# INTRODUCTION TO NON COMMUTATIVE DIFFERENTIAL GEOMETRY

A. Connes
Institut des Hautes
Études Scientifiques
35, Route des Chartres
F-91440 Bures-Sur-Yvette
France

This is the introduction to a series of papers in which we shall extend the calculus of differential forms and the de Rham homology of currents beyond their customary framework of manifolds, in order to deal with spaces of a more elaborate nature, such as,

a) the space of leaves of a foliation,

b) the dual space of a finitely generated non-abelian discrete group (or Lie group),

c) the orbit space of the action of a discrete group (or Lie group) on a manifold.

What such spaces have in common is to be, in general, badly behaved as point sets, so that the usual tools of measure theory, topology and differential geometry lose their pertinence. These spaces are much better understood by means of a canonically associated algebra which is the group convolution algebra in case b). When the space $V$ is an ordinary manifold, the associated algebra is commutative. It is an algebra of complex-valued functions on $V$, endowed with the pointwise operations of sum and product.

A smooth manifold $V$ can be considered from different points of view such as

α) Measure theory (i.e. $V$ appears as a measure space with a fixed measure class), β) Topology (i.e. $V$ appears as a locally compact space),

γ) Differential geometry (i.e. $V$ appears as a smooth manifold).

Each of these structures on $V$ is fully specified by the corresponding algebra of functions, namely:

α) The commutative von Neumann algebra $L^{\infty}(V)$ of classes of essentially bounded measurable functions on $V$,

β) The $C^*$-algebra $C_0(V)$ of continuous functions on $V$ which vanish at infinity,

γ) The algebra $C_c^{\infty}(V)$ of smooth functions with compact support.

It has long been known to operator algebraists that measure theory and topology extend far beyond their usual framework to:

A) The theory of weights and von Neumann algebras.

B) $C^*$- algebras, K theory and index theory.
Let us briefly discuss these two fields,

A) The theory of weights and von Neumann algebras.

To an ordinary measure space $(X,\mu)$ correspond the von Neumann algebra $L^{\infty}(X,\mu)$ and the weight $\varphi$ :

$$\varphi(f) = \int_X f d\mu \quad \forall f \in L^{\infty}(x,\mu)^+ .$$

Any pair $(M,\varphi)$ of a commutative von Neumann algebra $M$ and weight $\varphi$ is obtained in this way from a measure space $(X,\mu)$. Thus the place of ordinary measure theory in the theory of weights on von Neumann algebras is similar to that of commutative algebras among arbitrary ones. This is why A) is often called non-commutative measure theory.

Non-commutative measure theory has many features which are trivial in the commutative case. For instance to each weight $\varphi$ on a von Neumann algebra $M$ corresponds canonically a one-parameter group $\sigma_t^{\varphi} \in \mathrm{Aut}\, M$

of automorphisms of $M$, its modular automorphism group. When $M$ is commutative, one has $\sigma_t^{\varphi}(x) = x$, $\forall x \in M$, and for any weight $\varphi$ on $M$. We refer to [13] for a survey of non-commutative measure theory.

## B) C*-algebras, K theory and index theory.

Gel'fand's theorem implies that the category of commutative C*-algebras and *-homomorphisms is dual to the category of locally compact spaces and proper continuous maps.

Non-commutative C*-algebras have first been used as a tool to construct von Neumann algebras and weights, exactly as in ordinary measure theory, where the Riesz representation theorem [38], Theorem 2.14, enables to construct a measure from a positive linear form on continuous functions. In this use of C*-algebras the main tool is positivity. The fine topological features of the "space" under consideration do not show up. These fine features came into play thanks to Atiyah's topological K-theory [2]. First the proof of the periodicity theorem of R. Bott shows that its natural set up is non-commutative Banach algebras (cf. [46]). Two functors $K_0, K_1$ (with values in the category of abelian groups) are defined and any short exact sequence of Banach algebras gives rise to an hexagonal exact sequence of $K$ groups. For $A = C_0(X)$, the commutative C*-algebra associated to a locally compact space $X$, $K_j(A)$ is (in a natural manner) isomorphic to $K^j(X)$, the $K$ theory with compact supports of $X$. Since (cf. [41]) for a commutative Banach algebra $B$, $K_j(B)$ depends only upon the Gel'fand spectrum of $B$, it is really the C*-algebra case which is most relevant.

