theory in as simple a fashion as possible.

We begin by recalling the basic set-up of Hodge theory. The goal of this theory is to associate to an m-dimensional complex manifold X (for simplicity we will take X a submanifold of $\mathbb{P}^N$) a linear-algebraic invariant, as follows. To begin with, we can for each n associate to X its $n^{\underline{th}}$ topological cohomology group modulo torsion $H_{\mathbb{Z}} = H^n(X,\mathbb{Z})/\text{tors}$, or its complexification the $n^{\underline{th}}$ deRham cohomology group $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C} = H^n_{DR}(X)$. We can also associate the cup product in cohomology; or rather, since we are only dealing with one group at a time, the bilinear pairing

$$Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \longrightarrow \mathbb{Z}$$

defined by

$$Q(\alpha,\beta) = \int_X \alpha \cup \beta \cup \omega^{m-n}$$

where ω is the restriction to X of the generator of $H^2(\mathbb{P}^N,\mathbb{Z})$. Of course, these are invariants of the underlying differentiable manifold of X, and do not reflect its complex structure. What does determine the complex structure of X is the decomposition of the complexified tangent spaces to X into holomorphic and anti-holomorphic parts; or, equivalently, the decomposition of the space $A^n(X)$ of differential forms of degree n on X by type:

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}$$

Naturally, this data is too cumbersome to carry around, but here we are in luck: by the Hodge theorem, this decomposition descends to the level of cohomology. Precisely, if we let $H^{p,q} = H^{p,q}(X) \subset H^n_{DR}(X)$ be the subspace of classes representable by forms of type (p,q), we get a decomposition

$$H^n_{DR}(X) = H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$$

satisfying the obvious relations

$$\overline{H^{p,q}} = H^{q,p}$$

and

$$Q(H^{p,q}, H^{p',q'}) \equiv 0 \quad \text{unless} \quad p+p' = q+q' = n .$$

The package of data introduced so far -- a lattice $H_{\mathbb{Z}}$ with integral bilinear form Q and decomposition $H_{\mathbb{Z}} \otimes \mathbb{C} = \oplus H^{p,q}$ satisfying these relations -- we call a Hodge structure of weight n associated to X. It is an object that is on one hand essentially finite, and that on the other hand we may hope will reflect the geometry of X.

Now, whenever we associate to a geometric object a (presumably simpler) invariant, two questions arise: to what extent does the invariant actually determine the original object; and to what extent can we read off directly from the invariant answers to naive questions about the geometry of the object. In the present circumstances, the first question translates into the Torelli problem, which asks when the members of a given family of varieties (e.g., curves of genus g, hypersurfaces of degree d in $\mathbb{P}^n$) are determined by their Hodge structures; or the "generic" or "birational" Torelli problem, which asks when this is true for a general member of the family. The Torelli problem has been answered in a number of cases (e.g., for curves of genus g it was proved classically by Torelli; and the generic Torelli for hypersurfaces was proved recently by Donagi); it remains very much an open question in general.

The most famous example of a question in Hodge theory along the lines of the second sort above is of course the Hodge conjecture. It is not hard to see that if $Y \subset X$ is an analytic subvariety of codimension k, its fundamental class must lie in the subspace $H^{k,k} \subset H^{2k}(X,\mathbb{C})$. The Hodge conjecture asks whether the converse is true: that is, whether a class $\gamma \in H^{k,k} \cap H_{\mathbb{Z}}$ is necessarily a rational linear combination of classes of subvarieties.

The simplest case of Hodge theory is its application to curves, and here by any standards it is successful. To the Hodge structure

$(H_{\mathbb{Z}}, Q, H^{1,0} \oplus H^{0,1})$ of a curve we associate the projection Λ of $H_{\mathbb{Z}}$ to $H^{0,1}$ (traditionally represented by the period matrix: we choose a basis for $H_{\mathbb{Z}}$ normalized with respect to Q and write out the $(0,1)$-components of these vectors in a $g \times 2g$ matrix Ω) and then the complex torus $H^{0,1}/\Lambda = J(C)$, called the Jacobian of C. This in turn gives rise to a host of subvarieties of $J(C)$ and theta-functions that reflect and elucidate the geometry of C.

