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# The Bloch-Wigner-Ramakrishnan polylogarithm function

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*To Hans Grauert*

The polylogarithm function

$$Li_m(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^m}\qquad (x\in\mathbb{C},\ |x|\leqq 1,\ m\in\mathbb{N})$$

appears in many parts of mathematics and has an extensive literature [2]. It can be analytically extended to the cut plane $\mathbb{C}\backslash[1,\infty)$ by defining $Li_m(x)$ inductively as $\int_0^x Li_{m-1}(z)z^{-1}dz$ but then has a discontinuity as $x$ crosses the cut. However, for $m=2$ the modified function

$$D(x)=\Im(Li_2(x))+\arg(1-x)\log|x|$$

extends (real-) analytically to the entire complex plane except for the points $x=0$ and $x=1$ where it is continuous but not analytic. This modified dilogarithm function, introduced by Wigner and Bloch [1], has many beautiful properties. In particular, its values at algebraic argument suffice to express in closed form the volumes of arbitrary hyperbolic 3-manifolds and the values at $s=2$ of the Dedekind zeta functions of arbitrary number fields (cf. [6] and the expository article [7]). It is therefore natural to ask for similar real-analytic and single-valued modification of the higher polylogarithm functions $Li_m$. Such a function $D_m$ was constructed, and shown to satisfy a functional equation relating $D_m(x^{-1})$ and $D_m(x)$, by Ramakrishnan [3]. His construction, which involved monodromy arguments for certain nilpotent subgroups of $GL_m(\mathbb{C})$, is completely explicit, but he does not actually give a formula for $D_m$ in terms of the polylogarithm. In this note we write down such a formula and give a direct proof of the one-valuedness and functional equation. We will also:

i) prove a formula (generalizing a formula of Bloch for $m=2$) expressing certain infinite sums of the $D_m$ as special values of Kronecker double series related to $L$-series of Hecke characters,

ii) describe a relation between the $D_m(x)$ and certain Green's functions for the unit disc, and

iii) discuss the conjecture that the values at $s=m$ of the Dedekind zeta function $\zeta_F(s)$ for an arbitrary number field $F$ can be expressed in terms of values of $D_m(x)$ with $x \in F$.

The last relationship, which seems to be the most interesting property of the higher polylogarithm functions, is closely connected with algebraic $K$-theory and in fact leads to a conjectural description of higher $K$-groups of fields, as will be discussed in more detail in a later paper [9].

## 1. Definition of the function $D_m(x)$

For $m \in \mathbb{N}$ and $x \in \mathbb{C}$ with $|x| \leqq 1$ define

$$L_m(x) = \sum_{j=1}^{m} \frac{(-\log|x|)^{m-j}}{(m-j)!} Li_j(x),$$

$$D_m(x) = \begin{cases} \Im(L_m(x)) & (m \text{ even}), \\ \Re(L_m(x)) + \dfrac{(\log|x|)^m}{2m!} & (m \text{ odd}). \end{cases}$$

**Proposition 1.** *$D_m(x)$ can be continued real-analytically to $\mathbb{C} \backslash \{0, 1\}$ and satisfies the functional equation $D_m\left(\frac{1}{x}\right) = (-1)^{m-1} D_m(x)$.*

*Remarks.* Ramakrishnan's $D_m$ is equal to ours for $m$ even but is just $\Re(L_m(x))$ for $m$ odd. We have included the extra term $(\log|x|)^m/2m!$ for $m$ odd in order to make the functional equation as simple as possible (Ramakrishnan's function satisfies $D_m(1/x) = D_m(x) + (\log|x|)^m/m!$ for $m$ odd), but at the cost of making the function discontinuous at 0 in this case. (For $m$ even, $D_m$ extends to a continuous function on the extended plane $\mathbb{C} \cup \{\infty\}$, vanishing on $\mathbb{R} \cup \{\infty\}$.) The definition of $D_m$ here also differs by a factor $(-1)^{[(m+1)/2]}$ from the normalization given in [7], which was chosen to give a simpler relation between $\partial D_m/\partial z$ and $D_{m-1}$. The functions $D_1(x)$ and $D_2(x)$ are equal to $-\log|x^{1/2} - x^{-1/2}|$ and $D(x)$, respectively.

*Proof.* As mentioned in the introduction, we can continue $Li_m(x)$ analytically to the cut plane $\mathbb{C} \backslash [1, \infty)$ by successive integration along, say, radial paths from 0 to $x$. The two branches just below and just above the cut then continue across the cut. Write $\Delta$ for the difference of these two analytic functions in their common region of definition (say, in the range $|\arg(x-1)| < \varepsilon$, where $\varepsilon$ is small). Since $Li_1(x) = \log \frac{1}{1-x}$ for $|x| < 1$, we have $\Delta Li_1 = 2\pi i$, and it then follows from the formula $x Li'_m(x) = Li_{m-1}(x)$ that

$$\Delta Li_m(x) = 2\pi i (\log x)^{m-1}/(m-1)!$$

for each $m \geqq 1$. (This is well-defined in the region in question: we take the branch of $\log x$ which vanishes at $x=1$.) Consequently,

$$\Delta L_m(x) = 2\pi i \sum_{j=1}^{m} \frac{(-\log|x|)^{m-j}}{(m-j)!} \frac{(\log x)^{j-1}}{(j-1)!} = \frac{2\pi i}{(m-1)!} \left(\log \frac{x}{|x|}\right)^{m-1}.$$

Since $\log \frac{x}{|x|}$ is pure imaginary, this is real for $m$ even and pure imaginary for $m$ odd.

