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Higher Weil–Petersson Volumes of Moduli Spaces of Stable n-Pointed Curves

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Dedicated to the memory of Claude Itzykson

Abstract: Moduli spaces of compact stable n-pointed curves carry a hierarchy of cohomology classes of top dimension which generalize the Weil–Petersson volume forms and constitute a version of Mumford classes. We give various new formulas for the integrals of these forms and their generating functions.

0. Introduction

Let $\overline{M}_{g,n}$ be the moduli space of stable n-pointed curves of genus g. The intersection theory of these spaces is understood in the sense of orbifolds, or stacks. The algebro-geometric study of the Chow ring of $\overline{M}_{g,0}$ was initiated by D. Mumford.

The following important version of Mumford classes on $\overline{M}_{g,n}$ was introduced in [AC]. Let $p_n : \mathscr{C}_n \to \overline{M}_{g,n}$ be the universal curve, $x_i \subset \mathscr{C}_n,\ i = 1,\dots,n$, the images of the structure sections, $\omega_{\mathscr{C}/M}$ the relative dualizing sheaf. Put for $a \geqq 0$,

$$\omega_n(a) = \omega_{g,n}(a) := p_{n*}\left(c_1\left(\omega_{\mathscr{C}/M}\left(\sum_{i=1}^{n} x_i\right)\right)^{a+1}\right) \in H^{2a}(\overline{M}_{g,n}, \mathbb{Q})^{\mathbb{S}_n}\,. \qquad (0.1)$$

(We use here the notation of [KMK; AC] denote these classes κ_i. We will mostly omit g in our notation but not n).

The class $\omega_{g,n}(1)$ is actually $\frac{1}{2\pi^2}[v^{\mathrm{WP}}_{g,n}]$, where $v^{\mathrm{WP}}_{g,n}$ is the Weil–Petersson 2-form so that

$$\int\limits_{\overline{M}_{g,n}} \omega_{g,n}(1)^{3g-3+n} = (2\pi^2)^{3g-3+n} \times \text{WP-volume of } \overline{M}_{g,n}\,. \qquad (0.2)$$

(see [AC], end of Sect. 1). Generally, we will call *higher* WP-*volumes* the integrals of the type

$$\int\limits_{\overline{M}_{g,n}} \omega_{g,n}(1)^{m(1)} \cdots \omega_{g,n}(a)^{m(a)} \dots\,, \qquad \sum_{a\geqq 1} am(a) = 3g-3+n\,.$$

The objective of this paper is to derive several formulas for these volumes and their generating functions.

For genus zero, we prove a recursive formula and a closed formula for them. The latter formula represents each higher volume as an alternating sum of multinomial coefficients. It generalizes to the higher genus, with the multinomial coefficients replaced by the correlation numbers $\langle\tau_{d_1}\cdots\tau_{d_n}\rangle$ which are computable via Witten–Kontsevich's theorem [W, K1]. In the genus zero case, however, we can give an even better formula for the higher WP-volumes. We first encode these values via a generating function in infinitely many variables and translate the recursions for them into an infinite system of non-linear differential equations for this generating function. It then turns out that the inverse power series of (a slightly modified version of) this generating function satisfies a system of *linear* differential equations, which can then be solved explicitly.

We will now explain our results for the case of the classical WP-volumes (0.2) of the genus zero moduli spaces first calculated by P. Zograf [Z]. Put $v_3 = 1$ and

$$v_n := \int_{\overline{M}_{0n}} \omega_n(1)^{n-3}, \quad n \geqq 4\,. \tag{0.3}$$

The main result of [Z] is:

$$v_n = \frac{1}{2}\sum_{i=1}^{n-3} \frac{i(n-i-2)}{n-1}\binom{n-4}{i-1}\binom{n}{i+1} v_{i+2}\, v_{n-i}\,, \quad n \geqq 4\,. \tag{0.4}$$

(The factor 1/2 was inadvertently omitted when this formula was quoted in [KMK]). Consider the generating functions

$$\Phi(x) = \sum_{n=3}^{\infty} \frac{v_n}{n!(n-3)!}\, x^n\,, \qquad h(x) = \Phi'(x) = \sum_{n=3}^{\infty} \frac{v_n}{(n-1)!(n-3)!}\, x^{n-1}\,. \tag{0.5}$$

Then (0.4) directly translates into the differential equation

$$xh'' - h' = (xh' - h)\,h''\,. \tag{0.6}$$

Notice that according to [Ma], the generating series (0.5) arise in the Liouville gravity models.

In the first part of our paper we generalize both (0.4) and (0.5) to arbitrary WP-volumes of genus zero: see Theorems 1.2.1 and 1.6.1. We follow the scheme of proof explained in [KMK], Sect. 3. The second series of our results gives more explicit formulas which specialize to the case of v_n in the following way: for $n \geqq 4$,

$$v_n = \sum_{k=1}^{n-3} \frac{(-1)^{n-3-k}}{k!} \sum_{\substack{m_1,\dots,m_k>0\\ m_1+\cdots+m_k=n-3}} \binom{n-3}{m_1,\dots,m_k}\binom{n-3+k}{m_1+1,\dots,m_k+1}\,. \tag{0.7}$$

(Actually, we prove an analog of (0.7) for arbitrary genus, but as we have already mentioned the multinomial coefficients are then replaced by the less well understood numbers $\langle\tau_{d_1}\cdots\tau_{d_n}\rangle$). Then, using either this explicit formula or by inverting the system of differential equations, we obtain a formula for the generating function of the WP-volumes as the inverse power series of a very simple power series. For the case of the v_n this goes as follows: Differentiating the differential equation (0.6) gives $h'h''' = xh''^3$ or (setting $y = h'$) $yy'' = xy'^3$. This is now cubic rather than quadratic, but if we interchange the roles of x and y then it miraculously transforms into the Bessel equation $y\frac{d^2x}{dy^2}+x=0$, leading to the explicit solution of (0.4) via an inverted modified Bessel function:

$$y=\sum_{n=3}^{\infty}\frac{v_n}{(n-2)!\,(n-3)!}\,x^{n-2}\Leftrightarrow x=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m!\,(m-1)!}\,y^m\,. \tag{0.8}$$

It is tempting to see this as another tiny bit of the general "mirror phenomenon" first observed for Calabi–Yau threefolds.

As a corollary of (0.8) we get the asymptotics

$$v_{n+3}\approx\frac{(2n)!}{C^n}\left(1.3620537\cdots-\frac{0.131\cdots}{n}+\frac{0.019\cdots}{n^2}-\cdots\right), \tag{0.9}$$

where $C=2.496918339\cdots$ is a constant that can be expressed in terms of Bessel functions and their derivatives. The existence of such an asymptotic formula–for all genera, and with a constant C independent of the genus–was conjectured by Claude Itzykson (P. Zograf, private communication).

It is interesting to compare (0.9) to the asymptotics of the Euler characteristic

$$\chi(\overline{M}_{0,n+3})\approx\frac{1}{\sqrt{n+2}}\left(\frac{n+2}{e^2-2e}\right)^{n+\frac{3}{2}} \tag{0.10}$$

(cf. [M], p. 403). One more problem in the same spirit is to study the asymptotic structure of the representation of $\mathbb{S}_n$ on $H^*(\overline{M}_{0n})$ with respect to the Plancherel measure, in the same sense as it was done for the regular representation in [LS] (cf. also [VK]). A relevant generating function was recently calculated by E. Getzler and M. Kapranov, cf. [G].

The paper is structured as follows. In Sect. 1 we prove the recursive relations and the differential equations for the generating function in genus zero. In Sect. 2 we derive by a different method explicit formulas for higher WP-volumes and prove the analogue of the inversion formula (0.8). Finally, Sect. 3 briefly explains our main motivation for studying WP-volumes: they and their generating function naturally arise in the theory of the so-called Associativity Equations and the operation of tensor product in the category of algebras over the $\{H_*(\overline{M}_{0,n+1})\}$-operad. (For more details see [KMK].) Finally, in a short appendix we make some remarks about the asymptotic formulas (0.9) and (0.10) and about the Betti numbers of the spaces $\overline{M}_{0,n+3}$.

1. Recursive relations and differential equations for the generating function

1.1. Notation. Consider the semigroup N^{∞} of sequences $\mathbf{m}=(m(1),m(2),\ldots)$, where $m(a)$ are nonnegative integers, $m(a)=0$ for sufficiently large a.

We put

$$V_g(\mathbf{m}):=\frac{1}{\left(\sum_{a\geqq 1}am(a)\right)!}\int_{\overline{M}_{g,n}}\prod_{a\geqq 1}\frac{\omega_{g,n}(a)^{m(a)}}{m(a)!}\in\mathbb{Q}\,, \tag{1.1}$$

interpreting the r.h.s. as zero unless $\sum_{a\geqq 1}am(a)=\dim\overline{M}_{g,n}=3g-3+n$. In the rest of this section $g=0$ and we write $V(\mathbf{m})$ instead of $V_0(\mathbf{m})$.

In shorter versions of expressions like (1.1) we will use notation of the type

$$|\mathbf{m}| := \sum_{a\geqq 1} am(a), \qquad \|\mathbf{m}\| := \sum_{a\geqq 1} m(a), \qquad \mathbf{m}! := \prod_{a\geqq 1} m(a)!,$$
$$\omega_n^{\mathbf{m}} = \prod_{a\geqq 1} \omega_{g,n}(a)^{m(a)}, \qquad \mathbf{s}^{\mathbf{m}} = \prod_{a\geqq 1} s_a^{m(a)}, \tag{1.2}$$

where $\mathbf{s} = (s_1, s_2, \ldots)$ is a family of independent formal variables or complex numbers. For instance, we have $V(\mathbf{m}) = \int \omega^{\mathbf{m}}/\mathbf{m}!\,|\mathbf{m}|!$ in this notation.

1.2. A recursive Formula for $V(\mathbf{m})$. Put

$$K(n_1,\ldots,n_a) := \frac{1}{n_1(n_1+n_2)\cdots(n_1+\cdots+n_a)} \tag{1.3}$$

and denote by $\delta_a \in N^\infty$ the sequence with 1 at the a^{th} place and zeros elsewhere.

1.2.1. Theorem. *For any* $\mathbf{m}$ *and* $a \geqq 1$, *we have*:

$$(m(a)+1)V(\mathbf{m}+\delta_a) = (|\mathbf{m}|+a+1) \sum_{\mathbf{m}=\sum_{i=1}^{a+1}\mathbf{m}_i} K(n_1,\ldots,n_a)\prod_{i=1}^{a+1} V(\mathbf{m}_i), \tag{1.4}$$

where in each summand of (1.4),

$$(n_1,\ldots,n_a) := (|\mathbf{m}_1|,\ldots|\mathbf{m}_a|) + (2,1,\ldots,1) \tag{1.5}$$

(*notice the absence of* $|\mathbf{m}_{a+1}|$). *These relations uniquely define* $V(\mathbf{m})$ *starting with* $V(0) = 1$.