For higher-dimensional varieties, the application of Hodge theory has been less successful, for which there are perhaps two reasons. The first of these is the apparent absence of any reasonably natural geometric and/or analytic object associated to a Hodge structure in general. Looking at the case of curves, one sees that it is exactly through the geometry of the Jacobian, and the analysis of the theta-function, that Hodge theory is useful. Unfortunately, no analogous objects have been found in general.

The second factor is simply this: that only in a very few cases can one ever hope to determine explicitly the Hodge structure of a given variety. To be specific, consider the case of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ given by a homogeneous polynomial $F(Z) = 0$ of degree d. By the Lefschetz theorem, all the cohomology of X below the middle dimension (and hence above it as well) is at most one-dimensional, so we focus on $H^n(X)$. We can immediately identify one of the Hodge groups: $H^{n,0}(X)$, the space of holomorphic n-forms on X, may be realized as Poincaré residues of $(n+1)$-forms on $\mathbb{P}^{n+1}$ with poles along X; explicitly,

$$\omega = \operatorname{Res}\left(\frac{G(Z_0,\ldots,Z_{n+1})\, d\left(\frac{Z_1}{Z_2}\right)\wedge\ldots\wedge d\left(\frac{Z_{n+1}}{Z_0}\right)\cdot Z_0^{n+1}}{F(Z_0,\ldots,Z_n)}\right)$$

$$= \frac{G(Z_0,\ldots,Z_{n+1})\, d\left(\frac{Z_1}{Z_0}\right)\wedge\ldots\wedge d\left(\frac{Z_i}{Z_0}\right)\wedge\ldots\wedge d\left(\frac{Z_{n+1}}{Z_0}\right)\cdot Z_0^{n}}{\frac{\partial F}{\partial Z_i}(Z_0,\ldots,Z_{n+1})}$$

for $G(Z)$ a homogeneous polynomial of degree $d-n-1$. Thus

$$H^{n,0} = S_{d-n-1}$$

where S is the graded ring $\mathbb{C}[Z_0,\ldots,Z_{n+1}]$. Similarly, the other Hodge groups of X may be realized as residues of forms on $\mathbb{P}^{n+1}$ with higher-order poles on X (actually, we get in this way just the primitive cohomology $H^n_{pr}(X)$, which here means the classes orthogonal to ω). We obtain an identification

$$H^{n-k,k}_{pr}(X) = (S/J)_{(k+1)d-n-1}$$

where $J \subset S$ is the Jacobian ideal of X, that is, the homogeneous ideal generated by the partial derivatives of X.

We have thus found the vector space decomposition $H^n_{pr}(X) = \oplus H^{n-k,k}_{pr}(X)$. The problem is, it is impossible in general to identify in these terms the lattice $H_{\mathbb{Z}}$ of integral classes. Indeed, this has been done only in the presence of a large automorphism group acting on X, e.g., for Fermat hypersurfaces. Thus, for example, if one is given a particular hypersurface of even dimension $n = 2k$, it is impossible to determine in general $H^{k,k}(X) \cap H^n(X,\mathbb{Z})$, or when two such X have the same Hodge structure. Simply put, we cannot find the lattice; but without the lattice we have no invariants.

One solution of this difficulty appears at first to be moving in the wrong direction, toward increased difficulty. One considers not just a variety X, but a family of varieties $\{X_b\}_{b\in B}$ parametrized by a variety B, of which $X = X_0$ is a member; we assume $0 \in B$ is a smooth point. Locally around X_0, then, we can identify the lattices $H^n(X_b,\mathbb{Z})/\text{tors}$ with a single lattice $H_{\mathbb{Z}}$ and the vector spaces $H^n(X_b,\mathbb{C})$ with $H_{\mathbb{C}}$ correspondingly. We then consider the spaces $H^{n-k,k}(X_b)$ -- or the associated

$$F^k = \bigoplus_{\ell=0}^{k} H^{n-\ell,\ell}(X_b) \text{ --}$$

as variable subspaces of $H_{\mathbb{C}}$. The basic facts then are:

i) The map ϕ_k from B (or a neighborhood of $0 \in B$) to the Grassmannian sending b to $F^k(X_b) \subset H_{\mathbb{C}}$ is holomorphic; and

ii) In terms of the identification of the tangent space to the Grassmannian at $\Lambda \subset H$ with $\text{Hom}(\Lambda,H/\Lambda)$, the image under $\delta_k = d\phi_k$ of any tangent vector to B at 0 carries F_k into F_{k+1}/F_k. We thus arrive at a collection of maps $\delta_k : T_0B \longrightarrow \text{Hom}(H^{n-k,k}(X),H^{n-k-1,k+1}(X))$. By equality of mixed partials, they satisfy the relations

$$(*) \qquad \delta_{k+1}(v)\circ\delta_k(w) = \delta_{k+1}(w)\circ\delta_k(v) \qquad \forall \ v,w \in T$$

and since the spaces $F_k(X_b)$ satisfy the relation $Q(F_k,F_{n-k-1}) \equiv 0$ for all b, we have