Hence $\Re(i^{m+1}L_m(x))$ is one-valued, proving the first assertion of the proposition.

To prove the second, it will be convenient to introduce the generating function $\mathscr{L}(x;t)=\sum_{m=1}^{\infty} L_m(x)t^{m-1}$. For $|x|<1$, $|t|<1$ we have

$$\begin{aligned}\mathscr{L}(x;t)&=\sum_{j\geq 1,k\geq 0}\frac{(-\log|x|)^k}{k!}Li_j(x)t^{j+k-1}=|x|^{-t}\sum_{j=1}^{\infty}Li_j(x)t^{j-1}\\&=|x|^{-t}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{t^{j-1}}{n^j}x^n=|x|^{-t}\sum_{n=1}^{\infty}\frac{x^n}{n-t}\end{aligned}$$

or

$$\mathscr{L}(re^{i\theta};t)=\sum_{n=1}^{\infty}\frac{r^{n-t}}{n-t}e^{in\theta}=\int_0^r\frac{u^{-t}du}{e^{-i\theta}-u}\qquad(0\leq r<1),$$

where we have written $\frac{r^{n-t}}{n-t}$ as $\int_0^r u^{n-t-1}du$ and summed the geometric series under the integral sign. The integral converges also for $r\geq 1$ and immediately gives the extension to the cut plane $|\arg(1-z)|<\pi$. Since the integrand has a simple pole of residue $-e^{it\theta}$ at $u=e^{-i\theta}$, we again see that the difference between the two branches of $L_m(re^{i\theta})$ near the cut is $2\pi i^m\theta^{m-1}/(m-1)!$, giving the one-valuedness of $D_m$ as before. In terms of $\mathscr{L}(x;t)$, the functional equation can be stated as the assertion that $\mathscr{L}(re^{i\theta};t)+\mathscr{L}(re^{-i\theta};-t)+\frac{1}{t}r^t$ is unchanged when $r$ is replaced by $r^{-1}$. But for $0<t<1$ we have

$$\begin{aligned}\mathscr{L}(re^{i\theta};t)+\mathscr{L}(re^{-i\theta};-t)+\frac{r^t}{t}&=\int_0^r\frac{u^{-t}du}{e^{-i\theta}-u}+\int_0^r\frac{v^t dv}{e^{i\theta}-v}+\int_{r^{-1}}^{\infty}u^{-t-1}du\\&=\left(\int_0^{\infty}-\int_r^{\infty}-\int_{r^{-1}}^{\infty}\right)\frac{u^{-t}du}{e^{-i\theta}-u}\qquad(v=u^{-1}).\end{aligned}$$

This makes the desired symmetry obvious.

## 2. The functions $D_{a,b}(x)$ and Kronecker double series

It is clear from the definition that the Bloch-Wigner function $D(x)$ goes to 0 like $|x|\log|x|$ as $x\to 0$, and from the functional equation that $D(x)=O(|x|^{-1}\log|x|)$ as $x\to\infty$. Hence, for a complex number $q$ of absolute value strictly less than 1 and any complex number $x$, the doubly infinite series

$$D(q;x)=\sum_{l=-\infty}^{\infty}D(q^l x)$$

converges with exponential rapidity. Clearly $D(q;x)$ is invariant under $x\mapsto qx$, so it is in fact a function on the elliptic curve $\mathbb{C}^\times/q^{\mathbb{Z}}$. In other words, if we write $q=e^{2\pi i\tau}$ with $\tau$ in the complex upper half-plane and $x=e^{2\pi iu}$ with $u\in\mathbb{C}$, then $D(q;x)$ depends only on the image of $u$ in the quotient of $\mathbb{C}$ by the lattice $L=\mathbb{Z}\tau+\mathbb{Z}$. In [1], Bloch computed the Fourier development of this non-holomorphic elliptic function. Actually, he found that $D(x)$ should be supplemented by adding an imaginary part $-iJ(x)$, where

$$J(x)=\log|x|\log|1-x| \qquad (x\in\mathbb{C},\ x\neq 0,1).$$

The function $J(x)$ is small as $|x|\to 0$ but large as $|x|\to\infty$, so we cannot form the series $\sum_{l\in\mathbb{Z}} J(q^l x)$ as we did with $D$. However, using the functional equation $J(x^{-1})=-J(x)+\log^2|x|$ we find after a short calculation that the function