1.2.2. A particular case of (1.4). Applying (1.4) to $V(m) := V(m,0,0,\ldots)$ and $a = 1$ we get:

$$\begin{aligned}(m+1)V(m+1) &= (m+2)\sum_{m=m_1+m_2} \tfrac{1}{m_1+2} V(m_1)V(m_2)\\ &= \tfrac{1}{2}(m+2)\sum_{m=m_1+m_2}\left(\tfrac{1}{m_1+2}+\tfrac{1}{m_2+2}\right) V(m_1)\,V(m_2)\end{aligned}$$

so that

$$\frac{V(m+1)}{m+3} = \frac{1}{2}\frac{(m+2)(m+4)}{(m+1)(m+3)}\sum_{m=m_1+m_2}\frac{V(m_1)}{m_1+2}\,\frac{V(m_2)}{m_2+2}.$$

On the other hand, Zograf's recursive relations (0.4) can be rewritten as

$$\frac{(n-2)v_n}{(n-3)!(n-1)!} = \frac{1}{2}\frac{(n-2)n}{(n-3)(n-1)}\sum_{\substack{n+2=p+q\\ p,q\geqq 3}}\frac{(p-2)v_p}{(p-3)!(p-1)!}\,\frac{(q-2)v_q}{(q-3)!(q-1)!}.$$

These relations agree for $V(n-3) = v_n/(n-3)!^2$ which is the correct formula in view of (0.3) and (1.1).

1.2.3. Another special case. Let us use (1.4) to compute $V(\mathbf{m})$ for $|\mathbf{m}| \leqq 2$. We have from (1.4) for $\mathbf{m} = 0$:

$$V(\delta_a) = (a+1)K(2,1,\ldots,1) = \frac{1}{a!}. \tag{1.6}$$

But already in the next case, the consistency of the two possible formulas obtained from (1.4) for $V(\delta_a + \delta_b)$ is not evident a priori.

Put $\mathbf{m} = \delta_b$. There are $a+1$ partitions of δ_b contributing to (1.4): in the k^{th} partition $\mathbf{m}_k = \delta_b$, $\mathbf{m}_j = 0$ for $j \neq k$, $\prod_{i=1}^{a+1} V(\mathbf{m}_i) = 1/b!$,

$$(n_1, \ldots, n_a) = \begin{cases} (2,1,\ldots,1) + b\delta_k & \text{for } k \leqq a, \\ (2,1,\ldots,1) & \text{for } k = a+1\,, \end{cases}$$

so that

$$K(n_1, \ldots, n_a) = \frac{1}{(a+b+1)!} \frac{(b+k)!}{k!}$$

and

$$2V(\delta_a + \delta_b) = (a+b+1) \sum_{k=1}^{a+1} \frac{(b+k)!}{(a+b+1)!k!} \frac{1}{b!} = \frac{1}{(a+b)!} \sum_{k=1}^{a+1} \binom{b+k}{k}\,.$$

This does not look symmetric in a, b, but of course it is:

$$V(\delta_a + \delta_b) = \frac{1}{2(a+b)!} \left[\binom{a+b+2}{a+1} - 1 \right]. \tag{1.7}$$

We will generalize (1.6) and (1.7) below to all n (and g).

We will now start proving Theorem 1.2.1. We will use some of the formalism of [KMK]. For any stable n-tree σ, denote by $\varphi_\sigma : \overline{M}_\sigma \hookrightarrow \overline{M}_{0n}$ the embedding of the corresponding closed stratum. We recall that the image of a generic point of $\overline{M}_\sigma$ parametrizes a curve whose dual tree is σ. The set of vertices $v \in V_\sigma$ of this tree bijectively corresponds to the set of irreducible components of the curve. The edges $e \in E_\sigma$ "are" singular points of the curve, and the unpaired flags (tails or leaves) are in a bijection with the labelled points, that is, with $\{1, \ldots, n\}$. For $v \in V_\sigma$, we denote by $|v|$ the number of flags incident to v.

For any stable n-tree σ, we put (with notation (1.2))

$$\Omega_n(\mathbf{m}, \sigma) = \int_{\overline{M}_\sigma} \frac{\varphi_\sigma^*(\omega_n^{\mathbf{m}})}{\mathbf{m}!}\,, \tag{1.8}$$

interpreting this as zero unless $n - 3 - |\mathbf{m}| = \operatorname{codim} \varphi_\sigma(\overline{M}_\sigma) = |E_\sigma|$. If σ_n is an one-vertex n-tree, we write $\Omega_n(\mathbf{m}) := \Omega_n(\mathbf{m}, \sigma_n)$. Notice that $\Omega_n(a)$ from [KMK] is $\left(\frac{n-3}{a}\right)!\,\Omega_n(\frac{n-3}{a}\delta_a)$ in our present notation. The numbers $V(\mathbf{m})$ in (1.1) are $\Omega_n(\mathbf{m})/|\mathbf{m}|!$.

1.3. Lemma. *We have*

$$\Omega_n(\mathbf{m}, \sigma) = \sum_{\substack{(\mathbf{m}_v | v \in V_\sigma): \\ \mathbf{m} = \sum \mathbf{m}_v}} \prod_{v \in V_\sigma} \Omega_{|v|}(\mathbf{m}_v)\,, \tag{1.9}$$

where the sum in r.h.s. is taken over all partitions of $\mathbf{m}$ *indexed by vertices of* σ.

Proof. This follows from the crucial fact that $\omega_n(a)$ form what is called a "logarithmic CohFT" in [KMK], Sect. 3, i.e. satisfy the additivity property established in

[AC], (1.8), for any genus:

$$\varphi_\sigma^*(\omega_n(a)) = \sum_{w\in V_\sigma} \mathrm{pr}_w^*(\omega_{|w|}(a)), \tag{1.10}$$

where we identify $\overline{M}_\sigma$ with $\prod_{w\in V_\sigma} \overline{M}_{0,|w|}$ and pr_w means the respective projection. (Notice that although these identifications are defined only up to the action of $\prod_{w\in V_\sigma} \mathbb{S}_{|w|}$, the classes $\mathrm{pr}_w^*(\omega_{|w|}(a))$ do not depend on their choice, being $\mathbb{S}_{|w|}$-invariant.)

Hence

$$\begin{aligned}
\int_{\overline{M}_\sigma} \varphi_\sigma^*(\omega_n^{\mathbf{m}}) &= \int_{\Pi_{v\in V_\sigma}\overline{M}_{0,|v|}} \prod_{a\geqq 1} \left(\sum_{w\in V_\sigma} \mathrm{pr}_w^*(\omega_{|w|}(a))\right)^{m(a)} \\
&= \int_{\Pi_{v\in V_\sigma}\overline{M}_{0,|v|}} \prod_{a\geqq 1} \sum_{\substack{(m_w(a)|w\in V_\sigma):\\ m(a)=\sum_w m_w(a)}} \frac{m(a)!}{\Pi_w m_w(a)!} \prod_{w\in V_\sigma} \mathrm{pr}_w^*(\omega_{|w|}(a))^{m_w(a)} \\
&= \int_{\Pi_{v\in V_\sigma}\overline{M}_{0,|v|}} \sum_{\substack{(\mathbf{m}_w|w\in V_\sigma):\\ \mathbf{m}=\sum_w \mathbf{m}_w}} \frac{\mathbf{m}!}{\Pi_w \mathbf{m}_w!} \prod_{w\in V_\sigma} \mathrm{pr}_w^*(\omega_{|w|})^{\mathbf{m}_w} \\
&= \sum_{\substack{(\mathbf{m}_w|w\in V_\sigma):\\ \mathbf{m}=\sum_w \mathbf{m}_w}} \mathbf{m}! \prod_{v\in V_\sigma} \int_{\overline{M}_{0,|v|}} \frac{\omega_{|v|}^{\mathbf{m}_v}}{\mathbf{m}_v!}\,. \qquad \square
\end{aligned}$$

1.4. Calculation of $\omega_n(a)$ via strata classes. For a fixed $n \geqq 3$ and $a \geqq 1$ consider labelled $(a+1)$-partitions,

$$S:\ \underline{n} := \{1,\dots,n\} = S_1 \amalg \dots \amalg S_{a+1}\,.$$

Denote by $\tau(S)$ the n-tree with $V_{\tau(S)} = \{v_1,\dots,v_{a+1}\}$, and edges connecting v_i to v_{i+1} for $i = 1,\dots,a$, and unpaired flags (numbered by) S_i put at v_i. The stability condition for $\tau(S)$ and S is:

$$n_i := |S_i| \geqq 2 \text{ for } i = 1,\ a+1; \quad \geqq 1 \text{ for } i = 2,\dots,a\,. \tag{1.11}$$

In the following proof, all partitions are stable. Denote by $m(S)$ the dual cohomology class of the cycle $\varphi_{\tau(S)}(\overline{M}_{\tau(S)})$ in $\overline{M}_{0n}$.

1.4.1. Lemma. *We have*

$$\omega_n(a) = \sum_{S:\underline{n}=S_1\amalg\cdots\amalg S_{a+1}} \frac{(n_1-1)(n_{a+1}-1)n_1\cdots n_{a+1}}{n(n-1)} K(n_1,\dots,n_a)\, m(S)\,, \tag{1.12}$$

where $K(n_1,\dots,n_a)$ is defined in (1.3).

1.4.2. Notation. To state some intermediate formulas we will need some of the notation of [KMK]. Let $T_n(a)$ be the set of n-trees with a edges. For any flag f denote by $\beta(f)$ the set of tails of the branch of f and $S(f)$ the set of their labels.

Then to any set of flags T we associate the set $S(T) := \bigcup_{f\in T} S(f) \subset \{1,\dots,n\}$. If $\{S(T_1), S(T_2)\}$ is a partition of $\{1,\dots,n\}$ we use the shorthand notation D_{T_1,T_2} for $D_{S(T_1),S(T_2)}$. Let τ be an n-tree and let e be one of its edges, $\partial e = \{v_1, v_2\}, \sigma_e$ the corresponding partition $S_1 \amalg S_2$ and D_e the corresponding divisor. Choosing flags $\{i,j\} \in F(v_1)$ and $\{j,k\} \in F(v_2)$, we have [KM] the following formula:

$$D_e m(\tau) = -\sum_{\substack{T:\{i,j\}\subset T\subset F(v_1)\\ 2\leqq|T|\leqq|F(v_1)|-1}} D_{T,F(v_2)\amalg F(v_1)\setminus T}\, m(\tau) - \sum_{\substack{T:\{k,l\}\subset T\subset F(v_2)\\ 2\leqq|T|\leqq|F(v_2)|-1}} D_{T,F(v_1)\amalg F(v_2)\setminus T}\, m(\tau)\,. \tag{1.13}$$

1.4.3. Definition. *A tree is called linear if each vertex has at most two incident edges. An orientation of a linear tree is a labelling of its vertices by $\{1,\dots,|V(\tau)|\}$ such that v_i and v_{i+1} are connected by an edge for $i = 1,\dots,|E(\tau)|$.*

We denote by $LT_n(a)$ the set of stable linear n-trees with a edges modulo isomorphism. Given a geometrically oriented linear tree we number its vertices in the positive direction.

1.4.4. Remark. The oriented linear trees in $LT_n(a)$ are in 1-1 correspondence with labeled $a+1$-partitions $S : \underline{n} := \{1,\dots,n\} = S_1 \amalg \cdots \amalg S_{a+1}$ which satisfy (1.11).

