International Summer School on Modular Functions

BONN 1976

MODULAR FORMS WHOSE FOURIER COEFFICIENTS INVOLVE ZETA-FUNCTIONS OF QUADRATIC FIELDS

D. Zagier

TABLE OF CONTENTS

Introduction

For over a hundred years it has been known that there exist identities expressing the coefficients of certain modular forms as finite sums involving class numbers of imaginary quadratic fields; these identities, the so-called "class number relations," arose classically in the theory of complex multiplication but have reappeared since in several other contexts, e.g. in the Eichler-Selberg formula for the traces of Hecke operators and in the calculation of intersection numbers of curves on Hilbert modular surfaces [8]. Recently Cohen [3], using Shimura's theory of modular forms of half-integral weight, constructed modular forms whose Fourier coefficients are given by finite sums similar to those occurring in the class number relations, but with the class numbers replaced by values of Dirichlet L-series (or equivalently, of zeta functions of quadratic number fields) at integral arguments. In this paper we construct modular forms whose Fourier coefficients are given by infinite sums of zeta functions of quadratic fields, now at an arbitrary complex argument. The result includes both the classical class number relations and the modular forms constructed by Cohen, and further provides an expression for the latter as linear combinations of Hecke eigenfunctions $f(z)$, the coefficients being certain values of the associated Rankin zeta functions $\sum_{n=1}^{\infty} \frac{a(n)^2}{n^s}$ (where $f|T(n) = a(n)f$). From this we obtain formulas for the values of the Rankin zeta function at integral values within the critical strip, a typical identity being

$$(1) \qquad \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{20}} = \frac{2}{245} \frac{4^{20}}{20!} \pi^{29} \frac{\zeta(9)}{\zeta(18)} (\Delta,\Delta),$$

where $\zeta(s)$ is the Riemann zeta function, $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$ the discriminant function, and (Δ,Δ) the Petersson product of Δ with itself. As another corollary of the main identity we obtain a new proof of a recent result of Shimura [21] on the holomorphy of the Rankin zeta function. Finally, by combining the method developed in this paper with the results of [24] we obtain applications to the Doi-Naganuma lifting from modular forms of Nebentypus to Hilbert modular forms in two variables and also to the mapping in the other direction which was constructed in [8] in terms of the intersection numbers of modular curves on Hilbert modular surfaces. In particular, we give partial results in the direction of the conjecture made in [8] that these two maps are adjoint to one another with respect to the Petersson scalar product.

In § 1 we describe the main result of the paper, namely the construction of a modular form whose Fourier coefficients are infinite linear combinations of zeta functions of quadratic fields (with Legendre functions as coefficients) and whose Petersson product with an arbitrary Hecke eigenform is the corresponding Rankin zeta function. We also show how this can be used to obtain identities for special values of the Rankin zeta function like the one cited above and discuss the relationship between these identities and other known or conjectured results on the values at integral arguments of Dirichlet series associated to cusp forms. In § 2 we reduce the proof of the main result to the evaluation of an integral involving kernel functions for Hecke operators. This integral is calculated in § 3, while § 4 contains the properties of zeta-functions and Legendre functions which are needed to deduce identities like (1) above. In § 5 we describe an alternate method for proving such identities by expressing the product of a theta series and an Eisenstein series of half-integral weight as an infinite linear combination of Poincaré series. The applications to Hilbert modular forms are contained in § 6.

Note: The identities expressing $\sum a(n)^2 n^{-s}$ for special integral values of s in terms of (f,f) and values of the Riemann zeta function have been discovered independently by Jacob Sturm (Thesis, Princeton 1977).

§ 1 Identities for the Rankin zeta function

We use the following notation:

$H = \{z = x + iy \mid y > 0\}$ the upper half-plane, $dV = \frac{dx\,dy}{y^2}$ the invariant metric on H,

$j_k(\gamma,z) = (cz+d)^{-k}$ $\quad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}),\ k \in \mathbb{Z},\ z \in H)$,

$(f|_k\gamma)(z) = j_k(\gamma,z)\, f(\gamma z)$ $\quad$ (f any function on H).

Throughout §§ 1 - 4 we restrict ourselves to modular forms for the full modular group $\Gamma = SL_2(\mathbb{Z})/\{\pm 1\}$; the results could be generalized to arbitrary congruence subgroups, but this would involve considerable technical complication and no essentially new ideas. We denote by k an even integer >2, by S_k the space of cusp forms of weight k on Γ, equipped with the Petersson scalar product

$$(f,g) = \int_{\Gamma\backslash H} f(z)\,\overline{g(z)}\,y^k\,dV \qquad (f,\ g \in S_k),$$

and by $\{f_i\}_{1\le i\le \dim S_k}$ the basis of S_k consisting of normalized Hecke eigenforms, with

$$f_i(z) = \sum_{n=1}^{\infty} a_i(n)\,q^n, \quad a_i(1) = 1, \quad f_i|T(n) = a_i(n)\,f_i$$

(where as usual $q = e^{2\pi i z}$). For each normalized Hecke eigenform $f(z) = \sum a(n)\,q^n$ we set

$$(2)\qquad D_f(s) = \prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p\bar{\alpha}_p p^{-s})^{-1}(1-\bar{\alpha}_p^2 p^{-s})^{-1} \qquad (\mathrm{Re}(s) > k),$$

where the product is over all primes and α_p, $\bar{\alpha}_p$ are defined by

$$\alpha_p + \bar{\alpha}_p = a(p), \quad \alpha_p\,\bar{\alpha}_p = p^{k-1}$$

(by Deligne's theorem, previously the Ramanujan-Petersson conjecture, the numbers α_p and $\bar{\alpha}_p$ are complex conjugates). The function $D_f(s)$ is related to the Rankin zeta function by

(3) $$D_f(s) = \frac{\zeta(2s-2k+2)}{\zeta(s-k+1)} \sum_{n=1}^{\infty} \frac{a(n)^2}{n^s}$$

and hence, by the results of Rankin [17], has a meromorphic continuation to the entire complex plane, satisfies the functional equation

(4) $$D_f^*(s) = 2^{-s}\pi^{-3s/2}\,\Gamma(s)\;\Gamma\Big(\frac{s-k+2}{2}\Big)D_f(s) = D_f^*(2k-1-s),$$

and is related to the norm of f in the Petersson metric by

(5) $$(f,f) = \frac{(k-1)!}{2^{2k-1}\,\pi^{k+1}}\; D_f(k).$$

For the statement of the main identity we will also need a certain zeta function, defined as follows. Let Δ be any discriminant, i.e. $\Delta \in \mathbb{Z}$ and $\Delta \equiv 0$ or $1 \pmod 4$. We consider binary quadratic forms

$$\phi(u,v) = au^2 + buv + cv^2 \qquad (a,\ b,\ c \in \mathbb{Z})$$

with discriminant $|\phi| = b^2 - 4ac = \Delta$. The group Γ operates on the set of such forms by $\gamma\circ\phi(u,v) = \phi(au+cv,\ bu+dv)$ $(\gamma = \binom{a\ b}{c\ d} \in \Gamma)$, the number of equivalence classes being finite if $\Delta \neq 0$. We define

(6) $$\zeta(s,\Delta) = \sum_{\substack{\phi \bmod \Gamma \\ |\phi| = \Delta}} \;\; \sum_{\substack{(m,n)\in \mathbb{Z}^2/\mathrm{Aut}(\phi) \\ \phi(m,n) > 0}} \frac{1}{\phi(m,n)^s} \qquad (\mathrm{Re}(s) > 1),$$

where the first sum is over all Γ-equivalence classes of forms ϕ of discriminant Δ and the second over inequivalent pairs of integers with respect to the group of units $\mathrm{Aut}(\phi) = \{\gamma \in \Gamma \,|\, \gamma\phi = \phi\}$ of the form. If Δ is the discriminant of a (real or imaginary) quadratic field K, then $\zeta(s,\Delta)$ coincides with the Dedekind zeta function $\zeta_K(s)$ (the first sum corresponds to the ideal classes of K, the second to the ideals in a given class, with $\phi(m,n) = N(\mathfrak{a})$), while $\zeta(s,\Delta)$ for $\Delta = 1$ and $\Delta = 0$ is equal to $\zeta(s)^2$ and to $\zeta(s)\,\zeta(2s-1)$, respectively. If $\Delta = Df^2$, with D equal either to 1 or to the discriminant of a quadratic field and f a natural number, then $\zeta(s,\Delta)$ differs from $\zeta(s,D)$ only by a finite Dirichlet series. Thus in all cases $\zeta(s,\Delta)$ is divisible by the Riemann zeta function, i.e.

(7) $\zeta(s,\Delta) = \zeta(s)\, L(s,\Delta)$

where $L(s,\Delta)$ is an entire function of s (unless Δ is a perfect square, in which case $L(s,\Delta)$ has a simple pole at $s = 1$ with residue $\frac{1}{2}$ if $\Delta = 0$ and residue 1 otherwise).

Finally, for real numbers Δ and t satisfying $\Delta < t^2$ and $s \in \mathbb{C}$ with $\frac{1}{2} < \mathrm{Re}(s) < k$ we define

$$(8)\qquad \begin{aligned} I_k(\Delta,t;s) &= \int_0^\infty \int_{-\infty}^\infty \frac{y^{k+s-2}}{(x^2 + y^2 + ity - \frac{1}{4}\Delta)^k}\, dx\, dy \\ &= \frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)} \int_0^\infty \frac{y^{k+s-2}}{(y^2 + ity - \frac{1}{4}\Delta)^{k-\frac{1}{2}}}\, dy , \end{aligned}$$

where the second integral converges absolutely for $1 - k < \mathrm{Re}(s) < k$ (unless $\Delta = 0$, in which case we need $\frac{1}{2} < \mathrm{Re}(s) < k$) and can be expressed in terms of the associated Legendre function $P_{s-1}^{k-1}\left(\frac{t}{\sqrt{\Delta}}\right)$ (see § 4). We can now formulate the main result.

<u>Theorem 1</u>: <u>Let</u> $k > 2$ <u>be an even integer</u>. <u>For</u> $m = 1,2,\ldots$ <u>and</u> $s \in \mathbb{C}$ <u>set</u>

$$(9)\qquad \begin{aligned} c_m(s) = m^{k-1} &\sum_{t=-\infty}^{\infty} \left[I_k(t^2 - 4m,t;s) + I_k(t^2 - 4m, -t;s)\right] L(s,t^2 - 4m) \\ &+ \begin{cases} (-1)^{k/2} \dfrac{\Gamma(k+s-1)\ \zeta(2s)}{2^{2s+k-3}\ \pi^{s-1}\ \Gamma(k)}\ u^{k-s-1} & \underline{\text{if}}\ \ m = u^2,\ u > 0, \\ 0 & \underline{\text{if}}\ \ m\ \ \underline{\text{is not a perfect square}}, \end{cases} \end{aligned}$$

<u>where</u> $L(s,\Delta)$ <u>and</u> $I_k(\Delta,t;s)$ <u>are defined by equations</u> (6), (7) <u>and</u> (8). <u>Then</u>

i) <u>The series</u> (9) <u>converges absolutely and uniformly for</u> $2-k< \mathrm{Re}(s)< k-1$;

ii) <u>The function</u>

$$(10)\qquad \Phi_s(z) = \sum_{m=1}^{\infty} c_m(s)\ e^{2\pi i m z} \qquad (z \in H,\ 2-k < \mathrm{Re}(s)< k - 1)$$

<u>is a cusp form of weight</u> k <u>for the full modular group</u>;

iii) <u>Let</u> $f \in S_k$ <u>be a normalized Hecke eigenform. Then the Petersson product</u>

of Φ_s *and* f *is given by*

$$(11)\qquad (\Phi_s, f) = C_k \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} D_f(s+k-1),$$

where $D_f(s)$ *is defined by* (2) *and*

$$(12)\qquad C_k = \frac{(-1)^{k/2}\pi}{2^{k-3}(k-1)}.$$

We must say a few words concerning assertion i). If $t^2 - 4m$ is a perfect square, then $L(s, t^2 - 4m)$ has a pole at $s = 1$, as mentioned above. However, for $t^2 - 4m \geq 0$ the coefficient $[I_k(t^2 - 4m, t; s) + I_k(t^2 - 4m, -t; s)]$ has a simple zero at $s = 1$ (or any other odd integral value between 0 and k), as we will show in § 4, so the expression $[I_k(t^2 - 4m, t; s) + I_k(t^2 - 4m, -t; s)]$ $L(s, t^2 - 4m)$ makes sense even at $s = 1$, and the sum of these numbers as t runs from $-\infty$ to ∞ is absolutely convergent. Similarly, if m is a square then the second member of (9) has a simple pole at $s = \frac{1}{2}$, but in this case the terms $t = \pm 2\sqrt{m}$ in the first sum involve the function

$$(13)\qquad I_k(0;t;s) + I_k(0,-t;s) = 2\pi(-1)^{k/2} \cos\frac{\pi s}{2} \frac{\Gamma(s-\frac{1}{2})\,\Gamma(k-s)}{\Gamma(k)\,\Gamma(\frac{1}{2})} |t|^{s-k}$$

which also has a simple pole at $s = \frac{1}{2}$, and the two poles cancel; then i) states that the sum of the other terms of the series (9) is finite. Thus the expression defining $c_m(s)$ is holomorphic in the region $2-k < \mathrm{Re}(s) < k-1$. From equation (11) we deduce that $D_f(s+k-1)$ is also holomorphic in this region. On the other hand, the Euler product defining $D_f(s)$ is absolutely convergent for $\mathrm{Re}(s) > k$, so $D_f(s)$ is certainly holomorphic in this half-plane and, by the functional equation, also in the half-plane $\mathrm{Re}(s) < k-1$. Theorem 1 therefore implies the following result, which was proved by Shimura [21] in 1975 by a different method.

Corollary 1 (Shimura): *The function* $D_f(s)$ *defined by* (2) *has a holomorphic continuation to the whole complex plane.*

Secondly, we observe that statement iii) of Theorem 1 characterizes the cusp form Φ_s, since the space S_k is complete with respect to the Petersson metric. Indeed, since the eigenfunctions f_i form an orthogonal basis of S_k, equation (11) is equivalent to

$$\Phi_s(z) = C_k \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{i=1}^{\dim S_k} \frac{1}{(f_i,f_i)} D_{f_i}(s+k-1)\, f_i(z) \tag{14}$$

or to

$$c_m(s) = C_k \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{i=1}^{\dim S_k} \frac{a_i(m)}{(f_i,f_i)} D_{f_i}(s+k-1)\ . \tag{15}$$

In particular, if we take $s = 1$ and use formula (5), we find

$$c_m(1) = \frac{\pi}{2}\ C_k \sum_{i=1}^{\dim S_k} a_i(m). \tag{16}$$

On the other hand, the Fourier coefficients $a_i(m)$ of the functions f_i are at the same time their eigenvalues for the m^{th} Hecke operator $T(m)$, so

$$\sum_{i=1}^{\dim S_k} a_i(m) = \operatorname{Tr}(T(m),\ S_k).$$

Thus Theorem 1 includes as a special case a formula for the trace of $T(m)$. To see that this agrees with the well-known formula of Selberg and Eichler, we must investigate the various terms of (9) for $s = 1$. If $t^2 - 4m$ is negative, then (as we will show in § 4)

$$\begin{aligned} &C_k^{-1}\, m^{k-1} \left[I_k(t^2-4m,t;1) + I_k(t^2-4m,t;1)\right] \\ &\qquad = -\frac{1}{4}\sqrt{4m-t^2}\ \ p_{k,1}(t,m), \end{aligned} \tag{17}$$

where

$$\begin{aligned} p_{k,1}(t,m) &= \text{coefficient of } x^{k-2} \text{ in } \frac{1}{1 - tx + mx^2} \\ &= \frac{\rho^{k-1} - \bar\rho^{k-1}}{\rho - \bar\rho} \qquad (\rho + \bar\rho = t,\ \rho\bar\rho = m) \end{aligned} \tag{18}$$

and

$$L(1,t^2-4m) = \pi(4m-t^2)^{-\frac{1}{2}}\, H(4m-t^2), \tag{19}$$

where (with the same notation as in (6))

$$(20)\qquad H(n) = \sum_{\substack{\phi \bmod \Gamma \\ |\phi|=-n}} \frac{1}{\mathrm{Aut}(\phi)} \qquad (n > 0).$$

(This equals $2\sum' \frac{h(-n/f^2)}{w(-n/f^2)}$, where the sum ranges over $f > 0$ such that $f^2|n$ and $-n/f^2$ is congruent to 0 or 1 modulo 4 and $h(\Delta)$, $w(\Delta)$ denote the class number and number of units, respectively, of the order in $\mathbb{Q}(\sqrt{\Delta})$ of discriminant Δ.) If $t^2 - 4m > 0$, then the coefficient $I_k(t^2-4m,t;s) + I_k(t^2-4m,-t;s)$ vanishes at $s = 1$, as mentioned above, so the contribution of (9) is 0 unless $L(s,t^2-4m)$ has a pole at $s = 1$, i.e. unless t^2-4m is a perfect square. In this case, we will show that

$$(21)\qquad \begin{aligned} &\lim_{s\to 1} C_k^{-1} m^{k-1} \left[I_k(t^2-4m,t;s) + I_k(t^2-4m,-t;s)\right] L(s,t^2-4m) \\ &\qquad = -\frac{\pi}{4} \left(\frac{|t| - u}{2}\right)^{k-1} \qquad (t^2-4m = u^2,\ u > 0). \end{aligned}$$

Notice that there are only finitely many t with t^2-4m a perfect square, and that they are in 1:1 correspondence with the positive divisors of m:

$$t^2 - 4m = u^2 \iff m = dd',\ d,\ d' = \frac{|t| \pm u}{2}.$$

Therefore the series (9) for $s = 1$ becomes a finite sum and we obtain

<u>Corollary 2</u> (Eichler, Selberg): <u>For</u> $k > 2$ <u>an even integer,</u> $m > 1$,

$$\mathrm{Tr}(T(m), S_k) = -\frac{1}{2} \sum_{\substack{t\in\mathbb{Z} \\ t^2<4m}} p_{k,1}(t,m)\, H(4m-t^2) - \frac{1}{2} \sum_{\substack{dd'=m \\ d,d'>0}} \min(d,d')^{k-1}$$
$$+ \begin{cases} \frac{k-1}{12}\, u^{k-2} & \underline{\text{if}}\ m = u^2,\ u > 0, \\ 0 & \underline{\text{if}}\ m\ \underline{\text{is not a perfect square}}, \end{cases}$$

<u>where</u> $p_{k,1}(t,m)$ <u>and</u> $H(4m-t^2)$ <u>are defined by equations</u> (18) <u>and</u> (20).

It is perhaps worth remarking that we could have obtained the trace formula by specializing Theorem 1 to $s = 0$ instead of $s = 1$. At $s = 0$, the coefficient $[I_k(t^2-4m,t;s) + I_k(t^2-4m,-t;s)]$ does not vanish for any t, but

$L(t^2-4m,0)$ is zero whenever t^2-4m is positive and not a square, so again we get a finite sum.

If we specialize Theorem 1 to $s = r$ (or $s = 1 - r$), where r is an odd integer between 1 and $k-1$, then again the terms with $t^2-4m>0$ vanish (including those for which t^2-4m is a perfect square, if $r > 1$), and the series defining $c_m(s)$ reduces to a finite sum. In this case we recover the modular forms constructed by Cohen [3]. We recall his result.