$$J(q;x)=\sum_{l=0}^{\infty} J(q^l x)-\sum_{l=1}^{\infty} J(q^l x^{-1})+\frac{\log^3|x|}{3\log|q|}-\frac{\log^2|x|}{2}+\frac{\log|x|\log|q|}{6}$$
$$(q,x\in\mathbb{C},\ |q|<1)$$

is invariant under $x\mapsto qx$, so descends to the elliptic curve $\mathbb{C}^\times/q^{\mathbb{Z}}\simeq\mathbb{C}/L$ as before. Bloch's result can then be written

$$D(q;x)-iJ(q;x)=\frac{i}{\pi}\Im(\tau)^2\sum_{m,n}{}'\frac{\sin(2\pi(n\xi-m\eta))}{(m\tau+n)^2(m\bar{\tau}+n)},$$

where $q=e^{2\pi i\tau}$, $x=e^{2\pi iu}$ with $u=\xi\tau+\eta$ ($\xi,\eta\in\mathbb{R}/\mathbb{Z}$) and the sum is over all pairs of integers $(m,n)\neq(0,0)$. This is a classical series studied by Kronecker (see for instance Weil's book [5]). The special case when $\tau$ is quadratic over $\mathbb{Q}$ and $\xi$ and $\eta$ are rational numbers occurs in evaluating $L$-series of Hecke grossencharacters of type $A_0$ and weight 1 at $s=2$. To get other weights and other special values, we have to study series of the same type but with other powers of $m\tau+n$ and $m\bar{\tau}+n$ in the denominator. In this section we will prove the analogue of Bloch's formula for such series, the function $D(x)-iJ(x)$ being replaced by a suitable linear combination of the Ramakrishnan functions $D_m(x)$.

To define these combinations, we will need combinatorial coefficients, and we begin by defining these. For integers $a,m,r$ with $1\leq a,\ m\leq r$ let $c_{a,m}^{(r)}$ denote the coefficients of $x^{a-1}$ in the polynomial $(1-x)^{m-1}(1+x)^{r-m}$. These coefficients are easily computed by the recursion $c_{a,m}^{(r)}=c_{a,m}^{(r-1)}+c_{a-1,m}^{(r-1)}$ or by the closed formula

$$c_{a,m}^{(r)}=\sum_{h=1}^{a}(-1)^{h-1}\binom{m-1}{h-1}\binom{r-m}{a-h}.$$

They have the symmetry properties

$$\begin{gathered} c_{a,m}^{(r)}=(-1)^{a-1}c_{a,r+1-m}^{(r)}=(-1)^{m-1}c_{r+1-a,m}^{(r)},\\ \binom{r-1}{m-1}c_{a,m}^{(r)}=\binom{r-1}{a-1}c_{m,a}^{(r)}, \end{gathered} \tag{1}$$

the former being obvious and the latter a consequence of the identity

$$\sum_{a=1}^{r}\sum_{m=1}^{r}\binom{r-1}{m-1}c_{a,m}^{(r)}x^{a-1}y^{m-1}=(1+x+y-xy)^{r-1}.$$

The definition of $c_{a,m}^{(r)}$ is equivalent to saying that the $r\times r$ matrix $C_r=(c_{a,m}^{(r)})_{a,m=1,\dots,r}$ gives the transition between the bases $\{t^{r-1},t^{r-2}u,\dots,tu^{r-2},u^{r-1}\}$ and

$$\{(t+u)^{r-1},(t+u)^{r-2}(t-u),\dots,(t+u)(t-u)^{r-2},(t-u)^{r-1}\}$$

of the space of homogeneous polynomials of degree $r-1$ in two variables $t$ and $u$. The fact that the matrix $\begin{pmatrix}-1&1\\1&1\end{pmatrix}$ has square 2 implies that

$$C_r^{-1}=2^{-r+1}C_r. \tag{2}$$

We will also need the formulas

$$\left.\begin{aligned}\sum_{m=k}^{r}\binom{r-k}{m-k}c_{a,m}^{(r)}&=(-1)^{a-1}\binom{k-1}{a-1}2^{r-k}\\ \sum_{m=k}^{r}(-1)^{m-1}\binom{r-k}{m-k}c_{a,m}^{(r)}&=(-1)^{r-a}\binom{k-1}{r-a}2^{r-k}\end{aligned}\right\}\quad(1\leqq a,k\leqq r) \tag{3}$$

(the expressions on the right being 0 for $k<a$ or $k<r+1-a$, respectively) and

$$\sum_{\substack{m=1\\ m\text{ odd}}}^{r}\binom{r}{m}c_{a,m}^{(r)}=2^{r-1}\qquad(1\leqq a\leqq r). \tag{4}$$