1.4.5. Tautological classes and the $\omega_n(a)$. In the proof of the lemma we will consider some additional classes in $H^*(\overline{M}_{g,n},\mathbb{Q})$. Let $\xi_i : \overline{M}_{g,n} \to \mathscr{C}_n$ be the structure sections of the universal curve. Put as in [AC]:

$$\Psi_{n,i} := \xi_i^*(c_1(\omega_{\mathscr{C}/M})) \in H^2(\overline{M}_{g,n},\mathbb{Q})\,. \tag{1.14}$$

Here we will need them only for $g = 0$; see Sect. 2 for any genus.

Identify $C \to \overline{M}_{0n}$ with the forgetful morphism $p_n : \overline{M}_{0,n+1} \to \overline{M}_{0n}$. Then $\xi_i(\overline{M}_{0n})$ becomes the divisor $D_i = D_{\{i,n+1\}\{1,\dots,\hat{i},\dots,n\}}$ in $\overline{M}_{0,n+1}$ and

$$\Psi_{n+1,i} = \varphi_{D_i}^*(-D_i^2)\,,$$

where $\varphi_{D_i}^*$ denotes the pullback onto the divisor D_i. We know from [AC]:

$$\omega_{n-1}(a) = p_{n-1*}(\Psi_{n,n}^{a+1})\,.$$

Combining these two formulas we obtain:

$$\omega_{n-1}(a) = p_{n-1*} \circ \varphi^*((-1)^{a+1} D_i^{a+2})\,. \tag{1.15}$$

To derive (1.15) notice that

$$\Psi_{n,i} = \sum_{n\in S\subset\{1,\dots,n\}} \frac{(n-|S|)(n-|S|-1)}{(n-1)(n-2)} D_{S,\{1,\dots,n\}\setminus S}$$

(see [KMK]) so that we have

$$\begin{aligned}\Psi^{a+1}_{n,n} &= \left(\sum_{n\in S\in\{1,\dots,n\}} \frac{(n-|S|)(n-|S|-1)}{(n-1)(n-2)} D_{S,\{1,\dots,n\}\setminus S}\right)^{a+1} \\ &= \varphi^*_{D_n}\left(\left(\sum_{\{n,n+1\}\subset S\in\{1,\dots,n+1\}} \frac{(n-|S|)(n-|S|-1)}{(n-1)(n-2)} D_{S,\{1,\dots,n+1\}\setminus S}\right)^{a+1} D_n\right) \\ &= \varphi^*_{D_n}((-1)^{a+1}D_n^{a+2}) .\end{aligned}$$

1.4.6. A calculation. Denote by $D_nLT(k)$ the set of oriented linear $(n+1)$-trees with a edges, whose monomials are divisible by D_n and whose orientation is given by calling v_{k+1} the trivalent vertex with the two tails n and $n+1$. Also take S to be the set of the flags of the other vertex v_k of the edge corresponding to D_n without the flag belonging to that edge. Then

$$D_n^k = \sum_{\tau\in LDT_n(a)} \frac{|v_1|(|v_1|-1)}{(n-1)(n-2)} \prod_{i=2}^{k-1} \frac{|v_i|-2}{\sum_{j=i}^{k}(|v_j|-2)} m(\tau) . \tag{1.16}$$

We will prove (1.16) by induction using the following versions of (1.13). Let τ be a tree, which has an edge e corresponding to D_n, then call v_2 the vertex with $F(v_2)=\{n,n+1,f_e\}$, where f_e is the flag corresponding to e.

Averaging the formula (1.13) over the set S of all flags of v_1 without the flag belonging to e we obtain:

$$D_nm(\tau) = -\sum_{\substack{T\subset S\\ 2\leqq|T|\leqq|F(v1)|-2}} \frac{|T|(|T|-1)}{|S|(|S|-1)} D_{T,\{n,n+1\}\amalg S\setminus T} m(\tau) . \tag{1.17}$$

Fixing *one* particular flag f of S and averaging over the rest we obtain:

$$D_nm(\tau) = -\sum_{\substack{f\in T\subset S\\ 2\leqq|T|\leqq|F(v1)|-2}} \frac{|T|-1}{|S|-1} D_{T,\{n,n+1\}\amalg S\setminus T} m(\tau) . \tag{1.18}$$

Now for $k=1$ the formula (1.16) is clear and for $k=2$ it is a consequence of (1.18). For $k>2$, we have

$$\begin{aligned}D_n^k = D_n D_n^{k-1} &= D_n \sum_{\tau\in D_nLT(k-1)} \frac{|v_1|(|v_1|-1)}{(n-1)(n-2)} \prod_{i=2}^{k-2} \frac{|v_i|-2}{\sum_{j=i}^{k-1}(|v_j|-2)} m(\tau) \\ &= \sum_{\tau\in D_nLT(k-1)} \sum_{f\in T\subset S} \frac{|v_1|(|v_1|-1)}{(n-1)(n-2)} \\ &\quad\times \prod_{i=2}^{k-2} \frac{|v_i|-2}{\sum_{j=i}^{k-1}(|v_j|-2)} \frac{|T|-1}{|v_{k-1}|-2} D_{T,\{n,n+1\}\amalg S\setminus T} m(\tau) ,\end{aligned} \tag{1.19}$$

where we have used (1.18) with the distinguished flag being the unique flag of S belonging to an edge. This guarantees that the sum will again run over linear trees. In the second sum there is one edge inserted at the vertex v_{k-1} giving two new vertices v', v'' with $|v'| + |v''| = |v_{k-1}| + 2$. Giving v', v'' the labels $k-1$ and k and labelling the old vertex v_k with $k+1$ in the second sum we obtain the desired result (1.16).

1.4.7. Proof of the Lemma. What remains to be calculated is $p_{n-1*} \circ \varphi^*_{D_n}$ of the above formula for D_n^{a+2}. The only nonzero contributions come from trees $\tau \in D_n LT(a+2)$ with $|v_k| = 3$, so that exactly one of the flags is a tail. Hence after push forward and pullback the sum will run over oriented linear trees with the induced orientation given by the image of v_i with a distinguished flag at the vertex v_{k-2}. Summing first over the possible distinguished flags amounts to multiplication by $|v_{k-1}|$. We obtain:

$$\omega_n(a) = \sum_{\text{oriented } \tau \in LT_n(a)} \frac{(|v_{a+1}| - 1)(|v_1| - 1)}{n} \prod_{i=1}^{a+1} \frac{|v_i| - 2}{\sum_{j=i}^{a+1} (|v_j| - 2) + 1} m(\tau) ,$$

which using Remark 1.4.4 can be rewritten as a sum over partitions

$$\omega_n(a) = \sum_{S:S_1 \amalg \cdots \amalg S_{a+1}} \frac{n_1 n_{a+1}}{n} (n_1 - 1) n_2 \cdots n_a (n_{a+1} - 1) \frac{1}{n-1} K(n_a + 1, \ldots, n_2) m(S)$$

with $n_i = |v_i| - 1$ for $i = 1, a+1$ and $n_i = |v_i| - 2$ for $i = 2, \ldots, a$, which is equivalent to (1.12).

1.4.8. Remark. Instead of using (1.18) in the induction one can successively apply (1.17). In this case one obtains a formula for $\omega_n(a)$ involving all boundary strata. Since not necessarily linear trees cannot be handled using only partitions, the associated generating functions and recursion relations become very complicated.

1.5. Proof of Theorem 1.2.1. In view of (1.8), we have

$$\Omega_n(\mathbf{m} + \delta_a) = \int_{\overline{M}_{0n}} \prod_{b \geq 1} \frac{\omega_n(b)^{m(b)}}{m(b)!} \wedge \frac{\omega_n(a)}{m(a) + 1} . \tag{1.20}$$

Instead of wedge multiplying by $\omega_n(a)$ we can integrate the product $\boldsymbol{\omega}_n^{\mathbf{m}}/\mathbf{m}!$ over the cycle obtained by replacing $m(S)$ by $\varphi_{\tau(S)}(\overline{M}_{\tau(S)})$ in the r.h.s. of (1.12). The separate summands then can be calculated using (1.8) and (1.9). The net result is:

$$(m(a) + 1)\Omega_n(\mathbf{m} + \delta_a) = \sum_{S:\underline{n} = S_1 \amalg \ldots \amalg S_{a+1}} \frac{(n_1 - 1)(n_{a+1} - 1) n_1 \ldots n_{a+1}}{n(n-1)}$$

$$\times K(n_1, \ldots, n_a) \sum_{\mathbf{m} = \mathbf{m}_1 + \cdots + \mathbf{m}_{a+1}} \Omega_{n_1+1}(\mathbf{m}_1) \Omega_{n_{a+1}+1}(\mathbf{m}_{a+1}) \prod_{i=2}^{a} \Omega_{n_i+2}(\mathbf{m}_i) . \tag{1.21}$$

Now, the product of Ω's vanishes unless

$$|\mathbf{m}_i| = n_i - 2 \quad \text{for } i = 1,\ a+1, \qquad |\mathbf{m}_i| = n_i - 1 \quad \text{for } i = 2, \ldots, a \tag{1.22}$$

so that $n = |\mathbf{m} + \delta_a| + 3$. Hence we can make the exterior summation over vector $(a+1)$-partitions of $\mathbf{m}$, and for a fixed $(\mathbf{m}_i)$ sum over the set of $(a+1)$-partitions of $\underline{n}$ satisfying (1.22). Since the coefficients in (1.20) depend only on (n_i) rather than (S_i), we can then replace the summation over (S_i)'s by multiplication by $\frac{n!}{n_1!\dots n_{a+1}!}$. This leads to

$$(m(a)+1)\frac{\Omega_n(\mathbf{m}+\delta_a)}{|\mathbf{m}+\delta_a|!} = (n-2) \sum_{\mathbf{m}=\mathbf{m}_1+\dots+\mathbf{m}_{a+1}} K(n_1,\dots,n_a) \prod_{i=1}^{a+1} \frac{\Omega_{|\mathbf{m}_i|+3}(\mathbf{m}_i)}{|\mathbf{m}_i|!},$$

which is equivalent to (1.4) in view of (1.8) and (1.1). □

Remark. We do not know whether for $g \geqq 1$ the classes $\omega_n(a)$ belong to the span of the boundary strata and if yes, what might be a generalization of (1.20). Therefore we cannot extend the recursive relations (1.4) to arbitrary genus.

1.6. The differential equation for a generating function. Put

$$F(x;\mathbf{s}) = F(x;s_1,s_2,\dots) := \sum_{\mathbf{m}} V(\mathbf{m}) x^{|\mathbf{m}|}\, \mathbf{s}^{\mathbf{m}} \in \mathbb{Q}[\mathbf{s}][[x]] \qquad (1.23)$$

and denote $\partial_a = \partial/\partial s_a$, $\partial_x = \partial/\partial x$. Then the recursion (1.4) is equivalent to:

1.6.1. Theorem. *F satisfies the following system of differential equations*:

$$\partial_a F = \partial_x \left(\sum_{k=0}^{a} (-1)^k \frac{F^{2k+1}}{(\partial_x F)^{k+1}} \partial_{a-k} F \right), \quad a = 1,2,\dots, \qquad (1.24)$$

where we put $\partial_0 = x\,\partial_x$. It is the unique solution of this system in $1 + x\mathbb{Q}[\mathbf{s}][[x]]$ with $F(x;\mathbf{0}) = 0$.