(**) $$Q(\delta_k(v)(\alpha),\beta)+Q(\alpha,\delta_{n-k-1}(v)(\beta)) = 0$$

$$\forall\ \alpha \in H^{n-k,k}(x)\ ,\ \beta \in H^{k+1,n-k-1}(x)\ ,\ v \in T\ .$$

We now define an infinitesimal variation of Hodge structure (IVHS) to be just this collection of data: that is, a quintuple $(H_{\mathbb{Z}},Q,H^{p,q},T,\delta_q)$ in which $(H_{\mathbb{Z}},Q,H^{p,q})$ is a Hodge structure, T a vector space, and

$$\delta_q : T \longrightarrow \mathrm{Hom}(H^{p,q},H^{p-1,q+1})$$

maps satisfying (*) and (**) above. By what we have just said, to every member $X = X_0$ of a family of varieties $\{X_b\}$ we have associated such an object.

Two key observations here are the following:

i) The infinitesimal variation of Hodge structure associated to a family is in general computable; or at least as computable as the Hodge structures associated to the members. For example, going back to our example of hypersurfaces, if we let $X \subset \mathbb{P}^{n+1}$ be smooth with equation $F(Z) = 0$, the tangent space at X to the family of hypersurfaces of degree d up to projective isomorphism is just the space S_d of homogeneous polynomials of degree d, modulo the Jacobian ideal. (A variation of X in $\mathbb{P}^{n+1}$ is given by $F+\varepsilon G$ for $G \in S_d/\mathbb{C}F$; if $G = \Sigma a_{ij}X_i \frac{\partial F}{\partial X_j}$ this corresponds to first order to the motion of X along the 1-parameter group e^{tA} of automorphisms of $\mathbb{P}^{n+1}$). Thus $T = (S/J)_d$; and the maps

$$\delta_k : (S/J)_d \longrightarrow \mathrm{Hom}((S/J)_{(k+1)d-n-1},(S/J)_{(k+2)d-n-1})$$

turn out to be nothing but polynomial multiplication.

It should be noted here that this in itself has some nice consequences: for example, while we are as indicated earlier unable to determine $H_{k,k}(X) \cap H^{2k}(X,\mathbb{Z})$ for any given hypersurface in $\mathbb{P}^{n+1}$, $n = 2k$, the fact that for $d \geq n+1$ the map

$$(S/J)_d \times (S/J)_{kd-n-1} \longrightarrow (S/J)_{(k+1)d-n-1}$$

is surjective immediately implies that for general X , $H^{k,k}_{pr}(X) \cap H^{2k}(X,\mathbb{Z}) = 0$, and so $H^{k,k}(X) \cap \overline{H^{2k}(X,\mathbb{Z})} = \mathbb{Z}$. Thus on a general hypersurface every algebraic subvariety is homologous to a rational multiple of a complete intersection. In particular in case $n = 2$ this yields the famous

Theorem (Noether; Lefschetz): a surface $S \subset \mathbb{P}^3$ of degree $d \geq 4$, having general moduli, contains no curves other than complete intersections $S \cap T$ with other surfaces.

2) The second key point is this: that even without the lattice $H_{\mathbb{Z}}$, an infinitesimal variation of Hodge structure will in general possess non-trivial invariants, and will give rise to geometric objects. These of course come from the maps δ_k which, being trilinear objects, have lots of accessible invariants (e.g. their associated determinantal varieties).