For $r \geqslant 1$, r odd, Cohen defines an arithmetical function $H(r, N)$ which generalizes the class number function $H(N) = H(1, N)$ introduced above. The function $H(r, N)$ is defined as $\zeta(1-2r)$ if $N = 0$ and as a simple rational multiple of $\pi^{-r} \sum_{n=1}^{\infty} \left(\frac{-N}{n}\right) n^{-r}$ if $N > 0$, $N \equiv 0$ or $3 \pmod 4$. It is related to the function $L(s,\Delta)$ defined above by

$$(22) \qquad H(r, N) = \frac{(-1)^{(r-1)/2}\,(r-1)!}{2^{r-1}\,\pi^r}\, N^{r-\frac{1}{2}}\, L(r,-N) \qquad (r \geqslant 1 \text{ odd}, \ N \geqslant 0)$$

or, even more simply, by

$$(23) \qquad H(r, N) = L(1-r, -N) \qquad (r \geqslant 1 \text{ odd}, \ N \in \mathbb{Z}).$$

Then:

<u>Theorem</u> (Cohen [3], Theorem 6.2): <u>Let</u> $3 \leqslant r \leqslant k-1$, r <u>odd</u>, k <u>even</u>, <u>and set</u>

$$(24) \qquad C_{k,r}(z) = \sum_{m=0}^{\infty} \left(\sum_{\substack{t \in \mathbb{Z} \\ t^2 \leqslant 4m}} p_{k,r}(t,m)\, H(r,4m-t^2) \right) e^{2\pi i m z} \qquad (z \in H),$$

<u>where</u> $p_{k,r}(t,m)$ <u>is the polynomial defined by</u>

$$(25) \qquad p_{k,r}(t,m) = \text{coefficient of } x^{k-r-1} \text{ in } \frac{1}{(1-tx+mx^2)^r}$$

<u>(Gegenbauer polynomial). Then</u> $C_{k,r}$ <u>is a modular form of weight</u> k <u>for the full modular group. If</u> $r < k-1$, <u>it is a cusp form.</u>

We shall show in § 4 that, for $r = 1,3,5,\ldots,k-1$,

$$C_k^{-1}\, m^{k-1}\left[I_k(t^2-4m,t;r)+I_k(t^2-4m,-t;r)\right]$$

(26)
$$=\begin{cases}(-\tfrac{1}{4})^{(r+1)/2}\,(4m-t^2)^{r-\frac12}\,\dfrac{\Gamma(k-r)\,\Gamma(r)}{\Gamma(k-1)}\,p_{k,r}(t,m) & \text{if } t^2<4m,\\ 0 & \text{if } t^2\geqslant 4m.\end{cases}$$

Together with (22), this shows that the series $\Phi_r(z)$ defined by (10) is a multiple of the function (24) if $3\leqslant r\leqslant k-3$. For $r=k-1$ we are on the edge of the strip in which the series (9) is absolutely convergent. We will show in § 4 that

(27)
$$\lim_{s\to k-1} c_m(s) = \frac{(-1)^{\frac{k}{2}+1}\,\pi^k}{2^{k-1}\,(k-1)!}\sum_{t^2\leqslant 4m} H(k-1,4m-t^2)$$
$$-\,\frac{2\pi}{k-1}\,\frac{\Gamma(k-\frac12)\,\Gamma(\frac12)}{\Gamma(k)}\,\frac{\zeta(2k-2)}{\zeta(k)}\,\sigma_{k-1}(m)$$

(where $\sigma_{k-1}(m)=\sum_{d\mid m} d^{k-1}$ as usual), so that in this case the cusp form $\tilde{\Phi}_{k-1}(z)=\lim_{s\to k-1}\Phi_s(z)$ is a linear combination of Cohen's function $C_{k,k-1}$ and the Eisenstein series of weight k. Thus Cohen's theorem is a consequence of statement ii) of Theorem 1, while statement iii) implies the following result:

<u>Theorem 2</u>: <u>Let</u> r, k <u>be integers with</u> $3\leqslant r\leqslant k-1$, r <u>odd</u>, k <u>even</u>. <u>The Petersson product of the modular form</u> $C_{k,r}$ <u>defined by</u> (24) <u>with an arbitrary Hecke eigenform</u> $f\in S_k$ <u>is given by</u>

$$(28)\qquad (f,\,C_{k,r}) = -\,\frac{(r+k-2)!\,(k-2)!}{(k-r-1)!}\,\frac{1}{4^{r+k-2}\,\pi^{2r+k-1}}\,D_f(r+k-1),$$

<u>where</u> $D_f(s)$ <u>is the function defined by</u> (2).

Since the Fourier coefficients of $C_{k,r}$ are rational numbers, $C_{k,r}$ is a linear combination of eigenforms with algebraic coefficients, and we deduce :

<u>Corollary</u>: <u>Let</u> f <u>be a Hecke eigenform in</u> S_k. <u>The values of</u> $D_f(s)/\pi^{2s-k+1}$ <u>for</u> $s=k,\ k+2,\ k+4,\ldots,\ 2k-2$ <u>are algebraic multiples of</u> $(f,\,f)$.

(The case $s=k$ is a consequence of equation (5) rather than (28).) By virtue of the functional equation (4), the numbers $D_f(s)/\pi^s(f,f)$ $(s=1,3,5,\ldots,k-1)$

are also algebraic.

Example: For $k = 12$, the only normalized eigenform in S_k is the discriminant function

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)\, q^n.$$

The number r in Theorem 2 must be 3, 5, 7, 9 or 11. By computing the first few Fourier coefficients of Cohen's functions $C_{k,r}$ we find

$$C_{12,3} = -\frac{180}{7}\Delta, \qquad C_{12,5} = -210\Delta, \qquad C_{12,7} = -1120\Delta,$$

$$C_{12,9} = -20736\Delta, \quad C_{12,11} = -\frac{77683}{12\times 23} E_{12} - \frac{7 \times 10!}{23 \times 691}\Delta$$

where $E_{12} = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)\, q^n$ is the normalized Eisenstein series. Thus from (28) we get five identities like (1), namely

$$(29) \qquad \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s} = a_s \frac{4^s}{s!} \pi^{2s-11} \frac{\zeta(s-11)}{\zeta(2s-22)} (\Delta,\Delta) \qquad (s = 14, 16, 18, 20, 22)$$

with

$$a_{14} = 1, \quad a_{16} = \frac{1}{6}, \quad a_{18} = \frac{1}{30}, \quad a_{20} = \frac{2}{245}, \quad a_{22} = \frac{77}{31786} = \frac{7 \times 11}{2 \times 23 \times 691}.$$

The numerical values of the series on the left-hand side of (29), calculated by taking 250 terms of the series, are

1.06544, 1.0109865184, 1.00239992152, 1.00056976587, 1.00013948615.

Substituting any of these values (except the first, where the series converges too slowly to give 12-digit accuracy) into (29) we obtain the numerical value

$$(\Delta,\Delta) = 1.035\ 362\ 056\ 79 \times 10^{-6}$$

for (the square of) the norm of Δ in the Petersson metric. The previously published valued $1.035\ 290\ 481\ 79 \times 10^{-6}$ (Lehmer [12]), obtained by integrating $|\Delta(z)|^2\, y^{10}$ numerically, is false in the 5^{th} decimal place.

Finally, we make a few general remarks about values of Dirichlet series attached to modular forms. The series $D_f(s)$ can be thought of as the "symmetric square" of the Mellin transform

$$(30)\qquad L_f(s) = \sum_{n=1}^{\infty} a(n)\, n^{-s} = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\bar{\alpha}_p p^{-s})^{-1} \quad (\mathrm{Re}(s) >> 0)$$

of f, which is an entire function of s with the functional equation

$$(31)\qquad L_f^*(s) = (2\pi)^{-s}\,\Gamma(s)\,L_f(s) = (-1)^{k/2}\,L_f^*(k-s).$$

By the theorem of Eichler-Shimura-Manin on periods of cusp forms (cf. Chapter V of [11]), the ratios $L_f^*(1) : L_f^*(3) : \ldots : L_f^*(k-1)$ and $L_f^*(2) : L_f^*(4) : \ldots : L_f^*(k-2)$ are algebraic (and in fact belong to the number field generated by the Fourier coefficients of f). For $f = \Delta$, for example, there are real numbers ω_+ and ω_- with

$$(32)\qquad \begin{aligned} &L_\Delta^*(1) = L_\Delta^*(11) = \frac{192}{691}\,\omega_+,\ L_\Delta^*(3) = L_\Delta^*(9) = \frac{16}{135}\,\omega_+,\ L_\Delta^*(5) = L_\Delta^*(7) = \frac{8}{105}\,\omega_+,\\ &L_\Delta^*(2) = L_\Delta^*(10) = \frac{384}{5}\,\omega_-,\ L_\Delta^*(4) = L_\Delta^*(8) = 40\,\omega_-\ ,\ L_\Delta^*(6) = 32\,\omega_-; \end{aligned}$$

where by calculating the values of $L_\Delta(10)$ and $L_\Delta(11)$ (which are the most rapidly convergent of the series) numerically we find

$$\omega_+ = 2.144\ 606\ 670\ 68 \times 10^{-2},\ \omega_- = 4.827\ 748\ 001 \times 10^{-5}.$$

On the other hand, Rankin ([18], Theorem 4) showed that for any normalized eigenform $f \in S_k$ and any even integer q with $\frac{k}{2} + 2 \leqslant q \leqslant k-4$ one has

$$(33)\qquad L_f^*(q)L_f^*(k-1) = (-1)^{q/2}\,2^{k-3}\,\frac{B_q}{q}\,\frac{B_{k-q}}{k-q}\,(f, E_q\,E_{k-q})\ ,$$

where E_i is the normalized Eisenstein series and the B_i are Bernoulli numbers, so the product of the two independent periods of L_f^* is an algebraic multiple of (f, f). For $f = \Delta$, for example, (33) says

$$L^*(11)\,L^*(8) = \frac{7680}{691}\,(\Delta,\Delta)$$

or, using (32), that

$$\omega_+\,\omega_- = (\Delta,\Delta).$$

We can therefore restate the Corollary to Theorem 2 as saying that the values of $D_f(s)$ for $s = 1,3,5,\dots,k-1,\ k,\ k+2,\dots,2k-2$ are of the form

$$(\text{algebraic number})\cdot\omega_+\omega_-\pi^n,$$

while the result of Eichler-Shimura-Manin says that the values of $L_f(s)$ for $s = 1,2,\dots,k-1$ are of the form $(\text{alg.})\cdot\omega_+\pi^n$ or $(\text{alg.})\cdot\omega_-\pi^n$. Both statements fit into a general philosophy of Deligne that, if $L(s) = \sum c_n n^{-s}$ is any "motivated" Dirichlet series (i.e. one arising from a natural mathematical object such as a number field, a Galois representation, an algebraic variety, or a modular form) and satisfies a functional equation of the form

$$L^*(s) \;=\; \gamma(s)\,L(s) \;=\; w\,L^*(C-s)$$

with some Γ-factor $\gamma(s)$, then the value of $L(s)$ at any integral value of s for which neither s nor $C-s$ is a pole of $\gamma(s)$ should be given by a "closed formula" $L(s) = A\cdot\omega$, where A is algebraic and ω is a "period" about which something nice can be said (for instance, the twisted functions $L_\chi(s) = \sum c_n\chi(n)\,n^{-s}$ should have values $A_\chi\cdot\omega$ with the <u>same</u> period ω, and the algebraic numbers A_χ should have nice p-adic properties as χ varies). Now the series $L_f(s)$ and $D_f(s)$ are just the first two cases of the Dirichlet series

$$L_{m,f}(s) \;=\; \prod_p \prod_{i=0}^{m} \left(1-\alpha_p^{\,i}\,\bar{\alpha}_p^{\,m-i}\,p^{-s}\right)^{-1} \qquad (\mathrm{Re}(s) \gg 0)$$

attached to the symmetric powers of the representation associated to f, and these functions are conjectured [19] to be holomorphic and to satisfy the functional equations

$$L^*_{m,f}(s) \;=\; \gamma_m(s)\,L_{m,f}(s) \;=\pm\, L^*_{m,f}((k-1)m+1-s),$$

$$\gamma_m(s) \;=\; \begin{cases} (2\pi)^{-rs}\displaystyle\prod_{j=0}^{r-1}\Gamma(s-j(k-1)) & \text{if } m = 2r-1,\\[2ex] \pi^{-s/2}\,\Gamma\!\left(\dfrac{s}{2}-\left[\dfrac{r(k-1)}{2}\right]\right)\gamma_{2r-1}(s) & \text{if } m = 2r.\end{cases}$$

In a letter to the author (February 1976), Serre suggested that, in accordance with the above philosophy, the values of $L_{m,f}(s)$ may be given by a formula of the type $L_{m,f}(s) = (\text{alg.}).\omega_+^a\,\omega_-^b\,\pi^n$, probably with a+b=m, possibly with $|a-b|\leqslant 1$, for those integral values of s for which $\gamma_m(s)$ and $\gamma_m((k-1)m+1-s)$ are finite. For $f=\Delta$ and m=3 or 4 this would mean that there are identities

$$L_{3,\Delta}(s) = A\omega_\pm^2\omega_\mp\pi^n \ (s=18,19,20,21,22), \quad L_{4,\Delta}(s) = A\omega_+^2\omega_-^2\pi^n \ (s=24,26,28,30,32)$$

with $A\in\mathbb{Q}$, $n\in\mathbb{N}$. (We have given only those values of s for which the Dirichlet series converge absolutely.) However, the numerical computation of the values in question (done by G.Köckritz and R.Schillo on the IBM 370/168 at Bonn University, using 32-digit accuracy and over 1000 terms of the Euler products) did not lead to any simple values of A and n satisfying these formulas. At the Corvallis conference (July 1977), Deligne gave a revised and sharper conjecture for the values of $L_{m,f}(s)$: if f is an eigenform with rational Fourier coefficients (i.e. k=12,16,18,20,22 or 26), then one should have

$$L_{2r-1,f}(s) = (\text{rat.}).(2\pi)^{rs-\frac{r(r-1)}{2}(k-1)}\, C_\pm^{\frac{r(r+1)}{2}}\, C_\mp^{\frac{r(r-1)}{2}} \qquad (r-1<\tfrac{s}{k-1}\leqslant r, (-1)^s=\pm 1),$$

$$L_{2r,f}(s) = \begin{cases} (\text{rat.}).(2\pi)^{rs-\frac{r(r-1)}{2}(k-1)}\,(C_+C_-)^{\frac{r(r+1)}{2}} & (r-1<\frac{s}{k-1}\leqslant r, \ \ s \text{ odd}), \\ (\text{rat.}).(2\pi)^{(r+1)s-\frac{r(r+1)}{2}(k-1)}\,(C_+C_-)^{\frac{r(r+1)}{2}} & (r<\frac{s}{k-1}\leqslant r+1, \ s \text{ even}), \end{cases}$$

where C_+ and C_- are real numbers depending on f but not on r or s. For k=12, $f=\Delta$, and m=1 or 2, for instance, we have

s	$(2\pi)^{-s}\Gamma(s)L_{1,\Delta}(s)$	
6	$1/2\times3\times5$	C_+
7	$1/2^2\times7$	C_-
8	$1/2^3\times3$	C_+
9	$1/2\times3^2$	C_-
10	$2/5^2$	C_+
11	$2\times3^2\times5/691$	C_-

s	$(2\pi)^{-2s+11}\Gamma(s)L_{2,\Delta}(s)$	
12	$1/2$	C_+C_-
14	$1/2\times7$	C_+C_-
16	$1/2^5\times3$	C_+C_-
18	$1/2^2\times3^3\times5$	C_+C_-
20	$1/2\times5^2\times7^2$	C_+C_-
22	$7/2^2\times23\times691$	C_+C_-

where $C_+ = 2^6\times3\times5\ \omega_-$, $C_- = 2^5/3\times5\ \omega_+$. The computer calculation gives $C_+\approx 0.046\ 346\ 380\ 811\ 850\ 816\ 182\ 4$, $C_-\approx 0.045\ 751\ 608\ 975\ 539\ 581\ 74$, $C_+C_- = 2^{11}(\Delta,\Delta)\approx 0.002\ 120\ 421\ 492\ 335\ 249\ 248\ 968\ 328\ 831\ 438$

and suggests overwhelmingly the following identities (in accordance with Deligne's general conjecture) for $m = 3$ and 4 :

s	$(2\pi)^{-2s+11}\Gamma(s)L_{3,\Delta}(s)$	
18	$2^2/5$	$c_+^3c_-$
19	$3/7$	$c_+c_-^3$
20	$1/5$	$c_+^3c_-$
21	$5/7^2$	$c_+c_-^3$
22	$2\times3/5\times23$	$c_+^3c_-$

s	$(2\pi)^{-3s+33}\Gamma(11)^{-1}\Gamma(s)\Gamma(s-11)L_{4,\Delta}(s)$	
24	$2^5\times3^2$	$c_+^3c_-^3$
26	$2^5\times3\times5$	$c_+^3c_-^3$
28	$2^2\times23\times691/7^2$	$c_+^3c_-^3$
30	$2^3\times653$	$c_+^3c_-^3$
32	$2\times3\times34891/7$	$c_+^3c_-^3$

§ 2. An integral representation for the coefficients $c_m(s)$

In proving Theorem 1, we will reverse the order of the statements i) - iii). For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the numbers $D_f(s+k-1)$ are finite (since the series in (3) is absolutely convergent in the half-plane $\mathrm{Re}(s) > k$) and so there exists a unique cusp form $\tilde{\Phi}_s \in S_k$ satisfying

$$(34)\qquad (\tilde{\Phi}_s,\ f) = c_k \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} D_f(s+k-1)$$

for all eigenforms $f \in S_k$, namely the function given by the right-hand side of equation (14). We define $\tilde{c}_m(s)$ $(m = 1,2,\dots.)$ as the m^{th} Fourier coefficient of $\tilde{\Phi}_s$ (= the expression on the right-hand side of (15)) and must show that $\tilde{c}_m(s) = c_m(s)$. To do this, we will write $\tilde{c}_m(s)$ as an integral involving a certain kernel function ω_m which was first introduced by Petersson.

We recall the definition of the kernel function. As in § 1, we fix an even integer $k > 2$ which will be omitted from the notations. For $m = 1,2,\dots$ set

$$(35)\qquad \omega_m(z,z') = \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc=m}} \frac{1}{(czz'+dz'+az+b)^k} \qquad (z,\ z' \in H).$$

The series converges absolutely and therefore defines a function holomorphic in both variables, and one can see easily that it transforms like a modular form of weight k with respect to the action of Γ on each variable separately. One also checks easily that ω_m is a cusp form.

Proposition 1 (Petersson [16]): The function $C_k^{-1} m^{k-1} \omega_m(z, -\overline{z'})$ (C_k as in equation (3)) is the kernel function for the m^{th} Hecke operator with respect to the Petersson metric, i.e.

$$(36) \qquad C_k^{-1} m^{k-1} \int_{\Gamma\backslash H} f(z)\, \overline{\omega_m(z, -\overline{z'})}\, y^k\, dV = (f|T(m))(z') \qquad (\forall\, f \in S_k,\ z' \in H).$$

Equivalently, $\omega_m(z,z')$ has the following representation as a linear combination of Hecke eigenforms:

$$(37) \qquad m^{k-1}\, \omega_m(z,z') = C_k \sum_{i=1}^{\dim S_k} \frac{a_i(m)}{(f_i,f_i)}\, f_i(z)\, f_i(z').$$

Proof: The equivalence of (36) and (37) is immediate from the fact that the eigenforms f_i form an orthogonal basis of S_k. Also, it is easily seen that $m^{k-1}\, \omega_m(z,z')$ is obtained from $\omega_1(z,z')$ by applying the Hecke operator $T(m)$ with respect to (say) the first variable, so it suffices to prove (36) for $m = 1$.

We can write (35) for $m = 1$ in the form

$$\omega_1(z,z') = \sum_{ad-bc=1} \frac{1}{\left(z' + \frac{az+b}{cz+d}\right)^k} (cz+d)^{-k}.$$

For fixed $c, d \in \mathbb{Z}$ with $(c,d) = 1$, the pairs of integers a, b with $ad-bc = 1$ are all of the form $a_o + nc,\ b_o + nd$ ($n \in \mathbb{Z}$), where a_o, b_o is any fixed solution. Thus

$$\omega_1(z,z') = \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{1}{(cz+d)^k} \sum_{n=-\infty}^{\infty} \left(z' + \frac{a_o z + b_o}{cz+d} + n\right)^{-k}.$$

Using the identity

$$(38) \qquad \sum_{n=-\infty}^{\infty} (\tau + n)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1}\, e^{2\pi i r\tau} \qquad (\tau \in H),$$

we find

$$(39) \qquad \omega_1(z,z') = 2\, \frac{(2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1}\, G_r(z)\, e^{2\pi i r z'},$$

where $G_r(z)$ is the Poincaré series

$$(40) \qquad G_r(z) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k} e^{2\pi i r \frac{a_o z+b_o}{cz+d}} \qquad (r = 1,2,\ldots, \quad z \in H)$$

(with a_o, b_o again representing any integers with $a_o d - b_o c = 1$; in a more invariant notation $G_r(z) = \sum_{\gamma\in\Gamma_\infty\setminus\Gamma} j_k(\gamma,z)\, e^{2\pi i r\gamma z}$, where the summation is over representatives for the right cosets of $\Gamma_\infty = \{\pm \left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) \mid n \in \mathbb{Z}\}$ in Γ). But, as is well known (see, for example [6], p. 37), G_r is a cusp form of weight k and satisfies

$$(41) \qquad (f,G_r) = \frac{(k-2)!}{(4\pi r)^{k-1}}\, a(r) \qquad \text{for } f(z) = \sum_{n=1}^{\infty} a(n)\, q^n \in S_k$$

(this is proved in the same way as Rankin's identity below). Equation (36) for $m = 1$ follows immediately from equation (39) and (41). (For a different proof of Proposition 1, not using Poincaré series, see [25] .)