Proof. Put $H_0(x;\mathbf{s}) = x$ and

$$H_a(x;\mathbf{s}) := x \sum_{\mathbf{m}_1,\dots,\mathbf{m}_a} K(n_1,\dots,n_a) \prod_{i=1}^{a} V(\mathbf{m}_i) x^{|\mathbf{m}_i|+1}\, \mathbf{s}^{\mathbf{m}_i}$$

for $a = 1, 2, \dots$, where the summation is over a-tuples of vectors $\mathbf{m}_i \in N^\infty$. In particular, we have $H_1(x;\mathbf{s}) = \int_0^x \xi F(\xi;\mathbf{s}) d\xi$. Multiply (1.4) by $x^{|\mathbf{m}+\delta_a|}\mathbf{s}^{\mathbf{m}}$ and sum over all $\mathbf{m}$. Taking into account that in each summand

$$n_1 + \dots + n_a = |\mathbf{m}| + a + 1 = |\mathbf{m} + \delta_a| + 1,$$
$$(n_1 + \dots + n_a) K(n_1,\dots,n_a) = K(n_1,\dots,n_{a-1}),$$

we obtain

$$\partial_a F(x;\mathbf{s}) = \partial_x \big(H_a(x;\mathbf{s}) F(x;\mathbf{s})\big), \quad a \geqq 1. \qquad (1.25)$$

A similar calculation shows that

$$\partial_x H_a(x;\mathbf{s}) = H_{a-1}(x;\mathbf{s}) F(x;\mathbf{s}), \quad a \geqq 1. \qquad (1.26)$$

Combining these two identities, we get

$$\partial_a F = \partial_x H_a \cdot F + H_a \cdot \partial_x F = H_{a-1} \cdot F^2 + H_a \cdot \partial_x F, \quad a \geqq 1$$

or

$$H_a = \frac{\partial_a F}{\partial_x F} - H_{a-1}\frac{F^2}{\partial_x F}\,. \tag{1.27}$$

By induction starting with $H_0 = x$ we obtain from here

$$H_a = \sum_{k=0}^{a} (-1)^k \frac{F^{2k}\,\partial_{a-k}F}{(\partial_x F)^{k+1}} \tag{1.28}$$

(recall that $\partial_0 := x\partial_x$). Substituting this into (1.25) gives (1.24).

For the uniqueness, we reverse the argument. Suppose that $F(x) = F(x;\mathbf{s})$ satisfies (1.24) and define H_a by (1.28). Then $H_0(x) = x$ (by the definition of $\partial_0 F$) and H_a satisfies (1.27) for $a \geqq 1$, while (1.24) says that $\partial_a F = \partial_x(H_a F)$. Combining these equations gives (1.25). By assumption $F(x;\mathbf{s})$ has a Taylor series $\sum_{n=0}^{\infty} A_n(\mathbf{s})x^n$ where $A_0(\mathbf{s}) \equiv 1$ and $A_n(\mathbf{s})$ for $n \geqq 1$ is a polynomial with no constant term. Equation (1.28) then shows that $H_a(0) = 0$, after which the equation $\partial_x(H_a) = H_{a-1}F$ gives inductively $H_a = \frac{x^{a+1}}{(a+1)!} + \cdots$, where the coefficient of x^{a+n+1} is a weighted homogeneous polynomial in $A_1,\ldots,A_n$ of weight n. The equation $\partial_a F = \partial_x(H_a F)$ then gives a formula for all derivatives $\partial_a A_n(\mathbf{s})$ $(a = 1,2,\ldots)$ as polynomials in $A_1(\mathbf{s}),\ldots,A_{n-1}(\mathbf{s})$. This fixes $A_n(\mathbf{s})$ inductively up to a constant which is uniquely determined by the normalizing condition $A_n(\mathbf{0}) = 0$. In this argument we have implicitly assumed that $\partial_x F$ is an invertible power series (i.e. that $A_1(\mathbf{s})$ is not identically 0) in order to make sense of (1.24) in the ring of power series, but what we have really proved does not need this assumption, namely, that (F, H_a) is the unique solution in elements of $\mathbb{Q}[\mathbf{s}][[x]]$ of the system of differential equations (1.25), (1.26) subject to the normalizing conditions $F(0,\mathbf{s}) = F(x;\mathbf{0}) = 1$, $H_0(x) = x, H_a(0) = 0$.

1.6.2. Example. In the special case $\mathbf{s} = (s,0,0,\ldots)$, the function (1.23) reduces to $F(x;\mathbf{s}) = f(xs)$ with $f(x) = \sum_{m\geqq 0} V(m)x^m$, $V(m) = V(m,0,0,\ldots)$ as in 1.2.2. Then Eq. (1.24) (with $a = 1$) becomes

$$0 = \partial_x\left(\frac{F}{\partial_x F}\,\partial_s F - \frac{xF^3}{(\partial_x F)^2}\,\partial_x F\right) - \partial_s F = \frac{\partial}{\partial x}\left[\frac{x}{s}\,f(xs) - \frac{x}{s}\,\frac{f(xs)^3}{f'(xs)}\right] - \frac{\partial}{\partial s}\,f(xs)$$

and if we write $f = y'$ then we see that this is (up to a power of s) the derivative of the differential equation $y = xy'^3/y''$ for the function $y = \sum \frac{v(n)x^{n-2}}{(n-2)!(n-3)!}$ discussed in the Introduction.

2. Explicit Formulas and the Inversion of the Generating Function

2.1. Notation. In this section we fix a value of the genus $g \geqq 0$. We keep g in the notation for $\overline{M}_{g,n}$ and $V_g(\mathbf{m})$ but skip it elsewhere. To state our explicit formulas we must introduce some additional classes in $H^*(\overline{M}_{g,n},\mathbb{Q})$. Recall that

$$\Psi_{n,i} := \xi_i^*(c_1(\omega_{\mathscr{C}/M})) \in H^2(\overline{M}_{g,n},\mathbb{Q})\,, \tag{2.1}$$

where $\xi_i : \overline{M}_{g,n} \to \mathscr{C}_n$ are the structure sections of the universal curve.

After Witten [W], the integrals of top degree monomials in $\Psi_{n,i}$ are denoted

$$\langle\tau_{a_1}\cdots\tau_{a_n}\rangle = \int\limits_{\overline{M}_{g,n}} \Psi_{n,1}^{a_1}\cdots\Psi_{n,n}^{a_n}\,. \tag{2.2}$$

Below we will express $V_g(\mathbf{m})$ via these numbers. For $g = 0$, they are just multinomial coefficients:

$$\langle\tau_{a_1}\cdots\tau_{a_n}\rangle_{g=0} = \frac{(a_1+\cdots+a_n)!}{a_1!\cdots a_n!} \tag{2.3}$$

(see e.g. [K2], p. 354). The structure of a generating series for all $\langle\tau_{a_1}\cdots\tau_{a_n}\rangle$ and all g was predicted by Witten [W], and Kontsevich identified it as a "matrix Airy function,", cf. [K1] and below.

More generally, we will consider the relative integrals of the type (2.2). For $k \geqq l$, denote by $\pi_{k,l} : \overline{M}_{g,k} \to \overline{M}_{g,l}$ the morphism forgetting the last $k-l$ points. For any $a_1,\ldots,a_p \geqq 0$ define

$$\omega_n(a_1,\ldots,a_p) := \pi_{n+p,n*}(\Psi_{n+p,n+1}^{a_1+1}\cdots\Psi_{n+p,n+p}^{a_p+1}) \in H^{2(a_1+\cdots+a_p)}(\overline{M}_{g,n},\mathbb{Q})\,. \tag{2.4}$$

Notice that whenever $a_1+\cdots+a_p = \dim\overline{M}_{g,n}$, we have also $(a_1+1)+\cdots+(a_p+1) = \dim\overline{M}_{g,n+p}$, and then

$$\int\limits_{\overline{M}_{g,n}} \omega_n(a_1,\ldots,a_p) = \int\limits_{\overline{M}_{g,n+p}} \Psi_{n+p,n+1}^{a_1+1}\cdots\Psi_{n+p,n+p}^{a_p+1} = \langle\tau_0^n\tau_{a_1+1}\cdots\tau_{a_p+1}\rangle\,. \tag{2.5}$$

2.2. Theorem. *For any $g, n, a_1,\ldots,a_p, a_i \geqq 0$ we have*

$$\omega_n(a_1)\ldots\omega_n(a_p) = \sum_{k=1}^{p}\frac{(-1)^{p-k}}{k!}\sum_{\substack{\{1,\ldots,p\}=S_1\amalg\cdots\amalg S_k\\ S_i\neq\emptyset}}\omega_n\left(\sum_{j\in S_1}a_j,\ldots,\sum_{j\in S_k}a_j\right)\,. \tag{2.6}$$

Equivalently, for any $\mathbf{m}\in N^\infty\setminus\{\bar{0}\}$, $p = \|\mathbf{m}\|$,

$$\frac{(-1)^p}{\mathbf{m}!}\omega_n^{\mathbf{m}} = \sum_{k=1}^{p}\frac{(-1)^k}{k!}\sum_{\substack{\mathbf{m}=\mathbf{m}_1+\cdots+\mathbf{m}_k\\ \mathbf{m}_i\neq 0}}\frac{{}_n(|\mathbf{m}_1|,\ldots,|\mathbf{m}_k|)}{\mathbf{m}_1!\cdots\mathbf{m}_k!}\,. \tag{2.7}$$

The proof consists of a geometric and a combinatorial part.

The first one is summarized in [AC], (1.12), (1.13) and given here in a slightly different notation. It was previously obtained by C. Faber and D. Zagier (unpublished).

2.2.1. Lemma. *We have*

$$\omega_n(a_1,\ldots,a_p) = \sum_{\sigma\in\mathbb{S}_p}\prod_{\mathrm{o}\in\mathrm{o}(\sigma)}\omega_n\left(\sum_{j\in\mathrm{o}}a_j\right)\,, \tag{2.8}$$

where $\mathrm{o}(\sigma)$ denotes the set of the cycles of σ acting on $\{1,\ldots,p\}$, i.e. the orbits of the cyclic group $\langle\sigma\rangle$.

E. Albarello and M. Cornalba ([AC]) show that (2.8) formally follows with the help of the push-pull formula from an identity going back to Witten [W]

$$\pi_{n+1,n*}(\Psi_{n+1,1}^{a_1}\cdots\Psi_{n+1,n}^{a_n}\Psi_{n+1,n+1}^{a_{n+1}+1}) = \Psi_{n,1}^{a_1}\cdots\Psi_{n,n}^{a_n}\omega_n(a_{n+1}) \tag{2.9}$$

and an identity for which a geometric proof is supplied in [AC]:

$$\omega_n(a) = \pi^*_{n,n-1}(\omega_{n-1}(a)) + \Psi^a_{n,n}\,. \tag{2.10}$$

We will not repeat their argument here.