We leave the proofs to the reader (hint: expand $(1-x)^{k-1}\{1+x\pm(1-x)\}^{r-k}$ for $0\leqq k\leqq r$). As numerical examples to illustrate properties (1)–(4) we give the $c_{a,m}^{(r)}$ for $r=6$ and 7:

$$C_6=\begin{pmatrix}1&1&1&1&1&1\\5&3&1&-1&-3&-5\\10&2&-2&-2&2&10\\10&-2&-2&2&2&-10\\5&-3&1&1&-3&5\\1&-1&1&-1&1&-1\end{pmatrix},$$

$$C_7=\begin{pmatrix}1&1&1&1&1&1&1\\6&4&2&0&-2&-4&-6\\15&5&-1&-3&-1&5&15\\20&0&-4&0&4&0&-20\\15&-5&-1&3&-1&-5&15\\6&-4&2&0&-2&4&-6\\1&-1&1&-1&1&-1&1\end{pmatrix}.$$

We now define for integers $a,b\geqq 1$ and $x\in\mathbb{C}$

$$D_{a,b}(x)=2\sum_{m=1}^{r}c_{a,m}^{(r)}D_m^*(x)\frac{(-\log|x|)^{r-m}}{(r-m)!}+\frac{(-2\log|x|)^r}{2r!}\qquad(r=a+b-1),$$

where $D_m^*(x)=D_m(x)$ for $m$ odd, $D_m^*(x)=iD_m(x)=\frac{1}{2}[L_m(x)-\overline{L_m(x)}]$ for $m$ even.

**Proposition 2.** (i) *$D_{a,b}$ is a one-valued real-analytic function on $\mathbb{C}\backslash[1,\infty)$ and satisfies the functional equation*

$$D_{a,b}\left(\frac{1}{x}\right)=(-1)^{r-1}D_{a,b}(x)+\frac{(2\log|x|)^r}{r!}.$$

(ii) *$D_{a,b}$ is given in terms of the polylogarithm by*

$$D_{a,b}=(-1)^{a-1}\sum_{k=a}^{r}2^{r-k}\binom{k-1}{a-1}\frac{(-\log|x|)^{r-k}}{(r-k)!}Li_k(x)$$
$$+(-1)^{b-1}\sum_{k=b}^{r}2^{r-k}\binom{k-1}{b-1}\frac{(-\log|x|)^{r-k}}{(r-k)!}\overline{Li_k(x)}.$$

(iii) *The function defined for $q, x\in\mathbb{C}$ with $|q|<1$ by*

$$D_{a,b}(q;x)=\sum_{l=0}^{\infty}D_{a,b}(q^lx)+(-1)^{r-1}\sum_{l=1}^{\infty}D_{a,b}(q^lx^{-1})$$
$$+\frac{(-2\log|q|)^r}{(r+1)!}B_{r+1}\left(\frac{\log|x|}{\log|q|}\right)$$

*($B_{r+1}(x)=(r+1)$st Bernoulli polynomial) is invariant under $x\mapsto qx$.*

*Proof.* Statement (i) follows immediately from Proposition 1 and statement (ii) from equations (3) and (4). For (iii), we note first that the infinite sum converges absolutely for any $x$, because $D_{a,b}(x)=O(|x|\log^{a+b}|x|)$ as $|x|\to 0$. Hence $D_{a,b}(q;x)$ makes sense. Using (i) and the property $B_{r+1}(x+1)-B_{r+1}(x)=(r+1)x^r$, we find

$$D_{a,b}(q;x)-D_{a,b}(q;qx)=D_{a,b}(x)-(-1)^{r-1}D_{a,b}(x^{-1})$$
$$-\frac{(-2\log|q|)^r}{(r+1)!}(r+1)\left(\frac{\log|x|}{\log|q|}\right)^r=0.$$

This completes the proof of the proposition.

Notice that we can use the inversion formula (2) to write

$$D_m^*(x)\frac{(-\log|x|)^n}{n!}=\sum_{\substack{a,b\geqq 1\\a+b=r+1}}c_{m,a}^{(r)}\left\{2^{-r}D_{a,b}(x)-\frac{(-\log|x|)^r}{2r!}\right\}$$
$$(m\geqq 1, n\geqq 0, r=m+n);$$

in particular, the Ramakrishnan functions $D_m$ are linear combinations of the $D_{a,b}$. We could therefore have equally well defined the functions $D_{a,b}$ directly by the formula in (ii) and taken them rather than the functions $D_m$ as the primitive objects of study. The proof of the analytic continuation can be given directly from (ii) by the same method as in the proof of Proposition 1: using

$$\Delta Li_k(x)=2\pi i(\log x)^{k-1}/(k-1)!$$

and the binomial theorem, one finds easily that $\Delta D_{a,b}=0$.

Part (iii) of the proposition says that the function $D_{a,b}(q;e^{2\pi iu})$ is a (non-holomorphic) elliptic function of $u$. Our goal is to compute the Fourier development of this function.