Secondly, Brown, Douglas and Fillmore have classified (cf. [8]) short exact sequences of C*-algebras of the form:

$$0 \to \mathcal{K} \to A \to C(X) \to 0$$

where $\mathcal{K}$ is the C*-algebra of compact operators in Hilbert space, and $X$ is a compact space. They have shown how to construct a group from such extensions. When $X$ is a finite dimensional compact metric space, this group is naturally isomorphic to $K_1(X)$, the Steenrod $K$ homology of $X$, cf. [19],[24].

Since the original classification problem of extensions did arise as an internal question in operator and C*-algebra theory, the work of Brown, Douglas and Fillmore made it clear that K theory is an indispensable tool even for studying C*-algebras per se. This fact was further emphasized by the role of K theory in the classification of C*-algebras which are inductive limits of finite dimensional ones (cf. [7] [22] [21]),and in the work of Cuntz and Krieger on C*-algebras associated to topological Markov chains [18].

Finally the work of the Russian school, of Miscenko and Kasparov in particular, ([30] [26] [27] [28]), on the Novikov conjecture, has shown that the K theory of non-commutative C*-algebras plays a crucial role in the solution of classical problems in the theory of non-simply-connected manifolds. For such a space $X$, a basic homotopy invariant is the $\Gamma$-equivariant signature $\sigma$ of its universal covering $\tilde{X}$, where $\Gamma = \pi_1(X)$ is the fundamental group of $X$. This invariant $\sigma$ lies in the K group, $K_0(C^*(\Gamma))$, of the group C*- algebra $C^*(\Gamma)$.

The K theory of C*-algebras, the extension theory of Brown, Douglas and Fillmore and the Ell theory of Atiyah ([1]) are all special cases of Kasparov's bivariant functor $KK(A,B)$. Given two $\mathbb{Z}/2$ graded C*-algebras $A$ and $B$, $KK(A,B)$ is an abelian group whose elements are homotopy classes of Kasparov A-B bimodules (cf. [26] [27]).

After this quick overview of measure theory and topology in the non-commutative framework, let us be more specific about the algebras associated to the "spaces" occuring in a) b) c) above.

a) Let $V$ be a smooth manifold, $F$ a smooth foliation of $V$. The measure theory of the leaf space "V/F" is described by the von Neumann algebra of the foliation (cf.[10][11][12]). The topology of the leaf space is described by the C*-algebra $C^*(V,F)$ of the foliation (cf. [11] [12] [43]).

b) Let $\Gamma$ be a discrete group. The measure theory of the (reduced) dual space $\hat{\Gamma}$ is described by the von Neumann algebra $\lambda(\Gamma)$ of operators in the Hilbert space $\ell^2(\Gamma)$ which are invariant under right translations. This von Neumann algebra is the weak closure of the group ring $\mathbb{C}\Gamma$ acting in $\ell^2(\Gamma)$ by left translations.

The topology of the (reduced) dual space $\hat{\Gamma}$ is described by the C*-algebra $C_r^*(\Gamma)$, the norm closure of $\mathbb{C}\Gamma$ in the algebra of bounded operators in $\ell^2(\Gamma)$.

b') For a Lie group $G$ the discussion is the same, with $C_c^\infty(G)$ instead of $\mathbb{C}\Gamma$.

c) Let $\Gamma$ be a discrete group acting on a manifold $W$. The measure theory of the "orbit space" $W/\Gamma$ is described by the von Neumann algebra crossed product $L^\infty(W) \rtimes \Gamma$ (cf. [33]).