The passage from (2.8) to (2.6) and (2.7) is a formal inversion result which we will prove here in an axiomatized form because it is useful in other contexts as well.

Let R be a commutative $\mathbb{Q}$-algebra generated by some elements $\omega(a)$, where a runs over all elements of an additive semigroup A.

2.2.2. Lemma. *Define elements* $\omega(a_1,\ldots,a_p) \in R$ *for* $p \geqq 2$, $a_i \in A$ *recursively by*

$$\omega(a_1,\ldots,a_p) = \omega(a_1,\ldots,a_{p-1})\omega(a_p) + \sum_{i=1}^{p-1} \omega(a_1,\ldots,a_i + a_p,\ldots,a_{p-1})\,. \tag{2.11}$$

Then $\{\omega(a_1,\ldots,a_p) \mid p \geqq 1\}$ *span* R *as a linear space. They can be expressed via monomials in* $\omega(a)$ *by the following universal identity (coinciding with* (2.8)):

$$\omega(a_1,\ldots,a_p) = \sum_{\sigma\in\mathbb{S}_p} \prod_{\mathfrak{o}\in\mathfrak{o}(\sigma)} \omega\left(\sum_{j\in\mathfrak{o}} a_j\right)\,. \tag{2.12}$$

In particular, $\omega(a_1,\ldots,a_p)$ *are symmetric in* $a_1,\ldots,a_p$.

Conversely, monomials in $\omega(a)$ *can be expressed via these elements by the universal formula (coinciding with* (2.6)):

$$\omega(a_1)\cdots\omega(a_p) = \sum_{k=1}^{p} \frac{(-1)^{p-k}}{k!} \sum_{\substack{\{1,\ldots,p\}=S_1\amalg\cdots\amalg S_k \\ S_i\neq\emptyset}} \omega\left(\sum_{j\in S_1} a_j,\ldots,\sum_{j\in S_k} a_j\right)\,. \tag{2.13}$$

If $A = \{1,2,3,\ldots\}$, *we have also*

$$\frac{(-1)^p}{\mathbf{m}!}\boldsymbol{\omega}^{\mathbf{m}} = \sum_{k=1}^{p} \frac{(-1)^k}{k!} \sum_{\substack{\mathbf{m}=\mathbf{m}_1+\cdots+\mathbf{m}_k \\ \mathbf{m}_i\neq 0}} \frac{\omega(|\mathbf{m}_1|,\ldots,|\mathbf{m}_k|)}{\mathbf{m}_1!\cdots\mathbf{m}_k!}\,, \tag{2.14}$$

where $p = \|\mathbf{m}\|$ *as in* (2.7) *and* $\boldsymbol{\omega}^{\mathbf{m}} = \omega(1)^{m(1)}\omega(2)^{m(2)}\cdots$.

Furthermore, $\omega(a)$ *are algebraically independent iff* $\omega(a_1,\ldots,a_p)$ *are linearly independent. In this case* R *is graded by* A *via* $\deg\omega(a_1,\ldots,a_p) = a_1+\cdots+a_p$.

Example. The elements $\omega(a_1,\ldots,a_p)$ are given for $p=2$ by

$$\omega(a,b) = \omega(a)\omega(b) + \omega(a+b), \qquad \omega(a)\omega(b) = \omega(a,b) - \omega(a+b)\,,$$

and for $p=3$ by

$$\begin{aligned}\omega(a,b,c) &= \omega(a)\omega(b)\omega(c) + \omega(a+b)\omega(c) + \omega(a+c)\omega(b) \\ &\quad +\omega(b+c)\omega(a) + 2\omega(a+b+c), \\ \omega(a)\omega(b)\omega(c) &= \omega(a,b,c) - \omega(a+b,c) - \omega(a+c,b) \\ &\quad -\omega(b+c,a) + \omega(a+b+c)\,.\end{aligned}$$

Proof of Lemma 2.2.2. The following identity shows by induction in p that (2.11) and (2.12) are equivalent:

$$\sum_{\sigma\in\mathbb{S}_{p+1}}\prod_{o\in o(\sigma)}\omega\left(\sum_{j\in o}a_j\right)=\omega(a_{p+1})\left[\sum_{\sigma\in\mathbb{S}_p}\prod_{o\in o(\sigma)}\omega\left(\sum_{j\in o}a_j\right)\right]$$
$$+\sum_{i=1}^{p}\sum_{\tau\in\mathbb{S}_p}\prod_{o\in o(\tau)}\omega\left(\sum_{j\in o}a_j+\delta_{ij}a_{p+1}\right). \qquad (2.15)$$

To convince yourself of the validity of (2.15) look at the following bijective map from the l.h.s. monomials in $\omega(a)$ to the r.h.s. monomials. If a l.h.s. monomial is indexed by $\sigma\in\mathbb{S}_{p+1}$ for which $\sigma(p+1)=p+1$, we get it in the r.h.s. for $\tau=$ restriction of σ to $\{1,\dots,p\}$. Otherwise $p+1$ belongs to an orbit of σ of cardinality $\geqq 2$; deleting $p+1$ from this cycle, we get a permutation $\tau\in\mathbb{S}_p$ and a number $i=\sigma(p+1)\leqq p$ producing exactly the needed monomial in the second sum.

One can similarly pass from (2.11) to (2.13) and backwards. For example, assume that (2.13) is already proved for some p. Then we have

$$[\omega(a_1)\cdots\omega(a_p)]\omega(a_{p+1})$$
$$=\left(\sum_{k=1}^{p}\frac{(-1)^{p-k}}{k!}\sum_{\substack{\{1,\dots,p\}=S_1\amalg\dots\amalg S_k\\ S_i\neq\emptyset}}\left(\sum_{j\in S_1}a_j,\dots,\sum_{j\in S_k}a_j\right)\right)\omega(a_{p+1})$$
$$=\sum_{k=1}^{p}\frac{(-1)^{p-k}}{k!}\sum_{\substack{\{1,\dots,p\}=S_1\amalg\dots\amalg S_k\\ S_i\neq\emptyset}}\left(\omega\left(\sum_{j\in S_1}a_j,\dots\sum_{j\in S_k}a_j,a_{p+1}\right)\right.$$
$$\left.-\sum_{i=1}^{p}\omega\left(\sum_{j\in S_1}\left(a_j+\delta_{ij}a_{p+1},\dots,\sum_{j\in S_k}(a_j+\delta_{ij}a_{p+1}\right)\right)\right). \qquad (2.16)$$

Now essentially the same combinatorics as above govern a correspondence between the summands in (2.16) and those in the r.h.s. of (2.13) for $p+1$ arguments which is

$$\sum_{k=1}^{p+1}\frac{(-1)^{p+1-k}}{k!}\sum_{\substack{\{1,\dots,p+1\}=S_1\amalg\cdots\amalg S_k\\ S_i\neq\emptyset}}\omega\left(\sum_{j\in S_1}a_j,\dots,\sum_{j\in S_k}a_j\right).$$

Namely, any ordered k-partition $\{1,\dots,p\}=S_1\amalg\cdots\amalg S_k$ can be enhanced to $k+1$ ordered $(k+1)$-partitions of $\{1,\dots,p+1\}$ containing $\{p+1\}$ as a separate part, and to k ordered k-partitions of $\{1,\dots,p+1\}$ for which $p+1$ is put into one of the S_i's.

It remains to rewrite (2.13) in the form (2.14), when $A=\{1,2,3,\dots\}$. To this end, notice that if $\delta_{a_1}+\cdots+\delta_{a_p}=\mathbf{m}$, we have

$$\omega(a_1)\cdots\omega(a_p)=\prod_{a\geqq 1}\omega(a)^{m(a)}=\boldsymbol{\omega}^{\mathbf{m}},\quad p=\|\mathbf{m}\|$$

and $m(a) = \text{card}\{j \mid a_j = a\}$. Any set partition $\{1,\ldots,p\} = S_1 \amalg \cdots \amalg S_k$, $S_i \neq 0$, produces a vector partition

$$\mathbf{m} = \mathbf{m}_1 + \cdots + \mathbf{m}_k, \quad \mathbf{m}_i = (m_i(a)) \neq 0, \quad m_i(a) = \text{card}\{j \mid a_j = a\}\,.$$

We have $\sum_{j\in S_i} a_j = |\mathbf{m}_i|$, and each $\mathbf{m} = \mathbf{m}_1 + \cdots + \mathbf{m}_k$ comes from

$$\prod_{a\geq 1} \frac{m(a)!}{m_1(a)!\cdots m_k(a)!} = \frac{\mathbf{m}!}{\mathbf{m}_1!\cdots\mathbf{m}_k!}\,.$$

set partitions. This finishes the proof of Lemma 2.2.2 and Theorem 2.2. For a further discussion of this combinatorial setting, cf. 2.6 below.

As a corollary, we get:

2.3. Corollary. *We have for* $p = \|\mathbf{m}\|$, $3g-3+n = |\mathbf{m}|$:

$$V_g(\mathbf{m}) = \frac{1}{|\mathbf{m}|!}\sum_{k=1}^{p}\frac{(-1)^{p-k}}{k!}\sum_{\substack{\mathbf{m}=\mathbf{m}_1+\cdots+\mathbf{m}_k\\ \mathbf{m}_i\neq 0}}\frac{\langle\tau_0^n\tau_{|\mathbf{m}_1|+1}\cdots\tau_{|\mathbf{m}_k|+1}\rangle}{\mathbf{m}_1!\cdots\mathbf{m}_k!}\,. \tag{2.17}$$

In particular, if $g = 0$, *then*

$$V(\mathbf{m}) = \sum_{k=1}^{p}(-1)^{p-k}\binom{|\mathbf{m}|+k}{k}\sum_{\substack{\mathbf{m}=\mathbf{m}_1+\ldots+\mathbf{m}_k\\ \mathbf{m}_i\neq 0}}\frac{1}{\prod_{k=1}^{p}(|\mathbf{m}_i|+1)!\mathbf{m}_i!} \tag{2.18}$$

Proof. Combine (2.6), (2.7), (2.5) and (2.3).

2.3.1. A special case. Putting in (2.16) $\mathbf{m} = (n-3,0,0,\ldots)$ and multiplying by $(n-3)!^2$, we get the following formula for Zograf's numbers (0.3):

$$v_n = \sum_{k=1}^{n-3}(-1)^{n-3-k}\frac{(n-3+k)!(n-3)!}{k!}\sum_{\substack{n-3=m_1+\cdots+m_k\\ m_i\neq 0}}\frac{1}{\Pi_{i=1}^{k}(m_i+1)!\,m_i!}\,,$$

which is equivalent to (0.7).

We now proceed to the generalization of (0.8).

2.4. Theorem. *In the ring of formal series of one variable with coefficients in* $\mathbb{Q}[\mathbf{s}] = \mathbb{Q}[s_1,s_2,\ldots]$ *we have the following inversion formula*:

$$y = \sum_{|\mathbf{m}|\geq 0} V(\mathbf{m})\frac{x^{|\mathbf{m}|+1}}{|\mathbf{m}|+1}\mathbf{s}^{\mathbf{m}} \Leftrightarrow x = \sum_{|\mathbf{m}|\geq 0}\frac{y^{|\mathbf{m}|+1}}{(|\mathbf{m}|+1)!}\frac{(-\mathbf{s})^{\mathbf{m}}}{\mathbf{m}!}\,. \tag{2.19}$$

There are two ways to prove this theorem, one starting from the explicit formula (2.18) and the other using the differential equation for the generating function $F(x;\mathbf{s})$ derived in Sect. 1. Since we do not know which proof, if either, may be generalizable to the higher genus case, we will give both.