The other main ingredient for the proof of Theorem 1 is Rankin's integral representation of the function (3), namely

$$(42) \qquad \zeta(2s)\, \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^{s+k-1}} = \int_{\Gamma\setminus H} |f(z)|^2\, E(z,s)\, y^k\, dV$$

(valid for any cusp form $f(z) = \sum a(n)\, q^n \in S_k$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$), where $E(z,s)$ is the Epstein zeta-function

$$(43) \qquad E(z,s) = \frac{1}{2}\, y^s \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{|mz+n|^{2s}} \qquad (z = x+iy \in H,\ s \in \mathbb{C},\ \mathrm{Re}(s) > 1)$$

(here $\sum'$ denotes a sum over non-zero pairs of integers). This is a special case of a more general identity, namely that

$$(44) \qquad \int_{\Gamma\setminus H} h(z)\, E(z,s)\, dV = \zeta(2s) \int_0^{\infty}\int_0^1 h(x+iy)\, y^{s-2}\, dx\, dy$$

for any Γ-invariant function h on the upper half-plane for which the integrals in question converge absolutely. To see this, we write each pair of integers m, n in (43) as rc, rd with $r \geqslant 1$ and $(c,d) = 1$ and note that there is a 2 : 1 correspondence between the pairs c, d and the right cosets of Γ_∞ in Γ, so

$$E(z,s) = \frac{1}{2} \sum_{r=1}^{\infty} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{y^s}{r^{2s}|cz+d|^{2s}} = \zeta(2s) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \mathrm{Im}(\gamma z)^s .$$

Also, if F is a fundamental domain for the action of Γ on H, then $\bigcup_{\gamma \in \Gamma_\infty \setminus \Gamma} \gamma F$ is a fundamental domain for the action of Γ_∞. Hence

$$\begin{aligned}\int_{\Gamma\setminus H} h(z)\, E(z,s)\, dV &= \zeta(2s) \int_F \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \mathrm{Im}(\gamma z)^s\, h(\gamma z)\, dV \\ &= \zeta(2s) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \int_{\gamma F} \mathrm{Im}(z)^s\, h(z)\, dV \\ &= \zeta(2s) \int_{\Gamma_\infty \setminus H} h(z)\, \mathrm{Im}(z)^s\, dV,\end{aligned}$$

and (44) follows by choosing the fundamental domain $\{z \in H \mid 0 \leqslant x < 1\}$ for the action of Γ_∞. Equation (44) says that $\int_{\Gamma\setminus H} h(z)\, E(z,s)\, dV$ is $\zeta(2s)$ times the Mellin transform $\int_0^\infty h_o(y)\, y^{s-2}\, dy$ of the "constant term" $h_o(y)$ in the Fourier expansion

$$h(z) = \sum_{n=-\infty}^{\infty} h_n(y)\, e^{2\pi i n x}$$

of the function h (which is Γ-invariant and hence periodic). Equation (42) now follows by taking for h the Γ-invariant function

$$h(z) = y^k |f(z)|^2 = y^k \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a(n)\, \overline{a(m)}\, e^{2\pi i(n-m)x} e^{-2\pi(n+m)y}$$

with $h_o(y) = y^k \sum |a(n)|^2\, e^{-4\pi n y}$.

If f is a Hecke eigenform, then the series in (42) is related to $D_f(s)$ by equation (3) (note that a(n) is real in this case, so $a(n)^2 = |a(n)|^2$). Therefore (42) permits us to deduce the meromorphy of $D_f(s)$ and the two formulas (4) and (5) from the corresponding properties of E(z,s), namely that E(z,s) extends meromorphically to the whole s-plane with a simple pole of residue $\frac{\pi}{2}$

(independent of z!) at $s = 1$ as its only singularity and satisfies the functional equation

$$(45) \qquad E^*(z,s) = \pi^{-s}\,\Gamma(s)\,E(z,s) = E^*(z,1-s).$$

Putting together equations (37), (3) and (42), we obtain the integral representation

$$(46) \qquad \zeta(s)\,\tilde{c}_m(s) = m^{k-1} \int_{\Gamma\backslash H} \omega_m(z,-\bar{z})\,E(z,s)\,y^k\,dV \qquad (m = 1,2,\ldots,\quad s \in \mathbb{C})$$

for the function $\tilde{c}_m(s)$ defined by the right-hand side of (15). In the next paragraph we will compute the integral on the right-hand side of (46), thereby completing the proof of Theorem 1.

§ 3. Calculation of $\int_{\Gamma\backslash H} \omega_m(z,-\bar{z})\,E(z,s)\,y^k\,dV$

The computation of the integral in equation (46) will be carried out by a method similar to that used in [25] for the simpler integral

$$\int_{\Gamma\backslash H} \omega_m(z,-\bar{z})\,y^k\,dV$$

(which, by virtue of Proposition 1 above, equals $C_k m^{-k+1}$ times the trace of the Hecke operator $T(m)$ on S_k). The extra factor $E(z,s)$ in the integrand will actually simplify both the formal calculation and the treatment of convergence, which was handled incorrectly in [25] (see Correction following this paper).

The definition of $\omega_m(z,z')$, equation (35), involves a sum over all matrices of determinant m. We split up this sum according to the value of the trace of the matrix and observe that there is a $1 : 1$ correspondence between matrices of trace t and determinant m and binary quadratic forms of discriminant $t^2 - 4m$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \phi(u,v) = cu^2 + (d-a)\,uv - bv^2,$$
$$\phi(u,v) = au^2 + buv + cv^2 \longmapsto \begin{pmatrix} \frac{1}{2}(t-b) & -c \\ a & \frac{1}{2}(t+b) \end{pmatrix} .$$

Therefore

$$y^k\,\omega_m(z,-\bar z) = \sum_{t=-\infty}^{\infty} \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc=m\\ a+d=t}} \frac{y^k}{(c|z|^2+d\bar z-az-b)^k}$$

$$(47)\qquad = \sum_{t=-\infty}^{\infty} \sum_{|\phi|=t^2-4m} R_\phi(z,t),$$

where the inner sum is over all quadratic forms ϕ of discriminant t^2-4m and where we have written

$$(48)\qquad R_\phi(z,t) = \frac{y^k}{(a|z|^2+bx+c-ity)^k} \qquad (z = x+iy\in H,\ \ t\in\mathbb{R})$$

for a form ϕ, $\phi(u,v) = au^2+buv+cv^2$. The sum (47) converges absolutely for all $z\in H$, and we have

<u>Proposition 2</u>: <u>For</u> $s\in\mathbb{C}$ <u>with</u> $s\neq 1$ <u>and</u> $2-k<\mathrm{Re}(s)<k-1$, <u>we have</u>

$$\sum_{t=-\infty}^{\infty} \int_{\Gamma\backslash H} |E(z,s)|\,\Bigl|\sum_{|\phi|=t^2-4m} R_\phi(z,t)\Bigr|\,dV < \infty .$$

By virtue of this proposition, which we will prove at the end of the section, we may substitute (47) into (46) and interchange the order of summation and integration to obtain

$$\zeta(s)\,\tilde c_m(s) = m^{k-1} \sum_{t=-\infty}^{\infty} \int_{\Gamma\backslash H} \sum_{|\phi|=t^2-4m} R_\phi(z,t)\,E(z,s)\,dV \quad (2-k<\mathrm{Re}(s)<k-1).$$

Theorem 1 is then a consequence of the following result, which is of interest in its own right.

<u>Theorem 3</u>: <u>Let</u> k <u>be an even integer</u> >2, Δ <u>a discriminant</u> (i.e. $\Delta\in\mathbb{Z}$, $\Delta\equiv 0$ <u>or</u> $1 \pmod 4$)), t <u>a real number with</u> $t^2>\Delta$. <u>For each binary quadratic form</u> ϕ <u>of discriminant</u> Δ <u>let</u> $R_\phi(t,z)$ $(z\in H)$ <u>be the function defined by</u> (48). <u>Then for</u> $s\in\mathbb{C}$ <u>with</u> $s\neq 1$, $1-k<\mathrm{Re}(s)<k$,

$$\int_{\Gamma\backslash H}\left(\sum_{|\phi|=\Delta} R_\phi(z,t)\right) E(z,s)\,dV$$

(49)

$$= \zeta(s,\Delta)\,\{I_k(\Delta,t;s) + I_k(\Delta,-t;s)\} \;+$$

$$\begin{cases} (-1)^{k/2}\dfrac{\Gamma(s+k-1)\zeta(s)\zeta(2s)}{(2\pi)^{s-1}\Gamma(k)}\,|t|^{-s-k+1} & \text{if } \Delta = 0, \\ 0 & \text{if } \Delta \neq 0, \end{cases}$$

<u>where</u> $\zeta(s,\Delta)$ <u>and</u> $I_k(\Delta,t;s)$ <u>are given by</u> (6) <u>and</u> (8), <u>respectively.</u>

<u>Proof:</u> We observe first that

$$R_{\gamma\phi}(z,t) = R_\phi({}^t\gamma z,t) \qquad (\gamma\in\Gamma,\ {}^t\gamma = \text{transpose of } \gamma),$$

so that the (absolutely convergent) series $\sum_{|\phi|=\Delta} R_\phi(z,t)$ defines a function in the upper half-plane which is invariant under Γ. Moreover, this function is $O(y^{1-k})$ as $y = \mathrm{Im}(z) \to \infty$, as we will show in the proof of Proposition 2 below, while $E(z,s) = O(y^{\max(\sigma,1-\sigma)})$ for $y \to \infty$ ($\sigma = \mathrm{Re}(s)$). Hence the integral on the left hand side of (49) makes sense and is holomorphic (for $s \neq 1$) in the range specified. On the other hand, $\zeta(s,\Delta)$ also has a holomorphic continuation for all $s \neq 1$ and the integral defining $I_k(\Delta,t;s)$ converges for $1-k < \sigma < k$ (unless $\Delta = 0$, in which case the integral has a pole at $s = \frac{1}{2}$ compensating the pole coming from $\zeta(2s)$ in the expression on the right-hand side of (49)). It therefore suffices to prove (49) under the assumption $1 < \sigma < k$ and then extend the result to $1-k < \sigma < k$ by analytic continuation.

Suppose, then, that $\mathrm{Re}(s) > 1$. Written out in full, the expression on the left-hand side of (49) is

$$(50)\qquad \frac{1}{2}\int_{\Gamma\backslash H}\ \sum_{|\phi|=\Delta}\ \sideset{}{'}\sum_{m,n\in\mathbb{Z}} R_\phi(z,t)\,\frac{y^s}{|mz+n|^{2s}}\,dV.$$

The action of Γ on $z \in H$ permutes the terms of this sum, transforming the form ϕ and the pair $\pm(m,n) \in (\mathbb{Z}^2 - \{(0,0)\})/\{\pm 1\}$ in such a way that $\phi(n,-m)$ remains invariant. In particular, the sum of the terms with $\phi(n,-m) > 0$ in the integrand of (50) is Γ-invariant. Also, the group Γ acts freely on the set of pairs $(\phi, (m,n))$ with $\phi(n,-m) > 0$. Therefore, ignoring convergence for the moment, we have

$$\frac{1}{2} \int_{\Gamma\backslash H} \sum_{|\phi|=\Delta} \sideset{}{'}\sum_{\substack{m,n \\ \phi(n,-m)>0}} R_\phi(z,t) \frac{y^s}{|mz+n|^{2s}} \, dV$$

(51)

$$= \sum_{\phi} \sideset{}{'}\sum_{\substack{\pm(m,n) \\ \phi(n,-m)>0 \\ \text{mod } \Gamma}} \int_H R_\phi(z,t) \frac{y^s}{|mz+n|^{2s}} \, dV.$$

Making the substitution $z \mapsto \dfrac{nz - \frac{1}{2}bn + cm}{-mz + an - \frac{1}{2}bm}$ (which maps H to H if $an^2 - bnm + cm^2 > 0$), we find

$$\int_H \frac{y^k}{(a|z|^2+bx+c-ity)^k} \frac{y^s}{|mz+n|^{2s}} \, dV = \frac{1}{(an^2-bnm+cm^2)^s} \int_H \frac{y^{k+s} \, dV}{(|z|^2-\frac{1}{4}\Delta-ity)^k}$$

so that the right-hand side of (51) is equal to $\zeta(s,\Delta)\ I_k(\Delta,t;s)$. Since the sum defining $\zeta(s,\Delta)$ and the integral defining $I_k(\Delta,t;s)$ converge absolutely for $1 < \mathrm{Re}(s) < k$, it follows a posteriori that the expression on the left-hand side of (51) was absolutely convergent in this range. The terms with $\phi(n,-m) < 0$ can be treated in a similar manner (or simply by observing that $R_{-\phi}(z,t) = R_\phi(z,-t)$) and contribute $\zeta(s,\Delta)\ I_k(\Delta,-t;s)$.

Finally, we must treat the terms in (50) with $\phi(n,-m) = 0$. They occur only if Δ is a perfect square. These terms are not absolutely convergent in (50) (if we replace each $R_\phi(z,t)$ by its absolute value, then the sum in the integrand converges for each z but the integral diverges). We argue as in the proof of equation (44). First, by removing the greatest common divisor of m and n, we can write (50) as $\zeta(2s)$ times the corresponding sum with the extra condition $(m,n) = 1$. Since any relatively prime pair of integers (m,n) is Γ-equivalent to the pair $(0,1)$ by an element of Γ which is well-defined up to left multiplication by an element of Γ_∞, the terms of (50) with $\phi(n,-m) = 0$ give

$$\zeta(2s) \int_{\Gamma_\infty \backslash H} \sum_{\substack{|\phi|=\Delta \\ \phi(1,0)=0}} R_\phi(z,t)\, y^s\, dV$$

$$= \zeta(2s) \int_{\Gamma_\infty \backslash H} \sum_{\substack{a,b,c\, \in \mathbb{Z} \\ b^2-4ac=\Delta \\ a=0}} \frac{y^k}{(a|z|^2+bx+c-ity)^k}\, y^s\, dV$$

$$(52) \qquad = \zeta(2s) \int_0^\infty \int_0^1 \sum_{b^2=\Delta} \sum_{c=-\infty}^{\infty} \frac{1}{(bx-ity+c)^k}\, y^{s+k-2}\, dx\, dy.$$

The sum over c can be evaluated using (38) and equals

$$\frac{(2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1}\, e^{2\pi i r(\pm bx + i|t|y)}$$

if $\pm t > 0$ (note that $t^2 > \Delta$ and Δ is a square, so $t \neq 0$). If $\Delta \neq 0$, then this expression involves only terms $e^{2\pi i n x}$ with $n \neq 0$, so the integral (52) is identically zero. For $\Delta = 0$, the expression (52) becomes

$$\zeta(2s)\, \frac{(2\pi i)^k}{(k-1)!} \int_0^\infty \int_0^1 \sum_{r=1}^{\infty} r^{k-1} e^{-2\pi r|t|y}\, y^{s+k-2} dx\, dy = \frac{(2\pi i)^k}{(k-1)!} \zeta(2s)\zeta(s)\, \frac{\Gamma(s+k-1)}{(2\pi|t|)^{s+k-1}}$$

This completes the proof of Theorem 3.

<u>Proof of Proposition 2</u>: We choose for $\Gamma \backslash H$ the standard fundamental domain $\{z \,\big|\, |z| \geqslant 1,\ |x| \leqslant \frac{1}{2}\}$. Since $E(z,s) = O(y^\sigma + y^{1-\sigma})$ as $y = \mathrm{Im}(z) \to \infty$, where $\sigma = \mathrm{Re}(s)$ it will suffice to show that

$$(53) \qquad \sum_{t=-\infty}^{\infty} \int_1^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \Bigg| \sum_{|\phi|=t^2-4m} R_\phi(z,t) \Bigg|\, y^{\sigma-2}\, dx\, dy < \infty,$$

for $\sigma < k-1$. Also, the above proof shows that the integral occurring in (53) is finite for each fixed value of t (even in the larger range $\sigma < k$), so we can ignore the finitely many values of t for which t^2-4m is a perfect square. If t^2-4m is not a square, then

$$(54)\qquad \begin{aligned}\sum_{|\phi|=t^2-4m} R_\phi(z,t) &= 2\,\mathrm{Re}\left(\sum_{\substack{b^2-4ac=t^2-4m\\ a>0}} \frac{y^k}{(a|z|^2+bx+c-ity)^k}\right)\\ &= 2\,y^k\,\mathrm{Re}\left(\sum_{a=1}^{\infty}\frac{1}{a^k}\sum_{\substack{b(\mathrm{mod}\ 2a)\\ b^2\equiv t^2-4m\ (\mathrm{mod}\ 4a)}}\ \sum_{n=-\infty}^{\infty}\left[(x+\frac{b}{2a}+n)^2+y^2-\frac{ity}{a}-\frac{t^2-4m}{4a^2}\right]^{-k}\right).\end{aligned}$$

But it is easily shown that

$$\sum_{n=-\infty}^{\infty}\left[(x+n)^2+L^2\right]^{-k} = O(L^{1-2k})$$

uniformly for $x \in \mathbb{R}$ and $L \in \mathbb{C}$ with $\mathrm{Re}(L)$ bounded away from 0. We apply this with $L^2 = y^2 - ity - \frac{t^2-4m}{4a^2} = (y-\frac{it}{2a})^2 + \frac{m}{a^2}$. Since m is fixed and $a \geqslant 1$, $t \to \infty$, $y \to \infty$ in the sum (54), we can write

$$\sum_{n=-\infty}^{\infty}\left[(x+\frac{b}{2a}+n)^2+y^2-\frac{ity}{a}-\frac{t^2-4m}{4a^2}\right]^{-k} = O\left((y-\frac{it}{2a})^{-2k+1}\right) = O\left((y^2+t^2/a^2)^{-k+\frac{1}{2}}\right).$$

Also, the number of solutions $b(\mathrm{mod}\ 2a)$ of the congruence $b^2 \equiv t^2-4m \pmod{4a}$ is $O(a^{\varepsilon})$ as $a \to \infty$ for any $\varepsilon > 0$. Therefore (54) gives the estimate

$$\sum_{|\phi|=t^2-4m} R_\phi(z,t) = O\left(y^k\sum_{a=1}^{\infty} a^{\varepsilon-k}\,(y^2+t^2/a^2)^{-k+\frac{1}{2}}\right),$$

where the constant implied by $O(\ )$ depends on m and k but not on y or t. This expression is $O(y^{1-k})$ in the range $y \geqslant t$ and $O(y^{-\varepsilon}t^{-k+1+\varepsilon})$ for $t > y$, as one checks by splitting up the sum according as $a \leqslant t/y$ or $a > t/y$. Hence

$$\int_1^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\sum_{|\phi|=t^2-4m} R_\phi(z,t)\right| y^{\sigma-2}\,dx\,dy = O\left(t^{1-k+\varepsilon}\int_1^{t} y^{\sigma-2+\varepsilon}dy + \int_t^{\infty} y^{-k+\sigma-1}dy\right)$$

$$= O(t^{\sigma-k}),$$

so the sum (53) converges for $k-\sigma > 1$.

§ 4. Properties of the functions $\zeta(s,\Delta)$ and $I_k(\Delta,t;s)$

In order to deduce from Theorem 1 the various corollaries discussed in § 1, in particular the trace formula and the formula for the Petersson product of an eigenform with the modular forms constructed by Cohen, we will need various properties of the functions $\zeta(s,\Delta)$ and $I_k(\Delta,t;s)$ defined by Equations (6) and (8). We begin with the zeta-function.