To illustrate the use of this, consider the generic Torelli theorem for hypersurfaces. The application of IVHS to this problem is based on the following trick: for any map $f : X \longrightarrow Y$ of varieties, the condition that f is birational onto its image, i.e. that

$$\text{for general } p \in X, \nexists\, q \in X : q \neq p, f(q) = f(p)$$

is in fact equivalent to the a priori weaker statement

$$\text{for general } p \in X, \nexists\, q \in X :$$

$$q \neq p, f(q) = f(p) \text{ and } \mathrm{Im}(f_*)_q = \mathrm{Im}(f_*)_p .$$

In our present circumstances, this equivalence means that

A general hypersurface of degree d in $\mathbb{P}^{n+1}$ is determined by its Hodge structure

$\iff$

A general hypersurface of degree d in $\mathbb{P}^{n+1}$ is determined by its infinitesimal variation of Hodge structure.

Thus, to prove the generic Torelli theorem for hypersurfaces, Donagi shows that from the data of the vector spaces

$$(S/J)_{(k+1)d-n-1}, (S/J)_d$$

and the multiplication maps

$$\delta_{k-1} : (S/J)_d \times (S/J)_{kd-n-1} \longrightarrow (S/J)_{(k+1)d-n-1}$$

$$Q : (S/J)_{(k+1)d-n-1} \times (S/J)_{(n-k+1)d-n-1} \longrightarrow (S/J)_{(n+2)d-n-1} \cong \mathbb{C}$$

one can reconstruct the entire ring S/J , and from this the hypersurface X . This suffices to establish the theorem; again, it should be observed that at no point in the argument is the lattice $H_{\mathbb{Z}}$ mentioned.

Donagi's argument is a beautiful one, but this is not the place to reproduce it. Let me instead conclude by giving a similar and easier example of the use of IVHS: to prove the generic Torelli theorem for curves of genus $g \geq 5$.

Of course, the Torelli theorem has been proved many times over, in as strong a form as one could wish. One common characteristic of the proofs, however, is that they all make essential use of the geometry of the Jacobian and its subvarieties. A natural question if one is studying higher-dimensional Torelli theorems is: does there exist a proof of the Torelli for curves that avoids the use of the Jacobian? The answer to this is unknown to me; however, using IVHS we can give a very short proof of the generic Torelli in genus $g \geq 5$ as follows.

The tangent space, at a curve C , to the family of all curves is dual to the space $H^0(C,K^2)$ of quadratic differentials on C . The IVHS associated to C in this family thus consists of the Hodge structure of C , together with a map

$$\delta : H^0(C,K^2) \longrightarrow \mathrm{Hom}(H^{1,0},H^{0,1}) .$$

Here the relation (*) above is trivial; while the relation (**) says that in terms of the identification of $H^{0,1}$ with $(H^{1,0})^*$ given by Q , the image of δ lies in the subspace $\mathrm{Sym}^2(H^{1,0})^* \subset \mathrm{Hom}(H^{1,0},(H^{1,0})^*)$, i.e.

$$\delta : H^0(C,K^2)^* \longrightarrow \mathrm{Sym}^2(H^0(C,K)^* .$$

The transpose of δ is now easy to identify: it is the map

$${}^t\delta : \mathrm{Sym}^2 H^0(C,K) \longrightarrow H^0(C,K^2)$$

that simply takes a quadratic polynomial $P(\omega_1,\ldots,\omega_g)$ in the holomorphic differentials on C and evaluates it as a quadratic differential on C . In particular, the kernel of ${}^t\delta$ is just the vector space of quadratic polynomials vanishing on the image of the canonical curve $C \subset \mathbb{P}H^0(C,K)^* = \mathbb{P}^{g-1}$; since it is well known that a general canonical curve of genus $g \geq 5$ is the intersection of the quadrics containing it, we can recover the curve C . Explicitly, in terms of the infinitesimal variation of Hodge structure $(H_{\mathbb{Z}}, Q, H^{1,0} \oplus H^{0,1}, T, S)$ associated to C , we have

$$C = \mathbb{P}\{ Z \in H^{0,1} : Q(Z,\lambda(Z)) = 0 \quad \text{for all} \quad \lambda : H^{0,1} \longrightarrow H^{1,0} \\ \text{such that} \quad \mathrm{trace}(\lambda \circ \delta(v)) = 0 \quad \text{for all} \quad v \in T \}$$

and this suffices to establish generic Torelli.

References

[1] R. Donagi, J. Carlson, M. Green, P. Griffiths and J. Harris, Compositio Math 50(1983).

[2] C. Peters and J. Steenbrink, Infinitesimal variations of Hodge structure and the ϕ generic Torelli problem for projective hypersurfaces, in Birkhauser Progress in Mathematics series number 39. (1984)