The situation is summarized in the following table:

| Space | $V$ | $V/F$ | $\hat{\Gamma}$ | $\hat{G}$ | $W/\Gamma$ |
|---|---|---|---|---|---|
| Measure theory | $L^\infty(V)$ | v.N.algebra of $(V,F)$ | $\lambda(\Gamma)$ | $\lambda(G)$ | $L(W) \rtimes \Gamma$ |
| Topology | $C_0(V)$ | $C^*(V,F)$ | $C_r^*(\Gamma)$ | $C_r^*(G)$ | $C_0(W) \rtimes \Gamma$ |

It is a general principle (cf. [3] [14] [4]) that for families of elliptic operators $(D_y)_{y\in Y}$ parametrized by a "space" $Y$ such as those occuring above, the index of the family is an element of $K_0(A)$, the $K$ group of the C*-algebra associated to $Y$. For instance the $\Gamma$-equivariant signature of the universal covering $X$ of a compact oriented manifold is the $\Gamma$-equivariant index of the elliptic signature operator on $X$. We are in case b) and $\sigma \in K_0(C_r^*(\Gamma))$ . The obvious problem then is to compute $K(A)$ for the C*-algebras of the above spaces, and then the index of families of elliptic operators.

After the breakthrough of Pimsner and Voiculescu ([34]) in the computation of $K$ groups of crossed products, and under the influence of the Kasparov bivariant theory, the general program of computation of the $K$ groups of the above spaces (i.e. of the associated C*-algebras) has undergone rapid progress in the last years ([12] [43] [31] [32] [45] [44]).

So far, each new result confirms the validity of the general conjecture formulated in [4]. In order to state it briefly, we shall deal only with case c) above. We also assume that $\Gamma$ is discrete and torsion free , cf. [4] for the general case. By a familar construction of algebraic topology a space such as $W/\Gamma$, the orbit space of a discrete group action, can be realized as a simplicial complex, up to homotopy. One lets $\Gamma$ act freely and properly on a contractible space $E\Gamma$, and forms the homotopy quotient $W \times_\Gamma E\Gamma$ which is a meaningful space even when the quotient topological space $W/\Gamma$ is pathological. In case b) ($\Gamma$ acting on $W = \{pt\}$) this yields the classifying space $B\Gamma$. In case a), see [12] for the analoguous construction. In [4] (using [12] and [14]) a map $\mu$ is defined from the twisted K homology $K_{*,\tau}(W \times_\Gamma E\Gamma)$ to the K group of the C*-algebra $C_0(W)\rtimes\Gamma$. The conjecture is that this map $\mu$ is always an ismorphism.

$$\mu : K_{*,\tau}(W \times_\Gamma E\Gamma) \to K_*(C_0(W) \rtimes \Gamma)$$

At this point it would be tempting to advocate that the space $W \times_\Gamma E\Gamma$ gives a sufficiently good description of the topology of $W/\Gamma$ and that we can dispense with $C^*$ algebras. However, it is already clear in the simplest cxamples that the C*-algebra $A = C_0(W) \rtimes \Gamma$ is a finer description of the "topological space" of orbits. For instance, with $W = S^1$ and $\Gamma = \mathbb{Z}$, the actions given by two irrational rotations $R_{\theta_1}, R_{\theta_2}$ yield isomorphic C*-algebras if and only if $\theta_1 = \pm\theta$ ([34] [35]) and Morita equivalent C*-algebras iff $\theta_1$ and $\theta_2$ belong to the same orbit of the action of $PSL(2,\mathbb{Z})$ on $P_1(\mathbb{R})$[36]. On the contrary, the homotopy quotient is independent of $\theta$ (and is homotopic to the two torus).