2.4.1. First proof: Explicit formula. From (2.18) we have for any $\mu \geqq 1$:

$$\sum_{\mathbf{m}:|\mathbf{m}|=\mu} V(\mathbf{m})\mathbf{s}^{\mathbf{m}} = \sum_{k=1}^{\infty}(-1)^k \binom{\mu+k}{k} \left(\sum_{|\mathbf{m}|>0} \frac{(-\mathbf{s})^{\mathbf{m}}}{(|\mathbf{m}|+1)!\mathbf{m}!}\right)^k \Bigg|_{\text{degree } \mu}$$

$$= \left(\sum_{\mathbf{m}} \frac{(-\mathbf{s})^{\mathbf{m}}}{(|\mathbf{m}|+1)!\mathbf{m}!}\right)^{-\mu-1} \Bigg|_{\text{degree } \mu}$$

$$= \text{coeff of } y^{\mu} \text{ in } \left(\frac{x(y)}{y}\right)^{-\mu-1},$$

where

$$x = x(y;\mathbf{s}) := \sum_{|\mathbf{m}|\geqq 0} \frac{y^{|\mathbf{m}|+1}}{(|\mathbf{m}|+1)!} \frac{(-\mathbf{s})^{\mathbf{m}}}{\mathbf{m}!} \in \mathbb{Q}[\mathbf{s}][[y]]$$

is the power series occurring on the right-hand side of (2.19) and we have used the binomial identity $\sum_{k=1}^{\infty}(-1)^k \binom{\mu+k}{k} z^k = (1+z)^{-(\mu+1)} - 1$. But for any power series $x(y) = \sum_{r\geqq 1} b_r y^r$, $b_1 \neq 0$, we have

$$\text{coeff. of } y^{\mu} \text{ in } \left(\frac{x(y)}{y}\right)^{-\mu-1}$$
$$= \operatorname{res}_{y=0}\left(\frac{1}{y^{\mu+1}}\left(\frac{y}{x(y)}^{\mu+1}\right) dy\right) = \operatorname{res}_{x=0}\left(\frac{1}{x^{\mu+1}} \frac{dy(x)}{dx}\, dx\right)$$
$$= \text{coeff. of } x^{\mu} \text{ in } \frac{dy(x)}{dx} = \text{coeff. of } \frac{x^{\mu+1}}{\mu+1} \text{ in } y(x),$$

where $y(x)$ is the power series obtained by formal inversion of $x = x(y)$ (which is possible since $b_1 \neq 0$). Applying this to our case, we find that the inverse series of $x(y)$ is given by

$$y(x) = \sum_{\mu\geqq 0} \frac{x^{\mu+1}}{\mu+1} \sum_{|\mathbf{m}|=\mu} V(\mathbf{m})\mathbf{s}^{\mathbf{m}},$$

which is precisely the expression on the left of (2.19).

2.4.2. Second proof: Differential equation. In 1.6 we characterized the generating series (1.23) as the unique power series $F(x;\mathbf{s}) \in 1 + x\mathbb{Q}[\mathbf{s}][[x]]$ with $F(x;\mathbf{0}) = 0$ for which there are power series $H_a(x,\mathbf{s})$ satisfying

$$H_0(x) = x, \qquad H_a(0) = 0, \qquad \partial_x(H_a) = H_{a-1}F, \qquad \partial_a F = \partial_x(H_a F) \quad (a \geqq 1). \tag{2.20}$$

Write $y = \int_0^x F(\xi;\mathbf{s})d\xi = x + \cdots$ for the integral of F with respect to x. This is an invertible power series, so we can also write $x = x(y) = y + \cdots$ and define power series $h_a(y) = h_a(y;\mathbf{s}) \in y\mathbb{Q}[\mathbf{s}][[y]]$. In terms of h_a, we can also write $x = x(y) = y + \cdots$ and define power series $h_a(y) = h_a(y;\mathbf{s}) \in y\mathbb{Q}[\mathbf{s}][[y]]$ by $h_a(y) = H_a(x(y))$. In terms of h_a, the first three equations in (2.20) become

$$h_0(y) = x(y), \qquad h_a(0) = 0, \qquad h_a'(y) = h_{a-1}(y) \tag{2.21}$$

while the last equation in (2.20) becomes

$$\partial_a x(y,\mathbf{s}) = -h_a(y) \quad (a \geqq 1)\,. \tag{2.22}$$

(To see this, first integrate the last equation in (2.20) to get $\partial_a y = FH_a$, where the meaning of $\partial_a y$ is that we differentiate in s_a keeping x constant. But differentiating the identity $x(y(x;\mathbf{s});\mathbf{s}) \equiv x$ with resepct to s_a gives $\partial_a x + \partial_a y \cdot \partial_y x = 0$, where $\partial_a x$ is the derivative of x with respect to s_a keeping y constant, so $\partial_a x = -F^{-1}\partial_a y = -H_a(x) = -h_a(y)$ as claimed).

Equations (2.21) and (2.22) form a system of linear differential equations replacing the non-linear system (2.20). They can be combined into a system of linear differential equations for the single function $x(y)$, namely

$$\frac{\partial^2 x}{\partial s_1 \partial y} = -x\,, \qquad \frac{\partial^2 x}{\partial s_a \partial y} = \frac{\partial x}{\partial s_{a-1}} \qquad (a \geqq 2)\,. \tag{2.23}$$

But in fact this is not needed because we can solve the system immediately. Write

$$x = x(y) = x(y,\mathbf{s}) = \sum_{n=0}^{\infty} B_n(\mathbf{s}) \frac{y^{n+1}}{(n+1)!} \tag{2.24}$$

with $B_0(\mathbf{s}) = 1$ and $B_n(\mathbf{0}) = 0$ for all $n \geqq 1$. The solution of (2.21) is immediately seen to be

$$h_a(y,\mathbf{s}) = \sum_{n=0}^{\infty} B_n(\mathbf{s}) \frac{y^{n+a+1}}{(n+a+1)!}\,,$$

and (2.22) then says that $\partial_a B_n = -B_{n-a}$ ($= 0$ if $a > n$), giving successively $B_1 = -s_1$, $B_2 = \frac{1}{2}s_1^2 - s_2, \dots$ and in general

$$B_n(s) = \sum_{m_1+2m_2+\cdots=n} (-1)^{m_1+m_2+\cdots} \frac{s_1^{m_1} s_2^{m_2} \cdots}{m_1!\, m_2! \cdots}\,. \tag{2.25}$$

Substituting (2.25) into (2.24) gives the expansion on the r.h.s. of (2.19).

2.4.3. Remarks. Equation (2.19) specializes to (0.8) if we put $\mathbf{s} = (1,0,0,\dots)$, in which case the r.h.s. of (2.19) is a Bessel function. In general, this r.h.s. cannot be expressed in terms of standard functions, but the series is easily summed after applying to it the formal Laplace transform, since from (2.24) and (2.25) we get immediately

$$\eta^{-2} \int_0^\infty e^{-y/\eta} x(y)\, dy = e^{-s_1\eta - s_2\eta^2 - \cdots}\,. \tag{2.26}$$

Conversely, by integrating once by parts and making the change of variables from y to $x = x(y)$, we find the dual formula

$$\eta^{-1} \int_0^\infty e^{-y(x)/\eta}\, dx = e^{-s_1\eta - s_2\eta^2 - \cdots}\,. \tag{2.27}$$

These formulas allow us to get analytic information about the generating series $y(x)$ (and hence about the numbers $V(\mathbf{m})$ when we specialize $(s_1, s_2, \dots)$ suitably. A theoretical reinterpretation of them will be given in Theorem 3.4.2 below.

It would be interesting to extend (2.17) including the contributions of all genera and eventually of all combinatorial cohomology classes as in [K1]. Below we collect some remarks in this direction.

2.5. Kontsevich's formulas. We start with Kontsevich's formula ([K1], Sect. 3.1) for the correlation numbers $\langle\tau_{d_1}\dots\tau_{d_n}\rangle$ which can be used to calculate algorithmically $V_g(\mathbf{m})$ with the help of (2.17). It has the following structure.

Fix (g,n), put $d=3g-3+n$, and choose n independent variables $\lambda_1,\dots,\lambda_n$. Then

$$\sum_{\substack{d=d_1+\dots+d_n\\ d_i\geqq 0}}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\prod_{i=1}^{n}\frac{(2d_i-1)!!}{\lambda_i^{2d_i+1}}=\sum_{\Gamma\in G_{g,n}}\frac{2^{-|V_\tau|}}{|\mathrm{Aut}\,\tau|}\prod_{e\in E_\tau}\frac{2}{\lambda'(e)+\lambda''(e)}\,. \qquad (2.28)$$

Here $G_{g,n}$ is the set of the isomorphism classes of triples $\Gamma=(\tau,c,f)$ where:

i) τ is a connected graph with all vertices $v\in V_\tau$ of valency 3 and no tails;

ii) c is a family of cyclic orders on all $F_\tau(v)$, where $F_\tau(v)$ is the set of flags adjoining v;

iii) f is a bijection between $\{1,\dots,n\}$ and the set of all cycles of τ. We recall that a cycle is a cyclically ordered sequence of edges (without repetitions) $(e_1,e_2,\dots,e_k)$ such that for every i, e_i and e_{i+1} have a common vertex v_i and the flag (e_i,v_i) follows the flag (e_{i+1},v_i) in the sense of c;

iv) for any edge $e\in E_\tau$, $\{\lambda'(e),\lambda''(e)\}=\{\lambda_a,\lambda_b\}$, where $\{a,b\}\subset\{1,\dots,n\}$ are the f-labels of the two cycles to which e belongs.

If τ is embedded into a closed Riemann surface X oriented compatibly with c, the cycles of τ become the boundaries of the oriented connected components of $X\setminus|\tau|$ (2-cells). Then f labels these cells, and $\{a,b\}$ become the labels of the cells adjoining e.

A paradoxical property of (2.28) which does not allow to read off $\langle\tau_{d_1}\dots\tau_{d_n}\rangle$ directly from this identity is the cancellation of poles at $\lambda_a=-\lambda_b$ in the r.h.s., not at all evident a priori even in the simplest case $g=0$, $n=3$:

$$\frac{\langle\tau_0\tau_0\tau_0\rangle}{\lambda_1\lambda_2\lambda_3}=\frac{1}{\lambda_1\lambda_2\lambda_3}=\frac{1}{\lambda_1(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)}+\frac{1}{\lambda_2(\lambda_2+\lambda_3)(\lambda_2+\lambda_1)}$$
$$+\frac{1}{\lambda_3(\lambda_3+\lambda_1)(\lambda_3+\lambda_2)}+\frac{1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)}\,.$$

A generating function incorporating all stable (g,n) that can be summed using (2.28) and the standard techniques of perturbation theory is

$$W(s_0,s_1,\dots)=\left\langle\exp\left(\sum_{a=0}^{\infty}s_a\tau_a\right)\right\rangle=\sum_{n_i\geqq 0}\langle\tau_0^{n_0}\tau_1^{n_1}\cdots\rangle\frac{s_0^{n_0}s_1^{n_1}\cdots}{n_0!\,n_1!\cdots}\,. \qquad (2.29)$$

Kontsevich's theorem states that (2.29) is an asymptotic expansion of (the logarithm of) a matrix Airy function.