**Theorem 1.** *Write* $q=e^{2\pi i\tau}$, $x=e^{2\pi iu}$ *with* $\tau$ *in the complex upper half-plane and* $u=\xi\tau+\eta\in\mathbb{C}$, $\xi,\eta\in\mathbb{R}$. *Then*

$$D_{a,b}(q;x)=\frac{(\tau-\bar\tau)^r}{2\pi i}\sum_{m,n}{}'\frac{e^{2\pi i(n\xi-m\eta)}}{(m\tau+n)^a(m\bar\tau+n)^b}.$$

*Proof.* Since $D_{a,b}(e^{2\pi i\tau};e^{2\pi i(\xi\tau+\eta)})$ is invariant under $\xi\mapsto\xi+1$, we can develop it into a Fourier series $\sum_{n\in\mathbb{Z}}\lambda_n e^{2\pi in\xi}$ with

$$\begin{aligned}\lambda_n&=\int_0^1 e^{-2\pi in\xi}D_{a,b}(e^{2\pi i\tau};e^{2\pi i(\xi\tau+\eta)})d\xi\\&=\int_0^\infty e^{-2\pi in\xi}D_{a,b}(e^{2\pi i(\xi\tau+\eta)})d\xi+(-1)^{r-1}\int_0^\infty e^{2\pi in\xi}D_{a,b}(e^{2\pi i(\xi\tau-\eta)})d\xi\\&\quad+\frac{(4\pi\Im(\tau))^r}{(r+1)!}\int_0^1 e^{-2\pi in\xi}B_{r+1}(\xi)d\xi,\end{aligned}$$

where we have substituted for $D_{a,b}$ the expression defining it and then in the first two terms combined the sum over $l$ and the integral from 0 to 1 into a single integral from 0 to $\infty$ by the substitution $l\pm\xi\to\xi$. It is well-known [and easily shown by repeated integration by parts, using $B_j'=jB_{j-1}$ and $B_j(1)=B_j(0)$ for $j\neq1$] that the last integral is equal to 0 for $n=0$ and to $-(r+1)!/(2\pi in)^{r+1}$ for $n\neq0$. Substituting for $D_{a,b}(x)$ from part (ii) of the proposition, we find

$$\begin{aligned}\lambda_n=(-1)^{a-1}\sum_{k=a}^{r}2^{r-k}\binom{k-1}{a-1}\frac{(2\pi\Im(\tau))^{r-k}}{(r-k)!}\int_0^\infty \xi^{r-k}[Li_k(e^{2\pi i(\xi\tau+\eta)})e^{-2\pi in\xi}\\+(-1)^{r-1}Li_k(e^{2\pi i(\xi\tau-\eta)})e^{2\pi in\xi}]d\xi+\begin{pmatrix}a\leftrightarrow b\\\tau\leftrightarrow-\bar\tau\end{pmatrix}-\frac{(-2i\Im(\tau))^r}{2\pi i}n^{-r-1},\end{aligned}$$

where the second term denotes the result of interchanging $a$ and $b$ and replacing $\tau$ by $-\bar\tau$ in the first term and the last term is to be omitted if $n=0$. The two arguments of $Li_k$ in the integrand are less than 1 in absolute value, so we can replace $Li_k$ by its definition as a power series, obtaining

$$\begin{aligned}&\int_0^\infty \xi^{r-k}[Li_k(e^{2\pi i(\xi\tau+\eta)})e^{-2\pi in\xi}+(-1)^{r-1}Li_k(e^{2\pi i(\xi\tau-\eta)})e^{2\pi in\xi}]d\xi\\&=\sum_{m=1}^\infty\frac{1}{m^k}\int_0^\infty\{e^{2\pi i[(m\tau-n)\xi+m\eta]}+(-1)^{r-1}e^{2\pi i[(m\tau+n)\xi-m\eta]}\}\xi^{r-k}d\xi\\&=\sum_{m=1}^k\frac{1}{m^k}\frac{(r-k)!}{(-2\pi i)^{r+1-k}}\left\{\frac{e^{2\pi im\eta}}{(m\tau-n)^{r+1-k}}+(-1)^{r-1}\frac{e^{-2\pi im\eta}}{(m\tau+n)^{r+1-k}}\right\}\\&=\frac{(-1)^k(r-k)!}{(2\pi i)^{r+1-k}}\sum_{m\neq0}\frac{e^{-2\pi im\eta}}{m^k(m\tau+n)^{r+1-k}},\end{aligned}$$

where we have used the formula $\int_0^\infty e^{-\lambda\xi}\xi^l d\xi = l!\lambda^{-l-1}$ for $\Re(\lambda)>0$. Hence

$$2\pi i\lambda_n = (-1)^b \sum_{k=a}^{r} \binom{k-1}{a-1} (2i\Im(\tau))^{r-k} \sum_{m\neq 0} \frac{e^{-2\pi i m\eta}}{m^k(m\tau+n)^{r+1-k}} + \binom{a\leftrightarrow b}{\tau\leftrightarrow -\bar{\tau}} - \frac{2(-2i\Im(\tau))^r}{n^{r+1}}.$$

Applying the easily checked identity

$$(-1)^a \sum_{k=a}^{r} \binom{k-1}{a-1} \frac{(X-Y)^{r-k}}{X^{r+1-k}} + \sum_{k=b}^{r} \binom{k-1}{b-1} \frac{(X-Y)^{r-k}}{Y^{r+1-k}} = \frac{(X-Y)^r}{X^b Y^a} \qquad (r=a+b-1)$$

to $X=m\tau+n$, $Y=m\bar{\tau}+n$, we find

$$2\pi i\lambda_n = (2i\Im(\tau))^r \sum_{\substack{m\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{e^{-2\pi i m\eta}}{(m\tau+n)^a(m\bar{\tau}+n)^b}.$$

This proves the theorem.