Proposition 3: *Let $\zeta(s,\Delta)$ be the zeta-function defined by (6), where $\Delta \in \mathbb{Z}$, $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$. Then*

i) $\zeta(s,\Delta) = \zeta(2s) \sum_{a=1}^{\infty} \frac{n(a)}{a^s}$, *where* $n(a)$ *is the number of solutions* $b \pmod{2a}$ *of the congruence* $b^2 \equiv \Delta \pmod{4a}$.

ii) $\zeta(s,\Delta)$ *has a meromorphic continuation to the whole complex plane and, if* $\Delta \neq 0$, *satisfies the functional equation*

$$\gamma(s,\Delta)\,\zeta(s,\Delta) = \gamma(1-s,\Delta)\,\zeta(1-s,\Delta), \quad \text{where}$$

$$\gamma(s,\Delta) = \begin{cases} (2\pi)^{-s}\,|\Delta|^{s/2}\,\Gamma(s) & \text{if } \Delta < 0 \\ \pi^{-s}\,\Delta^{s/2}\,\Gamma(\frac{s}{2})^2 & \text{if } \Delta > 0 \end{cases}$$

iii) $\zeta(s,\Delta)$ *can be expressed in terms of standard Dirichlet series as follows:*

$$\zeta(s,\Delta) = \begin{cases} 0 & \text{if } \Delta \equiv 2 \text{ or } 3 \pmod 4 \\ \zeta(s)\,\zeta(2s-1) & \text{if } \Delta = 0 \\ \zeta(s)L_D(s) \sum_{d|f} \mu(d)\,(\frac{D}{d})d^{-s}\sigma_{1-2s}(\frac{f}{d}) & \text{if } \Delta \equiv 0 \text{ or } 1 \pmod 4, \Delta \neq 0, \end{cases}$$

where if $\Delta \equiv 0$ *or* $1 \pmod 4$, $\Delta \neq 0$ *we have written* $\Delta = Df^2$ *with* $f \in \mathbb{N}$, D *the discriminant of* $\mathbb{Q}(\sqrt{\Delta})$, $(\frac{D}{\cdot})$ *the Kronecker symbol*, $L_D(s) = \sum_{n=1}^{\infty} (\frac{D}{n})\, n^{-s}$ *the associated L-series, and* $\sigma_\nu(m) = \sum_{\substack{d|m \\ d>0}} d^\nu$ $(m \in \mathbb{N})$, $\nu \in \mathbb{C})$. *In particular, the function* $L(s,\Delta)$ *defined by* (7) *is entire except for a simple pole (of residue* $\frac{1}{2}$ *if* $\Delta = 0$ *and* 1 *if* $\Delta \neq 0$) *if* Δ *is a square.*

iv) <u>For</u> $\Delta < 0$, <u>the values of</u> $L(s,\Delta)$ <u>at</u> $s = 1$ <u>and</u> $s = 0$ <u>are given by</u>

$$L(1,\Delta) = \frac{\pi}{\sqrt{|\Delta|}}, \qquad L(0,\Delta) = \frac{\pi}{\sqrt{|\Delta|}} H(|\Delta|),$$

<u>where</u> $H(n)$ <u>is the class number defined by equation</u> (20). <u>More generally, if</u> r <u>is a positive odd integer then</u> $L(r,\Delta)$ <u>and</u> $L(1-r,\Delta)$ <u>are given by equations</u> (22) <u>and</u> (23), <u>where</u> $H(r,N)$ <u>is the function defined in</u> [3].

<u>Proof</u>: i) This identity is equivalent to the main theorem of the theory of binary quadratic forms (cf. [10], Satz 203), according to which $n(a)$ is the number of $SL_2(\mathbb{Z})$ - inequivalent primitive representations of a by binary quadratic forms of discriminant Δ. We can prove it directly by arguing as for the proof of (44) or (52): Let Φ denote the set of binary quadratic forms of discriminant Δ and $X = (\mathbb{Z}^2 - \{(0,0)\})/\{\pm 1\}$. For $\phi \in \Phi$ and $\pm(m,n) \in X$ set $\phi \cdot x = \phi(n,-m)$. Then Γ acts on $\Phi \times X$ preserving the pairing $\phi \cdot x$ and we can write (6) as

$$\zeta(s,\Delta) = \sum_{\phi\in\Phi/\Gamma} \sum_{x\in X/\Gamma_\phi} (\phi\cdot x)^{-s} = \sum_{(\phi,x)\in(\Phi\times X)/\Gamma} (\phi\cdot x)^{-s}$$

$$= \sum_{x\in X/\Gamma} \sum_{\phi\in\Phi/\Gamma_x} (\phi\cdot x)^{-s}$$

where Γ_ϕ, Γ_x denote the isotropy groups of ϕ and x in Γ. The orbits of X under Γ are in 1:1 correspondence with the natural numbers, since $\pm(m,n)$ is Γ-equivalent to $\pm(0,r)$ (r = g.c.d of m and n), and the isotropy group of $\pm(0,r)$ is Γ_∞. Hence

$$\zeta(s,\Delta) = \sum_{r=1}^{\infty} \sum_{\phi\in\Phi/\Gamma_\infty} \frac{1}{\phi(r,0)^s}$$

$$= \zeta(2s) \sum_{a=1}^{\infty} \sum_{\substack{b \,(\mathrm{mod}\ 2a) \\ b^2 \equiv \Delta \,(\mathrm{mod}\ 4a)}} a^{-s}.$$

ii) This follows from iii) and the functional equations of $\zeta(s)$ and $L_D(s)$. However, we can also deduce it from Theorem 3 together with the easily-proved functional equations of $E(z,s)$ (equation (45)) and $I_k(\Delta,t;s)$ (Proposition 4, iii)), so that Theorem 3 gives as a corollary new proofs for the functional equations of the zeta functions of both real and imaginary quadratic fields.

iii) This can be deduced without difficulty from i). The details are given in [8], Prop. 2, pp. 69 - 71 (our $n(a)$ is denoted there by $r_D^*(f,a)$, where $\Delta = Df^2$).

iv) From iii) and the Dirichlet class-number formula we get

$$\begin{aligned} L(1,\Delta) &= L_D(1) \sum_{d|f} \mu(d) \left(\tfrac{D}{d}\right) d^{-1} \sigma_{-1}(f/d) \\ &= \frac{2\pi}{\sqrt{|D|}} \frac{h(D)}{w(D)} \sum_{cd|f} \mu(d) \left(\tfrac{D}{d}\right) \frac{c}{f} \\ &= \frac{2\pi}{\sqrt{|\Delta|}} \frac{h(D)}{w(D)} \sum_{e|f} e \prod_{p|e} \left(1 - \left(\tfrac{D}{p}\right) p^{-1}\right) \qquad (e=cd) \\ &= \frac{2\pi}{\sqrt{|\Delta|}} \sum_{e|f} \frac{h(De^2)}{w(De^2)} = \frac{\pi}{\sqrt{|\Delta|}} H(|\Delta|). \end{aligned}$$

The general case follows similarly from ii) and iii) and the formula given by Cohen in [3], c), p. 273.

<u>Proposition 4</u>: <u>Let</u> Δ, t <u>be real numbers with</u> $\Delta < t^2$. <u>Then</u>

i) <u>The first integral in</u> (8) <u>converges absolutely for</u> $s \in \mathbb{C}$ <u>with</u>

$$k > \mathrm{Re}(s) > \begin{cases} 1-k & \text{if } \Delta < 0 \\ 0 & \text{if } \Delta > 0 \\ 1/2 & \text{if } \Delta = 0 \end{cases}$$

<u>and is then equal to the second integral.</u>

ii) <u>For</u> $\Delta \neq 0$, <u>the second integral in</u> (8) <u>converges for</u> $s \in \mathbb{C}$ <u>with</u> $1-k < \mathrm{Re}(s) < k$. <u>The function</u> $I_k(\Delta,t;s)$ <u>which it defines has a meromorphic continuation to all</u> s <u>whose only singularities are simple poles at</u> $s = k, k+1, k+2, \ldots$ <u>and</u> $s = -k+1, -k, -k-1, \ldots,$ <u>and which satisfies the functional equation</u>

$$I_k(\Delta,t;s) = \left(\tfrac{1}{4}|\Delta|\right)^{s-\frac{1}{2}} I_k(\Delta,t;1-s) \qquad \text{if } \Delta < 0,$$

$$\left[I_k(\Delta,t;s)+I_k(\Delta,-t;s)\right] = \cot\frac{\pi s}{2} \left(\tfrac{1}{4}\Delta\right)^{s-\frac{1}{2}} \left[I_k(\Delta,t;1-s)+I_k(\Delta,-t;1-s)\right] \quad \text{if } \Delta > 0.$$

iii) <u>For</u> $\Delta = 0$, $\pm t > 0$ <u>one has</u>

$$I_k(0,t;s) = e^{\pm\frac{i\pi}{2}(s-k)} \frac{\Gamma(\frac{1}{2})\,\Gamma(s-\frac{1}{2})\,\Gamma(k-s)}{\Gamma(k)} |t|^{-k+s}$$

iv) <u>For</u> $\Delta < 0$ <u>and</u> $0 < r < k$, <u>one has</u>

$$\left(\frac{t^2-\Delta}{4}\right)^{k-1} \left[I_k(\Delta,t;1-r) + I_k(\Delta,-t;1-r)\right] = \left(-\frac{1}{4}\right)^{\frac{k-r-1}{2}} \pi \frac{\Gamma(k-r)\Gamma(r)}{\Gamma(k)} p_{k,r}\left(t,\frac{t^2-\Delta}{4}\right)$$

<u>where</u> $p_{k,r}$ <u>is the polynomial defined by</u> (25).

v) <u>For</u> $\Delta > 0$

$$\frac{\sqrt{\Delta}}{2} I_k(\Delta,t;0) = i\ \mathrm{sign}\,(t)\ I_k(\Delta,t;1) = \frac{(-1)^{k/2}\pi}{k-1} \frac{1}{(|t| + \sqrt{\Delta})^{k-1}} .$$

<u>Proof</u>: i) The integrand $y^{k+s-2}(|z|^2 + ity - \frac{1}{4}\Delta)^{-k}$ $(z = x + iy \in H)$ has no poles in the upper half-plane H but grows on the boundary of H like

$$\begin{cases} |z|^{-k+\sigma-2} & \text{as } z \to i\infty \\ y^{k+\sigma-2} & \text{as } y \to 0 \\ |z-a|^{\sigma-2} & \text{as } z \to a \text{ if } \Delta = 4a^2 > 0 \\ y^{k+\sigma-2}(x^2+y)^{-k} & \text{as } z \to 0 \text{ if } \Delta = 0 \end{cases}$$

where $\sigma = \mathrm{Re}(s)$. The assertions about the convergence follow. The equality of the two integrals in (8), granted the convergence, is a consequence of the identity

$$(55) \qquad \int_{-\infty}^{\infty} (x^2+a)^{-\nu}\,dx = \frac{\Gamma(\nu-\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\nu)} a^{\frac{1}{2}-\nu} \qquad (a \in \mathbb{C} \smallsetminus (-\infty,0],\ \mathrm{Re}(\nu) > \tfrac{1}{2}).$$

ii) Set

$$(56) \qquad I_{k,s}(z) = \int_0^{\infty} \frac{x^{k+s-2}\,dx}{(x^2+2xz+1)^{k-1/2}} \qquad (1-k < \mathrm{Re}(s) < k,\ z \in \mathbb{C} - (-\infty,-1]) ;$$

this is related to the standard Legendre function $\mathcal{P}^{\mu}_{\nu}(z)$ ("associated Legendre function of the first kind") by

$$I_{k,s}(z) = \frac{2^{1-k}\,\Gamma(\frac{1}{2})}{\Gamma(k-\frac{1}{2})}\;\Gamma(k-1+s)\,\Gamma(k-s)\;(z^2-1)^{-\frac{k-1}{2}}\,P_{-s}^{1-k}(s)\qquad \left(z \in \mathbb{C}-(-\infty,+1]\right)$$

(cf. [5], 3.7 (33), p. 160). For $\Delta < 0$ the substitution $y = \frac{1}{2}\sqrt{|\Delta|}\,x$ in (8) gives

$$(57)\qquad I_k(\Delta,t;s) = (\tfrac{1}{4}|\Delta|)^{\frac{s-k}{2}}\,\frac{\Gamma(k-\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(k)}\;I_{k,s}\left(\frac{it}{\sqrt{|\Delta|}}\right)\;.$$

For $\Delta > 0$ we can also express $I_k(\Delta,t;s)$ in terms of $I_{k,s}(z)$. Indeed, since $t^2 > \Delta > 0$ we have $t \neq 0$. If t is positive, then the poles of the integrand in the second integral of (8) lie on the negative real axis, and by shifting the path of integration to the positive imaginary axis and substituting $y = \frac{1}{2}\sqrt{\Delta}\,x$ we obtain

$$(58a)\quad I_k(\Delta,t;s) = (\tfrac{1}{4}\Delta)^{\frac{s-k}{2}}\,e^{\frac{i\pi}{2}(s-k)}\,\frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)}\;I_{k,s}\left(\frac{t}{\sqrt{\Delta}}\right)\qquad(\Delta > 0,\ t > 0).$$

Similarly, if $t < 0$

$$(58b)\quad I_k(\Delta,t;s) = (\tfrac{1}{4}\Delta)^{\frac{s-k}{2}}\,e^{-\frac{i\pi}{2}(s-k)}\,\frac{\Gamma(k-\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(k)}\;I_{k,s}\left(\frac{|t|}{\sqrt{\Delta}}\right)\qquad(\Delta > 0,\ t < 0).$$

The assertions about $I_k(\Delta,t;s)$ ($\Delta \neq 0$) now follow at once from the corresponding properties of $I_{k,s}(z)$: the function $I_{k,s}(z)$ satisfies the functional equation $I_{k,s}(z) = I_{k,1-s}(z)$ (as one sees by making the substitution $x \mapsto x^{-1}$ in (56)) and has a meromorphic continuation to the whole s-plane whose only singularities are simple poles of residue $-d_n(z)$ and $+d_n(z)$ at $s = k + n$ and $s = 1 - k - n$, $n \geqslant 0$, where $d_n(z)$ is the polynomial of degree n defined by the asymptotic expansion

$$(1 + 2xz + x^2)^{-k+\frac{1}{2}} \sim \sum_{n=0}^{\infty} d_n(z)\,x^n \qquad (x \to 0).$$

iii) For $\Delta = 0$, the same argument as for $\Delta > 0$ gives

$$I_k(0,t;s) = e^{(\mathrm{sign}\ t)\cdot i\pi(s-k)/2}\,\frac{\Gamma(k-\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(k)}\int_0^\infty \frac{x^{s-3/2}\,dx}{(x+|t|)^{k-\frac{1}{2}}}$$

which is equivalent to the formula given.

iv) We have to prove that

$$\left(\frac{t^2-\Delta}{4}\right)^{k-1}\int_{-\infty}^{\infty}\frac{y^{k-r-1}}{(y^2+iyt-\frac{1}{4}\Delta)^{k-3/2}}\,dy = (-\tfrac{1}{4})^{k-r-1}\ \frac{\Gamma(\frac{1}{2})\Gamma(r)\Gamma(k-r)}{\Gamma(k-\frac{1}{2})}\ p_{k,r}\left(t,\ \frac{t^2-\Delta}{4}\right)$$

This follows by comparing the coefficients of u^{k-r-1} in the two sides of the identity

$$\int_{-\infty}^{\infty}(y^2+iyt-\tfrac{\Delta}{4}+2iy\,\tfrac{t^2-\Delta}{4}u)^{-r-\frac{1}{2}}\,dy = \frac{\Gamma(r)\Gamma(\frac{1}{2})}{\Gamma(r+\frac{1}{2})}\left(\frac{t^2-\Delta}{4}\right)^{-r}\left(1+tu+\frac{t^2-\Delta}{4}u^2\right)^{-r},$$

which in turn can be proved by taking

$$a = \frac{t^2-\Delta}{4}\,(1 + tu + \frac{t^2-\Delta}{4}u^2), \qquad \nu = r + \frac{1}{2}$$

in (55) and making the substitution $x = y + i(\frac{t}{2} + \frac{t^2-\Delta}{4}u)$.

v) These formulas (which are equivalent to one another by virtue of (58) and the functional equation $I_{k,s}(z) = I_{k,1-s}(z)$) follow from the identities

$$\int_0^{\infty}\frac{dy}{(y^2+ity-\frac{1}{4}\Delta)^{3/2}} = \frac{4}{t^2-\Delta}\ \frac{y-\frac{1}{2}it}{\sqrt{y^2+ity-\frac{1}{4}\Delta}}\ \Bigg|_0^{\infty} = -\frac{4}{\sqrt{\Delta}}\ \frac{1}{\sqrt{\Delta}+|t|}$$

$$\int_0^{\infty}\frac{y\,dy}{(y^2+ity-\frac{1}{4}\Delta)^{3/2}} = \frac{4}{t^2-\Delta}\ \frac{-\frac{1}{2}ity+\frac{1}{4}\Delta}{\sqrt{y^2+ity-\frac{1}{4}\Delta}}\ \Bigg|_0^{\infty} = \frac{-2i\ \mathrm{sign}\ (t)}{\sqrt{\Delta}+|t|}$$

by differentiating $k-2$ times with respect to t. This completes the proof of Proposition 4.

We can now prove the various assertions made in § 1 about special values of the series $c_m(s)$ defined in (9). Consider first $s = 1$. The contribution of the (finitely many) terms in (9) with $t^2 < 4m$ can be calculated from equations (17) and (19), which are special cases of Prop. 4, iv), and Prop. 3, iii), respectively. The contribution of the (finitely many) terms with $t^2 - 4m$ a non-zero square is given by (21), which is a consequence of Prop. 3, iii) and Prop. 4, ii) and v). The contribution of the two terms with $t^2 - 4m = 0$ when m is a square can be calculated from equation (13) (which follows from Prop. 4, iii)) and the equation $L(s, 0) = \zeta(2s-1)$ (Prop. 3, iii)). Finally, the (infinitely many) terms with $t^2 - 4m$ a positive non-square in (9) give 0 for $s = 1$ because $L(1, t^2 - 4m)$ is finite and $I_k(t^2-4m,t;1) + I_k(t^2-4m,-t;1)$ vanishes (by virtue of the functional equation, Prop.4,ii)). Putting all of this into the formula for $c_m(1)$ we obtain from (16) the Eichler-Selberg trace formula.

For $s = r \in \{3,5,7,\dots, k-3\}$ the calculation is even easier, since the terms in (9) with $t^2 - 4m = u^2 > 0$ now give no contribution (the factor $I_k(t^2-4m,t;s) + I_k(t^2-4m,-t;s)$ is again 0 because of the functional equation, but $L(s, t^2-4m)$ is now finite). From equations (9), (22) (= Prop. 3, iv)) and (26) (which is a consequence of ii) and iv) of Prop. 4) we obtain

$$
\begin{aligned}
c_m(r) &= C_k \cdot (-\tfrac{1}{4})^{\frac{r+1}{2}} \frac{\Gamma(k-r)\Gamma(r)}{\Gamma(k-1)} (-1)^{\frac{r-1}{2}} \frac{2^{r-1}\pi^r}{\Gamma(r)} \sum_{t^2<4m} p_{k,r}(t,m)\, H(r,4m-t^2) \\
&\quad + \begin{cases} (-1)^{\frac{k}{2}+1} \dfrac{(k+r-2)!\,\pi^{r+1} u^{k-r-1}}{2^{k-2}(k-1)!\,(2r-1)!}\, \zeta(1-2r) & \text{if } m = u^2 \\ 0 & \text{if } m \neq \text{square} \end{cases} \\
&= -\tfrac{1}{4}\, C_k \frac{\Gamma(k-r)}{\Gamma(k-1)} \pi^r \sum_{t^2\leq 4m} p_{k,r}(t,m)\, H(r,4m-t^2)
\end{aligned}
\tag{59}
$$

or (with the notations of (10) and (24))

$$\Phi_r(z) = -\tfrac{1}{4}\, C_k \frac{\Gamma(k-r)}{\Gamma(k-1)} \pi^r\, C_{k,r}(z) \qquad (r = 3,5,\dots,k-3)$$

This together with Theorem 1 shows that $C_{k,r}$ is a cusp form of weight k whose Petersson product with an arbitrary Hecke eigenform f is given by (28).