We will skip the description of this function because we were unable to find a sensible operator processing $W(s_0,s_1,\dots)$ into a generating series for the WP-volumes.

Instead we will show that the formalism of Lemma 2.2.2 has a nice self-reproducing property in the language of formal series, but in order to use it in our context, a different generating series for $\langle\tau_{d_1}\cdots\tau_{d_n}\rangle$ is needed.

2.6. A remark on Lemma 2.2.2. In the situation of this lemma, assume that $A = \{0,1,2,\dots\}$ and put for $p \geqq 1$,

$$U(t_1,\dots,t_p) := \sum_{a_1,\dots,a_p\geqq 0} \omega(a_1,\dots,a_p)\frac{t_1^{a_1}}{a_1!}\cdots\frac{t_p^{a_p}}{a_p!}\,. \tag{2.30}$$

2.6.1. Lemma. *We have*

$$U(t_1,\dots,t_p) = U(t_1,\dots,t_{p-1})U(t_p) + \sum_{i=1}^{p-1} U(t_1,\dots,t_i+t_p,\dots,t_{p-1})\,. \tag{2.31}$$

The proof is a straightforward calculation using (2.11).

2.6.2. Corollary. *We have*

$$U(t_1)\cdots U(t_p) = \sum_{k=1}^{p}\frac{(-1)^{p-k}}{k!}\sum_{\substack{\{1,\dots,p\}=S_1\amalg\cdots\amalg S_k\\ S_i\neq\emptyset}} U\left(\sum_{j\in S_1} t_j,\dots,\sum_{j\in S_p} t_j\right)\,. \tag{2.32}$$

For the proof, apply Lemma 2.22 to $A = \bigoplus_{i=1}^{p}\mathbb{Z}t_i$, $R[[t_i]]$, and $U(t)$ instead of $\omega(a)$.

We can try to use (2.32) in the case $R = H^*(\overline{M}_{g,n}), \omega(a) = \omega_{g,n}(a)$. The l.h.s. of (2.32) after integration becomes a polynomial with coefficients which are WP-volumes multiplied by some factorials, whereas the r.h.s. becomes a similar polynomial which is a linear combination of correlation numbers.

3. Potential of the Invertible Cohomological Field Theories

Here we explain following [KMK] that the genus zero generating function for higher WP-volumes is the third derivative of the potential of a generic invertible CohFT written in coordinates additive with respect to the tensor multiplication.

3.1. Definition. *Let k be a ground field of characteristic zero, H a finite dimensional $\mathbb{Z}_2$-graded linear space over k, and g an even nondegenerate symmetric pairing on H represented by its dual element $\Delta \in H\otimes H$.*

A structure of the tree level Cohomological Field Theory on (H,g) is given by a sequence of maps

$$I_n : H^{\otimes n} \to H^*(\overline{M}_{0n},k), \quad n \geqq 3\,,$$

which are $\mathbb{S}_n$-covariant and satisfy the following identities. For any 2-partition σ : $\{1,\dots,n\} = S_1 \amalg S_2$, $n_i = |S_i| \geqq 2$ and the respective embedding of the boundary divisor $\varphi_\sigma : \overline{M}_{0,n_1+1}\times\overline{M}_{0,n_2+1}\to\overline{M}_{0,n}$ we have

$$\varphi_\sigma^*(I_n(\gamma_1\otimes\cdots\otimes\gamma_n)) = \varepsilon(\sigma)(I_{n_1+1}\otimes I_{n_2+1})\left(\bigotimes_{j\in S_1}\gamma_j\otimes\Delta\otimes\left(\bigotimes_{j\in S_2}\gamma_j\right)\right)\,,$$

where $\varepsilon(\sigma)$ is the sign of σ acting on the odd $\gamma_i \in H$.

This notion was introduced in [KM]. One of its most interesting instances is quantum cohomology: a canonical structure of CohFT on ($H^*(V,k)$, *Poincaré pairing*) for any smooth projective (or C^∞ compact symplectic) manifold V.

3.2. Tensor product. Let $\{H',g',I'_n\}$ and $\{H'',g'',I''_n\}$ be two CohFT's. Put $H = H'\otimes H''$ and $g = g'\otimes g''$. We can define a CohFT on (H,g) by

$$I_n(\gamma'_1\otimes\gamma''_1\otimes\cdots\otimes\gamma'_n\otimes\gamma''_n) := \varepsilon(\gamma',\gamma'')I'_n(\gamma'_1\otimes\cdots\otimes\gamma'_n)\wedge I''_n(\gamma''_1\otimes\cdots\otimes\gamma''_n)\,.$$

In the context of quantum cohomology, this product serves as a Künneth formula.

3.3. Potential. The potential of a CohFT (H,g,I_n) is the formal series $\Phi\in k[[H^\vee]]$ which in terms of coordinates w.r.t. a basis (Δ_a) of H can be written as

$$\Phi(x) := \sum_{n\geqq 3}\frac{1}{n!}\int_{\overline{M}_{0n}} I_n\left(\left(\sum x^a\Delta_a\right)^{\otimes n}\right)\,. \tag{3.1}$$

The main result about $\Phi(x)$ proved in [KM] and [KMK] is the following theorem.

3.3.1. Theorem. *The map $(H,g,I_n)\mapsto\Phi$ establishes a bijection between the following objects*:

a) *The structures of a* CohFT *on* (H,g),

b) *The solutions of the Associativity Equations in* $k[[H^\vee]]$ *modulo terms of degree* $\leqq 2$.

We recall that the Associativity Equations are

$$\forall a,b,c,d:\sum_{ef}\partial_a\partial_b\partial_e\Phi\cdot g^{ef}\partial_f\partial_c\partial_d\Phi = (-1)^{\tilde{x}_a(\tilde{x}_b+\tilde{x}_c)}\sum_{ef}\partial_b\partial_c\partial_e\Phi\cdot g^{ef}\partial_f\partial_a\partial_d\Phi\,,$$

where $(g^{ef}) = (g(\Delta_e,\Delta_f))^{-1}, \partial_a = \partial/\partial x^a, \tilde{x}^a =$ the $\mathbb{Z}_2$-parity of x^a and Δ_a.

3.4. The moduli space of one-dimensional CohFT's. We can think naively of CohFT's as forming an infinite dimensional algebraic variety, with the tensor product defining a structure of a semigroup on it. In view of 3.3.1, it is natural to try to understand the properties of the potential as a function on the moduli space. In particular, we would like to understand how to express the potential function $\Phi_{\mathscr{A}'\otimes\mathscr{A}''}$ associated to the tensor product of two CohFT's $\mathscr{A}' = (H',g',I')$ and $\mathscr{A}'' = (H'',g'',I'')$ in terms of the potential functions $\Phi_{\mathscr{A}'}$ and $\Phi_{\mathscr{A}''}$.

As a special case, let us consider CohFT structures on one-dimensional spaces. Such a theory will be invertible with respect to the tensor product if the map I_3 from $H^{\otimes 3}\cong k$ to $H^*(\overline{M}_{03},k)\cong k$ is an isomorphism. We will call the theory normalized if we have a chosen basis of length one, $H = k\Delta_0$, $g(\Delta_0,\Delta_0) = 1$, and $I_3(\Delta_0\otimes\Delta_0\otimes\Delta_0) = 1$. (or equivalently $I_n(\Delta_0^{\otimes n}) = 1_n +$ terms of dimension > 0 for all $n\geqq 3$, where $1_n\in H^0(\overline{M}_{0n},k)$ is the fundamental class). The potential function (3.1) has the form

$$\Phi_{\mathscr{A}}(x) = \sum_{n=3}^{\infty} C_n\frac{x^n}{n!}\,, \tag{3.2}$$

where $C_3 = 1$ but the other coefficients are arbitrary by virtue of Theorem 3.3.1, since the Associativity Equations are empty in this case. Thus the space $\mathbf{CohFT_1}(k)$ of all normalized and invertible 1-dimensional CohFT's is canonically isomorphic to $\frac{1}{6}x^3 + x^4 k[[x]]$ and has canonical coordinates C_n $(n \geqq 4)$, and we would like to describe the tensor product in terms of these coordinates.

These 1-dimensional CohFT structures were studied in [KMK] and a different set of coordinates was given. For $s_1, s_2, \ldots \in k$ there is an element $\mathscr{A}(\mathbf{s}) \in \mathbf{CohFT_1}(k)$ given by

$$I_n(\Delta_0^{\otimes n}) = \omega_n[s_1, s_2, \ldots] := \exp\left(\sum_{a=1}^{\infty} s_a \omega_n(a)\right) \qquad (n \geqq 3)\,, \tag{3.3}$$

in the notation of our paper and [KMK]. Then it was shown that the map $\mathbf{s} \mapsto \mathscr{A}(\mathbf{s})$ gives a bijection between $k^{\mathbb{N}}$ and $\mathbf{CohFT_1}(k)$ and that $\mathscr{A}(\mathbf{s}') \otimes \mathscr{A}(\mathbf{s}'') \cong \mathscr{A}(\mathbf{s}' + \mathbf{s}'')$, i.e.:

3.4.1. Theorem [KMK]. *The parameters $(s_1, s_2, \ldots)$ form a coordinate system on the space of normalized 1-dimensional* CohFT'*s. The tensor product becomes addition in these coordinates.*

Denote by $\Phi(x; \mathbf{s})$ the potential associated to the theory (3.3). The connection with what we have done in this paper is that the third derivative of the potential $\Phi(x; \mathbf{s})$ associated to the theory (3.3) is just our generating function for higher WP-volumes. Indeed, Definition (3.1) gives

$$\Phi(x; s_1, s_2, \ldots) = \sum_{n=3}^{\infty} \frac{x^n}{n!} \int_{\overline{M}_{0n}} \sum_{\sum a m(a) = n-3} \prod_{a \geqq 1} \omega_n(a)^{m(a)} \frac{s_a^{m(a)}}{m(a)!}\,,$$

and the third derivative of this is obviously the function $F(x; \mathbf{s}) = \sum_{\mathbf{m}} V(\mathbf{m}) x^{|\mathbf{m}|} \mathbf{s}^{\mathbf{m}}$ defined in 1.6. We now use this connection to describe both the tensor product and the coordinates on the space of invertible 1-dimensional CohFT's explicitly.