## 3. $D_m$ and the Green's function of the unit disc

Let $\mathfrak{H}=\{z=x+iy\in\mathbb{C}\,|\,y>0\}$ denote the upper half-plane and for each positive integer $k$ define a function $G_k^{\mathfrak{H}}: \mathfrak{H}\times\mathfrak{H}\backslash(\text{diagonal})\to\mathbb{R}$ by

$$G_k^{\mathfrak{H}}(z,z') = -2Q_{k-1}\left(1+\frac{|z-z'|^2}{2yy'}\right) \qquad (z=x+iy,\ z'=x'+iy'\in\mathfrak{H}).$$

Here $Q_n(t)$ $(n\geqq 0)$ is the $n^{\text{th}}$ Legendre function of the second kind:

$$Q_0(t)=\frac{1}{2}\log\frac{t+1}{t-1}, \qquad Q_1(t)=\frac{t}{2}\log\frac{t+1}{t-1}-1, \qquad Q_2(t)=\frac{3t^2-1}{4}\log\frac{t+1}{t-1}-\frac{3}{2}t$$

and in general $Q_n(t)=P_n(t)Q_0(t)-R_n(t)$ where $P_n(t)$ and $R_n(t)$ are the unique polynomials of degree $n$ and $n-1$, respectively, making $Q_n(t)\sim\frac{2^n n!^2}{(2n+1)!}t^{-n-1}$ for $t\to\infty$. The function $G_k^{\mathfrak{H}}$ is real-analytic on $\mathfrak{H}\times\mathfrak{H}\backslash(\text{diagonal})$, has a singularity of type

$$G_k^{\mathfrak{H}}(z,z')=\log|z-z'|^2+\text{continuous} \qquad (z'\to z)$$

along the diagonal, and satisfies the partial differential equation $\Delta_z G_k^{\mathfrak{H}}=\Delta_{z'} G_k^{\mathfrak{H}} = k(1-k)G_k^{\mathfrak{H}}$, where $\Delta_z=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ denotes the hyperbolic Laplace operator. Moreover, by virtue of the defining property of $Q_{k-1}$, it is small enough at infinity that the series

$$G_k^{\mathfrak{H}/\mathbb{Z}}(z,z')=\sum_{n=-\infty}^{\infty} G_k^{\mathfrak{H}}(z,z'+n)$$

converges and has properties similar to those of $G_k^{\mathfrak{H}}$, but now with $z$ and $z'$ in $\mathfrak{H}/\mathbb{Z}$. This "Green's function" is studied [in connection with the analogously defined functions $G_k^{\mathfrak{H}/\Gamma}$, where $\Gamma$ is a subgroup of finite index in $PSL(2,\mathbb{Z})$] in [8] and is shown there to be closely related to Ramakrishnan's modified polylogarithm function. We content ourselves with stating the result, referring to [8] for the proof.

**Theorem 2.** *Let $k\in\mathbb{N}$, $z=x+iy$, $z'=x'+iy'\in\mathfrak{H}$. Then*

$$G_k^{\mathfrak{H}/\mathbb{Z}}(z,z')=\sum_{n=1}^{k} f_{k,n}(2\pi y, 2\pi y')\left[D_{2n-1}(q/q')-D_{2n-1}(q\bar{q}')\right],$$

*where $q=e^{2\pi i z}$, $q'=e^{2\pi i z'}$ and*

$$f_{k,n}(u,v)=2^{1-2k}(uv)^{1-k}\sum_{\substack{r,s\geq 0\\ r+s=k-n}}\frac{(2k-2-2r)!}{r!(k-1-r)!}\,\frac{(2k-2-2s)!}{s!(k-1-s)!}u^{2r}v^{2s}.$$

Note that the symmetry of $G_k^{\mathfrak{H}/\mathbb{Z}}$ in its two arguments is reflected by the two symmetry properties $D_{2n-1}(x)=D_{2n-1}(x^{-1})=D_{2n-1}(\bar{x})$. The map $z\to q$ identifies $\mathfrak{H}/\mathbb{Z}$ with the punctured unit disc $\{q\in\mathbb{C}\,|\,0<|q|<1\}$, but the right-hand side of the formula in the theorem now makes sense for any $q, q'\in\mathbb{C}^\times$ (with $2\pi y, 2\pi y'$ replaced by $-\log|z|, -\log|z'|$) and represents some kind of Green's function on $\mathbb{C}^\times\times\mathbb{C}^\times$.

## 4. $D_m$ and special values of Dedekind zeta functions

The Bloch-Wigner dilogarithm function $D(x)$ is related in a very beautiful way to special values of Dedekind zeta functions. Specifically, we have the following theorem.