For $s = r = k - 1$ the same calculation shows that the value of the series (9) is given by equation (59), but the function (10) is no longer a modular form since we have left the region of convergence. On the other hand, it follows from ii) and iii) of Theorem 1 that the function

$$\tilde{\Phi}_{k-1}(z) = \lim_{s\to k-1} \Phi_s(z)$$

is a cusp form of weight k satisfying

$$(\tilde{\Phi}_{k-1}, f) = C_k \frac{\Gamma(2k-2)}{(4\pi)^{2k-2}} D_f(2k-2) \tag{60}$$

for each Hecke eigenform $f \in S_k$. We want to show that the m^{th} Fourier coefficient $\tilde{c}_m(k-1)$ of $\tilde{\Phi}_{k-1}$ is given by equation (27). Each term of the series (9) is continuous at $s = k-1$, and each term with $|t| > 2\sqrt{m}$ has the limit 0 as $s \to k-1$. Therefore

$$\tilde{c}_m(k-1) = c_m(k-1)+\lim_{\varepsilon\to 0}\Big(m^{k-1}\sum_{|t|>2\sqrt{m}}\big[I_k(t^2-4m,t;k-1-\varepsilon)+I_k(t^2-4m,-t;k-1-\varepsilon)\big] \times L(k-1-\varepsilon,t^2-4m)\Big). \tag{61}$$

By (59) the first term on the right is equal to the first term on the right-hand side of (27). From (58) we find

$$\begin{aligned} &I_k(t^2-4m,t;k-1-\varepsilon) + I_k(t^2-4m,-t;k-1-\varepsilon) = \\ &= 2\cos\frac{\pi(1+\varepsilon)}{2}\cdot\left(\frac{t^2-4m}{4}\right)^{-\frac{1+\varepsilon}{2}} \frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)} I_{k,k-1-\varepsilon}\left(\frac{|t|}{\sqrt{t^2-4m}}\right) \\ &= -2\pi \frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)} I_{k,k-1}(1)\ \ \varepsilon t^{-1-\varepsilon}(1+O(\varepsilon) + O(t^{-1})), \end{aligned}$$

with

$$I_{k,k-1}(1) = \int_0^\infty \frac{x^{2k-3}dx}{(x^2+2x+1)^k} = \int_0^1 u^{2k-3}\,du = \frac{1}{2k-2} \qquad \left(u = \frac{x}{x+1}\right).$$

Also $L(t^2-4m,k-1-\varepsilon) = L(t^2-4m, k-1) + O(\varepsilon)$, with both terms uniformly bounded in t. Therefore the second term in (61) equals

$$-\frac{\pi}{k-1}\frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)} m^{k-1} \lim_{\varepsilon\to 0}\Big(\varepsilon\sum_{|t|>2\sqrt{m}} \frac{1}{t^{1+\varepsilon}} L(k-1,\ t^2-4m)\Big). \tag{62}$$

On the other hand, Prop. 3)i) gives

$$L(k-1,t^2-4m) = \frac{\zeta(2k-2)}{\zeta(k-1)} \sum_{a=1}^{\infty} \frac{1}{a^{k-1}} \#\{b \pmod{2a} \mid b^2 \equiv t^2-4m \pmod{4a}\}$$

$$= \frac{\zeta(2k-2)}{\zeta(k-1)} \sum_{a=1}^{\infty} \frac{1}{a^{k-1}} \#\{d \pmod{a} \mid d(t-d) \equiv m \pmod{a}\},$$

where in the last line we have set $d = \frac{t-b}{2}$. The condition $d(t-d) \equiv m \pmod{a}$ depends only on the residue class of $t \pmod{a}$, and for a fixed residue class $t_o \pmod{a}$ one has

$$\lim_{\varepsilon\to 0} \left(\varepsilon \sum_{\substack{|t|>2\sqrt{m} \\ t\equiv t_o \pmod{a}}} \frac{1}{t^{1+\varepsilon}} \right) = \frac{2}{a} \lim_{\varepsilon\to 0} \varepsilon\zeta(1+\varepsilon) = \frac{2}{a},$$

so (62) equals

$$(63) \qquad -\frac{2\pi}{k-1} \frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)} \frac{\zeta(2k-2)}{\zeta(k-1)} m^{k-1} \sum_{a=1}^{\infty} \frac{\rho(a)}{a^k},$$

where $\rho(a) = \#\{d,d' \pmod{a} \mid dd' \equiv m \pmod{a}\}$. The function $a \mapsto \rho(a)$ is multiplicative and for a prime power $a = p^\nu$ is given by $\rho(p^\nu) = (p-1)p^{\nu-1}$ if $p \nmid m$ and by

$$\rho(p^\nu) = \begin{cases} (\nu+1)\, p^\nu - \nu p^{\nu-1} & \text{if } 0 \leqslant \nu \leqslant \mu \\ (\mu+1)\, (p^\nu - p^{\nu-1}) & \text{if } \nu > \mu \end{cases}$$

in general, where p^μ is the largest power of p dividing m. Hence

$$\sum_{a=1}^{\infty} \frac{\rho(a)}{a^k} = \prod_{p\nmid m} \left(1+ \frac{p-1}{p^k} + \frac{(p-1)p}{p^{2k}} + \dots\right) \prod_{\substack{p^\mu\| m \\ \mu\geqslant 1}} \left(1 + \frac{2p-1}{p^k} + \frac{3p^2-2p}{p^{2k}} + \dots + \frac{(\mu+1)p^\mu - \mu p^{\mu-1}}{p^{\mu k}}\right.$$

$$\left. + \frac{(\mu+1)p^{\mu+1} - (\mu+1)p^\mu}{p^{(\mu+1)k}} + \dots\right)$$

$$= \prod_p \frac{1-p^{-k}}{1-p^{1-k}} \cdot \prod_{\substack{p^\mu\| m \\ \mu\geqslant 1}} (1 + p^{1-k} + p^{2(1-k)} + \dots + p^{\mu(1-k)})$$

$$= \frac{\zeta(k-1)}{\zeta(k)} \sigma_{1-k}(m),$$

and substituting this into (63) we obtain the second term in equation (27). Therefore

$$\tilde{\Phi}_{k-1}(z) = \sum_{m=1}^{\infty} \tilde{c}_m(k-1)e^{2\pi imz} = \frac{(-1)^{k/2+1}\pi}{2^{k-1}(k-1)!} C_{k,k-1}(z)$$

$$(64) \qquad - \frac{(-1)^{k/2}(2\pi)^{1-k}}{k-1} \Gamma(k-\tfrac{1}{2})\Gamma(\tfrac{1}{2})\ \zeta(2k-2)\ E_k(z)$$

$$= \frac{(-1)^{k/2+1}\pi^k}{2^{k-1}(k-1)!} \left[C_{k,k-1}(z) - \zeta(3-2k) \quad E_k(z) \right] ,$$

where $C_{k,k-1}(z)$ is defined by (24) and

$$E_k(z) = 1 + \frac{(-1)^{k/2}(2\pi)^k}{\Gamma(k)\zeta(k)} \sum_{m=1}^{\infty} \sigma_{k-1}(m)\ e^{2\pi imz}$$

is the normalized Eisenstein series of weight k (the formula $H(k-1,0) = \zeta(3-2k)$ implies that the constant term on the right-hand side of (64) is zero). Equations (60) and (64) and the fact that E_k is orthogonal to all cusp forms imply that equation (28) holds even in the case $r = k - 1$, when $C_{k,r}$ is not a cusp form.

Finally, we should say something about the case $k = 2$. Up to now we have excluded this case because it presents the most awkward convergence questions and because there are no cusp forms of weight 2 on $SL_2(\mathbb{Z})$ anyway. However, the case $k = 2$ is also important, both for the generalization of Theorems 1 and 2 to congruence subgroups and for the applications to Hilbert modular forms given in § 6. For $k > 2$ the interesting range of values for s was $1 < \mathrm{Re}(s) < k - 1$, and the two extreme values $s = 1$ and $s = k - 1$ created extra terms (and extra difficulties) as given by formulas (21) and (27). For $k = 2$, the only interesting value is $s = 1$, and one has all of the convergence difficulties which occured previously for $s = 1$ and for $s = k - 1$, and some new difficulties due to the fact that the series expression (35) for the kernel function $\omega_m(z,z')$ is no longer absolutely convergent, so that at first sight the whole method of proof appears to break down. To get around this, one must define ω_m as $\lim_{\varepsilon\to 0} \omega_{m,\varepsilon}$, where

$$\omega_{m,\varepsilon}(z,z') = \sum_{ad-bc=m} (czz'+dz'+az+b)^{-k} \, |czz'+dz'+az+b|^{-\varepsilon}$$

("Hecke's trick"). As in Appendix 2 of [24], one can show that ω_m is a cusp form of weight 2 with the properties given by Proposition 1 (of course for $SL_2(\mathbb{Z})$ this simply means $\omega_m = 0$). Then one carries out the whole calculation of §§ 2 - 4 with $\omega_{m,\varepsilon}$ instead of ω_m, taking in Theorem 1 a value of s with $1 < \mathrm{Re}(s) < 1+\varepsilon$, and at the end lets ε tend to 0. I omit the calculation, which is awful. The result is as simple as one could hope: for $k = 2$ and $s = 1$ the m^{th} Fourier coefficient $\tilde{c}_m(1)$ of the cusp form $\tilde{\Phi}_1$ defined by (34) is given by the sum of the expression previously obtained for $k > 2$, $s = 1$ (i.e. for the trace formula) and of the extra contribution previously obtained for $k > 2$, $s = k-1$ (second term of (27)), i.e.

$$\tilde{c}_m(1) = \frac{\pi^2}{2}\left\{ \sum_{t^2 \leqslant 4m} H(4m-t^2) + \sum_{\substack{dd'=m \\ d,d'>0}} \min(d,d') - 2\sigma_1(m) \right\}.$$

Since $\tilde{\Phi}_1(z)$ is a cusp form of weight 2 on $SL_2(\mathbb{Z})$ all coefficients must be zero and we obtain the class number relation

$$\sum_{t^2 \leqslant 4m} H(4m-t^2) = \sum_{\substack{dd'=m \\ d>0}} \max\ (d,d')$$

due to Hurwitz [9]. If, however $\Gamma' \subset \Gamma$ is a congruence subgroup for which there are cusp forms of weight 2, then we obtain an expression for the trace of the Hecke operator $T(m)$ on $S_2(\Gamma')$.

§ 5. The series $\sum n^{k-1-s} G_{n^2}(z)$ and the convolution of L-series associated to modular forms

Let s be a complex number with $\mathrm{Re}(s) > 1$ and $\tilde{\Phi}_s$ the unique cusp form in S_k satisfying (34) for all normalized Hecke eigenforms $f \in S_k$. Our starting point for the calculation of the Fourier coefficients of $\tilde{\Phi}_s$ in §§ 2 - 3 was the identity (3) expressing $D_f(s)$ in terms of the Rankin zeta-function $\sum_{n=1}^{\infty} a(n)^2 n^{-s}$ (where $f = \sum a(n)\, q^n$). But $D_f(s)$ also satisfies the identity

$$D_f(s) = \zeta(2s-2k+2) \sum_{n=1}^{\infty} a(n^2) n^{-s},$$

as is well-known and easily verified using the multiplicative properties of the $a(n)$. Thus the equation defining $\tilde{\Phi}_s$ is equivalent to

$$(65) \qquad (\tilde{\Phi}_s, f) = C_k \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \zeta(2s) \sum_{n=1}^{\infty} a(n^2) n^{-s-k+1}$$

and since this equation is linear in the coefficients $a(n)$, it must hold for all cusp forms $f = \sum a(n)\, q^n \in S_k$, not just for eigenforms. Equation (65) determines $\tilde{\Phi}_s$ uniquely, and by comparing it with equation (41) we obtain the identity

$$(66) \qquad \tilde{\Phi}_s(z) = C_k \frac{\Gamma(s+k-1)}{(4\pi)^s\, \Gamma(k-1)} \zeta(2s) \sum_{n=1}^{\infty} n^{k-1-s} G_{n^2}(z)$$

expressing $\tilde{\Phi}_s$ as an infinite linear combination of Poincaré series.

It is now natural to ask whether one can obtain a proof of Theorem 1 (which states that $\tilde{\Phi}_s = \Phi_s$ for $\mathrm{Re}(s) < k-1$, where Φ_s is defined by (9) and (10)) by combining (66) with known facts about Poincaré series. Two methods suggest themselves:

1. One can substitute into (66) the formula for the m^{th} Fourier coefficient g_{rm} of $G_r(z)$, namely

$$g_{rm} = \delta_{rm} + 2\pi(-1)^{k/2}(m/r)^{\frac{k-1}{2}} \sum_{c=1}^{\infty} H_c(r,m)\, J_{k-1}\left(\frac{4\pi}{c}\sqrt{rm}\right)$$

(where δ_{rm} is the Kronecker delta, J_{k-1} a Bessel function, and

$$H_c(r,m) = \frac{1}{c} \sum_{\substack{a,d \pmod c \\ ad\equiv 1 \pmod c}} e^{2\pi i(ra+md)/c}$$

a Kloosterman sum), and try to show directly that the sum equals $c_m(s)$. I do not know whether this can be done, but it is amusing to note that the term δ_{rm} in the formula for g_{rm} produces in (66) exactly the extra contribution to $c_m(s)$ occurring in (9) when m is a square.

2. One can substitute into (66) the defining equation (40) of the Poincaré series and interchange the order of summation to obtain

$$(67) \qquad \tilde{\Phi}_s(z) = C_k \frac{\Gamma(s+k-1)}{(4\pi)^s\,\Gamma(k-1)} \zeta(2s) \sum_{\gamma\in\Gamma_\infty\setminus\Gamma} j_k(\gamma,z) \sum_{n=1}^{\infty} n^{k-1-s} e^{2\pi i n^2 \gamma z}$$

(this is certainly legitimate for $\mathrm{Re}(s) > 2$, since the double series is absolutely convergent in that region). Again, I have not been able to deduce from this that $\tilde{\Phi}_s = \Phi_s$ in general. But if $k - 1 - s$ is a non-negative even integer, then the series $\sum n^{k-1-s} e^{2\pi i n^2 z}$ occurring in (67) is (up to a factor) a derivative of the theta-series

$$\Theta(z) = \sum_{n=1}^{\infty} e^{2\pi i n^2 z}.$$

and therefore transforms nicely under the action of the modular group, and in this case it <u>is</u> possible to deduce from (67) the expression for $c_m(s)$ as a finite sum of values of zeta-functions, thus obtaining a different (and conceptually simpler) proof of Theorem 2 and of the identities for special values of $D_f(s)$ discussed in § 1. To present the idea as clearly as possible, we begin with the special case $s = k - 1$.

For $r = k - 1$, the modular form (24) figuring in Theorem 2 is given by

$$(68)\qquad C_{k,k-1}(z) = \sum_{m=0}^{\infty}\left(\sum_{\substack{t\in\mathbb{Z}\\ t^2\leqslant 4m}} H(k-1\ ,\ t^2-4m)\right)e^{2\pi imz} = (\theta\mathcal{H}_{k-1})|U_4,$$

where $\mathcal{H}_r$ $(r \geqslant 1)$ is defined by

$$\mathcal{H}_r(z) = \sum_{N=0}^{\infty} H(r,N)\, q^N \qquad\qquad (q = e^{2\pi iz})$$

and U_4 is the operator which sends $\sum a(n)\, q^n$ to $\sum a(4n)\, q^n$. In [3], Cohen proved that $\mathcal{H}_r$ is a modular form of weight $r+\frac{1}{2}$, namely

$$(69)\quad \mathcal{H}_r(z) = \zeta(1-2r)\left[E^{(4)}_{r+\frac{1}{2}}(z) + (1-i)(4z)^{-r-\frac{1}{2}}\, E^{(4)}_{r+\frac{1}{2}}\left(\frac{-1}{4z}\right)\right],$$

where

$$E^{(4)}_{r+\frac{1}{2}}(z) = \sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\setminus\Gamma_o(4)} \frac{(\frac{c}{d})\,(\frac{-4}{d})^{-1/2}}{(cz+d)^{r+1/2}}$$

is the Eisenstein series of weight $r+\frac{1}{2}$ on $\Gamma_o(4)$ (for conventions concerning modular forms of half-integral weight, see [20]). It follows that $\mathcal{H}_{k-1}(z)\,\Theta(z)$ is a modular form of weight k on $\Gamma_o(4)$ having the property that its m^{th} Fourier coefficient is 0 for all $m \equiv 2 \pmod 4$, and since one easily shows (directly or using results in [13]), that U_4 maps all forms on $\Gamma_o(4)$ with this property to forms on the full modular group, one obtains $C_{k,k-1} \in M_k(SL_2(\mathbb{Z}))$. We want to show how (67) implies that the cusp form $C_{k,k-1} - \zeta(3-2k)\, E_k$ is a multiple of $\widetilde{\Phi}_{k-1}$.

Equation (67) for $s = k-1$ can be written

$$\begin{aligned}\widetilde{\Phi}_{k-1}(z) &= c_k\, \frac{\Gamma(2k-2)}{(4\pi)^{k-1}\,\Gamma(k-1)}\,\zeta(2k-2) \sum_{\gamma\in\Gamma_\infty\setminus\Gamma} j_k(\gamma,z)\,(\tfrac{1}{2}\Theta(\gamma z) - \tfrac{1}{2}) \\ &= \frac{(-1)^{k/2}\pi^k}{2^{k-1}\,\Gamma(k)}\,\zeta(3-2k)\left[E_k(z) - \sum_{\gamma\in\Gamma_\infty\setminus\Gamma} j_k(\gamma,z)\,\Theta(\gamma,z)\right] \\ &= \frac{(-1)^{k/2}\,\pi^k}{2^{k-1}\,\Gamma(k)}\,\zeta(3-2k)\left[E_k(z) - \mathrm{Tr}_1^4\left(\sum_{\gamma\in\Gamma_\infty/\Gamma_o(4)} j_k(\gamma,z)\Theta(\gamma z)\right)\right].\end{aligned}$$

where $\mathrm{Tr}_1^4 : M_k(\Gamma_o(4)) \to M_k(SL_2(\mathbb{Z}))$ is the map defined by

$$\mathrm{Tr}_1^4(f) = \sum_{\gamma\in\Gamma_0(4)\backslash\Gamma} f|_k\gamma = \sum_{n \pmod 4} f|_k\begin{pmatrix}1&0\\n&1\end{pmatrix} + \sum_{n \pmod 2} f|_k\begin{pmatrix}0&-1\\1&2n\end{pmatrix}$$

(cf. [13]). Also, for $\gamma = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \Gamma_0(4)$,

$$\Theta(\gamma z) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2} (cz+d)^{1/2}\ \Theta(z),$$

so

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} j_k(\gamma,z)\Theta(\gamma z) = \Theta(z)\ E^{(4)}_{k-\frac{1}{2}}(z).$$

Therefore

$$\mathrm{Tr}_1^4\ (\Theta E^{(4)}_{k-\frac{1}{2}}) = E_k + (-1)^{\frac{k}{2}+1}\ \frac{2^{k-1}\ \Gamma(k)}{\pi^k\zeta(3-2k)}\ \tilde{\Phi}_{k-1}.$$

Equation (64) can now be obtained from this by using equations (68) and (69) and the explicit description of the way the series Θ and $E^{(4)}_{k-1/2}$ transform under the operation of $\begin{pmatrix}0&-1\\4&0\end{pmatrix}$ and of the matrices involved in the definitions of Tr_1^4 and U_4. We omit the details.

We observe that the argument used here for $\sum G_{n^2}(z)$ would apply to any series $\sum b(n)\ G_n(z)$, where the $b(n)$ are the Fourier coefficients of a modular form (here $\Theta(z)$). Since this principle is not very well known (although it was already used by Rankin in 1952), we give a general formulation of it, applicable also to forms of non-integral weight.