3.4.2. Theorem. *Define bijections*

$$\mathbf{CohFT_1}(k) \leftrightarrow \frac{x^3}{6} + x^4 k[[x]] \leftrightarrow 1 + \eta k[[\eta]]\,, \tag{3.4}$$

where the first map assigns to a theory $\mathscr{A}$ its potential $\Phi_{\mathscr{A}}(x)$ and the second map is defined by

$$\Phi(x) \leftrightarrow U(\eta) = \int_0^{\infty} e^{-\Phi''(\eta x)/\eta}\, dx \tag{3.5}$$

or alternatively by assigning to $\Phi(x) = \frac{1}{6}x^3 + \cdots$ the power series $U(\eta) = \sum_{n=0}^{\infty} B_n \eta^n$ where $x = \sum B_n \frac{y^{n+1}}{(n+1)!} = y + \cdots$ is the inverse power series of $y = \Phi''(x) = x + \cdots$. Then the tensor product of 1*-dimensional* CohFT's *corresponds to multiplication in $1 + \eta k[[\eta]]$: $U_{\mathscr{A}' \otimes \mathscr{A}''}(\eta) = U_{\mathscr{A}'}(\eta)\, U_{\mathscr{A}''}(\eta)$. The coefficients of $-\log U_{\mathscr{A}}(\eta)$ are the canonical coordinates of $\mathscr{A}$.*

Proof. Equations (2.25) and (2.26) tell us that the two descriptions of $U(\eta)$ given in the theorem agree and give $e^{-s_1\eta - s_2\eta^2 - \cdots}$ when $\Phi(x) = \Phi(x; \mathbf{s})$, and in view of Theorem 3.4.1 this implies the result in general.

3.5. Explicit formulas. Substituting (3.2) (with $C_3 = 1$) into (3.5), expanding, and integrating term by term gives the explicit formula

$$B_n = \sum_{\substack{n_4, n_5, \ldots \geqq 0 \\ n_4 + 2n_5 + \cdots = n}} \frac{(2n_4 + 3n_5 + \cdots)!}{2!^{n_4}\, 3!^{n_5} \cdots n_4!\, n_5! \cdots} (-C_4)^{n_4} (-C_5)^{n_5} \cdots$$

for the coefficients of $U(\eta)$ in terms of the coefficients of $\Phi(x)$, and the same argument applied to the inverse power series gives the reciprocal formula

$$C_n = \sum_{\substack{n_1, n_2, \ldots \geqq 0 \\ n_4 + 2n_5 + \cdots = n-3}} \frac{(2n_1 + 3n_2 + \cdots)!}{2!^{n_1}\, 3!^{n_2} \cdots n_1!\, n_2! \cdots} (-B_1)^{n_1} (-B_2)^{n_2} \cdots .$$

Combining these formulas with the identity $U_{\mathscr{A}' \otimes \mathscr{A}''}(\eta) = U_{\mathscr{A}'}(\eta)\, U_{\mathscr{A}''}(\eta)$ we obtain the explicit law for the tensor product of two normalized invertible CohFT's in terms of the coefficients of their potential functions:

$$\begin{aligned}
C_4 &= C_4' + C_4'' , \\
C_5 &= C_5' + 5C_4'C_4'' + C_5'' , \\
C_6 &= C_6' + (8\, C_4'^2 + C_5')\, C_4'' + C_4' (8\, C_4''^2 + C_5'') + C_6'' , \\
C_7 &= C_7' + (35\, C_4'\, C_5' + 14\, C_6')\, C_4'' + (61\, C_4'^2\, C_4''^2 + 33\, C_4'^2 C_5'' + 33\, C_5'\, C_4''^2 \\
&\qquad + 19\, C_5'\, C_5'') + C_4' (35\, C_4''\, C_5'' + 14\, C_6'') + C_7'' , \ldots
\end{aligned}$$

Finally, we observe that the values of the genus 0 Weil–Petersson volumes $V(\mathbf{m})$ can be calculated numerically from any of a number of formulas in this paper: the recursion relation (1.4), the differential equation (1.24), the closed formula (2.18), or the generating function formula (2.19). Here are the values up to $|\mathbf{m}| = 5$, expressed in terms of the generating function (1.23):

$$\begin{aligned}
F(x, \mathbf{s}) &= 1 + s_1 x + \left(5\, \frac{s_1^2}{2} + s_1\right) \frac{x^2}{2} + \left(61\, \frac{s_1^3}{6} + 9\, s_1 s_2 + s_3\right) \frac{x^3}{6} \\
&\quad + \left(1379\, \frac{s_1^4}{24} + 161\, \frac{s_1^2 s_2}{2} + 14\, s_1 s_3 + 19\, \frac{s_2^2}{2} + s_4\right) \frac{x^4}{24} \\
&\quad + \left(49946\, \frac{s_1^5}{120} + 4822\, \frac{s_1^3 s_2}{6} + 344\, \frac{s_1^2 s_3}{2} + 470\, \frac{s_1 s_2^2}{2} + 20\, s_1 s_4 \right. \\
&\quad \left. + 34\, s_2 s_3 + s_5\right) \frac{x^5}{120} + O(x^6) .
\end{aligned}$$

Note that the coefficient $\int_{\overline{M}_{0,n}} \omega^{\mathbf{m}}$ of $\frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \frac{x^{|\mathbf{m}|}}{|\mathbf{m}|!}$ is integral for every $\mathbf{m}$ since the cohomology classes $\omega_{g,n}(a)$ are integral for $g = 0$.

Appendix

In this appendix we make a few remarks on the asymptotics of the Betti numbers and Euler characteristics of the moduli spaces of $\overline{M}_{0n}$ and on the Weil–Petersson volumes(0.2). Set $P_1(q) = 1$ and for $n \geqq 2$ let

$$B_j(n) = \dim H^j(\overline{M}_{0,n+1}), \qquad P_n(q) = \sum_{j \geqq 0} B_j(n) q^j$$

be the Betti numbers and Poincaré polynomial, respectively, of $\overline{M}_{0,n+1}$. It was shown in [M] that the polynomials P_n satisfy the recursion

$$P_{n+1}(q) = P_n(q) + q^2 \sum_{m=2}^{n} \binom{n}{m} P_m(q) P_{n+1-m}(q) \qquad (n \geqq 1). \tag{A.1}$$

This is equivalent to the differential equation

$$\frac{\partial y}{\partial x} = \frac{1+y}{1-q^2(y-x)} \tag{A.2}$$

for the generating function

$$y = \sum_{n=1}^{\infty} P_n(q) \frac{x^n}{n!} = x + \frac{1}{2}x^2 + \frac{1+q^2}{6}x^3 + \frac{1+5q^2+q^4}{24}x^4 + \cdots.$$

The solution is given by the implicit equation

$$(1+y)^{q^2} = 1 + q^2 x + q^4 (y-x). \tag{A.3}$$

If we specialize to $q = 1$, then the solution of (A.2), or the limiting value of (A.3), is given by the implicit equation

$$x = 2y - (1+y)\log(1+y) = y - \sum_{n=2}^{\infty} \frac{(-1)^n y^n}{n(n-1)} \qquad (q = 1),$$

from which the asymptotic formula (0.10) for the values $\chi(\overline{M}_{0,n+1}) = P_n(1)$ follows easily. (The derivative dx/dy vanishes simply at $y = e-1$, $x = e-2$, so the power series expansion of $y(1,x)$ as a function of x has radius of convergence $e-2$ with a square-root singularity at $x = e-2$). The same method applies to the inversion formula (0.8) to give the asymptotic equation (0.9) with the constant C given by $C = 2\gamma_0 J_0'(\gamma_0)$, where $\gamma_0 = 2.4048255577$ is the smallest zero of the Bessel function $J_0(x)$ and the other constants in the expansion can be obtained by doing a more detailed analysis of the function $\sqrt{y}J_1(2\sqrt{y})$ near its maximum.

We can also use (A.1) and (A.3) to study the behavior of the Betti numbers $B_j(n)$ as a function of n for fixed j. From (A.1) we get

$$B_2(n) = 2^n - \tfrac{1}{2}(n^2+n+2),$$
$$B_4(n) = \tfrac{3}{2}3^n - \tfrac{1}{4}(n^2+5n+8)2^n + \tfrac{1}{24}(3n^4+2n^3+21n^2+22n+12),$$

and more generally

$$B_{2j}(n) = \sum_{l=0}^{j} p_{j,l}(n)(+1-l)^n \tag{A.4}$$

for some polynomials $p_{j,l}(n)$ of degree $2l$ in n. (The odd Betti numbers are 0). To see this, and to get information about the polynomials $p_{j,l}$, we observe that (A.4) is equivalent to the statement that for each j we have the generating function identity

$$\sum_{n=1}^{\infty} B_j(n)\frac{x^n}{n!} = A_j(x, e^x)$$

for some polynomial $A_j(x,u) \in \mathbb{Q}[u,x]$, the first few values being

$$A_0 = u-1, \qquad A_2 = u^2 - \left(1+x+\frac{x^2}{2}\right)u,$$
$$A_4 = \frac{3}{2}u^3 - (2+3x+x^2)u^2 + \left(\frac{1}{2}+2x+2x^2+\frac{5x^3}{6}+\frac{x^4}{8}\right)u\,.$$

The function y is then replaced by a power series $y(q,x,u) = \sum A_j(x,u)q^j$ in three variables, the previous power series being obtained by setting $u = e^x$, and the recursion (A.1) gives the same differential equation (A.2) as before but with the left-hand side replaced by $\frac{\partial y}{\partial x} + u\frac{\partial y}{\partial u}$. The solution is given by the implicit equation

$$\left(\frac{1+y}{ue^{-x}}\right)^{q^2} = 1 + q^2x + q^4(y-x)\,. \tag{A.5}$$

The polynomial $A_j(u,x)$ has degree $j+2$, where u and x are assigned degrees 2 and 1, respectively, and u divides A_j for $j \geqq 1$, so $q^2(1+y) = \Phi(q,qx,q^2u)$ with $\Phi(q,X,U) \in \mathbb{Q}[[q,X,U]]$. Then (A.5) gives

$$\Phi = U\,e^{-X/q}(1+qX+q^2\,\Phi+\mathrm{O}(q^3))^{1/q^2} = U\,e^{\Phi-X^2/2+\mathrm{O}(q)}\,,$$

so $\Phi(0,X,U)$ is a solution of $\Phi e^{-\Phi} = Ue^{-X^2/2}$. Inverting this gives

$$1+y = q^{-2}\,\Phi = q^{-2}\sum_{j=1}^{\infty}\frac{(j+1)^{j-1}}{j!}\,U^{j+1}\,e^{-(j+1)X^2/2}(1+\mathrm{O}(q))\,,$$

which with $u = Uq^{-2}$, $x = Xq^{-1}$ translates back as

$$A_j(x,u) = \sum_{l=0}^{j}\frac{(-1)^l}{2^l l!}\frac{(j-l+1)^{j-1}}{(j-l)!}\,u^{j-l+1}(x^{2l}+\mathrm{O}(x^{2l-1}))$$

or, for the polynomials $p_{j,l}$ defined in (A.4), as

$$p_{j,l}(n) = \frac{(-1)^l}{2^l l!}\frac{(j-l+1)^{j-2l-1}}{(j-l)!}\,n^{2l} + \mathrm{O}(n^{2l-1})\,.$$

In particular,

$$B_{2j}(n) = \frac{(j+1)^{n+j-1}}{j!} + \mathrm{O}\big(n^2 j^n\big)\,.$$

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