**Theorem 3.** *Let $F$ be an arbitrary algebraic number field, $d_F$ the discriminant of $F$, $r_1$ and $r_2$ the numbers of real and complex places $(r_1+2r_2=[F:\mathbb{Q}])$, and $\zeta_F(s)$ the Dedekind zeta function of $F$. Then $\zeta_F(2)$ is equal to $\pi^{2(r_1+r_2)}|d_F|^{-1/2}$ times a rational linear combination of $r_2$-fold products $D(x^{(r_1+1)})\dots D(x^{(r_1+r_2)})$ with $x\in F$.*

(Here $x^{(1)},\dots,x^{(r_1)}, x^{(r_1+1)},\dots,x^{(r_1+r_2)}, \overline{x^{(r_1+1)}},\dots,\overline{x^{(r_1+r_2)}}$ are the images of $x$ under the various embeddings $F\hookrightarrow\mathbb{C}$.)

This result was proved in [5] in a somewhat weaker form (it was asserted only that the $x$ could be chosen of degree $\leq 4$ over $F$, rather than in $F$ itself) by a geometric method: the value of $\zeta_F(2)$ was related to the volume of a hyperbolic $3r_2$-dimensional manifold (more precisely, a manifold locally isometric to $\mathfrak{H}_3^{r_2}$, where $\mathfrak{H}_3$ denotes hyperbolic 3-space) and this volume was then computed by triangulating the manifold into a union of $r_2$-fold products of hyperbolic tetrahedra whose volumes could be expressed in terms of the function $D(x)$. The more precise statement above comes from algebraic $K$-theory: the value of $\zeta_F(2)$ is related by a result of Borel to a certain "regulator" attached to $K_3(F)$, and this is calculated using results of Bloch, Levine, Suslin, and Mercuriev in terms of the Bloch-Wigner function. For details and references, see [4] or [7]. The $K$-theoretical proof in fact gives a somewhat stronger statement than the above theorem: the value of $|d_F|^{1/2}\zeta_F(2)/\pi^{2r_1+2r_2}$ is equal to an $r_2\times r_2$ determinant of rational linear combinations of values $D(x)$, rather than merely to a rational linear combination of $r_2$-fold combinations of such values.

As examples of Theorem 3, we have for $F=\mathbb{Q}(\sqrt{-7})$ ($d_F=-7$, $r_1=0$, $r_2=1$)

$$\zeta_F(2)=\frac{2^2\pi^2}{3\cdot 7^{3/2}}\left(2D\left(\frac{1+\sqrt{-7}}{2}\right)+D\left(\frac{-1+\sqrt{-7}}{4}\right)\right)$$

and for $F=\mathbb{Q}(\theta)$ with $\theta^3-\theta-1=0$ ($d_F=-23$, $r_1=r_2=1$)

$$\zeta_F(2)=\frac{2^3\pi^4}{3\cdot 23^{3/2}}D(\theta')=-\frac{2^2\pi^4}{3\cdot 23^{3/2}}D(-\theta'),$$

where $\theta'\left(=\frac{\theta}{2}\left(-1+\frac{i\sqrt{23}}{2\theta+3}\right),\text{ if }\theta\text{ is the real root}\right)$ denotes the conjugate of $\theta$ with $\Im(\theta')>0$.

We can now formulate

**Conjecture 1.** *Theorem 3 holds true for $\zeta_F(m)$ for all positive even $m$ with $\pi^{2(r_1+r_2)}$ replaced by $\pi^{m(r_1+r_2)}$ and with the function $D$ replaced by the function $D_m$. For $m$ odd a similar statement is true but with $\pi^{mr_2}$ instead of $\pi^{m(r_1+r_2)}$ and $D_m(x^{(1)})\dots D_m(x^{(r_1+r_2)})$ instead of $D_m(x^{(r_1+1)})\dots D_m(x^{(r_1+r_2)})$.*

The difference between the two cases $m$ even and $m$ odd is, on the one hand, that $D_m$ satisfies $D_m(\bar{x})=(-1)^{m-1}D_m(x)$ [so in particular $D_m(x)=0$ for $x$ real and $m$ even] and, on the other hand, that the order of vanishing of $\zeta_F(s)$ at $s=1-m$ for $m>1$ equals $r_2$ for $m$ even but $r_1+r_2$ for $m$ odd. Again we can make a more precise conjecture with an $r\times r$ determinant ($r=r_2$ or $r_1+r_2$) instead of simply a linear combination of $r$-fold products. Moreover, one can make a more general conjecture with Artin $L$-functions in place of Dedekind zeta functions. In particular, $\zeta_F(s)/\zeta(s)$ ($\zeta=\zeta_{\mathbb{Q}}$), which is a product of such $L$-series, should be a sum of $(r-1)$-fold products of values $D_m$. This statement makes sense also for $m=1$ and is true by the Dirichlet regulator formula (recall that $D_1$ is essentially the logarithm-of-the-absolute-value function), but even when $m=1$ the general conjecture for Artin $L$-series is unknown (Stark conjectures).