Let $\Gamma' \subset \Gamma$ be a congruence subgroup and $k > 0$ a real number. We consider multiplier systems $v : \Gamma' \to \{t \in \mathbb{C} \mid |t| = 1\}$ such that the automorphy factor

$$J(\gamma,z) = v(\gamma)^{-1}(cz+d)^{-k} \qquad (\gamma = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \Gamma',\quad z \in H)$$

satisfies the cocycle condition

$$J(\gamma_1\gamma_2,z) = J(\gamma_1,\gamma_2 z)\ J(\gamma_2,\ z) \qquad\qquad (\gamma_1,\gamma_2 \in \Gamma',\ z \in H)$$

and such that $v(\gamma) = 1$ for $\gamma \in \Gamma'_\infty = \Gamma' \cap \Gamma_\infty$ (then $J(\gamma,z)$ depends only on the coset of γ in $\Gamma'_\infty\backslash\Gamma'$, i.e. on the second row of γ). We write $M_k(\Gamma',v)$

$(S_k(\Gamma',v))$ for the spaces of modular forms (cusp forms) on Γ' which transform by

$$f(z) = J(\gamma,z)\, f(\gamma z) \qquad (\gamma \in \Gamma',\ z \in H).$$

(If $k \in \mathbb{Z}$, then v is a character on Γ' and this agrees with the usual notation; if $k \in \mathbb{Z} + \frac{1}{2}$, then our notation conflicts with that of [21] but has the advantage that the product of forms in $M_{k_1}(\Gamma',v_1)$ and $M_{k_2}(\Gamma',v_2)$ lies in $M_{k_1+k_2}(\Gamma',v_1v_2)$.) If $k > 2$, we have the Eisenstein series

$$E_k'(z) = \sum_{\gamma \in \Gamma_\infty' \backslash \Gamma'} J(\gamma,z) \in M_k(\Gamma',v)$$

and for each natural number n the Poincaré series

$$G_n'(z) = \sum_{\gamma \in \Gamma_\infty' \backslash \Gamma'} J(\gamma,z)\, e^{2\pi i n \gamma z/w} \in S_k(\Gamma',v),$$

where $w = [\Gamma_\infty : \Gamma_\infty']$ is the width of Γ'; the same proof as for (41) shows that

$$(70) \qquad (f,G_n') = \int_{\Gamma'\backslash H} f(z)\overline{G_n'(z)}\, y^k\, dV = \frac{\Gamma(k-1)w^k}{(4\pi n)^{k-1}}\, a(n)$$

for any form $f = \sum_{n=0}^{\infty} a(n)e^{2\pi i n z/w}$ in $M_k(\Gamma',v)$. With these notations we have:

<u>Proposition 5</u>: <u>Let</u> $J_i(\gamma,z) = v_i(\gamma)^{-1}(cz+d)^{-k_i}$ $(i = 1,2)$ <u>be two automorphy factors on</u> Γ', <u>where</u> k_1,k_2 <u>are real numbers with</u> $k_2 \geqslant k_1 + 2 > 2$. <u>Let</u> $f(z) = \sum_{n=1}^{\infty} a(n)\, e^{2\pi i n z/w}$ <u>and</u> $g(z) = \sum_{n=0}^{\infty} b(n)\, e^{2\pi i n z/w}$ <u>be modular forms in</u> $S_{k_1+k_2}(\Gamma',v_1v_2)$ <u>and</u> $M_{k_1}(\Gamma',v_1)$, <u>respectively, and</u> $E_{k_2}'(z)$ <u>the Eisenstein series in</u> $M_{k_2}(\Gamma',v_2)$ <u>as defined above. Then the Petersson product of</u> $g(z)\,E_{k_2}'(z)$ <u>and</u> $f(z)$ <u>is given by</u>

$$(71) \qquad (f,gE_{k_2}') = \frac{\Gamma(k_1+k_2-1)}{(4\pi)^{k_1+k_2-1}}\, w^{k_1+k_2} \sum_{n=1}^{\infty} \frac{a(n)b(n)}{n^{k_1+k_2-1}}\ .$$

<u>Proof</u>:
Set $k = k_1+k_2$, $v = v_1v_2$, $J(\gamma,z) = v(\gamma)(cz+d)^{-k} = J_1(\gamma,z)\, J_2(\gamma,z)$.
If $k_2 > k_1 + 2$, then

$$\begin{aligned} g(z)\, E'_{k_2}(z) &= \sum_{\gamma\in\Gamma'_\infty\backslash\Gamma'} J_2(\gamma,z)\, g(z) \\ &= \sum_{\gamma\in\Gamma'_\infty\backslash\Gamma'} J_2(\gamma,z)\, J_1(\gamma,z)\, g(\gamma z) \\ &= \sum_{\gamma\in\Gamma'_\infty\backslash\Gamma'} \sum_{n=0}^{\infty} b(n)\, J(\gamma,z)\, e^{2\pi i n\gamma z/w} \\ &= b(0)\, E'_k(z) + \sum_{n=1}^{\infty} b(n)\, G'_n(z), \end{aligned}$$

because the double series is absolutely convergent; if $k_2 = k_1 + 2$, we obtain the same equation by multiplying $J_2(\gamma,z)$ by $y^\varepsilon\,|cz+d|^{-2\varepsilon}$ and letting $\varepsilon \to 0$ (Hecke's trick). Equation (70) now implies the statement of the proposition. (The series in (71) converges for $k_2 > 2$ because $a(n) = O(n^{k_2/2})$ and $b(n) = O(n^{k_1-1})$.)

The method we have just described was used by Rankin (for forms on the full modular group) in [18]; his identity (33) is obtained by taking $k_1 = q$, $k_2 = k - q$, $g(z) = E_q(z) = 1 - \frac{2q}{B_q}\sum_{n=1}^{\infty}\sigma_{q-1}(n)\, e^{2\pi i n z}$ and $f = \sum a(n)\, e^{2\pi i n z} \in S_k$ an eigenform and using the identity

$$(72)\qquad \sum_{n=1}^{\infty} \frac{\sigma_r(n)\, a(n)}{n^s} = \frac{L_f(s)\, L_f(s-r)}{\zeta(2s-r-k+1)} \qquad (\mathrm{Re}(s) > r + \tfrac{k+1}{2}).$$

We remark that equation (71) is in fact true under weaker restrictions than those given. For example, Rankin's identity (33) is still valid for $q=\frac{k}{2}$ (the reader can check this for $k=12$, $f=\Delta$, using (32)). It is also worth remarking that the identity (33), together with the non-vanishing of $L_f(s)$ in the region of absolute convergence and the fact that the Hecke algebra acts on M_k with multiplicity 1, imply that for $q \geqslant \frac{k}{2} + 1$ the modular form $E_q(z)\, E_{k-q}(z)$ generates M_k as a module over the Hecke algebra. I do not know any elementary proof of this fact; a direct proof in the case $q = \frac{k}{2}$ would imply the non-vanishing of $L_f(k/2)$.

We now prove a generalization of Prop. 5 which can be used to give another proof of Theorem 2 (i.e. of the proportionality of $\tilde{\Phi}_r$ and $C_{k,r}$) and hence of the identities for $D_f(r+k-1)$, where r is an odd number satisfying

$1 < r < k-1$. To prove that $C_{k,r}$ is a modular form when $r < k-1$, Cohen defined bilinear operators $F_\nu = F_\nu^{k_1,k_2}$ $(\nu \in \mathbb{N},\ k_1,k_2 \in \mathbb{R})$ on smooth functions by the formula

$$(73)\qquad F_\nu(f_1,f_2)(z) = \sum_{\mu=0}^{\nu} (-1)^{\nu-\mu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)\,\Gamma(k_2+\nu)}{\Gamma(k_1+\mu)\,\Gamma(k_2+\nu-\mu)} \frac{\partial^\mu f_1}{\partial z^\mu} \frac{\partial^{\nu-\mu} f_2}{\partial z^{\nu-\mu}}$$

and showed ([3], Theorem 7.1) that

$$(74)\qquad F_\nu(f_1|_{k_1}\gamma,\ f_2|_{k_2}\gamma) = F_\nu(f_1,f_2)|_{k_1+k_2+2\nu}\,\gamma$$

for all $\gamma \in GL_2^+(\mathbb{R})$. From this it follows that if f_1 and f_2 are modular forms on some group Γ', with weights k_1 and k_2 and multiplier systems v_1 and v_2, respectively, then $F_\nu(f_1,f_2)$ is a modular form on Γ' of weight $k_1+k_2+2\nu$ and with multiplier system v_1v_2 and is a cusp form if $\nu > 0$. (Of course $F_o(f_1,f_2) = f_1f_2$.) The fact that $C_{k,r}$ $(r < k-1)$ is a cusp form of weight k on $SL_2(\mathbb{Z})$ then follows by the same argument as in the case $r = k-1$ from the identity

$$(75)\qquad C_{k,r} = (2\pi i)^{-\nu} \frac{\Gamma(\nu+r)}{\Gamma(r)\,\Gamma(k-r)} F_\nu(\Theta, \mathcal{H}_r)|U_4 \qquad \left(\nu = \frac{k-r-1}{2}\right)$$

We now give a formula for $F_\nu(f_1,f_2)$ when either F_1 or F_2 is an Eisenstein series; this proposition in conjunction with (69) and (75) can be used to give another proof of the identity

$$\tilde{\Phi}_r(z) = -\frac{1}{4} C_k \frac{\Gamma(k-r)}{\Gamma(k-1)} \pi^r C_{k,r}(z) \qquad (r = 3,5,\ldots, k-3),$$

which is equivalent to Theorem 2.

<u>Proposition 6:</u> <u>Let</u> k_1, k_2 J_1, J_2, g <u>and</u> E_{k_2} <u>be as in Proposition</u> 5, ν <u>a non-negative integer, and</u> $f(z) = \sum_{n=1}^{\infty} a(n)\, e^{2\pi i n z/w}$ <u>a cusp form in</u> $S_k(\Gamma',v)$, <u>where</u> $k = k_1+k_2+2\nu$ <u>and</u> $v = v_1v_2$. <u>Define</u> $F_\nu(g, E'_{k_2})$ <u>as in</u> (73). <u>Then</u>

$$(f,\ F_\nu(g,E'_{k_2})) = (2\pi i)^\nu \frac{\Gamma(k-1)\,\Gamma(k_2+\nu)}{(4\pi)^{k-1}\,\Gamma(k_2)} w^{k-\nu} \sum_{n=1}^{\infty} \frac{a(n)\,\overline{b(n)}}{n^{k_1+k_2+\nu-1}}$$

<u>Proof</u>: Let $g^{(\mu)}(z) = \frac{\partial^\mu g(z)}{\partial z^\mu}$. Then for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$

$$(76) \qquad g^{(\nu)}(\gamma z) = v_1(\gamma) \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)} c^{\nu-\mu} (cz+d)^{k_1+\nu+\mu} g^{(\mu)}(z).$$

This can be proved by induction on ν (the case $\nu = 0$ is just the transformation law of g), using the identities

$$g^{(\nu+1)}(\gamma z) = (cz+d)^2 \frac{d}{dz} g^{(\nu)}(\gamma z)$$

and

$$\binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)} (k_1+\nu+\mu) + \binom{\nu}{\mu-1} \frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu-1)} = \binom{\nu+1}{\mu} \frac{\Gamma(k_1+\nu+1)}{\Gamma(k_1+\mu)} ;$$

we leave the verification to the reader. Let $G_n'(z)$ be the Poincaré series in $S_k(\Gamma', v)$. Using the Fourier expansion

$$g^{(\nu)}(z) = \left(\frac{2\pi i}{w}\right)^\nu \sum_{n=0}^{\infty} n^\nu b(n) e^{2\pi i n z/w}$$

and (76) we obtain

$$\left(\frac{2\pi i}{w}\right)^\nu \sum_{n=0}^{\infty} n^\nu b(n) G_n'(z) = \sum_{\gamma \in \Gamma_\infty' \backslash \Gamma'} v(\gamma)^{-1} (cz+d)^{-k} g^{(\nu)}(\gamma z)$$

$$= \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)} g^{(\mu)}(z) \sum_{\gamma \in \Gamma_\infty' \backslash \Gamma'} v_2(\gamma)^{-1} c^{\nu-\mu} (cz+d)^{-k_2-\nu+\mu}$$

$$= \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)} g^{(\mu)}(z) \cdot (-1)^{\nu-\mu} \frac{\Gamma(k_2)}{\Gamma(k_2+\nu-\mu)} E_{k_2}'^{(\nu-\mu)}(z)$$

$$= \frac{\Gamma(k_2)}{\Gamma(k_2+\nu)} F_\nu(g, E_{k_2}'),$$

and this together with (70) implies the statement of the Proposition.

Applying Proposition 6 to the case $\Gamma' = \Gamma$, g an Eisenstein series, and f a Hecke eigenform, we obtain (using (72)) the following generalization of Rankin's identity (33):

<u>Corollary</u>: <u>Let</u> $k_1, k_2 \geqslant 4$ <u>be even integers with</u> $k_1 \neq k_2$ <u>and</u> E_{k_1}, E_{k_2}

<u>the normalized Eisenstein series of weight</u> k_1, k_2. <u>Let</u> ν <u>be a non-negative integer and</u> $f(z) = \sum a(n)\, e^{2\pi i n z}$ <u>a normalized eigenform in</u> S_k, $k = k_1 + k_2 + 2\nu$. <u>Then</u>

$$(77) \qquad (2\pi i)^{-\nu}\,(f, F_\nu(E_{k_1}, E_{k_2})) = (-1)^{k_2/2}\,\frac{2k_1}{B_{k_1}}\,\frac{2k_2}{B_{k_2}}\,\frac{\Gamma(k-1)}{2^{k-1}\,\Gamma(k-\nu-1)} \times L_f^*(k-\nu-1)\,L_f^*(k_2+\nu),$$

<u>where</u> B_{k_1}, B_{k_2} <u>are Bernoulli numbers and</u> $L_f^*(s)$ <u>is defined by equations</u> (30) <u>and</u> (31).

<u>Remarks</u>: 1. If $k_2 > k_1 \geqslant 4$, we prove (77) by applying Proposition 6 directly; if $k_1 > k_2$, we interchange the roles of k_1 and k_2, using the functional equation (31). As in the case $\nu = 0$, we observe that (77) remains valid also when $k_1 = k_2$.

2. Since $(2\pi i)^{-\nu} F_\nu(E_{k_1}, E_{k_2})$ has rational coefficients, the left-hand side of (77) is equal to the product of (f,f) with an algebraic number lying in the field generated by the Fourier coefficients of f. For any $k \geqslant 16$ there are sufficiently many triples (k_1, k_2, ν) satisfying the conditions of the Corollary to deduce that $L_f^*(a)\, L_f^*(b)$ is an algebraic multiple of (f,f) whenever a and b are integers of opposite parity satisfying $\frac{k}{2} < a,b < k$ (or simply $0 < a,b < k$ if we use (77) for $k_1 = k_2$). For example, if $k = 16$ and $f = \Delta_{16}$ is the unique normalized eigenform in S_{16}, we find

$$\frac{L_f^*(13)}{L_f^*(11)} = \frac{L_f^*(13)\,L_f^*(10)}{2^{14}(f,f)} \times \frac{2^{14}(f,f)}{L_f^*(10)\,L_f^*(15)} \times \frac{L_f^*(15)\,L_f^*(12)}{2^{14}(f,f)}$$

$$\times \frac{2^{14}(f,f)}{L_f^*(12)\,L_f^*(9)} \times \frac{L_f^*(9)\,L_f^*(14)}{2^{14}(f,f)} \times \frac{2^{14}(f,f)}{L_f^*(14)\,L_f^*(11)}$$

$$= \frac{1}{2.3.5.13} \times \frac{3617}{2.3.7} \times \frac{3.5.7}{3617} \times \frac{2^3.3^2.13}{1} \times \frac{1}{5.7^2} \times \frac{5.7.11}{3}$$

$$= \frac{22}{7}\,.$$

Thus we obtain a different proof of the result of Eichler-Shimura-Manin on periods of cusp forms mentioned in § 1.

3. For the six values of k with $\dim S_k = 1$, equation (77) takes the form

$$L_f^*(k_2)\, L_f^*(k-1) = (-1)^{\frac{k_2}{2}-1}\, 2^{k-2} \left(\frac{B_{k_1}}{k_1} + \frac{B_{k_2}}{k_2} - \frac{k}{B_k} \frac{B_{k_1}}{k_1} \frac{B_{k_2}}{k_2} \right) (f,f)$$

if $\nu = 0$ (Rankin [18], Theorem 5) and

$$L_f^*(k_2+\nu)\, L_f^*(k-\nu-1) = (-1)^{\frac{k_2}{2}-1}\, 2^{k-2} \frac{\Gamma(k-\nu-1)}{\Gamma(k-1)} \cdot \left((-1)^{\nu} \frac{\Gamma(k_1+\nu)}{\Gamma(k_1+1)} B_{k_1} + \frac{\Gamma(k_2+\nu)}{\Gamma(k_2+1)} B_{k_2} \right) (f,f)$$

if $\nu > 0$.

4. Proposition 6 is similar to a recent result of Shimura ([22], Theorem 2). Also, the method sketched in this section for proving Theorem 2 is related to the method used by J.Sturm (cf. note at the end of the introduction).

§ 6 The Doi-Naganuma lifting and curves on Hilbert modular surfaces

In 1969 Doi and Naganuma [4] constructed a "lifting" from modular forms on $SL_2(\mathbb{Z})$ to Hilbert modular forms on $SL_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a real quadratic field $K = \mathbb{Q}(\sqrt{D})$. Four years later, Naganuma [14] defined a similar lifting from $S_k(\Gamma_o(D), (\frac{D}{\cdot}))$ to $S_k(SL_2(\mathcal{O}))$; together, these maps give the subspace of $S_k(SL_2(\mathcal{O}))$ generated by eigenforms which are invariant under the action of $Gal(K/\mathbb{Q})$. In [24] the author constructed a "kernel function" for the Naganuma mapping, i.e. a function $\Omega(z, z'; \tau)$ of three variables which is a modular form of Nebentypus $(\frac{D}{\cdot})$ with respect to τ and a Hilbert modular form with respect to (z, z') and whose Petersson product with any modular form $f(\tau)$ of Nebentypus is the Naganuma lift $\hat{f}(z, z')$ of f. The m^{th} Fourier coefficient of $\Omega(z, z'; \tau)$ (with respect to τ) is a Hilbert modular form $\omega_{m,D}(z,z')$ defined by a series similar to that defining the function $\omega_m = \omega_{m,1}$ of § 2. By replacing ω_m by $\omega_{m,D}$ in the calculations of §§ 2 - 3 of this paper, we will obtain a theorem generalizing Theorem 1 and, as corollaries,

i) new proofs that certain functions constructed in [3] and in [8], given by Fourier expansions whose coefficients involve finite sums of values of L-series at integer arguments, are modular forms;

ii) characterization of these forms in terms of their Petersson product with Hecke eigenforms;

iii) proof that $(\hat{f},\hat{f})/(f,f)^2$ is an algebraic number for any eigenform $f \in S_k(\Gamma_o(D), (\frac{D}{\cdot}))$;

iv) partial proof of a conjecture made in [8] expressing the adjoint map of the Naganuma lifting (w.r.t. the Petersson product) in terms of intersection numbers of curves on the Hilbert modular surface $H^2/SL_2(\mathcal{O})$.

Some of the results have been obtained independently by T. Asai and T. Oda in the period since the Bonn conference. In particular, both iii) and iv) overlap with work of Oda.

We recall the result of [24]. We suppose throughout that the discriminant

D of K is $\equiv 1 \pmod 4$ and denote by k some positive even integer. The forms $\omega_{m,D}$ are defined (if $k > 2$) by

$$(78) \qquad \omega_{m,D}(z,z') = \sum_{\substack{a,b \in \mathbb{Z} \\ \lambda \in \vartheta^{-1} \\ \lambda\lambda' - ab = m/D}}' (azz' + \lambda z + \lambda' z' + b)^{-k} \quad (z,z' \in H,\ m=1,2,\ldots),$$

where λ' denotes the conjugate of λ and $\vartheta = (\sqrt{D})$ is the different of K; one checks without difficulty that $\omega_{m,D}$ is a cusp form of weight k for the Hilbert modular group $SL_2(\mathcal{O})$. The main result of [24] is that the function

$$(79) \qquad \Omega(z,z';\tau) = \sum_{m=1}^{\infty} m^{k-1}\,\omega_{m,D}(z,z')\,e^{2\pi i m\tau} \qquad (z,z',\tau \in H)$$

is a cusp form on $\Gamma_0(D)$ of weight k and Nebentypus $(\frac{D}{\cdot})$ with respect to the variable τ whose Petersson product with any other cusp form $f \in S_k(\Gamma_0(D);\,(\frac{D}{\cdot}))$ is given by

$$(80) \qquad \int_{\Gamma\backslash H} f(\tau)\,\Omega(z,z';-\bar{\tau})\,(\mathrm{Im}\ \tau)^k dV = C_k \sum_{\substack{\nu \in \vartheta^{-1} \\ \nu \gg 0}}' c((\nu)\vartheta)\,e^{2\pi i(\nu z + \nu' z')},$$

where C_k is given by (12) and $c(\mathfrak{n})$ ($\mathfrak{n}$ an integral ideal) is an explicitly given finite linear combination of the Fourier coefficients of f at the various cusps of $\Gamma_0(D)$. It is also shown that, if D is prime and f a normalized Hecke eigenform, then $\sum c((\nu)\vartheta)\,e^{2\pi i(\nu z + \nu' z')}$ equals the Naganuma lift $\hat{f}$ of f i.e. the coefficients $c(\mathfrak{n})$ are multiplicative and satisfy

$$(81) \qquad c(\mathfrak{p}) = \begin{cases} a(p) & \text{if } \mathfrak{p}\mathfrak{p}' = (p),\ (\frac{D}{p}) = 1, \\ a(p)^2 + 2p^{k-1} & \text{if } \mathfrak{p} = (p),\ (\frac{D}{p}) = -1, \\ a(p) + \overline{a(p)} & \text{if } \mathfrak{p}^2 = (p),\ (\frac{D}{p}) = 0, \end{cases}$$

$$(82) \qquad \sum_{\mathfrak{n}}' c(\mathfrak{n})\,N(\mathfrak{n})^{-s} = \left(\sum_{n=1}^{\infty}{}' a(n)n^{-s}\right)\left(\sum_{n=1}^{\infty}{}' \overline{a(n)}n^{-s}\right).$$

Asai [1] has shown that equations (81) and (82) still hold (for f an eigenform) when D is not prime.