Moreover, as we already mentioned, an important role of a "space" such as $Y = W/\Gamma$ is to parametrize a family of elliptic operators, $(D_y)_{y\in Y}$. Such a family has both a topological index $\mathrm{Ind}_t(D)$, which belongs to the twisted K homology group $K_{*,\tau}(W\times_\Gamma E\Gamma)$, and an analytic index $\mathrm{Ind}_a(D) = \mu(\mathrm{Ind}_t(D))$, which belongs to $K_*(C_0(W) \rtimes \Gamma)$ (cf. [4] [16]). But it is a priori only through $\mathrm{Ind}_a(D)$ that the analytic properties of the family $(D_y)_{y\in Y}$ are reflected. For instance, if each $D_y$ is the Dirac operator on a Spin Riemannian manifold $M_y$ of strictly positive scalar curvature, one has $\mathrm{Ind}_a(D) = 0$ (cf. [37][16]), but the equality $\mathrm{Ind}_t(D) = 0$ follows only if one knows that the map $\mu$ is injective (cf. [4][37][16]). The problem of injectivity of $\mu$ is an important reason for developing the analogue of de Rham homology for the above "spaces". Any closed de Rham current $C$ on a manifold $V$ yields a map $\varphi_C$ from $K^*(V)$ to $\mathbb{C}$

$$\varphi_C(e) = \langle C, \mathrm{ch}\, e\rangle \quad \forall e \in K^*(V)$$

where $\mathrm{ch}: K^*(V) \to H^*(V,\mathbb{R})$ is the usual Chern character.

Now , any "closed de Rham current" $C$ on the orbit space $W/\Gamma$ should yield a map $\varphi_C$ from $K_*(C_0(W) \rtimes \Gamma)$ to $\mathbb{C}$. The rational injectivity of $\mu$ would then follow from the existence, for each $\omega \in H^*(W \times_\Gamma E\Gamma)$, of a "closed current" $C(\omega)$ making the following diagram commutative,

$$\begin{array}{ccc} K_{*,\tau}(W \times_\Gamma E\Gamma) & \xrightarrow{\mu} & K_*((C_0(W) \rtimes \Gamma) \\ \Big\downarrow \mathrm{ch}_* & & \Big\downarrow \varphi_C(\omega) \\ H_*(W \times_\Gamma E\Gamma, \mathbb{R}) & \xrightarrow{\omega} & \mathbb{C} \end{array}$$

Here we assume that $W$ is $\Gamma$-equivariantly oriented so that the dual Chern character $\mathrm{ch}_*: K_{*,\tau} \to H_*$ is well defined (See [16]). Also, we view $\omega \in H^*(W \times_\Gamma E\Gamma, \mathbb{C})$ as a linear map from $H_*(W \times_\Gamma E\Gamma, \mathbb{R})$ to $\mathbb{C}$.

This leads us to the subject to our series of papers which is;

1. <u>The construction of de Rham homology for the above spaces;</u>

2. <u>Its applications to $K$ theory and index theory.</u>

The construction of the theory of currents, closed currents, and of the maps $\varphi_C$ for the above "spaces", requires two quite different steps. The first is <u>purely algebraic:</u>

One starts with an algebra $A$ over $\mathbb{C}$, which plays the role of $C^\infty(V)$, and one develops the analogue of de Rham homology, the pairing with the algebraic $K$ theory groups $K_0(A)$, $K_1(A)$, and algebraic tools to perform the computations. This step yields a contravariant functor $H^*_\lambda$ from non commutative algebras to graded modules over the polynomial ring $\mathbb{C}(\sigma)$ with a generator $\sigma$ of degree 2. In the definition of this functor the finite cyclic groups play a crucial role, and this is why $H^*$ is called <u>cyclic cohomology.</u> Note that it is a contravariant functor for algebras and hence a covariant one for "spaces". It is the subject of part II under the title,

## De Rham homology and non-commutative algebra

The second step involves analysis:

The non-commutative algebra $\mathcal{A}$ is now a dense subalgebra of a $C^*$-algebra $A$ and the problem is, given a closed current $C$ on $\mathcal{A}$ as above satisfying a suitable continuity condition relative to $A$, to extend $\varphi_C : K_0(\mathcal{A}) \to \mathbb{C}$ to a map from $K_0(A)$ to $\mathbb{C}$. In the simplest situation, which will be the only one treated in parts I and II, the algebra $\mathcal{A} \subset A$ is stable under holomorphic functional calculus (cf. Appendix 3 of part I) and the above problem is trivial to handle since the inclusion $\mathcal{A} \subset A$ induces an isomorphism $K_0(\mathcal{A}) \approx K_0(A)$. However, even to treat the fundamental class of $W/\Gamma$, where $\Gamma$ is a discrete group acting by orientation preserving diffeomorphisms on $W$, a more elaborate method is required and will be discussed in part V (cf. [16]). In the context of actions of discrete groups we shall construct $C(\omega)$ and $\varphi_{C(\omega)}$ for any cohomology class $\omega \in H^*(W \times_\Gamma E\Gamma, \mathbb{C})$ in the subring $R$ generated by the following classes:

a) Chern classes of $\Gamma$-equivariant (non unitary) bundles on $W$.

b) $\Gamma$-invariant differential forms on $W$.

c) Gel'fand Fuchs classes.

As applications of our construction we get (in the above context):

$\alpha$) If $x \in K_{*,\tau}(W \times_\Gamma E\Gamma)$ and $\langle ch_* x, \omega \rangle \neq 0$ for some $\omega$ in the above ring $R$ then $\mu(x) \neq 0$.

In fact we shall further improve this result by varying $W$; it will then apply also to the case $W = \{pt\}$, i.e. to the usual Novikov conjecture. All this will be discussed in part V, but see [16] for a preview.

$\beta$) For any $\omega \in R$ and any family $(D_y)_{y \in Y}$ of elliptic operators parametrized by $Y = W/\Gamma$, one has the index theorem.

$$\varphi_C(\mathrm{Ind}_a(D)) = \langle ch_* \mathrm{Ind}_t(D), \omega \rangle$$

When $Y$ is an ordinary manifold, this is the cohomological form of the Atiyah Singer index theorem for families ([3]).

It is important to note that, in all cases, the right hand side is computable by a standard recipe of algebraic topology from the symbol of $D$. The left hand side carries the analytic information such as vanishing, homotopy invariance,...

All these results will be extended to the case of foliations (i.e. when $Y$ is the leaf space of a foliation) in part VI.

As a third application of our analogue of de Rham homology for the above spaces we shall obtain index formulae for transversally elliptic operators; that is elliptic operators on the above "spaces" $Y$. In part IV we shall work out the pseudo-differential calculus for crossed products of a C*-algebra by a Lie group, (cf. [15]), thus yielding many non-trivial examples of elliptic operators on spaces of the above type. Let $A$ be the C*-algebra associated to $Y$, any such elliptic operator on $Y$ yields a finitely summable Fredholm module over the dense subalgebra $\mathcal{A}$ of smooth elements of $A$.

In part I we show how to construct canonically from such a Fredholm module a closed current on the dense subalgebra $\mathcal{A}$. The title of part I, the Chern character in K homology is motivated by the specialization of the above construction to the case when $Y$ is an ordinary manifold. Then the K homology $K_*(V)$ is entirely described by elliptic operators on $V$ ([6] [14]) and the association of a closed current provides us with a map,

$$K_*(V) \to H_*(V,\mathbb{C})$$

which is exactly the dual Chern character $ch_*$.

The explicit computation of this map $ch_*$ will be treated in part III as an introduction to the asymptotic methods of computations of cyclic cocycles which will be used again in part IV. As a corollary we shall, in part IV give completely explicit formulae for indices of finite difference, differential operators on the real line.

If $D$ is an elliptic operator on a "space" $Y$ and $C$ is the closed current $C = ch_*D$ (constructed in part I), the map $\varphi_C : K_*(A) \to \mathbb{C}$ makes sense and one has,

$$\varphi_C(x) = \langle x, [D] \rangle = \text{Index } D_x \quad \forall x \in K_*(A)$$

where the right hand side means the index of $D$ with coefficients in $x$, or equivalently the value of the pairing between $K$ homology and $K$ cohomology. The integrality of this value, Index $D_x \in \mathbb{Z}$, is a basic result which will be already used in a very efficient way in part I, to control $K_*(A)$.