As a special case, we make the very specific

**Conjecture 2.** *Let $F$ be a real quadratic field. Then $|d_F|^{1/2}\zeta_F(3)/\zeta(3)$ is a rational linear combination of differences $D_3(x)-D_3(x')$, $x\in F$.*

Here $x'$ denotes the conjugate of $x$ over $\mathbb{Q}$. Note that $\zeta(3)=D_3(1)$, so this is a strengthening of Conjecture 1 in this case. As numerical examples, we give

$$\frac{\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}\overset{?}{=}\frac{2^5}{3\cdot 5^{5/2}}\left(3\left[D_3\left(\frac{1+\sqrt{5}}{2}\right)-D_3\left(\frac{1-\sqrt{5}}{2}\right)\right]\right.$$
$$\left.-[D_3(2+\sqrt{5})-D_3(2-\sqrt{5})]\right)$$

and

$$\frac{\zeta_{\mathbb{Q}(\sqrt{2})}(3)}{\zeta(3)} \stackrel{?}{=} \frac{3}{5\cdot 2^{5/2}}([D_3(4+2\sqrt{2})-D_3(4-2\sqrt{2})]$$
$$-9[D_3(2+\sqrt{2})-D_3(2-\sqrt{2})]$$
$$-6[D_3(1+\sqrt{2})-D_3(1-\sqrt{2})]+9[D_3(\sqrt{2})-D_3(-\sqrt{2})]),$$

both true to at least 25 decimals. (These relations were found empirically by using the Lenstra-Lenstra-Lovasz lattice reduction algorithm to search numerically for linear relations between $|d_F|^{1/2}\zeta_F(3)/\zeta(3)$ and selected values of $D_3(x)-D_3(x')$, $x\in F$.)

That the *quotient* $\zeta_F/\zeta_{\mathbb{Q}}$ should be connected with the *differences* $D_m(x)-D_m(x')$ is a special case of a "Galois descent" property which we expect to hold in general, and which is known for the case $m=2$ by the $K$-theoretical work already cited (cf. [4] for details). Roughly speaking, this property implies that the $\mathbb{Q}$-vector space spanned by the $x\in F$ occurring in the conjecture should be invariant under the group of automorphisms of $F$ over $\mathbb{Q}$ and that the value of an (abelian or Artin) $L$-function factor of $\zeta_F$ at $s=m$ should be the determinant of a matrix of combinations of $D_m(x)$ with $x$ in the corresponding subspace. An example of how this works is provided by the case when $F$ is abelian over $\mathbb{Q}$. Here the assertion of Conjecture 1 is easy if we allow the arguments $x$ to be in the abelian closure $N=\mathbb{Q}(\zeta_f)$ ($f=$conductor of $F$), rather than in $F$ itself: $\zeta_F$ factors into a product of Dirichlet $L$-series $L(s,\chi)$ with $r_1+r_2$ even and $r_2$ odd Dirichlet characters $\chi$ modulo $f$ (of course, either $r_1$ or $r_2$ is zero), and the value of $L(m,\chi)$ is an algebraic multiple of $\pi^m$ if $\chi(-1)=(-1)^m$ and an algebraic linear combination of values of $D_m(x)$, $x^f=1$, in the opposite case. This gives the statement with an algebraic rather than rational combination of products of $D$-values, but a little more work shows that the algebraic multiples occurring combine correctly to give a rational multiple of $|d_F|^{1/2}$. The point is now that the set of $x$ occurring, and the coefficients with which they occur, are invariant under the action of Gal$(N/F)$. For instance, in the above case $F$ real quadratic, $m=3$, $f=d_F$, we have

$$d_F^{1/2}\zeta_F(3)/\zeta(3)=f^{1/2}L\left(3,\left(\frac{d_F}{\cdot}\right)\right)=\sum_{n=1}^{f-1}\left(\frac{d_F}{n}\right)Li_3(e^{2\pi in/f})=\sum_{n=1}^{f-1}\left(\frac{d_F}{n}\right)D_3(e^{2\pi in/f}),$$

and the conjugates of $e^{2\pi in/f}\in N$ over $F$ are exactly the $e^{2\pi in'/f}$ with $\left(\frac{d_F}{n}\right)=\left(\frac{d_F}{n'}\right)$.

By analyzing the structure of the numerical examples of Conjectures 1 and 2, one can get a more precise conjecture which actually predicts which linear combinations of products of polylogarithm values must be used in order to get zeta-values. Using it, it is easy to produce as many (conjectural) formulas involving polylogarithms and zeta-values as desired. In many cases, these seem to be new even for $F=\mathbb{Q}$, e.g.

$$\tfrac{67}{24}\zeta(3)\stackrel{?}{=}6D_3(\tfrac{2}{3})+3D_3(\tfrac{3}{4})-3D_3(\tfrac{1}{2})-D_3(\tfrac{8}{9})-2D_3(\tfrac{1}{3})+D_3(-\tfrac{1}{3}).$$

We will discuss the various versions of this conjecture, and its relation to algebraic $K$-theory, in a later paper [9].

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