Finally, for our generalization of Theorem 1 we must define the analogue of (2) for forms of Nebentypus. Let $f = \sum a(n)q^n \in S_k(\Gamma_o(D), (\frac{D}{\cdot}))$ and set

$$(83) \qquad D_f(s) = \prod_p (1-\alpha_p^2\, p^{-s})^{-1}(1-(\tfrac{D}{p})\alpha_p\bar\alpha_p\, p^{-s})^{-1}(1-\bar\alpha_p^2\, p^{-s})^{-1},$$

where $\alpha_p, \bar\alpha_p$ are defined by

$$(84) \qquad \alpha_p + (\tfrac{D}{p})\,\bar\alpha_p = a(p), \qquad \alpha_p\bar\alpha_p = p^{k-1}$$

or equivalently by

$$(85) \qquad \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_p \frac{1}{(1-\alpha_p\, p^{-s})(1-(\frac{D}{p})\,\bar\alpha_p\, p^{-s})} \,.$$

Then, with the same notations as in Theorem 1, we have:

Theorem 4: Let $D \equiv 1 \pmod 4$, $D > 1$, be a square-free integer and $k > 2$ an even integer. For $m = 1,2,\ldots$ and $s \in \mathbb{C}$, $2-k < \mathrm{Re}(s) < k-1$ set

$$(86) \qquad c_{m,D}(s) = m^{k-1}D^{\frac{1}{2}-s}\sum_{\substack{t\in\mathbb{Z}\\ t^2\equiv 4m \pmod D}}\left[I_k(t^2-4m,t;s)+I_k(t^2-4m,-t;s)\right]L(s,\tfrac{t^2-4m}{D})$$

$$+\begin{cases} (-1)^{k/2}\dfrac{\Gamma(s+k-1)\,\zeta(2s)}{2^{2s+k-3}\pi^{s-1}\,\Gamma(k)}\,u^{k-s-1} & \text{if } m=u^2,\ u>0,\\[2ex] 0 & \text{if } m \neq \text{square}.\end{cases}$$

Then the function

$$\Phi_{s,D}(z) = \sum_{m=1}^{\infty} c_{m,D}(s)\, e^{2\pi imz}$$

is a cusp form on $\Gamma_o(D)$ of weight k and Nebentypus $(\frac{D}{\cdot})$ and satisfies

$$(\Phi_{s,D}, f) = C_k\,\frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}}\; D_f(s+k-1)$$

for any normalized Hecke eigenform $f \in S_k(\Gamma_o(D), (\frac{D}{\cdot}))$.

Proof: We would like to imitate the proof of Theorem 1 in § 2 - 3 with $\omega_{m,D}$ instead of ω_m. We cannot use $\omega_{m,D}(z,-\bar{z})$ for this purpose, because $(z,z') \to (z,-\overline{z'})$ is not compatible with the action of the Hilbert modular group. We can get around this by using $\omega_{m,D}(\varepsilon z, \varepsilon'\bar{z})$ if K has a unit ε with $\varepsilon > 0 > \varepsilon'$ and in general by using the function $\omega_{-m,D}(z,\bar{z})$, where $\omega_{-m,D}$ is obtained by replacing m by $-m$ in (78) and is defined for $(z,z') \in H \times H_-$ (H_- = lower half-plane). Writing c instead of b in (78) and setting $\lambda = \frac{1}{2}(b + \frac{t}{\sqrt{D}})$, we obtain

$$\omega_{-m,D}(z,\bar{z}) = \sum_{\substack{a,c \in \mathbb{Z} \\ \lambda \in \mathfrak{d}^{-1} \\ \lambda\lambda' - ac = -m/D}} (a|z|^2 + \lambda z + \lambda'\bar{z} + c)^{-k}$$

$$= \sum_{\substack{a,b,c,t \in \mathbb{Z} \\ \frac{1}{4}(b^2 - t^2/D) - ac = -m/D}} (a|z|^2 + bx + c + \frac{ity}{\sqrt{D}})^{-k}$$

or, with the notation (48),

$$y^k \omega_{-m,D}(z,\bar{z}) = \sum_{\substack{t \in \mathbb{Z} \\ t^2 \equiv 4m \pmod{D}}} \sum_{|\phi| = \frac{t^2 - 4m}{D}} R_\phi(z,t) .$$

Theorem 3 (§ 3) now implies

$$(87) \quad c_{m,D}(s) = \zeta(s)^{-1} D^{-\frac{s+k-1}{2}} m^{k-1} \int_{\Gamma\backslash H} \omega_{-m,D}(z,\bar{z})\, E(z,s)\, y^k\, dV.$$

Hence

$$(88) \quad \Phi_{s,D}(\tau) = \zeta(s)^{-1} D^{-\frac{s+k-1}{2}} \int_{\Gamma\backslash H} \Omega_-(z,\bar{z};\tau)\, E(z,s)\, y^k\, dV,$$

where

$$\Omega_-(z,z';\tau) = \sum_{m=1}^{\infty} m^{k-1} \omega_{-m,D}(z,z')\, e^{2\pi i m\tau} \qquad (z,\ \tau \in H,\ \ z' \in H_-).$$

The function Ω_- has properties like those of Ω, namely it is a cusp form of Nebentypus with respect to τ and satisfies an equation like (80) but with the summation running over all $\nu \in \mathfrak{d}^{-1}$ such that $\nu > 0 > \nu'$. (This follows directly from the results of [24] if K has a unit ε with $\varepsilon > 0 > \varepsilon'$, since then $\Omega_-(z,z';\tau)$ equals $\Omega(\varepsilon z, \varepsilon' z';\tau)$, and can be proved without this assumption by making the obvious modifications in the proofs given in [24].) Therefore (88) implies that $\Phi_{s,D} \in S_k(\Gamma_0(D), (\frac{D}{\cdot}))$ and that

$$(\Phi_{s,D}, f) = C_k\, \zeta(s)^{-1}\, D^{-\frac{s+k-1}{2}} \int_{\Gamma\backslash H} h_f(z)\, E(z,s)\, dV$$

for any $f \in S_k(\Gamma_0(D), (\frac{D}{\cdot}))$, where

$$h_f(z) = y^k \sum_{\substack{\nu\in\mathfrak{d}^{-1} \\ \nu > 0 > \nu'}} \overline{c((\nu)\mathfrak{d})}\, e^{2\pi i(\nu z + \nu'\bar{z})} \qquad (z \in H).$$

The function $h_f(z)$ is $SL_2(\mathbb{Z})$-invariant because $\Omega_-(z,z';\tau)$ is a cusp form of weight k with respect to the action of $SL_2(\mathcal{O})$ on $(z,z') \in H \times H_-$. Therefore we can apply the general principle (44) to obtain

$$\int_{\Gamma\backslash H} h_f(z)\, E(z,s)\, dV = \zeta(2s) \int_0^\infty \int_0^1 \sum_\nu \overline{c((\nu)\mathfrak{d})}\, e^{2\pi i(\nu + \nu')x - 2\pi(\nu - \nu')y}\, y^{k+s-2} dx dy.$$

The only terms that contribute to this integral are those with $\nu + \nu' = 0$, i.e. $\nu = \frac{n}{\sqrt{D}}$ with $n \in \mathbb{N}$, and we obtain the identity

$$(89) \qquad (\Phi_{s,D}, f) = C_k \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \left(\frac{\zeta(2s)}{\zeta(s)} \sum_{n=1}^\infty \frac{\overline{c((n))}}{n^{s+k-1}} \right) ,$$

valid for all $f \in S_k(\Gamma_0(D), (\frac{D}{\cdot}))$. If f is an eigenform, then $\overline{c((n))} = c((n))$ and the series $\sum c((n))\, n^{-s}$ has an Euler product whose terms can be computed using (81); a short computation then shows that the expression in brackets in (89) equals $D_f(s+k-1)$.

We can now deduce several corollaries exactly as in the case $D = 1$. First of all, the functions $D_f(s)$ is entire (proved by Shimura, [21], Theorem 1) and satisfies a functional equation (proved by Asai [1], Theorem 3). Next, by taking $s = r \in \{3,5,\ldots, k-3\}$ and using (22) and (26), we find that

$$\Phi_{r,D}(z) = -\frac{\pi^r}{4} C_k \frac{\Gamma(k-r)}{\Gamma(k-1)} \quad C_{k,r,D}(z),$$

where

$$C_{k,r,D}(z) = \sum_{m=0}^{\infty} \left(\sum_{\substack{t \in \mathbb{Z} \\ t^2 \leqslant 4m \\ t^2 \equiv 4m \pmod D}} p_{k,r}(t,m)\, H(r, \frac{4m-t^2}{D}) \right) e^{2\pi i m z} .$$

For $r = k - 1$ we get an extra contribution which can be computed as in § 4, the only difference being that the multiplicative function $\rho(a)$ occurring in equation (63) must be replaced by the multiplicative function

$$\rho_D(a) = \#\{b,t \pmod{2a} \mid b^2 \equiv \frac{t^2-4m}{D} \pmod{4a}\}$$

$$= \# \{\lambda \in \mathcal{O} / a\mathcal{O} \mid \lambda\lambda' \equiv m \pmod a\} ,$$

which is calculated in [24] (Lemma 3, p. 27). We obtain an equation similar to (27) but with $\sigma_{k-1}(m)$ replaced by the m^{th} Fourier coefficient of the Eisenstein series $E_2^+(z)$ in the space $M_k^+(\Gamma_o(D),(\frac{D}{\cdot}))$ consisting of those modular forms in $M_k(\Gamma_o(D),(\frac{D}{\cdot}))$ whose m^{th} Fourier coefficient vanishes whenever m is not a quadratic residue of D ($M_k^+(\Gamma_o(D), (\frac{D}{\cdot}))$ is the subspace of $M_k(\Gamma_o(D),(\frac{D}{\cdot}))$ fixed under all Atkin-Lehner involutions). Therefore Theorem 4 implies the result of Cohen ([3], Theorem 6.2) that the functions $C_{k,r,D}$ are modular forms (cusp forms if $r < k - 1$) and at the same time gives a formula for the Petersson product of these functions with Hecke eigenforms. In particular, since $D_f(r+k-1) \neq 0$ for $r > 1$, we deduce from the "multiplicity 1" theorem that each $C_{k,r,D}$ generates the whole of $M_k(\Gamma_o(D), (\frac{D}{\cdot}))$ (resp. of $S_k(\Gamma_o(D), (\frac{D}{\cdot}))$ if $r < k - 1$) under the action of the Hecke algebra.

For $r = 1$, we again find an extra contribution (given by (21)) coming from the terms in (86) for which $\frac{t^2-4m}{D}$ is a perfect square. In contrast to the case $D = 1$, there are in general infinitely many such terms, in 1 : 1 correspondence with the integers of norm m in K, and we find

(90) $$\Phi_{1,D}(z) = -\frac{\pi}{4}\, c_k \cdot C_{k,1,D}(z) \qquad (k > 2)$$

with

$$(91)\qquad C_{k,1,D}(z) = \sum_{m=0}^{\infty}\left(\sum_{\substack{t\in\mathbb{Z}\\ t^2\leq 4m\\ t^2\equiv 4m \pmod D}} p_{k,1}(t,m)H\left(\frac{4m-t^2}{D}\right) + \frac{1}{\sqrt{D}}\sum_{\substack{\lambda\in\mathcal{O}\\ \lambda>0\\ \lambda\lambda'=m}}\min(\lambda,\lambda')^{k-1}\right)e^{2\pi imz}.$$

Finally, if $k = 2$ then we find (as in the case $D = 1$), that $\Phi_{1,D}$ equals $-\frac{\pi}{4}C_{2,1,D}$ plus a multiple of the Eisenstein series $E_2^+(z)$. Thus $C_{2,1,D}\in M_2^+(\Gamma_o(D),(\frac{D}{\cdot}))$ and $C_{k,1,D}\in S_k^+(\Gamma_o(D),(\frac{D}{\cdot}))$ for $k > 2$. This result is considerably harder to prove directly than the modularity of $C_{k,r,D}$ for $r > 1$, because the function $\mathcal{H}_r(z) = \sum H(r,N)q^N$ used by Cohen is no longer a modular form of half-integral weight when $r=1$. The fact that $C_{2,1,D}\in M_2^+(\Gamma_o(D),(\frac{D}{\cdot}))$ was proved in [8], Chapter 2.

Equation (90) together with Theorem 4 characterize the function $C_{k,1,D}$ by the formula

$$(92)\qquad (C_{k,1,D},f) = -\frac{4}{\pi}\,\frac{\Gamma(k)}{(4\pi)^k}\,D_f(k) \qquad (f\in S_k(\Gamma_o(D),(\tfrac{D}{\cdot}))\text{ an eigenform}).$$

To interpret this, we need some other product representations of $D_f(s)$. Let $a(n)$ and $c(\mathfrak{n})$ be the Fourier coefficients of f and of the Naganuma lift $\hat{f}$, respectively, α_p, $\bar{\alpha}_p$ the numbers defined by (85), and $A_{\mathfrak{p}}$ ($\mathfrak{p}\subset\mathcal{O}$ a prime ideal) the numbers defined by

$$(93)\qquad A_{\mathfrak{p}} = \alpha_p^{\,f} \qquad \text{if } N(\mathfrak{p}) = p^f.$$

Then (81), (82) are equivalent to the Euler product expansion

$$(94)\qquad \sum_{\mathfrak{n}} c(\mathfrak{n})\,N(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}}(1-A_{\mathfrak{p}}N(\mathfrak{p})^{-s})^{-1}\,(1-\overline{A_{\mathfrak{p}}}\,N(\mathfrak{p})^{-s})^{-1}$$

for the Mellin transform of $\hat{f}$, and by applying the identity (3) (or rather its analogues for forms of Nebentypus and for Hilbert modular forms) to f and $\hat{f}$ we obtain

$$(95)\qquad \sum_{\mathfrak{n}} c(\mathfrak{n})^2\,N(\mathfrak{n})^{-s} = \frac{\zeta_K(s-k+1)}{\zeta_K(2s-2k+2)}\,D_{\hat{f}}(s),$$

and

$$(96) \quad \sum_{n=1}^{\infty} |a(n)|^2 n^{-s} = \frac{\zeta(s-k+1)}{\zeta(2s-2k+2)} \prod_{p|D} (1 + p^{k-1-s})^{-1} D_f'(s)$$

where

$$D_{\hat f}(s) = \prod_{\wp} (1 - A_\wp^2 N(\wp)^{-s})^{-1} (1 - A_\wp \bar A_\wp N(\wp)^{-s})^{-1} (1 - \bar A_\wp^2 N(\wp)^{-s})^{-1},$$

and

$$D_f'(s) = \prod_p (1 - (\tfrac{D}{p}) a_p^2 p^{-s})^{-1} (1 - p^{k-1-s})^{-1} (1 - (\tfrac{D}{p}) \bar\alpha_p^2 p^{-s})^{-1}$$

(= the "twist" of D_f by $(\frac{D}{\cdot})$). On the other hand, using (93) we deduce after some trivial manipulations

$$(97) \qquad D_{\hat f}(s) = D_f(s) D_f'(s).$$

Thus $D_f(s)$ is, up to a simple factor, the ratio of the Rankin zeta functions associated to $\hat f$ and to f. The above formulas (and more general ones corresponding to Hilbert eigenforms which are not liftings) have also been observed by Asai [2].

Using the analogue of formula (5) for f and $\hat f$ (i.e. comparing the residues on the two sides of equations (95) and (96) at $s = k$ by Rankin's method) we obtain

$$D_f'(k) = \frac{2^{2k-1} \pi^{k+1}}{\Gamma(k)} D^{-1} (f,f),$$

$$D_{\hat f}(k) = \frac{2^{4k-1} \pi^{2k+2}}{\Gamma(k)^2} D^{-k-1} (\hat f, \hat f),$$

and hence, by (97),

$$D_f(k) = \frac{2^{2k} \pi^{k+1}}{\Gamma(k)} D^{-k} \frac{(\hat f,\hat f)}{(f,f)} .$$

Substituting this into (92), we obtain

<u>Theorem 5</u>: <u>Let</u> $D \equiv 1 \pmod 4$ <u>be a square-free integer</u> > 1 <u>and</u> k <u>an integer</u> $\geqslant 2$. <u>Then the function</u> $C_{k,1,D}(z)$ <u>defined by (91) is a modular form in</u> $M_k(\Gamma_o(D), (\frac{D}{\cdot}))$ <u>(a cusp form if</u> $k > 2$). <u>If</u> $f \in S_k(\Gamma_o(D), (\frac{D}{\cdot}))$ <u>is a Hecke</u>

eigenform and $\hat{f} \in S_k(SL_2(\mathcal{O}))$ its lift under the Naganuma mapping, then

$$(98) \qquad (C_{k,1,D},f) = -\frac{4}{D^k} \frac{(\hat{f},\hat{f})}{(f,f)} .$$

Since $C_{k,1,D}$ has rational Fourier coefficients we deduce

Corollary 1: Let D, f,$\hat{f}$ be as in the Theorem. Then $(\hat{f},\hat{f})/(f,f)^2$ is an algebraic number belonging to the field generated by the Fourier coefficients of f.

Doi informs me that this result has also been obtained recently by T. Oda.

Secondly, since $(\hat{f},\hat{f})$ and (f,f) are non-zero, we deduce from the "multiplicity 1" principle:

Corollary 2: The modular form $C_{k,1,D}$ together with its images under all Hecke operators span the space $S_k(\Gamma_0(D),(\frac{D}{\cdot}))$ (respectively the space $M_2(\Gamma_0(D),(\frac{D}{\cdot}))$ if $k=2$).

This corollary was conjectured in the case D prime, $k = 2$ by Hirzebruch and the author ([8], Conjecture 1', p. 108) in connection with the intersection behaviour of modular curves on Hilbert modular surfaces. We devote the rest of this section to a discussion of the relation between the above results and the results of [8].

We suppose from now on that D is a prime. For each $m \geqslant 1$ define a curve $T_m \subset H \times H$ by

$$T_m = \{(z,z') \mid \exists\, a,b, \in \mathbb{Z},\ \lambda \in \mathfrak{d}^{-1} \text{ with } ab - \lambda\lambda' = \frac{m}{D}\} ,$$

i.e. T_m is the union of the divisors of all of the expressions figuring in the definition of $\omega_{-m,D}$. The curve T_m is invariant under $SL_2(\mathcal{O})$, its image on $X = SL_2(\mathcal{O})\backslash H \times H$ being an affine algebraic curve (also denoted by T_m) each of whose components is isomorphic to the quotient of H by some arithmetic group. It was shown in [8] that

$$T_m \circ T_1 = \sum_{\substack{t\in\mathbb{Z} \\ t^2 \leq 4mm \\ t^2 \equiv 4m \pmod D}} H\left(\frac{4m-t^2}{D}\right)$$

if m is not a norm in K. In general, we must compactify X to a smooth surface $\widetilde{X}$ (by adding finitely many "cusps" and resolving all singularities on the resulting surface). Then the closure of T_m represents a homology class in $H_2(\widetilde{X})$ which we decompose as the sum of a class T_m^c in $\mathrm{Im}(H_2(X) \to H_2(\widetilde{X}))$ and of a linear combination of the classes represented by the curves of the singularity resolutions, and one has

$$T_1^c \circ T_m^c = \sum_{\substack{t\in\mathbb{Z} \\ t^2 \leq 4m \\ t^2 \equiv 4m \pmod D}} H\left(\frac{4m-t^2}{D}\right) + \frac{1}{\sqrt{D}} \sum_{\substack{\lambda\in\mathcal{O} \\ \lambda > 0 \\ \lambda\lambda' = m}} \min(\lambda, \lambda').$$

Therefore we can write

$$C_{2,1,D}(z) = -\frac{1}{12} + \sum_{m=1}^{\infty} (T_1^c \circ T_m^c)\, e^{2\pi i m z}.$$

The formula for $T_n^c \circ T_m^c$ $(n,m \in \mathbb{N}$ arbitrary) was also given in [8] and can be compactly summarized by

$$(99) \qquad \sum_{m=0}^{\infty} (T_n^c \circ T_m^c)\, e^{2\pi i m z} = (C_{2,1,D} \mid T^+(n))(z),$$

where T_o^c is defined as a certain multiple of the volume form on $\vec{X}$ and $T^+(n)$ is the composition of the n^{th} Hecke operator on $M_2(\Gamma_o(D), (\frac{D}{\cdot}))$ with the canonical projection $M_2(\Gamma_o(D), (\frac{D}{\cdot})) \to M_2^+(\Gamma_o(D), (\frac{D}{\cdot}))$. By using these intersection number formulas in combination with a direct analytical proof (by means of non-holomorphic modular forms of weight 3/2) of the fact that $C_{2,1,D} \in M_2(\Gamma_o(D), (\frac{D}{\cdot}))$, the following theorem was proved in [8]:

Theorem ([8], Chapter 3): For each homology class $K \in H_2(\widetilde{X}; \mathbb{C})$ the series

$$\Phi_K(z) = \sum_{m=0}^{\infty} (K \circ T_m^c)\, e^{2\pi i m z} \qquad (z \in H)$$

is a modular form in $M_2^+(\Gamma_o(D), (\frac{D}{\cdot}))$. The map $\Phi : K \to \Phi_K$ is injective on the

<u>subspace of</u> $H_2(\widetilde{X};\mathbb{C})$ <u>generated by the classes</u> T_n^c.