The aim of part I is to show that the construction of the Chern character $ch_*$ in $K$ homology dictates the basic definitions and operations - such as the suspension map $S$ - in cyclic cohomology. It is motivated by the previous work of Helton and Howe [23], Carey and Pincus [9] and Douglas and Voiculescu [20].

There is another, equally important, natural route to cyclic cohomology. It was taken by Loday and Quillen ([29]) and by Tsigan ([42]). Since the latter work is independent from ours, cyclic cohomology was discovered from two quite different points of view.

There is also a strong relation with the work of I. Segal [39] on quantized differential forms, which will be discussed in part IV and with the work of M. Karoubi on secondary characteristic classes [25], which is discussed in part II, Theorem 33.

Our results and in particular the spectral sequence of part II were announced in the conference on operator algebras held in Oberwolfach in September 1981 ([17]).

Besides parts I and II, which will soon appear in the IHES Publications, our set of papers will contain:

I. The Chern character in $K$ homology.
II. De Rham homology and non commutative algebra.
III. Smooth manifolds, Alexander Spanier cohomology and index theory.
IV. Pseudodifferential calculus for $C^*$ dynamical systems, index

References

[1] M.F. Atiyah, Global theory of elliptic operators, Proc. Internat. Conf. on functional analysis and related topics, Univ. of Tokyo Press, Tokyo (1970).

[2] M.F. Atiyah, K theory, Benjamin (1967).

[3] M.F. Atiyah and I. Singer, The index of elliptic operators IV, Ann. of Math. 93 (1971) p. 119-138.

[4] P. Baum and A. Connes, Geometric K theory for Lie groups and Foliations, Preprint IHES, 1982.

[5] P. Baum and A. Connes, Leafwise homotopy equivalence and rational Pontrjagin classes, Preprint IHES, 1983.

[6] P. Baum and R. Douglas, K homotopy and index theory, Operator algebras and applications, Proc. Symposia Pure Math. 38 (1982), part I, p. 117-173.

[7] O. Bratteli, Inductive limits of finite dimensional C*-algebras, Trans. AMS 171 (1972), p. 195-234.

[8] L.G. Brown, R. Douglas and P.A. Fillmore, Extensions of C*-algebras and K homology, Ann. of Math. (2) 105 (1977) p. 265-324.

[9] R. Carey and J.D. Pincus, Almost commuting algebras, K theory and operator algebras, Lecture Notes in Math. N°575, Springer Berlin-New York (1977).

[10] A.Connes, The von Neumann algebra of a foliation, Lecture Notes in Physics N°80 (1978) p. 145-151, Springer Berlin-New York.

[11] A. Connes, Sur la théorie non commutative de l'intégration, Algèbres d'opérateurs, Lecture Notes in Math. N°725, Springer Berlin-New York (1979).

[12] A. Connes, A Survey of foliations and operator algebras, Operator algebras and applications, Proc. Symposia Pure Math. 38 (1982) Part I, p. 521-628.

[13] A. Connes, Classification des facteurs, Operator algebras and applications, Proc. Symposia Pure Math. 38 (1982) Part II, p. 43-109.

[14] A. Connes and G. Skandalis, The longitudinal index theorem for foliations, to appear in Publ. R.I.M.S. Kyoto.

[15] A. Connes, C*-algèbres et géométrie différentielle, C.R.Acad. Sci. Paris, tome 290, Série I (1980).

[16] A. Connes, Cyclic cohomology and the transverse fundamental class of a foliation, Preprint IHES M/84/7.

[17] A. Connes, Spectral sequence and homology of currents for operator algebras. Math. Forschungsinstitut Oberwolfach Tagungsbericht 42/81, Funktionalanalysis und C*-Algebren , 27.9. - 3.10.1981.

[18] J. Cuntz and W. Krieger, A class of C*-algebras and topological Markov chains, Invent. Math. 56 (1980) p. 251 - 268.

[19] R. Douglas, C*-algebra extensions and K homology, Annals of Math. Studies, N°95, Princeton University Press 1980.