On the other hand, the map Φ is zero on the orthogonal complement of a certain subspace U of $H_2(\widetilde{X};\mathbb{C})$ (defined in [7], p. 91) containing all the classes T_n^c, with

$$\dim U = \left[\frac{D+19}{24}\right] = \dim M_2^+(\Gamma_o(D), (\tfrac{D}{\cdot})).$$

On the basis of this and of numerical calculations for $D < 200$ it was conjectured ([8], Conjecture 1, p. 108) that the subspace of $H_2(\widetilde{X};\mathbb{C})$ spanned by the classes T_n^c coincides with U and that the map $\Phi : U \to M_2^+(\Gamma_o(D), (\tfrac{D}{\cdot}))$ is an isomorphism. But $\Phi(T_n^c) = C_{2,1,D}| T^+(n)$ by equation (99), so Corollary 2 implies that the restriction of Φ to the space generated by the T_n^c is surjective, thus proving the conjecture. We state this result as

<u>Corollary</u> 3: <u>The subspace of</u> $H_2(\widetilde{X};\mathbb{C})$ <u>generated by the homology classes</u> T_n^c <u>has dimension</u> $\left[\frac{D+19}{24}\right]$ <u>and is mapped isomorphically onto</u> $M_2^+(\Gamma_o(D), (\tfrac{D}{\cdot}))$ <u>by</u> Φ.

By associating to a Hilbert cusp form $F \in S_2(SL_2(\mathcal{O}))$ the differential form

$$\frac{i}{2}\left[F(\varepsilon z, \varepsilon'\overline{z'})\ dz \wedge d\overline{z'} + F(\varepsilon z', \varepsilon'\overline{z})\ dz' \wedge d\overline{z}\right]$$

(ε = fundamental unit) and then applying the Poincaré duality map to the cohomology class represented by this form, one obtains an injective map

$$j : S_2(SL_2(\mathcal{O})) \to H_2(\widetilde{X};\mathbb{C})$$

(see [7] or [8] for details). Under this map, the codimension 1 subspace $U^o \subset U$ consisting of classes x with $xT_o^c = 0$ corresponds to the subspace $S_2^{sym} \subset S_2(SL_2(\mathcal{O}))$ generated by Hecke eigenforms F with $F(z,z') = F(z',z)$. Thus Φ can be identified with a map from S_2^{sym} to $S_2^+(\Gamma_o(D), (\tfrac{D}{\cdot}))$. On the other hand, one has the Naganuma lifting $\iota: f \to \hat{f}$ going the other way. It was conjectured in [8] (Conjecture 2, p. 109) that the two maps $\Phi \circ j : S_2^{sym} \to$

$S_2^+(\Gamma_o(D), (\frac{D}{\cdot}))$ and $\iota : S_2^+(\Gamma_o(D), (\frac{D}{\cdot})) \to S_2^{sym}$ are, up to a constant, adjoint maps with respect to the Petersson scalar product. From the definition of Φ via intersection numbers with the classes T_m^c and of ι via the Petersson product with $\Omega = \sum m\omega_{m,D}\, e^{2\pi i m\tau}$ one sees that this is equivalent to the statement

$$(100) \qquad j(m\omega_{m,D}) = (\text{const}) \cdot T_m^{co} \qquad (m = 1,2,\dots),$$

where

$$T_m^{co} = T_m^c - \frac{(T_m^c\, T_o^c)}{(T_o^c\, T_o^c)}\, T_o^c$$

is the component of T_m^c in U^o (equation (100) is conjecture 2' of [8], p. 110, except that there T_m^c was inadvertently written instead of T_m^{co}).

There are two partial results in the direction of (100) which can be deduced from Theorem 5. First of all, a formal calculation using (80) shows that for $k > 2$ the Petersson product of $m^{k-1}\omega_{m,D}$ and $n^{k-1}\omega_{n,D}$ equals $-\frac{c_k^2}{2} D^k$ times the m^{th} Fourier coefficient of $C_{k,1,D}|T^+(n)$, where $T^+(n) = S_k^+(\Gamma_o(D), (\frac{D}{\cdot})) \longrightarrow S_k^+(\Gamma_o(D), (\frac{D}{\cdot}))$ is the modified Hecke operator introduced above, while for $k = 2$ the same is true if we remove from $C_{2,1,D}$ a multiple of the Eisenstein series $E_2^+(z)$ to get a cusp form. Using (99) and the equation $j(\omega_{m,D}) \circ j(\omega_{n,D}) = -2(\omega_{m,D}, \omega_{n,D})$ ([8], p. 109, equation (17)), we can state this result for $k = 2$ as

$$j(m\omega_{m,D}) \circ j(n\omega_{n,D}) = \pi^2 D^2\; T_m^{co} \circ T_n^{co},$$

which is compatible with (100) and gives the value of the constant occurring there as $\pm\pi D$. Secondly, by letting $s \to 1$ in (87) and using

$$\lim_{s\to 1} E(z,s)/\,\zeta(s) = \frac{\pi}{2}$$

we obtain the formula

$$c_{m,D}(1) = \frac{\pi}{2} D^{-k/2} m^{k-1} \int_{\Gamma\backslash H} \omega_{m,D}(\varepsilon z, \varepsilon'\bar{z})\; y^k\, dV,$$

and for $k = 2$ this can be interpreted (using (90) and (99) and the fact that T_1

is the curve $\Gamma\backslash H \subset SL_2(\mathcal{O})\backslash H^2$, embedded by the diagonal map) as the statement

$$\pi D\; T_1 \cdot T_m^{co} \quad = \quad T_1 \circ j(m\omega_{m,D}),$$

which is again compatible with (100) and now gives the value of the constant exactly as πD. It should be possible to prove this statement with T_1 replaced by T_n using similar methods; by virtue of Corollary 3, this would suffice to establish (100) in full generality.

We end this section by proving the analogue of (100) for forms of higher weight. The principle is very general and should be applicable to any cycles on automorphic varieties (i.e. quotients of bounded symmetric domains by arithmetic groups) and automorphic forms which have the same formal relation to one another as T_m has to $\omega_{m,D}$. The proof we give would probably be carried over to the case of weight 2 (proving (100)), by using the definition of $\omega_{m,D}$ as

$$\lim_{s\to 0} \sum_{a,b,\lambda} (azz' + \lambda z + \lambda' z' + b)^{-2} \left| azz' + \lambda z + \lambda' z' + b\right|^{-s}$$

([24], Appendix 1) and carrying out the limit in the integrals.

Equation (100) (with the constant equal to πD) is equivalent to the formula

$$(101) \qquad (m\omega_{m,D}, F) \; = -\frac{\pi D}{2} \cdot i \int_{T_m} F(\varepsilon z, \varepsilon' \overline{z'})\; dz \wedge d\overline{z'} \qquad \left(\forall F \in S_2(SL_2(\mathcal{O}))\right),$$

because the right-hand side is just $-\frac{1}{2}\pi D$ times the intersections number of $j(F)$ with T_m (we can write T_m instead of T_m^c or T_m^{co} because $j(F)$ is orthogonal to the curves of the singularity resolutions and to the volume form T_o^c). Let

$$\mathcal{A} \; = \; \{A \in M_2(\mathcal{O}) \mid A^* \; = \; A'\},$$

where A^* and A' are defined for $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_2(\mathcal{O})$ as $\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix}$, respectively. The group $G = SL_2(\mathcal{O})/\{\pm 1\}$ acts on $\mathcal{A}$ by $M \circ A = M^* A M'$. Each $A \in \mathcal{A}$ with $\det A > 0$ defines a curve in $H \times H$, namely its graph $\{(z, Az) \mid z \in H\}$, and T_m consists of the images in $SL_2(\mathcal{O})\backslash H^2$ of these graphs for all A

with $\det A = m$. The components of ω_m correspond to the $SL_2(\mathcal{O})$-equivalence classes of A with $\det A = m$. Let $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ $(i = 1,\dots,r)$ denote representatives of these classes and G_i $(i = 1,\dots,r)$ the isotropy groups

$$G_i = \{ M \in G \mid M^* A_i M' = \pm A_i \}.$$

Then

$$T_m = \bigcup_{i=1}^{r} G_i \backslash H,$$

where the i^{th} component is embedded by $z \mapsto (z, A_i z)$, and $\omega_{m,D}$ is defined by

$$\omega_{m,D}(z,z') = D^{k/2} \sum_{i=1}^{r} \sum_{M \in G_i \backslash G} \phi_{M^* A_i M'}(\varepsilon z, \varepsilon' z')^{k/2} \qquad (k>2)$$

$$= \sum_{i=1}^{r} \omega_{m,D}^{(i)}(z,z')$$

where

$$\phi_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z,z') = (czz' - az + dz' - b)^{-2}$$

$$= (cz+d)^{-2} \left(z' - \frac{az+b}{cz+d}\right)^{-2}$$

(cf. [24], 11.4 - 5), and each function $\omega_{m,D}^{(i)}$ is in $S_k(SL_2(\mathcal{O}))$. For $k = 2$, one has a similar splitting of $\omega_{m,D}$ as $\sum \omega_{m,D}^{(i)}$. Then equation (100) states that

$$\sum_{i=1}^{r} (F, \omega_{m,D}^{(i)}) = -\pi D \sum_{i=1}^{r} \int_{G_i \backslash H} (c_i \bar{z} + d_i)^{-2} F(\varepsilon z, \varepsilon' \overline{A_i z})\, dx\, dy$$

for all $F \in S_2(SL_2(\mathcal{O}))$. The analogue we prove for forms of higher weight is the following.

<u>Theorem 6:</u> <u>Let</u> $k > 2$, $m \geqslant 1$ <u>and</u> $\omega_{m,D}$ <u>the Hilbert modular form</u> (78) <u>Then</u>

$$(F, \omega_{m,D}) = \tfrac{1}{2} C_k D^{k/2} \sum_{i=1}^{r} \int_{G_i \backslash H} (c_i \bar{z} + d_i)^{-k} F(\varepsilon z, \varepsilon' \overline{A_i z})\, y^k\, dV$$

<u>for all</u> $F \in S_k(SL_2(\mathcal{O}))$, i.e. <u>the Petersson product of</u> F <u>with</u> $m^{k-1}\omega_{m,D}$ <u>is proportional to the integral of</u> F <u>over the curve</u> T_m <u>in a suitable sense</u>.

Proof: We have

$$(F,\omega^{(i)}_{m,D}) = \iint_{G\backslash H^2} F(\varepsilon z, \varepsilon'\overline{z'})\ \omega^{(i)}_{m,D}(\varepsilon\bar{z}, \varepsilon' z')\quad y^k\, y'^k\, dV'\, dV$$

(102)

$$= D^{k/2} \iint_{G_i\backslash H^2} F(\varepsilon z, \varepsilon'\overline{z'})\quad \phi_{A_i}(\bar{z},z')^{k/2}\ y^k\, y'^k\, dV'dV\ .$$

Since G_i acts properly discontinuously on H, we can take for $G_i\backslash H^2$ a fundamental domain of the form $\mathcal{F} \times H$, where $\mathcal{F}$ is a fundamental domain for the action of G_i on H. Then the integral on the right-hand side of (102) equals

$$\int_{G_i\backslash H} (c_i\bar{z}+d_i)^{-k}\left(\int_H F(\varepsilon z,\varepsilon'\overline{z'})(z'-\overline{A_i z})^{-k}\ y'^k\ dV'\right)\ y^k\ dV. \tag{103}$$

But one has the identity

$$\int_H f(z)(\bar{z}-a)^{-k}\ y^k\ dV\ = \frac{1}{2}\ C_k\ f(a) \qquad\qquad (a \in H)$$

for any holomorphic function f on H with $\int_H |f(z)|^2\ y^k\ dV < \infty$ (cf. [25], p. 46), so the inner integral in (103) equals

$$\frac{1}{2}\ C_k\ F(\varepsilon z,\ \varepsilon'\overline{A_i z}).$$

This proves the theorem.

Remarks. 1. Theorem 6 is contained in recent work of T. Oda [15]. However, the explicit working out of his very general results for the case of the curve T_m has, so far as I know, not yet been given in the literature.

2. In the theorem we describe a way of integrating cusp forms of weight k over certain curves of X, whereas one would expect such an integral to make sense only for $k = 2$. Presumably there is some appropriate homology theory $\mathcal{H}_k(\widetilde{X})$ which has a natural pairing with $S_k(SL_2(\mathcal{O}))$ and such that the curves in question yield classes in $\mathcal{H}_k$. The bilinear form on $\mathcal{H}_k$ corresponding to the Petersson product in $S_k(SL_2(\mathcal{O}))$ should then have a geometrical interpretation, i.e. for two compact curves C_1 and C_2 which meet transversally the intersection number of

C_1 and C_2 in $\mathcal{H}_k$ should be a sum of local contributions from the intersection points of C_1 and C_2. The intersection number of T_n and T_m in $\mathcal{H}_k$ (assuming that n or m is not a norm and $(n,m) = 1$) must be given by

$$\sum_{\substack{t^2 < 4nm \\ t^2 \equiv 4nm \pmod D}} \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}} \; H\left(\frac{4nm-t^2}{D}\right),$$

(cf. (91)), where $\rho + \bar{\rho} = \frac{t}{\sqrt{nm}}$, $\rho\bar{\rho} = 1$. Here $\sum H(\frac{4nm-t^2}{D})$ is the number of intersection points of T_n and T_m, and from the description of the local geometry near such an intersection point given in Chapter I of [8] we find that the number ρ at a point $x \in T_n \cap T_m$ has a local description as the cross-ratio of the four tangent directions in the tangent space $T_x X$ given by $\partial_{z_1}, \partial_{z_2}$ and the directions of T_n and T_m at x. (This was suggested to me by Atiyah.)

3. For $k \geqslant 12$ the space $S_k^{sym} \subset S_k(SL_2(\mathcal{O}))$ is no longer the image of the Naganuma lifting ι but the direct sum of this image with the image of the Doi-Naganuma lifting

$$\iota_o \; : \; S_k(SL_2(\mathbb{Z})) \to S_k(SL_2(\mathcal{O})).$$

We can give a description of the adjoint map (w.r.t. the Petersson product) of ι_o in terms of intersection numbers as follows: The curve T_m is $\bigcup_{d^2|m} F_{m/d^2}$, where F_m is defined in the same way as T_m but with the condition that the triple (a, b, λ) not be divisible by any natural number > 1. In a recent thesis ("Kurven in Hilbertschen Modulflächen und Humbertsche Flächen im Siegel-Raum", Bonn 1977), H.-G. Franke proved that, for prime discriminants D, the curve F_m is irreducible if $D^2 \nmid m$ and has exactly two components if $D^2|m$. Call these two components F_m^+ and F_m^-; they are given by taking those triples (a, b, λ) in the definition of F_m for which $(\frac{a}{D}) + (\frac{b}{D})$ is positive or negative, respectively (note that $ab \equiv \lambda\lambda' \pmod D \Rightarrow (\frac{a}{D}) + (\frac{b}{D}) \neq 0$). Set

$$T_m^{\pm} = \bigcup_{f^2|m} F^{\pm}_{mD^2/f^2}$$

(so that T_m^+ and T_m^- are unions of components of T_{mD^2}, with $T_{mD^2} = T_m + T_m^+ + T_m^-$). We can break up $\omega_{mD^2,D}$ as $\omega_{m,D} + \omega^+_{m,D} + \omega^-_{m,D}$ in a parallel way, and the proof of Theorem 6 again gives an interpretation of $(F, \omega^{\pm}_{m,D})$ as an integral of F over $T_m^{\pm}$. The relation to the Doi-Naganuma mapping is given by

<u>Theorem 7:</u> <u>Let</u> $k > 2$. <u>The function</u>

$$\Omega^{o}(z,z';\tau) = \sum_{m=1}^{\infty} m^{k-1}\left[\omega^+_{m,D}(z,z') - \omega^-_{m,D}(z,z')\right] e^{2\pi i m\tau} \qquad (z,z',\tau \in \mathbb{H})$$

<u>is a cusp form of weight</u> k <u>on</u> $SL_2(\mathbb{Z})$ <u>with respect to</u> τ <u>and is, up to a factor, the kernel function of the Doi-Naganuma lifting</u> $\iota_o : S_k(SL_2(\mathbb{Z})) \to S_k(SL_2(\mathcal{O}))$.

I omit the proof, which is analogous to that in [24].

References

[1] T. Asai, On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution, J. Math. Soc. Japan 28 (1976) 48 - 61

[2] T. Asai, On certain Dirichlet series associated with Hilbert modular forms and Rankin's method, Math. Ann. 226 (1977) 81 - 94.

[3] H. Cohen, Sums involving the values at negative integers of L functions of quadratic characters, Math. Ann. 217 (1975) 271 - 285.

[4] K. Doi, H. Naganuma, On the functional equation of certain Dirichlet series, Inv. Math. 9 (1969) 1 - 14

[5] A. Erdelyi, W. Magnus, F. Oberhettinger, F. Tricomi, Higher Transcendental Functions, Vol. I, Mc Graw-Hill, New York-Toronto-London 1953.

[6] R. Gunning, Lectures on Modular Forms, Ann. of Math. Studies No. 48, Princeton Univ. Press, Princeton 1962.

[7] F. Hirzebruch, Kurven auf den Hilbertschen Modulflächen und Klassenzahlrelationen, in Classification of algebraic varieties and compact complex manifolds , Springer Lecture Notes No. 412, pp. 75 - 93.

[8] F. Hirzebruch, D. Zagier, Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus, Inv. Math. 36 (1976) 57 - 113

[9] A. Hurwitz, Über Relationen zwischen Klassenzahlen binärer quadratischer Formen von negativer Determinante, Math. Ann. 25 (1885). Math. Werke, Bd. II, pp. 8 - 50, Birkhäuser, Basel-Stuttgart 1963

[10] E. Landau, Vorlesungen über Zahlentheorie (Aus der elementaren Zahlentheorie), Hirzel, Leipzig 1927.

[11] S. Lang, Introduction to Modular Forms, Grundlehren der mathematischen Wissenschaften 222, Springer-Verlag, Berlin-Heidelberg-New York 1976

[12] D.H. Lehmer, Ramanujan's function $\tau(n)$, Duke Math. J. 10 (1943) 483 - 492.

[13] W. Li, New forms and functional equations, Math. Ann 212 (1975) 285 - 315.

[14] H. Naganuma, On the coincidence of two Dirichlet series associated with cusp forms of Hecke's "Neben"-type and Hilbert modular forms over a real quadratic

field, J. Math. Soc. Japan 25 (1973) 547 - 555.

[15] T. Oda, On modular forms associated with indefinite quadratic forms of signature (2, n-2), Math. Ann., to appear

[16] H. Petersson, Über eine Metrisierung der ganzen Modulformen, Jahresb. d. Deutschen Math. Verein. 49 (1939) 49 - 75.

[17] R. A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions, I, Proc. Camb. Phil. Soc. 35 (1939) 351 - 372.

[18] R.A. Rankin, The scalar product of modular forms, Proc. London Math. Soc. 2 (1952) 198 - 217

[19] J -P. Serre, Une interprêtation des congruences relatives à la fonction τ de Ramanujan, Sêminaire Delange-Pisot-Poitou, 9e année, 1967/8, n^{o} 14

[20] G. Shimura, On modular forms of half integral weight, Ann of Math. 97 (1973) 440 - 481

[21] G. Shimura, On the holomorphy of certain Dirichlet series, Proc. Lond. Math. Soc. 31 (1975) 79 - 98

[22] G. Shimura, The special values of the zeta functions associated with cusp forms, Comm. on Pure and Applied Math. 29 (1976) 783 - 804

[23] D. Zagier, On the values at negative integers of the zeta-function of a real quadratic field, L'Ens. Math. 22 (1976) 55 - 95.

[24] D. Zagier, Modular forms associated to real quadratic fields, Inv. Math. 30 (1975) 1 - 46

[25] D. Zagier, Traces des opêrateurs de Hecke, Sêminaire Delange-Pisot-Poitou, 17e année, 1975/6, n^{o} 23; = The Eichler-Selberg trace formula on $SL_2(\mathbb{Z})$, Appendix to Part I of [11], pp. 44 - 54.