[20] R. Douglas and D. Voiculescu, On the smoothness of sphere extensions, J. Operator Theory 6(1) (1981) p. 103.

[21] E.G. Effros, D.E. Handelman and C.L. Shen, Dimension groups and their affine representations, Amer. J. Math. 102 (1980) p. 385-407.

[22] G. Elliott, On the classification of inductive limits of sequences of semi-simple finite dimensional algebras, J. Alg. 38 (1976) p. 29-44.

[23] J. Helton and R. Howe, Integral operators, commutators, traces, index and homology, Proc. of Conf. on operator theory, Lecture Notes in Math. N°345, Springer Berlin-New York (1973).

[24] D.S. Kahn, J. Kaminker and C. Schochet, Generalized homology theories on compact metric spaces, Michigan Math. J. 24 (1977) p. 203-224.

[25] M. Karoubi, Connexions, courbures et classes caractêristiques en K theorie algêbrique, Canadian Math. Soc. Proc. Vol.2, part I (1982) p. 19-27.

[26] G. Kasparov, K functor and extensions of C*-algebras, Izv. Akad. Nauk SSSR, Ser. Mat. 44(1980) p. 571-636.

[27] G. Kasparov, K theory, group C*-algebras and higher signature, Conspectus, Chernogolovka (1983).

[28] G. Kasparov, Lorentz groups: K theory of unitary representations and crossed products, preprint, Chernogolovka, 1983.

[29] J.L. Loday and D. Quillen, Cyclic homology and the Lie algebra of matrices, C.R. Acad. Sci. Paris, Sêrie I, 296 (1983) p. 295-297.

[30] A.S. Miscenko, Infinite dimensional representations of discrete groups and higher signature, Math. USSR Izv. 8 (1974) p. 85-112.

[31] M.Penington, K theory and C*-algebras of Lie groups and Foliations, D. Phil. thesis, Michaelmas, Term. 1983, Oxford.

[32] M.Penington and R. Plymen, The Dirac operator and the principal series for complex semi-simple Lie groups, J. Funct. Analysis 53 (1983) p. 269-286.

[33] G. Pedersen, C*-algebras and their automorphism groups. Academic Press, New York (1979).

[34] M. Pimsner and D. Voiculescu, Exact sequences for K groups and Ext groups of certain cross-product C*-algebras. J. of operator theory 4 (1980), 93-118.

[35] M. Pimsner and D. Voiculescu, Imbedding the irrational rotation C* algebra into an AF algebra, J. of operator theory 4 (1980) 201-211.

[36] M. Rieffel, C*-algebras associated with irrational rotations, Pac. J. of Math. 93, N°2 (1981).

[37] J. Rosenberg, C* algebras, positive scalar curvature and the Novikov conjecture, Publ. Math. IHES, Vol. 58 (1984) p. 409-424.

[38] W. Rudin, Real and complex analysis, Mc. Graw Hill, New York (1966).

[39] I. Segal, Quantized differential forms, Topology 7 (1968) p. 147-172.

[40] I.M. Singer, Some remarks on operator theory and index theory, Lecture notes in Math. 575 (1977) p. 128-138, Springer New York.

[41] J.L. Taylor, Topological invariants of the maximal ideal space of a Banach algebra, Advances in Math. 19 (1976) N°2, p. 149-206.

[42] B.L. Tsigan, Homology of matrix Lie algebras over rings and Hochschild homology, Uspekhi Math. Nauk. Vol. 38 (1983) p. 217-218.

[43] A.M. Torpe, K theory for the leaf space of foliations by Reeb components, Mat. Institut, Odense Univ., Preprint (1982).

[44] A. Valette, K. Theory for the reduced C*-algebra of semisimple Lie groups with real rank one, Quarterly J. of Math., Oxford Sêrie 2, Vol. 35 (1984) p. 334-359.

[45] A. Wasserman, Une démonstration de la conjecture de Connes-Kasparov, To appear in C.R. Acad. Sci. Paris.

[46] R. Wood, Banach algebras and Bott periodicity, Topology 4 (1965-66) p. 371-389.